October 17th, 2015

Programme: Probabilistic Methods for Finance (Maggis/Burzoni)

- **Elements of Measure Theory (4 hours). Chapters 1,2,3 in [Wi91].** Algebra, σ-algebra, π system. The π-system $\pi(\mathbb{R})$ on $\mathbb{R}$, the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$; $\mathcal{B}_{\mathbb{R}} = \sigma(\pi(\mathbb{R}))$(*). Limit of a sequence of sets in terms of unions and intersections. Set functions and their properties: positivity, additivity, monotonicity, σ-additivity, finite measures, and σ-finite measures. Measurable space and measure space. Probabilities and elementary properties. Continuity from below and above. Equivalent condition for σ-additivity of a finite positive additive set function (*). Caratheodory extension Theorem. Distribution and probabilities on $\mathbb{R}$: a characterization (**). The Lebesgue measure on $([a,b], \mathcal{B}_{[a,b]})$ and $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Random variables. σ-algebra generated by random variables. Random variables on a measurable space generated by a finite partition (*). Measurability with respect to different σ-algebra. Operation with random variables. Distribution of a random variable, distribution function and density.

- **Lebesgue integral (8 hours).** The space $L^0(\Omega, \mathcal{F}, \mathbb{P})$, pointwise convergence, almost sure convergence, convergence in probability. The integral of a random variable and the expectation operator: from simple random variables to the general case. Properties of the expectation. Monotone Convergence and Dominated Convergence Theorems. The space $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and its norm. The integral on a set. Absolutely continuous meausres/probabilities and equivalent measures/probabilities (relation with No Arbitrage). The Radon-Nikodym Theorem. The computation of the expectation $E_{\mathbb{P}}[X]$ via the induced probability P_X (*). Independence of events/σ-algebras/random variables. Conditional probabilities. Relation between independence and expectations: the expectation of the product of two independent integrable random variables. Variance and Covariance. The space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and its norm. The space $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $p \in [1, +\infty]$ and their norms. Chebyshev Inequality (*) and weak law of large numbers, Jensen Inequality, Holder inequality, Minkowscki inequality. Properties of the norm and inclusion among L^p spaces.

- **Stochastic Processes (8 hours).** Conditional expectation: example of the binomial model and for finite σ-algebras. The Kolmogorov Theorem on the conditional expectation of an integrable $\mathcal{F}$-measurable random variable with respect to the sigma algebra $\mathcal{G} \subseteq \mathcal{F}$. Its proof via the Radon-Nikodym Theorem. Idea of the relation between conditional expectation and projection theorem in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Ten properties (*) of the conditional expectation. Filtration. Stochastic processes, adapted processes, trajectories of a process. Filtration generated by a process: examples and interpretation. Martingales, submartingales and supermartingales. Equivalent formulations of the condition $E_{\mathbb{P}}[M_t \mid \mathcal{F}_s] = M_s$(*). Examples of martingales: the sum of independent and zero mean r.v. (*); the product of independent r.v. with mean 1 (*); the conditional expectation process (*). The process $f(M)$ is a sub/super martingale if M is a martingale and f is convex/concave (*). Predictable processes and interpretation. The martingale transform $M_n = \sum_{i=1}^{n} H_i(S_i - S_{i-1})$ is a martingale (*) and interpretation. The Doob decomposition of a supermartingale as the sum of a martingale and a predictable decreasing process in the discrete time case (*). The Gaussian distribution. The Brownian

Motion W: definition and properties. W is a martingale (*); $W_t^2 - t$ is a martingale (*); the exponential martingale $Z_t = e^{-\frac{1}{2}\theta^2 t + \theta W_t}$(*).

- **Stochastic Calculus (10 hours).** Finite variation $V_{[0,T]}(f)$ of a function $f : [0,T] \to \mathbb{R}$. If f is differentiable then $V_{[0,T]}(f) = \int_0^T f'(s)ds$. Introduction to the Riemann-Stieltjes Integral. The integral with respect to a distribution function. The quadratic variation of a function $f : [0,T] \to \mathbb{R}$. If $f \in C[0,T] \cap BV[0,T]$ (or f is differentiable) then $V^2_{[0,T]}(f) = 0$ (*). The Riemann-Stieltjes stochastic integral: the ω by ω definition of the stochastic integral with respect to a process of finite variation. How to define $\int_0^T W_s dW_s$???. The quadratic variation of W (*). The trajectory of W are not of finite variation ($\mathbb{P}$ a.s.) and are not differentiable (*). Introduction to the Ito integral and financial interpretation. Construction of the Ito integral for simple processes: properties (*). The space $L^2([0,T]\times\Omega, \mathcal{B}_{[0,T]}\otimes\mathcal{F}_T, Leb\otimes\mathbb{P})$. Extension of the Ito integral to adapted processes in $L^2([0,T] \times \Omega, \mathcal{B}_{[0,T]} \otimes \mathcal{F}_T, Leb \otimes \mathbb{P})$. Properties of the Ito integral and the Ito isometry. The stochastic integral is a martingale (**). Ito processes: $dX_t = \mu_t dt + \sigma_t dW_t$. Ito's formula for $df(t, X_t)$. Examples. The martingale representation theorem (with respect to the brownian filtration) and interpretation in relation to complete markets. Exponential martingale: Novikov Lemma and Girsanov Theorem. The Black and Scholes model: finding the equivalent martingale measure. The pricing of a call option. Stochastic Differential Equations.

(the (*) means compulsory proofs, whereas the (**) means that the proof is only sketched.

References

[Wi91] Williams D., *Probability with martingales*, Cambridge University Press, 1991.