Bargaining and Competition in Small Markets
(very preliminary and incomplete)

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Abstract

This paper studies bargaining and competition in decentralized small dynamic markets. In our model, buyers and sellers arrive over time and make offers to each other. We characterize equilibria in the limit where bargaining frictions disappear, which exhibit Bertrand competition when the market is unbalanced and Rubinstein-like bargaining when the market is balanced. When the market grows, the transaction price converges to a “competitive price” proportional to the ergodic probability of the market having an excess demand. We investigate a limit of our model where the price process becomes a diffusion process, which allows us to recover some stylized empirical facts observed in many financial markets.

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1 Introduction

Many markets are characterized by being decentralized and having a relatively small but permanently changing number of traders in them. Examples include housing/rental markets in given locations, resale markets for durable goods with limited supply, job markets for specific occupations or markets for some financial assets. In such markets, prices are typically determined (or can be renegotiated) by offers and counter-offers in bilateral encounters between traders. As rejecting offers delays trade, each agent in the market is typically endowed with some bargaining power. Also, the option that each agent has to wait and meet other trading counterparties generates endogenous competition between agents within each side of the market.

This paper studies how bargaining and competition interact in a decentralized dynamic market with arrival of both buyers and sellers. In our model, sellers own one unit of a good and buyers have a unit demand. Once in the market, buyers and sellers stochastically meet and make price offers to each other until they trade. The arrival process is allowed to depend both on the number of each kind of traders in the market, and on an observable time-evolving state of the economy.

The model features a unique Markov equilibrium where, whenever trade is possible, it occurs, and we characterize it when the frequency with which agents in the market meet becomes increasingly high. In such an equilibrium, there is trade in every meeting between a seller and a buyer, since it is not possible that both traders are better off by delaying trade. When the market is unbalanced, there is Bertrand competition between agents on the long side of the market: the transaction price equals their (endogenous) continuation value. In this case, the threat of an agent on the short side of the market to wait for another meeting makes agents on the long side of the market behave competitively. When, instead, the market is balanced, the agents in the market engage in a stochastic Rubinstein-like bargaining game, with endogenous outside options given by the possibility of arrival of new agents. In this case, each agent obtains her continuation value from waiting plus a fraction of the remaining surplus from trade, which is proportional to the frequency with which agents on his or her side of the market make offers.

We provide a simple characterization of the continuation value of the agents in the market: it can be written as the payoff of a fictitious agent who obtains a flow payoff equal to all surplus from trade when he or she is on the short side of the market, and a fraction of it when the market is balanced. We use this result to analyze the effect that enlarging markets has on the price process. We do that by replicating our original market an increasing number of times which, in our model, corresponds to increasing proportionally.
the arrival rates of both sellers and buyers or, analogously, lowering their discount rate. We show that, in this limit, the price of the market is equal to a constant “competitive price”, which is proportional to the probability of the market having excess demand.

We analyze the large-market limit of our model, that is, the limit where the arrival rates of sellers and buyers into the market per unit of time become similar and big. As the frequency of arrival of traders in the market increases, the transaction price process approaches a diffusion process, with a drift pointing toward the price of a balanced market. The outcome of the model becomes “protocol-free” in this limit, that is, probabilities of making offers by the different agents in the market become irrelevant in determining the market price. Furthermore, we obtain that, in this limit, the ergodic average transaction price of the market is equal to the competitive price.

We apply our model to study a tractable market where buyers are short-lived and the arrival rate of both sellers and buyers is constant. In this case, the transaction price process approaches a geometric Brownian motion, with an ergodic distribution given by a truncated power-law function. Such an ergodic distribution exhibits “fat tails” (fatter than a Pareto distribution) before the truncation. Depending on the parameters of the model, the dynamics of prices may involve a high average price with infrequent “crashes” given by sharp drops in the price, or an average low price with infrequent “booms” given by sharp increases in the price. The volume of trade is positively related with changes in the price, since arrivals of buyers increase both, while the arrival of sellers only decreases the price. These findings are consistent with numerous empirical studies in financial markets, highlighting that stochastic demand and supply may be one of the causes driving them.

The organization of the paper is as follows. After this introduction, we review the literature related to our paper. Section 2 introduces our model and characterizes its unique Markov perfect equilibrium with trade. In Section 3 we analyze the limit of our model where the average number of traders in the market is increasingly large. Finally, Section 4 concludes. An appendix provides the proofs of the results in the previous sections.

1.1 Literature Review

The paper in the literature closest to ours is Taylor (1995). The paper analyzes a centralized market where buyers and sellers arrive over time at constant rates. In every period, agents on the short side of the market make offers, while in a balanced market each side makes an offer with probability $\frac{1}{2}$. The author characterizes the set of pure-strategy equilibria of the model. Coles and Muthoo (1998) consider a similar model where buyers and sellers arrive
in pairs, and they allow for heterogeneity in both buyers and goods. We deviate from their analysis by considering a decentralized market with random matching, and allowing for a general arrival process. This allows us to analyze the effect that the arrival process and bargaining asymmetries have on the price in the market, the properties of the ergodic price distribution, and the limit where the market is infinitely replicated. We further consider the limit where the market becomes large, which allows us to analyze the case where buyers and sellers differ in their discount and exit rates, and show that the stochastic arrival of traders in the market may help to explain, in part, some stylized empirical facts.

Our paper is also related to the literature on matching markets with arrival of buyers and sellers. Wolinsky (1988) considers a model where buyers’ values are redrawn from the same distribution whenever they face a new seller. De Fraja and Sákovics (2001) examine a similar dynamic market where goods are homogeneous. Satterthwaite and Shneyerov (2007, 2008) show that in the presence of persistent frictions the market converges to the static Walrasian equilibrium.

## 2 The Model

Time is continuous, $t \in \mathbb{R}_+$, with an infinite horizon. There is an infinite number of potential (male) buyers and (female) sellers. At a given moment in time, there are $B_t$ buyers and $S_t$ sellers in the market. The state (of the market) is defined to be $(B_t, S_t)$.

**Arrival process.** A buyer arrives in the market at a rate $\gamma_b(B_t, S_t)$, and a seller arrives in the market at a rate $\gamma_s(B_t, S_t)$. The total rate at which the state is exogenously updated is denoted $\gamma \equiv \gamma_b + \gamma_s$, and it is assumed to be bounded.

**Bargaining.** At a Poisson arrival rate $\lambda(B_t, S_t)$ nature selects one of the buyers and one of the sellers at random. Within the selection, the probability that the seller is chosen to make the offer is $\xi(B_t, S_t) \in (0, 1)$.

**Information.** We assume that the traders in the market observe the whole total previous history\(^1\).

**Strategies.** The strategy an agent (buyer or seller) is a map from the total history $h^t$ to a price offer distribution in $\Delta(\mathbb{R}_+)$ and an acceptance decision $\alpha : \mathbb{R}_+ \rightarrow [0, 1]$.

**Payoffs.** Both buyers and sellers discount the future at rate $\rho$. If a buyer and a seller trade at time $t$ at price $p$ they obtain, respectively, $e^{-\rho t} p$ and $e^{-\rho t} (1 - p)$. If they never

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\(^1\)We will focus in Markov perfect equilibria, which remain equilibria even the agents have less information as long as the current state of the market is known to them.
trade they both obtain 0. Both buyers and sellers are risk-neutral and expected-utility maximizers.

2.1 Equilibrium Concept

We restrict our attention to Markov equilibria where the strategies of the agents only depend on the state of the market. We further focus on “trade equilibria”, that is, on equilibria where within a meeting between a buyer and a seller, if there is an offer which both traders (weakly) prefer to not trading, trade happens with probability one.

Definition 2.1. A Markov perfect equilibrium (MPE) is an offer/acceptance decision by each of the type of traders which only depends on the current state of the market, \((B_t, S_t)\). An MPE is a trade equilibrium if, within each match, whenever there is a price acceptable for both the seller and the buyer, an acceptable price is offered and accepted for sure.

From now on we focus on trade MPEs, so we call them just equilibria.

2.2 Preliminary Analysis

Fix a state \((B_t, S_t)\). If \(B_t > 0\), we use \(V^b(B_t, S_t)\) to denote the continuation value of the buyers in the market, and if \(S_t > 0\), the continuation value of the sellers.

The first result establishes that, independently of the arrival process, there is always trade when two traders meet. The intuition is simple: waiting is costly for both trading counter-parties and, on expectation, they cannot both get better off from waiting, so there are acceptable prices in every meeting.

Lemma 2.1. There is always trade.

Proof. In this proof, as in the rest of the proofs, when the state of the market \((B, S)\) is clear, it is notationally convenient to not the dependence of the entry process on it. In particular, we use \(\gamma_b \equiv \gamma_b(B, S)\) and \(\gamma_s \equiv \gamma_s(B, S)\) to denote the rate at which a buyer and a seller arrives at state, respectively, and \(\gamma \equiv \gamma_b + \gamma_s\).

Let’s define, for each state \((B, S) \in \mathbb{N}^2\), \(V(B, S) \equiv V^b(B, S) + V^s(B, S)\). Assume, for the sake of contradiction, that \(V^* \equiv \sup_{(B, S) \in \mathbb{N}^2} V(B, S) > 1\). Let \((B, S)\) be such that \(V(B, S) > 1\), so there is no trade at \((B, S)\). Then, we have that

\[
V(B, S) \leq \frac{\gamma}{\gamma + \rho} E_\gamma[V(B', S')] \leq \frac{\gamma}{\gamma + \rho} \bar{V}^*,
\]
where $E_\gamma[\bar{V}(B', S')]$ is the expected value of $\bar{V}$ if the state changes due to an arrival of an agent into the market. This is a contradiction, since the supremum on the right hand side is bounded away from $\bar{V}^*$ for any pair $(B, S)$ (since $\gamma$ is bounded), while the left hand side can be chosen arbitrarily close to it.

Notice now that if a buyer and a seller meet when the state is $(B, S) \in \mathbb{N}^2$ and $\bar{V}(B, S) < 1$, then there is trade for sure. Indeed if, for example, the buyer is chosen to make an offer, any price in $(V_s(B, S), 1 - V_b(B, S))$ is accepted for sure by the seller, and is strictly preferred by the buyer to making an acceptable offer. The usual argument for take-it-or-leave-it offers implies that, in any equilibrium, if $\bar{V}(B, S) < 1$ and two traders meet, the trader making the offer offers the continuation value of the trading counter-party, and the trading counter-party accepts the offer for sure.

Finally, if a buyer and a seller meet when the state is $(B, S) \in \mathbb{N}^2$ and $\bar{V}(B, S) = 1$, then both traders are willing to trade at a price equal to $V_s(B, S)$, so they trade with probability one in a trade MPE.

An immediate consequence of Lemma 2.1 is that the population dynamics is independent of the equilibrium considered. Indeed, the number of buyers and sellers in the market is uniquely determined by the number of previous arrivals of each kind of trader and the number of meetings between them. The uniqueness of the population dynamics can be used to prove the uniqueness of equilibrium:

**Proposition 2.1.** There is a unique equilibrium.

**Proof.** Assume that there are two equilibria, and let us index the continuation payoffs as $(V^i_b, V^i_s)$, $i = 1, 2$. Let’s define, for all $N \in \{0\} \cup \mathbb{N}$,

$$\Delta_N \equiv \sup \left\{ \max \{|V^1_b(B, S) - V^2_b(B, S)| + |V^1_s(B, S) - V^2_s(B, S)|\} \mid B + S \leq N \right\}.$$ 

Assume, for the sake of contradiction, that $\Delta_N > 0$ for some $N$. Then, we have that

$$V^1_b(B, S) - V^2_b(B, S) = \frac{\lambda/B \left( \xi (V^2_s(B, S) - V^1_s(B, S)) + (1-\xi) (V^1_b(B, S) - V^2_b(B, S)) \right)}{\lambda + \gamma + \rho} + \frac{\lambda(B-1)/B \left( V^1_s(B-1, S-1) - V^2_s(B-1, S-1) \right)}{\lambda + \gamma + \rho} + \frac{\gamma}{\lambda + \gamma + \rho} \mathbb{E}[V^1_s(B', S') - V^2_s(B', S')] .$$

We can obtain a similar expression for $V^1_s(B, S) - V^2_s(B, S)$. Adding the two expressions, we have that

$$\Delta_N \leq \frac{\bar{\gamma}}{\bar{\gamma} + \rho} \Delta_{N+1} \leq \left( \frac{\bar{\gamma}}{\bar{\gamma} + \rho} \right)^{N'} \Delta_{N+N'} ,$$

where $\bar{\gamma}$ be an upper bound on $\gamma$. Since $\Delta_{N+N'} \leq 1$ for all $N'$, we have a contradiction. $\square$
2.3 Frictionless Limit

We now consider the limit where there the bargaining frictions disappear, that is, where the buyers and sellers in the market meet at a very high rate. This limit smoothes bargaining and permits us to study how the continuation payoff depends on the arrival process and bargaining protocol. As we know from Rubinstein (1982), the outcome from bargaining is not trivial in this limit: the threat of rejection is important on determining how surplus is split even when the cost waiting is small.

To take the limit where frictions disappear, we fix some function $\ell : \mathbb{N}^2 \rightarrow \mathbb{R}_{++}$ and $\bar{\lambda} \in \mathbb{R}_{++}$ such that $\lambda(B_t, S_t) \equiv \bar{\lambda} \ell(B_t, S_t)$. Then, fixing $\ell$, we let $\bar{\lambda}$ increase, and we compute the limit of the continuation values (and therefore equilibrium strategies) of the agents. The following proposition establishes that, when the market is imbalanced, agents on the short side of the market obtain all surplus from trade net of continuation value:

**Proposition 2.2** (Bertrand competition). If $B, S > 0$ and $B \neq S$ then, in the frictionless limit,

$$(V_s(B, S), V_s(B, S)) = \begin{cases} (V_b(B-S, 0), 1-V_b(B-S, 0)) & \text{if } B > S, \\ (1-V_s(0, S-B), V_s(0, S-B)) & \text{if } B < S. \end{cases}$$

If $B, S > 0$, the transaction price equals $V_s(0, S-B)$.

If the market is imbalanced, as bargaining frictions disappear, the trade outcome converges to the one of a competitive market: the agents on the long side of the market obtain their (endogenous) reservation value. The result follows from Lemma 2.1, as an agent on the long side of the market (for example a buyer) has the option of not trading and letting the market clear and obtain his or her continuation value (in this case obtain $V_b(B - S, 0)$). Such an agent cannot obtain a higher payoff since, by Lemma 2.1, agents on the short side of the market (for example sellers) have the option of wait until they is no other agent on their side of the market, and then offer the reservation price of their trade counter-parties.

**Equilibrium Payoffs**

The following result characterizes the continuation values of the agents in the market.

**Proposition 2.3.** In the frictionless limit,

$$V_s(B_0, S_0) = 1-V_b(B_0, S_0) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} (\mathbb{I}_{B_t > S_t} + \xi(1, 1) \mathbb{I}_{B_t = S_t}) \rho \, dt \right]. \quad (2.1)$$
Proof. Let, with some abuse of notation, \( V_b(B - S) \) denote the payoff of the buyers in the state \((B, S)\). To first obtain \( V_b(0) \), assume that \( B = S = 1 \), that is, there are only one buyer and one seller in the market. In this case, they bargain “à la Rubinstein”, taking in account the possibility of arrival of new agents. As Proposition ?? in the Appendix shows, the payoff for the buyer in this case is given by

\[
V_b(0) = V_b(1, 1) = (1 - \xi) \rho + \frac{\gamma_b}{\gamma_b + \gamma_s + \rho} V_b(1) + \frac{\gamma_s}{\gamma_b + \gamma_s + \rho} V_b(-1) .
\]

where \( \gamma_b \) and \( \gamma_s \) are, respectively, the probabilities of the arrival of a buyer and a seller in state \((1, 1)\). The first term on the right hand side of the previous expression plays the role of the usual Rubinstein payoff for bargaining (in the limit without bargaining frictions): the buyer gets a payoff proportional to the likelihood with which he makes the offer. The other two terms are related to the fact that, by rejecting an offer, not only there is some discounting, but also there is the possibility that a new trader will arrive.

When \( B > S \), buyers are indifferent on trading now or rejecting the offer and waiting. This implies that the equation that \( V_b \) follows in this case is given by

\[
V_b(B - S) = \frac{\gamma_b}{\gamma_b + \gamma_s + \rho} V_b(B - S + 1) + \frac{\gamma_s}{\gamma_b + \gamma_s + \rho} V_b(B - S - 1) .
\]

Finally, if \( 0 < B < S \), we can use Proposition 2.2 to obtain the following equation

\[
V_b(B - S) = \frac{\rho}{\gamma_b + \gamma_s + \rho} + \frac{\gamma_b}{\gamma_b + \gamma_s + \rho} V_b(B - S + 1) + \frac{\gamma_s}{\gamma_b + \gamma_s + \rho} V_b(B - S - 1) .
\]

Notice that \( V_b(\cdot) \) satisfies the same equation as the continuation value of a fictitious infinitely-lived agent who receives a payoff of 1 whenever \( B > S \), \( 1 - \xi \) when \( B = S \) and 0 when \( B < S \). Using the fact that \( V_s = 1 - V_b \) we have that our result holds.

Proposition 2.3 implies that the equilibrium price offer is independent of the probability with which sellers make offers, \( \xi \), except for its value when exactly one seller and one buyer are in the market, \( \xi(1, 1) \). This is intuitive: when the market is imbalanced, there is Bertrand competition on the long side of the market. Second, the only statistic important for determining price is the evolution of the sign of the net amount of sellers (or buyers) in the market. Intuitively, the “intensity” of competition between agents on the long side of the market is irrelevant for determining the price when the market is unbalanced: the price equals their reservation value independently of the number of them. Finally, notice that the continuation value of an agent does not depend the future transactions, but on the state of the market, even when such an state is such that there is no trader in the market. This indicates how the outside option of waiting shapes the bargaining power of traders: prices are determined by the continuation value that not trading provides to them.
Remark 2.1 (No Diamond’s paradox). Proposition 2.3 shows that, after having taken the limit of frictionless bargaining, the payoff of each agent in the market is bounded away from zero as long as the probability that his side of the market becomes the short side of the market is not zero. This may be surprising, since at it is well known from the Diamond (1971)’s paradox, in a bargaining model with one-sided offers, the side of the market making the offers obtains all trade surplus. In this case, the order of limits matters: we first take the frictionless limit, and then the limit of one-sided offers. This result does not hold if first we assume that \( \xi(B, S) \in \{0, 1\} \) for all \((B, S) \in \mathbb{N}^2\) and then we take the frictionless limit: in this case, an agent on the side of the market which makes all offers obtains all trade surplus.

Still, one could generalize bargaining protocol of our model by allowing matches of not only two agents, but more. In this case, for each state \((B, S)\), the meeting probability \(\lambda\) can be generalized to be a function from \(\{1, \ldots, B\} \times \{1, \ldots, S\}\) to \(\mathbb{R}_+\), where \(\lambda_{B,S}(B’, S’)\) now indicates the rate at which \(B’\) buyers and \(S’\) sellers meet. Analogously, \(\xi_{B,S}\) can be generalized to a function from \(\{1, \ldots, B\} \times \{1, \ldots, S\}\) to \([0, 1]\), where \(\xi_{B,S}(B’, S’)\) indicates the probability that the sellers make (simultaneous) offers when \(B’\) buyers and \(S’\) sellers meet. In this case, for example, if only sellers would make offers (\(\xi \equiv 1\)), they will Bertrand-compete when two or more of them meet a buyer, offering \(V_s(0, S - B)\) when \(S > B\).

Example 2.1 (no arrival of sellers). To illustrate Proposition 2.3, consider the case where no sellers enter in the market and where, at each state of the market, the arrival rate of buyers is positive. This may correspond, for example, to re-sale markets with a limited supply, such as limited editions (books, mobile phones, cars...) or housing units in housing projects in exclusive areas. Assume also, for simplicity, that initially there is no buyer in the market \(B_0 = 0\), and the initial number of sellers is \(S_0 > 0\).

When no sellers arrive, equation (2.1) can be written in terms of the (stochastic) time that it takes for the market to clear. Indeed, let \(\tau\) be the first time (i.e., infimum) where \(S_\tau = 0\). By Lemma 2.1, this corresponds to the first time when there is only one seller in the market and a buyer arrives. In this case, we have that when there is only one buyer and one seller in the market, the payoff of the seller (arising from the frictionless limit of the Rubinstein bargaining), is given by

\[
V_s(1, 1) = \frac{\gamma_b(1, 1)}{\gamma_b(1, 1) + \rho} = \frac{\xi(1, 1) \rho}{\gamma_b(1, 1) + \rho}. 
\]

As a result, we can write the payoff of a seller as \(V_s(0, S_0) = \frac{\gamma_b(1, 1) + \xi(1, 1) \rho}{\gamma_b(1, 1) + \rho} \mathbb{E}[e^{-\rho \tau}]\). The
transaction price at time $t$ if there are $S_t > 1$ sellers in the market equals $V_s(0, S_t - 1)$, so prices are increasing over time, highlighting that the outside option that sellers have to wait until they are monopolist becomes increasingly attractive.

**Replicating the Market**

We now consider the effect that enlarging the market has on the distribution of prices. This may correspond, for example, to the effect of webpages which gather information on local rental or housing prices. As a result, buyers can easily compare across markets, which may de facto transform them into a single market. Also, as we will see, this limit can be interpreted as lowering the discount-rate, that is, reducing the frictions which are generated in the model due to the fact that sometimes the market is imbalanced and, as a result, some traders have to wait for trading.

Before we analyze the limit of enlarging the market, we first present a result involving the continuation value of the sellers:

**Lemma 2.2.** The ergodic expected sellers’ continuation value is

$$\lim_{t \to \infty} \mathbb{E}[V_s(B_t, S_t)] = \lim_{t \to \infty} \mathbb{E}[\mathbb{I}_{B_t > S_t} + \xi(1,1) \mathbb{I}_{B_t = S_t}],$$

Notice that the long-run expected continuation value is independent of the discount rate. The reason is that, the average flow payoff of the fictitious agent used in the proof of Proposition 2.3 to obtain 2.3 is proportional to $\rho$ and, as a result, his or her expected continuation value is independent of it.

We fix an initial market, characterized by $(\gamma, \xi)$. For each $M > 1$, we replicate the market $M$ times, that is, we consider a market characterized which we $(M\gamma, \xi)$

2 As $M$ grows large, the arrival rate of both buyers and sellers increases, so the market evolves increasingly faster. The following result establishes the limit of the distribution of transaction prices:

**Proposition 2.4.** As $M \to \infty$, the ergodic distribution of the transaction prices of a market characterized by $(M\gamma, \xi)$ converges to a degenerated distribution at the price $p^* \equiv \lim_{t \to \infty} (\Pr(B_t > S_t) + \xi(1,1) \Pr(B_t = S_t)).$

Proposition 2.4 establishes that, as the market is replicated, the distribution of transaction prices converges to a degenerate “competitive” price. Such a competitive price is

2See Section 3.1 for a different limit where, as we change $M$, the arguments of $\gamma$ also change.
obtained by computing the long-run probability of finding the market in excess demand, and adding the probability that sellers make offers when the market is balanced.

The limit discussed in this section can be complementarily re-interpreted as, instead, taking the limit where agents get increasingly patient, that is, \( \rho \downarrow 0 \). This interpretation corresponds to lowering the frictions given by the fact that the market is sometimes unbalanced, so some of the traders have to wait to trade. So, when \( \rho \) is low even though each trader can wait (at a low waiting cost) to trade until he or she is on the short side of the market, the change in the price that this option gives is small.

## 3 Large Markets

We devote this section to studying the limit where the size of the market grows, that is, where the average number of traders in the market becomes, on average, increasingly large. In our model this corresponds to simultaneously increasing the arrival rate of both sellers and buyers (per unit of real time), but making them increasingly similar in relative terms.

The reason for studying the large-markets limit is twofold. First, it allows us to generalize our previous analysis and results to permitting sellers and buyers to have different discount rates, and to exogenously exit the market at different rates. This allows us to provide further insights on how the prices are determined by the arrival and exit processes and the agent’s payoffs. Second, it allows us to establish properties of the price process which are potentially testable. In particular, in Section \[\text{3.2}\] we obtain that our model can replicate some stylized empirical facts observed in different financial markets.

### 3.1 General Results

In this section we consider a general large market. To do this, we fix some Lipschitz-continuous functions \( \sigma : \mathbb{R} \to \mathbb{R}_{++} \) and \( \mu : \mathbb{R} \to \mathbb{R} \). For each \( K > 0 \) and each net state \( N \equiv S - B \in \mathbb{Z} \), we parametrize the rate at which the state increases by one unit (interpreted as the arrival of a seller) as \( \frac{1}{2} \sigma(N/K)^2 K^2 + \mu_s(N/K) K \), and the rate at which the state decreases by one unit as \( \frac{1}{2} \sigma(N/K)^2 K^2 + \mu_b(N/K) K \) (interpreted as the arrival of a buyer), and we define \( \mu \equiv \mu_s - \mu_b \). This specification ensures that, as the market grows, the price process does not become degenerated. Alternatively, an imbalanced arrival of traders would make the equilibrium price process degenerate towards 0 (if sellers arrive at a higher rate) or 1 (if buyers arrive at a higher rate).

The first result establishes that, in the limit where \( K \) increases, the number of traders
in the market follows a diffusion process:

**Lemma 3.1.** As \( K \) increases, \( n_t \equiv N_t/K \) approaches a continuous-time diffusion processes, \( \tilde{n}_t \), with drift equal to \( \mu(\tilde{n}_t) \) and with volatility equal to \( \sigma(\tilde{n}_t) \).

Most of the previous analysis carries on when the agents’ discount rates are different and we allow agents in the market at a certain rate (see the proof of Proposition 3.2). We can then generalize Proposition 2.3 to determine the continuation value of the sellers in the market for a net normalized state \( n_0 \) as follows. For any sequence pair of sequences \((K^i)_i\) and \((N^i_0)_i\) such that \( K^i \to \infty \) and \( N^i_0/K^i \to n_0 \), we have that:

\[
\lim_{i \to \infty} V_s(N^i_0) = \mathbb{E} \left[ \int_0^\infty \mathbb{I}_{\tilde{n}_t < 0} e^{\int_0^t \rho_{\theta(\tilde{n}_t)} \, dt'} \rho_{\theta(\tilde{n}_t)} \, dt \mid \tilde{n}_0 = n_0 \right],
\]

where \( \theta(n) = s \) if \( n > 0 \) and \( \theta(n) = b \) otherwise. As we see, the continuation payoff of a seller is decreasing in her discount rate \( \rho_s \), and increasing in the discount rate of the buyers, \( \rho_b \).

As \( K \) increases, the transaction price in each state \( N \) approximates \( V_s(N) \) (which is equal to \( V_s(N+1) \) if \( N > 0 \), and \( V_s(N-1) \) if \( N > 0 \)). So, we can write the transaction price as a function \( p \) of the normalized state \( n \) at which a transaction happens, and using standard stochastic calculus we have that \( p \) solves the following Hamilton-Jacobi-Bellman equation

\[
\rho_{\theta(n)} p(n) = \rho_{\theta(n)} \mathbb{I}_{n < 0} + \mu(n) p'(n) + \frac{1}{2} \sigma(n)^2 p''(n),
\]

satisfying \( \lim_{n \to \infty} p(n) = 0 \) and \( \lim_{n \to -\infty} p(n) = 1 \). Notice that the discount rate of the agents on the long side of the market is the one relevant to determine the price process.

Given that \( p(\cdot) \) is a strictly decreasing function, we can define \( \theta(\tilde{p}) \equiv s \) if \( \tilde{p} < p(0) \) and \( \theta(p) = b \) otherwise. We can then establish the following result about the drift of the price:

**Proposition 3.1.** The drift of the prices is \( \rho_{\theta(\tilde{p})} (\tilde{p}_t - \mathbb{I}_{\tilde{p}_t \geq p(0)}) \).

Our model predicts the price process’ drift points toward the price obtained when the market is balanced, \( p(0) \). This independent of the intensity with which agents arrive and leave in the market, or even the imbalance of their arrival/exit rates. The drift of the price is high when the price is not extreme, that is, when the market is likely to become

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3This is true path by path. For example, the contribution to the continuation value of the seller of a path where \( \tilde{n}_t > 0 \) if \( t \in (T_1, T_2) \) and \( \tilde{n}_t > 0 \) for \( t \in (0, T_1) \), \( (T_2, \infty) \), for \( 0 < T_1 < T_2 \) is \( e^{-\rho_s T_1} (1 - e^{-(T_2 - T_1) \rho_b}) \), the payoff of the seller clearly decreasing in \( \rho_s \), and increasing in \( \rho_b \).
balanced soon. When, instead, the price is extreme (which typically corresponds to a very unbalanced market), the drift of the price is small.

We finally establish that the ergodic average transaction price is independent of $\rho$.

**Proposition 3.2.** Assume $\rho_b = \rho_s = \rho$. Then, the expected price in the ergodic distribution of the market is independent of $\rho$.

Proposition 3.2 implies that the long-run average transaction price is equal to the competitive price. Indeed, Proposition 2.4 establishes that as $\rho$ decreases the ergodic distribution tends to a distributed degenerated at the competitive price. Our model then predicts that while changes in the interest rate affect the price distribution (when $\rho$ low is degenerated at competitive price, while when $\rho$ is large it is concentrated around 0 and 1), it has no effect on the expected transaction price.

**Constant Arrival**

As an example of the previous setting, consider the case where buyers and sellers arrive at constant rates, and they die at the same rate $\delta > 0$. In this case, the volatility of $\tilde{n}_t$ is constant, $\sigma(\tilde{n}_t) \equiv \tilde{\sigma}$, while its drift is $\mu(\tilde{n}_t) = \bar{\mu} - \bar{n}_t \delta$. In consequence, we have the following result regarding the ergodic price distribution:

**Corollary 3.1.** As $K$ increases, $n_t$ approaches a continuous-time diffusion processes, $\tilde{n}_t$, Ornstein-Uhlenbeck process with ergodic distribution is $\mathcal{N}(\frac{\mu}{\delta}, \frac{2\sigma^2}{\delta})$. As a result, the competitive price is equal to $p^* \equiv \Phi(-\frac{\mu}{2\tilde{\sigma}^2 \delta/\sigma})$, where $\Phi$ is the CDF of a normal distribution.

The previous corollary can be used to undertake some comparative statics on how the competitive price depends on the different parameters. If the excess of arrival of sellers, $\mu$, is large, the competitive price decreases. Conversely, a low arrival of traders (low $\bar{\sigma}$) in the market lowers the volatility of the size of the market and, as a result, makes the price degenerated, either towards 0 (when $\mu < 0$) or towards 1 (when $\mu < 0$). Finally, an increase in $\delta$ has a double effect: decreases in the same proportional amount both the average net number of sellers and its ergodic variance. So, since the standard deviation of the ergodic distribution of $\tilde{n}$ decreases less than its mean, we have that an increase in $\delta$ makes the competitive price closer to $\frac{1}{2}$.

Finally, let us illustrate the need for balancedness in the market in order to have a non-degenerate ergodic distribution. We do this by considering the limit where $\tilde{\sigma}$ gets small. This makes the arrival process becomes less uncertain and, as a result, more predictable:

**Corollary 3.2.** Assume $\mu < 0$. Then, as $\sigma \searrow 0$, $p(n) \to I_{n<0} + I_{n>0} (1 - n \delta/\mu)^{-\rho_s/\delta}$. 

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Corollary 3.2 establishes a result similar to the one obtained in Example 2.1 in the case of no arrival of new sellers. In both cases, the price equals to the expected time it takes to the market to have no buyer which, in the limit considered in Corollary 3.2, is a certain time.

3.2 Application: Large Market with Short-Lived Buyers

We now analyze the limit where the buyers are short lived. This limit is illustrative because as it allows us to obtain explicitly the ergodic distribution of the transaction prices in the market, and use it to obtain some comparative results. Also, this exercise allows us to recover some of the stylized empirical facts, and shed some light on how the arrival of traders in a market may help on explaining them.

The fact that buyers are short-lived can be interpreted as them having an alternative trading option outside the market. Indeed, it is not necessary for the analysis that $\rho_b$ is large, only that they have an outside option which is preferable to waiting in the market. For example, in financial markets, potential buyers of financial assets may have alternative investment opportunities (given by purchasing other assets). So, if a buyer arrives in the market and there is no seller willing to sell a given asset, she may opt for investing in another asset.

We take the limit of short-lived buyers by letting their discount rate, $\rho_b$, increase to infinity, and by increasing simultaneously their death rate. In this limit, the net supply in the market, $\tilde{n}_t$, is positive almost surely. Furthermore, the price (which equals the continuation payoff of a seller in the market) can be computed by as follows:

Lemma 3.2. Let $\tau$ be the first time $\tilde{n}_t$ hits 0. Then, for any $n > 0$, the price is given by $p(n) = \mathbb{E}[e^{-\rho_b \tau}\tilde{n}_0 = n]$.

From now on we restrict ourselves to the case where the arrival of both sellers and buyers is constant, and the death rate of the sellers is zero. Also, for notational convenience, we assume that $\mu_s = 0$, so $\mu = -\mu_b$. This allows us to characterize the processes governing both the number of sellers in the market and the price process in the following result:

Proposition 3.3. As $K$ increases, $n_t$ and $p_t$ approximate two continuous-time diffusion
processes, \( \tilde{n}_t \) and \( \tilde{p}_t \), respectively, characterized by the following stochastic equations:

\[
\begin{align*}
\text{d}\tilde{n}_t &= \begin{cases} 
\max\{0, -\mu_b \, dt + \sigma \, dB_t\} & \text{if } \tilde{n}_t = 0 \\
-\mu_b \, dt + \sigma \, dB_t & \text{if } \tilde{n}_t \neq 0 
\end{cases} \\
\text{d}\tilde{p}_t &= \begin{cases} 
\min\{0, \tilde{p}_t \rho_s \, dt + \tilde{p}_t \frac{\tilde{\sigma}^2}{\sigma} \, dB_t\} & \text{if } \tilde{p}_t = 1 \\
\tilde{p}_t \rho_s \, dt + \tilde{p}_t \frac{\tilde{\sigma}^2}{\sigma} \, dB_t & \text{if } \tilde{p}_t < 1 
\end{cases}
\end{align*}
\]

(3.1) (3.2)

where \( B_t \) is a standard one-dimensional Brownian motion on the canonical probability space, and where \( \tilde{\sigma}^2 \equiv \sqrt{\mu_b^2 + 2 \frac{\tilde{\sigma}^2}{\sigma}} - \mu_b \). If the normalized state of the market is \( \tilde{n}_t \), the price is \( \tilde{p}_t = e^{-\frac{\tilde{\sigma}^2}{\sigma} \tilde{n}_t} \).

The constant arrival and exit rate of agents in the market implies that the (normalized) number of sellers follows a random walk with negative drift truncated at 0. As a result, the transaction price converges to a truncated geometric Brownian motion with a positive drift. So, as the price increases, its expected change gets increasingly positive, and also it becomes more volatile.

**Comparative Statics**

Interpreting \( \rho_s \) as the (risk-free) interest rate, we can see the effect of an unexpected permanent increase in \( \rho_s \). If at some \( t \) there is an unexpected change from \( \rho_0 \) to \( \rho_1 \), with \( \rho_0 < \rho_1 \), the immediate effect is a jump down in the price. Indeed, given that \( n \) remains unchanged immediately after the jump up in \( \rho_s \), the sellers in the market discount more the time at which they can become a monopolist. The effect in the price process is an increase in the drift of \( \tilde{p} \) (interpreted as a high return in the asset) and a higher volatility of the asset (interpreted as higher risk). So, right after an increase in the interest rate, the market “crashes” (there is a sharp decrease in the price). Afterwards prices become lower on average and the asset becomes more risky but with yields a higher expected return.

Models where assets follow a geometric Brownian motion do not have an ergodic distribution of the asset price. In our case, instead, the price process is truncated by the valuation of the buyers, so the price of the asset is (normalized to be) always between 0 and 1. As a result, our model features an ergodic distribution of the transaction price, which is characterized in the following corollary (its proof is part of the proof of Proposition 3.3):

\[\text{It is easy to obtain from the explicit ergodic distribution of the price process in Corollary 3.3 that the expected price on the long run is equal to } \bar{p} \equiv \frac{2\mu_b}{2\rho_s + \tilde{\sigma}^2}, \text{ which is decreasing in } \rho_s. \text{ Also, both the ergodic expected return, } \rho_s \bar{p}, \text{ and the expected volatility, } \bar{p} \frac{\tilde{\sigma}^2}{\sigma} \text{ are increasing in } \rho_s.\]

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**Corollary 3.3.** As $K \to \infty$, if $\mu_b > 0$, the ergodic cumulative distribution function (CDF) for the transaction price converges to $F(p) = p^{2 \mu_b \tilde{\sigma}^2}$. 

Let’s analyze the skewness of the distribution of the transaction price established in the previous result. When $2 \mu_b = \tilde{\sigma}^2$ (which is equivalent to $\mu_b = \sqrt{\sigma^2 \rho_s / 4}$), the ergodic distribution of the transaction price is a uniform distribution in $[0,1]$, so high prices are as likely as low prices. If we increase the frequency of arrival of buyers ($2 \mu_b > \tilde{\sigma}^2$), the skewness of $F$ is negative, and the probability density function (PDF) of the price distribution is increasing. In this case, the prices in the market tend to be high, and the market clears frequently. Finally, when the arrival rate of buyers is low ($2 \mu_b < \tilde{\sigma}^2$) we have that the PDF of the price distribution is decreasing. As $\mu_b$ decreases towards 0, most of the mass is concentrated around 0, so the price tends to be low, with infrequent sharp increases in the price. Note that in this case the distribution has “fat tails”, that is, the rate at which the PDF decreases is a power function. 

We can finally analyze the correlation between volume of trade and the volatility of the price of the market.

**Corollary 3.4.** Fix $K > 0$ large enough. Then, the average amount of trade (parametrized by $\sigma^2$) is positively related with the volatility of the price. Furthermore, with high probability, the number of transactions is positively related the price of the good.

Let’s obtain some intuition about the first result in the previous corollary. Note that if the rates at which sellers and buyers arrive increase (i.e., if $\sigma^2$ increases) there is a direct effect on the volatility of the risk, since $\tilde{\sigma}^2$ is increasing in $\sigma^2$. Also, it is easy to see that, for example, doubling the “size” of the market (i.e. multiplying the arrival rates of sellers and buyers by 2) has the same effect on the price distribution as dividing the discount rate by 2. In this case, even though the market is bigger for a large $\sigma$, each seller benefits from the increase in the volatility of the number of sellers in the market, since the market also clears more frequently. So, since the PDF of the transaction price shifts to the right, this also increases the volatility of the price.

The intuition about the second result of Corollary 3.4 is the following. In our model, there is a transaction when a buyer arrives and there is at least one seller in the market. So, the arrival of a buyer implies an increase in both the price and the number of transactions.

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5Note that the tails (before they are truncated) are “fatter” than the standard Pareto distribution, where the PDF is $f(x) \sim x^{-\nu}$ with $\nu > 1$, while in our case $\nu = 1 - \frac{2 \mu_b \tilde{\sigma}^2}{\sigma^2} \in (0,1)$ (actually, $1/p$ follows a Pareto distribution with parameter $\frac{2 \mu_b \tilde{\sigma}^2}{\sigma^2} + 1 > 1$). In this case, the price of the assets in our model is derived from the frequency of “rare” events which generate high increases the price.
If a seller arrives, instead, the price decreases and there is no transaction. Only when buyers and sellers arrive in the same period (which happens with very low probability), the number of transactions increases while the price remains constant.

**Evidence**

The most usual explanation for the random behavior of the asset price in some financial markets is the efficient market hypothesis, which claims that such behavior is mostly due to the arrival of new information. Our model shows that this stochastic behavior may arise also from underlying state variables of the market like the number of traders active at each moment in time.

We establish that when the price is lower than its maximum, it follows a geometric Brownian motion, which is widely used to approximate the evolution of a high range of financial asset prices. As we mentioned before, the fact that our price process is truncated implies that there is an ergodic distribution of prices. Our model can be extended to allow for correlation in the arrival of buyers or dependence on the arrival probability and the price could potentially generate fat tails also in the returns (i.e., changes in the price) of the asset.

We find a positive relationship between the (expected or realized) amount of trade in the market and the change (or volatility) in the price of the asset. This relationship is consistent with empirical evidence, documented among others by French and Roll (1986), Karpoff (1987) and Gallant, Rossi, and Tauchen (1992). The standard rationale in the market microstructure literature (see O’Hara, 1997 for the review of this approach) for this relationship comes from the existence private information (which is revealed through the strategies via the amount of trade). In our model, high trade volumes are associated with decreases in the supply of assets, which increases the price.

*Remark 3.1 (short-lived sellers).* In this section we treat buyers and sellers asymmetrically, assuming that the first are short-lived and the second long-lived. It is easy to see that the same analysis can be carried out if their roles are reversed, that is, making buyers long-lived and sellers short-lived. Still, some of the analogous comparative results would have an inverted sign: increases in the discount rate would increase prices, and the average amount of trade would be negatively related with the changes in price and its volatility. This indicates that sellers may tend to remain longer in the market and, as a result, be more likely to be on the long side of the market. The is consistent with our interpretation

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6See, for example, Cuthbertson and Nitzsche (2005, ch. 3).
of short-livedness of the buyers as having access to an outside option (like buying other financial assets), while the seller of an asset can only sell this asset.

4 Conclusions

Our paper analyzes markets characterized by being decentralized and by having, at each moment in time, few active traders. In such markets, each agent enjoys some market power and, since payoff from rejecting offers depends on the likelihood of finding another trading counter-part, the expectations about the future evolution of the market conditions and the timing of the sale play a crucial role on determining the transaction price.

As we have shown, under a general arrival process, the continuation value of the buyers (and the price as a result) can be characterized by the expected discounted time where there is an excess demand in the market. This allows us to prove a convergence result: in the limit where the market is replicated an increasing number of times the price is constant and equal to a competitive price, given by the ergodic probability of the market having more sellers than buyers. Our results show that some of the regularities observed in some markets can be in part explained, in part, without the need to incorporate asymmetric information or heterogeneity of agents in the market.

The robustness of our results or their generalizations to extensions of our model are left future research. An interesting direction may be, for example, allowing for buyer and/or seller heterogeneity which, as Elliott and Nava (2016) show in a static setting, may lead to random bargaining outcomes even in the frictionless limit.

5 Appendix

Under construction.

References


