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On efficiency of mean-variance based portfolio selection in DC pension schemes

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Abstract

We consider the portfolio selection problem in the accumulation phase of a defined contribution (DC) pension scheme. We solve the mean-variance portfolio selection problem using the embedding technique pioneered by Zhou and Li (2000) and show that it is equivalent to a target-based optimization problem, consisting in the minimization of a quadratic loss function. We support the use of the target-based approach in DC pension funds for three reasons. Firstly, it transforms the difficult problem of selecting the individual's risk aversion coefficient into the easiest task of choosing an appropriate target. Secondly, it is intuitive, flexible and adaptable to the member's needs and preferences. Thirdly, it produces final portfolios that are efficient in the mean-variance setting.

We address the issue of comparison between an efficient portfolio and a portfolio that is optimal according to the more general criterion of maximization of expected utility (EU). The two natural notions of Variance Inefficiency and Mean Inefficiency are introduced, which measure the distance of an optimal inefficient portfolio from an efficient one, focusing on their variance and on their expected value, respectively. As a particular case, we investigate the quite popular classes of CARA and CRRA utility functions. In these cases, we prove the intuitive but not trivial results that the mean-variance inefficiency decreases with the risk aversion of the individual and increases with the time horizon and the Sharpe ratio of the risky asset.

Numerical investigations stress the impact of the time horizon on the extent of mean-variance inefficiency of CARA and CRRA utility functions. While at instantaneous level EU-optimality and efficiency coincide (see Merton (1971)), we find that for short durations they do not differ significantly. However, for longer durations – that are typical in pension funds – the extent of inefficiency turns out to be remarkable and should be taken into account by pension fund investment managers seeking appropriate rules for portfolio selection. Indeed, this result is a further element that supports the use of the target-based approach in DC pension schemes.

Keywords. Mean-variance approach, efficient frontier, expected utility maximization, defined contribution pension scheme, portfolio selection, risk aversion, Sharpe ratio.

JEL classification: C61, D81, G11, G23.

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1 Introduction

The crisis of international Pay As You Go public pension systems is forcing governments of most countries to cut drastically pension benefits of future generations and to encourage the development of fully funded pension schemes. It is well-known that the reforms undertaken in most industrialized countries give a preference towards defined contribution (DC) plans rather than defined benefit (DB) plans. Thus, defined contribution pension schemes will play a crucial role in the social pension systems and financial advisors of DC plans will be needing flexible decision making tools to be appropriately tailored to a member's needs, to help her making optimal and conscious choices. Given that the member of a DC pension scheme has some freedom in choosing the investment allocation of her fund in the accumulation phase, she has to solve a portfolio selection problem. Traditionally, the usual way to deal with it has been maximization of expected utility (EU) of final wealth. The literature on the accumulation phase of defined contribution pension schemes is full of examples of optimal investment strategies resulting from EU maximization. See, for instance, Battocchio and Menoncin (2004), Boulier, Huang and Taillard (2001), Cairns, Blake and Dowd (2006), Deelstra, Grasselli and Koehl (2003), Devolder, Bosch Princep and Dominguez Fabian (2003), Di Giacinto, Federico and Gozzi (2010), Gao (2008), Haberman and Vigna (2002), Xiao, Zhai and Qin (2007).

In the context of DC pension funds the problem of finding the optimal investment strategy that is mean-variance efficient, i.e. minimizes the variance of the final fund given a certain level of expected value of the fund has not been reported in published articles. This is not surprising and is mainly due to the fact that the exact and rigorous multi-period and continuous-time versions of the mean-variance problem have been produced only quite recently. The main reason of this delay in solving such a relevant problem, since Markowitz (1952) and Markowitz (1959), lies in the difficulty inherent in the extension from single-period to multi-period or continuous-time framework. In the portfolio selection literature the problem of finding the minimum variance trading strategy in continuous-time has been solved by Richardson (1989) via the martingale approach. The same approach has been used also by Bajeux-Besnainou and Portait (1998) in a more general framework. They also find the dynamic efficient frontier and compare it to the static single-period one. Regarding the use of stochastic control theory to solve a mean-variance optimization problem, a real breakthrough was introduced by Li and Ng (2000) in a discrete-time multi-period framework and Zhou and Li (2000) in a continuous-time model. They show how to transform the difficult problem into a tractable one, by embedding the original problem into a stochastic linear-quadratic control problem, that can then be solved through standard methods. These seminal papers have been followed by a number of extensions; in the financial literature see, for instance, Bielecky, Jin, Pliska and Zhou (2005) and references therein; in the actuarial literature see, for instance, Chiu and Li (2006), Wang, Xia and Zhang (2007) and Josa-Fombellida and Rincón-Zapatero (2008).

In this paper, we use the embedding technique introduced by Zhou and Li (2000) to solve the mean-variance portfolio selection problem in the accumulation phase of a DC plan. We show the equivalence between the mean-variance approach and a target-based approach, that consists in minimizing a quadratic loss function, based on the achievement of a final target. We support the target-based approach, and argue that this optimization criterion is suitable for active members of DC pension schemes, for three reasons. Firstly, it transforms the difficult problem of selecting the individual's risk aversion coefficient into the easiest task of choosing an appropriate target. Secondly, the approach is intuitive and largely flexible to be adapted to the member's needs and preferences. Thirdly, the resulting optimal portfolio is efficient in the mean-variance setting. We then address the delicate issue of comparison of efficient portfolios with portfolios that are optimal under criteria other than the M-V approach. In particular, we investigate how far is an optimal

portfolio from the corresponding M-V efficient one. This issue is interesting in its own nature, for in stochastic control problems the assessment of the exact distance of a sub-optimal solution from the optimal one is typically a difficult problem. Depending on whether the focus is on the variance of the portfolio or on its expected value, the natural notions of Variance Inefficiency (VI) and Mean Inefficiency (MI) are introduced. Then, the dependence of the inefficiency on the relevant parameters of the model is investigated. In a few meaningful examples, i.e. expected utility with CARA and with CRRA utility functions, it turns out that, interestingly, the inefficiency is decreasing with the risk aversion of the individual and is increasing with the time horizon and the Sharpe ratio of the risky asset. These relationships, though intuitive, are not so obvious to show. In fact, while they can be proven in a straightforward way in the exponential and logarithmic case, the proof in the power case turns out to be quite technical.

We end with a numerical example, aimed at showing, in the context of a DC pension scheme, the extent of inefficiency of optimal portfolios derived with CRRA and CARA utility functions with typical risk aversion coefficients. The most interesting result coming from the numerical investigations is related to the dependence of the inefficiency on the time horizon. While Merton (1971) showed that EU-optimality and M-V efficiency coincide at instantaneous level, here we find that for short durations (e.g. 1-2 years) EU-optimality and M-V efficiency do not differ substantially. In this cases, EU-optimal policies can be considered good approximations of M-V efficient policies. On the contrary, with long time horizons, typical for pension funds, the inefficiency increases remarkably and leads to final outcomes likely to be undesirable and inappropriate for the average pension fund member. This result further enhances the convenience of the target-based approach for DC pension funds, given also the fact that investors should care more about behaving efficiently on the entire time horizon, rather than in each single instant with myopically efficient strategies.

The remainder of the paper is organized as follows. In section 2, we introduce the mean-variance (M-V) approach and show that it is equivalent to the target-based (T-B) approach. In section 3, we outline the more general expected utility optimization approach and give the guidelines for comparison between an efficient optimal portfolio and a not-efficient optimal portfolio, introducing the Variance Inefficiency (VI) and the Mean Inefficiency (MI). In section 4, we show that in the cases of exponential, logarithmic and power utility functions the inefficiency decreases with the individual's risk aversion and increases with the time horizon and with the Sharpe ratio of the risky asset. We also assess the dependence of the inefficiency on the initial wealth and on the contribution rate. In section 5, we report a numerical example, aimed at showing in practical cases the extent of inefficiency by adopting popular utility functions in a DC pension plan. Section 6 concludes and outlines further research.

2 The mean-variance approach

Most of the results of this section can be found in Højgaard and Vigna (2007).

2.1 The model

A member of a defined contribution pension scheme is faced with the problem of how to invest optimally the fund at her disposal and the future contributions to be paid in the fund. The financial market available for her portfolio allocation problem is the Black-Scholes model (see e.g. Björk (1998)). This consists of two assets, a riskless one, whose price $B(t)$ follows the dynamics:

$$dB(t) = rB(t)dt,$$

where $r > 0$, and a risky asset, whose price dynamics $S(t)$ follows a geometric Brownian motion with drift $\lambda > r$ and diffusion $\sigma > 0$:

$$dS(t) = \lambda S(t)dt + \sigma S(t)dW(t),$$

where $W(t)$ is a standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$, with $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$.

The constant contribution rate paid in the unit time in the fund is $c \geq 0$. The proportion of portfolio invested in the risky asset at time t is denoted by $y(t)$. The fund at time t , $X(t)$, grows according to the following SDE:

$$\begin{aligned} dX(t) &= \{X(t)[y(t)(\lambda - r) + r] + c\}dt + X(t)y(t)\sigma dW(t) \\ X(0) &= x_0 \geq 0. \end{aligned} \tag{1}$$

The amount x_0 is the initial fund paid in the member's account, which can also be null, if the member has just joined the scheme with no transfer value from another fund. The member enters the plan at time 0 and contributes for T years, after which she retires and withdraws all the money (or converts it into annuity). The temporal horizon T is supposed to be fixed, e.g. T can be 20, 30 years, depending on the member's age at entry.

2.2 The mean-variance approach

In this section, we assume that the individual chooses the mean-variance approach for her portfolio selection problem. She then pursues the two conflicting objectives of maximum expected final wealth together with minimum variance of final wealth, namely she seeks to minimize the vector

$$[-E(X(T)), \text{Var}(X(T))].$$

Definition 1 An investment strategy $y(\cdot)$ is said to be admissible if $y(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbf{R})$.

Definition 2 The mean-variance optimization problem is defined as

$$\begin{aligned} \text{Minimize} \quad & (J_1(y(\cdot)), J_2(y(\cdot))) \equiv (-E(X(T)), \text{Var}(X(T))) \\ \text{subject to} \quad & \begin{cases} y(\cdot) \text{ admissible} \\ X(\cdot), y(\cdot) \text{ satisfy (1)}. \end{cases} \end{aligned} \tag{2}$$

An admissible strategy $\bar{y}(\cdot)$ is called an efficient strategy if there exists no admissible strategy $y(\cdot)$ such that

$$J_1(y(\cdot)) \leq J_1(\bar{y}(\cdot)) \quad J_2(y(\cdot)) \leq J_2(\bar{y}(\cdot)),$$

and at least one of the inequalities holds strictly. In this case, the point $(J_1(\bar{y}(\cdot)), J_2(\bar{y}(\cdot))) \in \mathbf{R}^2$ is called an efficient point and the set of all efficient points is called the efficient frontier.

The multiobjective optimization problem (2) can be equivalently reformulated in the following single objective optimization problem

$$\begin{aligned} \text{Minimize} \quad & (J(y(\cdot)), \alpha) \equiv [-E(X(T)) + \alpha \text{Var}(X(T))], \\ \text{subject to} \quad & \begin{cases} y(\cdot) \text{ admissible} \\ X(\cdot), y(\cdot) \text{ satisfy (1)}, \end{cases} \end{aligned} \tag{3}$$

where $\alpha > 0$. Notice that α is a measure of the risk aversion of the individual: the higher α the higher her risk aversion. It is well-known that it is not so straightforward to tackle problem (3) with standard stochastic control techniques. This is due to the fact that while the expectation operator possesses the "smoothing" property, the variance operator does not. However, Zhou and Li (2000) show that the difficult problem (3) can be approached by solving the standard LQ control problem:

$$\begin{aligned} & \text{Minimize} \quad (J(y(\cdot)), \alpha, \mu) \equiv E[\alpha X(T)^2 - \mu X(T)], \\ & \text{subject to} \quad \begin{cases} y(\cdot) \text{ admissible} \\ X(\cdot), y(\cdot) \text{ satisfy (1)}. \end{cases} \end{aligned} \quad (4)$$

Indeed, they show that if $\bar{y}(\cdot)$ is a solution of (3), then it is a solution of (4) with

$$\bar{\mu} = 1 + 2\alpha E(\bar{X}(T)) \quad (5)$$

where $\bar{X}(T)$ is the fund under optimal control. In solving the standard LQ control problem (4) we follow the approach presented in Zhou and Li (2000). To this aim, let us set:

$$\delta := \frac{\mu}{2\alpha} \quad (6)$$

and

$$Z(t) := X(t) - \delta.$$

The process $Z(t)$ follows the SDE:

$$\begin{aligned} dZ(t) &= \{(Z(t) + \delta)[y(t)(\lambda - r) + r] + c\}dt + (Z(t) + \delta)\sigma y(t)dW(t) \\ Z(0) &= x_0 - \delta \end{aligned} \quad (7)$$

It turns out that problem (4) is equivalent to solve

$$\begin{aligned} & \text{Minimize} \quad (J(y(\cdot)), \alpha) \equiv \left[\frac{1}{2}\alpha Z(T)^2\right], \\ & \text{subject to} \quad \begin{cases} y(\cdot) \text{ admissible} \\ Z(\cdot), y(\cdot) \text{ satisfy (7)}. \end{cases} \end{aligned} \quad (8)$$

This is a standard stochastic optimal control problem and Højgaard and Vigna (2007) have followed the dynamic programming approach to solve it. We refer the interested reader to the mentioned paper for the detailed derivation of the solution of problem (8) and, for space constraints, we here report only the optimal policy of the transformed problem (4). The feedback map for the optimal investment allocation at time t , given that the fund is x , is given by

$$\bar{y}(t, x) = -\frac{\lambda - r}{\sigma^2 x} \left[x - \delta e^{-r(T-t)} + \frac{c}{r} \left(1 - e^{-r(T-t)} \right) \right], \quad (9)$$

where δ is given by (6). The evolution of the fund under optimal control $\bar{X}(t)$ is easily obtained:

$$\begin{aligned} d\bar{X}(t) &= \left[(r - \beta^2)\bar{X}(t) + e^{-r(T-t)}(\beta^2\delta + \frac{\beta^2 c}{r}) + (c - \frac{\beta^2 c}{r}) \right] dt + \\ &+ \left[-\beta\bar{X}(t) + e^{-r(T-t)}(\beta\delta + \frac{\beta c}{r}) - \frac{\beta c}{r} \right] dW(t), \end{aligned} \quad (10)$$

where

$$\beta := \frac{\lambda - r}{\sigma}$$

is the Sharpe ratio of the risky asset. By application of Ito's lemma to (10), we obtain the SDE that governs the evolution of $\bar{X}^2(t)$:

$$\begin{aligned} d\bar{X}^2(t) &= [(2r - \beta^2)\bar{X}^2(t) + 2c\bar{X}(t) + \beta^2((\delta + \frac{c}{r})e^{-r(T-t)} - \frac{c}{r})^2]dt + \\ &- 2\beta\{\bar{X}^2(t) - [(\delta + \frac{c}{r})e^{-r(T-t)} - \frac{c}{r}]\bar{X}(t) + \frac{c}{r}\}dW(t). \end{aligned} \quad (11)$$

If we take expectations on both sides of (10) and (11), we find that the expected value of the optimal fund and the expected value of its square follow the linear ODE's:

$$\begin{aligned} dE(\bar{X}(t)) &= [(r - \beta^2)E(\bar{X}(t)) + e^{-r(T-t)}\beta^2(\delta + \frac{c}{r}) + (c - \frac{\beta^2 c}{r})]dt \\ E(\bar{X}(0)) &= x_0 \end{aligned}$$

and

$$\begin{aligned} dE(\bar{X}^2(t)) &= [(2r - \beta^2)E(\bar{X}^2(t)) + 2cE(\bar{X}(t)) + \beta^2((\delta + \frac{c}{r})e^{-r(T-t)} - \frac{c}{r})^2]dt \\ E(\bar{X}^2(0)) &= x_0^2. \end{aligned}$$

By solving the ODE's we find that the expected value of the fund under optimal control at time t is

$$E(\bar{X}(t)) = \left(x_0 + \frac{c}{r}\right) e^{-(\beta^2-r)t} + \left(\delta + \frac{c}{r}\right) e^{-r(T-t)} - \left(\delta + \frac{c}{r}\right) e^{-r(T-t)-\beta^2 t} - \frac{c}{r},$$

and the expected value of the square of the fund under optimal control at time t is:

$$\begin{aligned} E(\bar{X}^2(t)) &= \left(x_0 + \frac{c}{r}\right)^2 e^{-(\beta^2-2r)t} - \left(\delta + \frac{c}{r}\right)^2 e^{-2r(T-t)-\beta^2 t} - \frac{2c}{r} \left(\delta + \frac{c}{r}\right) e^{-r(T-t)} \\ &+ \frac{2c}{r} \left(\delta + \frac{c}{r}\right) e^{-r(T-t)-\beta^2 t} - \frac{2c}{r} \left(x_0 + \frac{c}{r}\right) e^{-(\beta^2-r)t} + \left(\delta + \frac{c}{r}\right)^2 e^{-2r(T-t)} + \frac{c^2}{r^2}. \end{aligned}$$

At terminal time T we have:

$$E(\bar{X}(T)) = \left(x_0 + \frac{c}{r}\right) e^{-(\beta^2-r)T} + \delta \left(1 - e^{-\beta^2 T}\right) - \frac{c}{r} e^{-\beta^2 T}, \quad (12)$$

and

$$E(\bar{X}^2(T)) = \left(x_0 + \frac{c}{r}\right)^2 e^{-(\beta^2-2r)T} + \delta^2 \left(1 - e^{-\beta^2 T}\right) - \frac{2c}{r} \left(x_0 + \frac{c}{r}\right) e^{-(\beta^2-r)T} + \frac{c^2}{r^2} e^{-\beta^2 T}. \quad (13)$$

We are now able to retrieve the optimal solution to the initial problem (3). To this aim, we plug (5) into (6) and then (12) into (6), to get:

$$\bar{\delta} = \frac{e^{\beta^2 T}}{2\alpha} + x_0 e^{rT} + \frac{c}{r}(e^{rT} - 1). \quad (14)$$

Before proceeding, we need to define a special function, that will be useful in the sequel:

$$\bar{x}_{0,t} := x_0 e^{rt} + \frac{c}{r}(e^{rt} - 1).$$

Evidently, $\bar{x}_{0,t}$ is the fund that one would have at time t by investing the whole portfolio in the riskless asset only, i.e. by adopting the null strategy. An important quantity, that will play a special role in the rest of the paper, is:

$$\bar{x}_0 := \bar{x}_{0,T} = x_0 e^{rT} + \frac{c}{r}(e^{rT} - 1), \quad (15)$$

that is the fund that would be available at time T by adoption of the null strategy from 0 to T . By plugging (14) into (12) and using (15), we get the expected optimal final fund in terms of α, β and \bar{x}_0 :

$$E(\bar{X}(T)) = \bar{x}_0 + \frac{e^{\beta^2 T} - 1}{2\alpha}. \quad (16)$$

It is easy to see that the expected optimal final fund is the sum of the fund that one would get with the null strategy plus a term, $\frac{e^{\beta^2 T} - 1}{2\alpha}$, that depends both on the goodness of the risky asset w.r.t. the riskless one and on the individual's risk aversion. Thus, the higher the Sharpe ratio of the risky asset, β , the higher the expected optimal final wealth, everything else being equal; the higher the member's risk aversion, α , the lower its mean. These are intuitive results.

Similarly, by plugging (14) into (9) and using (15), it is possible to write $\bar{y}(t, x)$ in this way:

$$\bar{y}(t, x) = -\frac{\beta}{\sigma x} \left(x - \bar{x}_{0,t} - \frac{e^{-r(T-t)+\beta^2 T}}{2\alpha} \right). \quad (17)$$

The amount $x\bar{y}(t, x)$ invested in the risky asset at time t is proportional to the difference between the fund x at time t and the fund, $\bar{x}_{0,t}$, available at time t with the null strategy, minus a term that depends on β^2 , α and the time to retirement. The higher the risk aversion, the lower the amount invested in the risky asset, and vice versa, which is an obvious result. It is clear from (17) that a necessary and sufficient condition for the fund to be invested at any time t in the riskless asset is $\alpha = +\infty$: the (extreme) strategy of investing the whole portfolio in the riskless asset is optimal if and only if the risk aversion is infinite.

Using (16) and (17) one can express the optimal investment strategy also in terms of the expected final wealth, in the following way:

$$\bar{y}(t, x) = -\frac{\lambda - r}{\sigma^2 x} \left[x - \left(E[\bar{X}(T)]e^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)}) \right) - \frac{e^{-r(T-t)}}{2\alpha} \right]. \quad (18)$$

The interpretation is that the amount $x\bar{y}(t, x)$ invested in the risky asset at time t is proportional to the difference between the fund x at time t and the amount that would be sufficient to guarantee the achievement of the expected value by adoption of the riskless strategy until retirement, minus a term that depends on α and the time to retirement. Expression (18) will be used in Theorem 4 in the next section to show the equivalency between the mean-variance and the target-based (T-B) approach.

In practical situations, when the risk aversion plays a role in the investor's decisions, expressions (12) and (13) allow one to choose her own profile risk/reward. In fact, as in classical mean-variance analysis, it is possible to express the variance - or the standard deviation - of the final fund in terms of its mean. The subjective choice of the profile risk/reward becomes easier if one is given the efficient frontier of feasible portfolios.

It can be shown (see Højgaard and Vigna (2007)) that the variance of the final wealth is

$$Var(\bar{X}(T)) = \frac{e^{-\beta^2 T}}{1 - e^{-\beta^2 T}} \left(\frac{e^{\beta^2 T} - 1}{2\alpha} \right)^2 = \frac{e^{\beta^2 T} - 1}{4\alpha^2}.$$

The variance is increasing if the Sharpe ratio increases, which is an expected result: in this case the investment in the risky asset is heavier, leading to higher variance. Obviously, the higher the risk aversion α , the lower the variance of the final fund, which is null if and only if $\alpha = +\infty$: in this case, the portfolio is entirely invested in the riskfree asset and

$$\bar{X}(T) = E(\bar{X}(T)) = \bar{x}_0.$$

The efficient frontier of portfolios in the mean-variance diagram is (see Højgaard and Vigna (2007) and also the Appendix):

$$Var(\bar{X}(T)) = \frac{1}{e^{\beta^2 T} - 1} (E(\bar{X}(T)) - \bar{x}_0)^2 \quad (19)$$

Therefore, the efficient frontier of portfolios in the mean-standard deviation diagram is:

$$E(\bar{X}(T)) = \bar{x}_0 + \left(\sqrt{e^{\beta^2 T} - 1} \right) \sigma(\bar{X}(T)). \quad (20)$$

Expectedly, the efficient frontier in the mean-standard deviation diagram is a straight line with slope $\sqrt{e^{\beta^2 T} - 1}$ which is called "price of risk" (see Luenberger (1998)): it indicates by how much the mean of the final fund increases if the volatility of the final fund increases by one unit. When $c = 0$, the efficient frontier coincides with that found by Richardson (1989), Bajoux-Besnainou and Portait (1998) and Zhou and Li (2000) for self-financing portfolios.

2.3 Quadratic loss function: the target-based approach

Although the M-V approach is certainly a good criterium for portfolio selection, its applicability in realistic situations may be not immediate. Indeed, given the efficient frontier (20), the less financially educated individuals may find it difficult to select the couple $(\sigma(\bar{X}(T)), E(\bar{X}(T)))$ corresponding to their preferences and needs. Even more difficult would be the task of selecting their appropriate coefficient α of risk aversion. Indeed, empirical economics provides little guidance as to how the degree of risk aversion should be measured. In the literature several experimental approaches are proposed, see for instance Holt and Laury (2002). Therefore, one of the aims of this section is to show that the M-V approach is equivalent to a more "user-friendly" approach, that is based on the achievement of a final target via minimization of a quadratic loss function. We shall call it the "target-based" (T-B) approach. We will prove this equivalency in Theorem 4 and stress its relevance in Remark 5. Notice that decision-making driven by targets' achievement is not only intuitive but also widely accepted and supported by the economics literature, see for instance the classical works of Kahneman and Tversky (1979) and Tversky and Kahneman (1992) on Prospect Theory and, more recently, Bordley and Li Calzi (2000) and Jin and Zhou (2009).

Thus, in this section we show the useful and expected result that in the framework outlined in section 2.1 the expected utility optimization approach with a quadratic loss function is equivalent to the mean-variance approach. Optimization of a quadratic loss or utility function is a typical approach in pension schemes. Examples of this approach can be found for instance in Boulier, Michel and Wisnia (1996), Boulier, Trussant and Florens (1995), Cairns (2000), Haberman and Sung (1994) for defined benefit pension funds, in Haberman and Vigna (2002), Gerrard, Haberman and Vigna (2004), Gerrard, Højgaard and Vigna (2010) for defined contribution pension schemes.

Højgaard and Vigna (2007) consider the problem of a member of a DC pension scheme who chooses a target value at retirement $F > 0$ and solves the following optimization problem:

$$\begin{aligned} & \text{Minimize} \quad (J(y(\cdot)), F) \equiv E[(X(T) - F)^2], \\ & \text{subject to} \quad \begin{cases} y(\cdot) & \text{admissible} \\ X(\cdot), y(\cdot) & \text{satisfy (1),} \end{cases} \end{aligned}$$

In these circumstances, in the remaining of this paper we shall say that the member solves the

portfolio selection problem with the target-based (T-B) approach¹. For the problem to be financially interesting, the final target F should be chosen big enough, i.e. such that

$$F > \bar{x}_0. \quad (21)$$

We can see from Gerrard et al. (2004) that the optimal investment strategy for the T-B approach is given on the following form²

$$y_{tb}(t, x) = -\frac{\lambda - r}{\sigma^2 x}(x - G(t)), \quad (22)$$

where

$$G(t) = Fe^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)}). \quad (23)$$

Let us notice that the function $G(t)$ represents a sort of target level for the fund at time t : should the fund $X(t)$ reach $G(t)$ at some point of time $t < T$, then the final target F could be achieved by adoption of the riskless strategy until retirement. However, it is not difficult to see that the achievement of $G(t)$, and therefore the sure achievement of the target, is prevented under optimal control by the construction of the solution. In order to show this, let us observe that the fund under optimal control $X^*(t)$ satisfies the following SDE:

$$dX^*(t) = [rG(t) + c + (\beta^2 - r)(G(t) - X^*(t))]dt + \beta(G(t) - X^*(t))dW(t). \quad (24)$$

As in previous work, let us define the process

$$Z(t) = G(t) - X^*(t). \quad (25)$$

Then

$$dZ(t) = G'(t)dt - dX^*(t) = (r - \beta^2)Z(t)dt - \beta Z(t)dW(t),$$

where in the last equality we have applied (23) and (24). We can see that the process $Z(t)$ follows a geometric Brownian motion given by

$$Z(t) = Z(0)e^{(r - \frac{1}{2}\beta^2)t - \beta W(t)}, \quad (26)$$

that is strictly positive if the starting point $Z(0) > 0$. However, the requirement (21) implies $Z(0) > 0$. Indeed,

$$Z(0) = G(0) - x_0 = Fe^{-rT} - \frac{c}{r}(1 - e^{-rT}) - x_0 = e^{-rT}(F - \bar{x}_0) > 0. \quad (27)$$

Therefore, the final fund is always lower than the target. This result is not new. A similar result was already found by Gerrard et al. (2004) and by Gerrard, Haberman and Vigna (2006) in the decumulation phase of a DC scheme: with a different formulation of the optimization problem and including a running cost, in both works they find that there is a "natural" time-varying target that acts as a sort of safety level for the needs of the pensioner and that cannot be reached under optimal control. Previously, in a different context, a similar result was found by Browne (1997): in a problem where the aim is to maximize the probability of hitting a certain upper boundary before ruin, when optimal control is applied the safety level (i.e. the minimum level of fund that guarantees fixed consumption by investing the whole portfolio in the riskless asset) can never be reached.

¹In a more general model, presented in Gerrard et al. (2004), the individual chooses a target function $F(t)$ so as to minimize

$$E \left[\int_0^T e^{-\rho t} \varepsilon_1 (X(t) - F(t))^2 dt + \varepsilon_2 e^{-\rho T} (X(T) - F(T))^2 \right]$$

Here, for consistent comparisons we eliminate the running cost and select only the terminal wealth problem.

²Notice that Gerrard et al. (2004) consider the decumulation phase of a DC scheme. The difference in the wealth equation is that in that case there are periodic withdrawals from the fund whereas here we have periodic inflows into the fund. Formally the equations are identical if one sets $-b_0 = c$.

Remark 3 The expected final fund can be rewritten in a meaningful way. In fact, from

$$Z(T) = G(T) - X^*(T) = F - X^*(T),$$

and using also (26), one has:

$$E(X^*(T)) = F - E(Z(T)) = F - (G(0) - x_0)e^{-(\beta^2-r)T} = e^{-\beta^2 T} \bar{x}_0 + (1 - e^{-\beta^2 T})F. \quad (28)$$

The expected final fund is a weighted average of the target and of the fund that one would obtain with the null strategy. The weights depend only on the Sharpe ratio of the risky asset and the time horizon.

We are now ready to state and prove a theorem that shows that the target-based approach and the mean-variance approach are equivalent. Namely, the T-B approach is M-V efficient and each point on the efficient frontier corresponds to the optimal solution of a T-B optimization problem.

Theorem 4 Assume that the financial market and the wealth equation are as described in section 2.1. Assume that the portfolio selection problem is solved via minimization of expected loss of final wealth at time T , with preferences described by the loss function $L(x)$. Let $X_L^*(T)$ be the final wealth under optimal control. Then,

- i) the couple $(\text{Var}(X_L^*(T)), E(X_L^*(T)))$ is mean-variance efficient if $L(x) = (F - x)^2$;
- ii) each point $(\text{Var}(\bar{X}(T)), E(\bar{X}(T)))$ on the efficient frontier as outlined in section 2.2 equation (19) is the solution of an expected loss minimization problem with loss function $L(x) = (F - x)^2$.

Proof i) We first set

$$E(\bar{X}(T)) = E(X^*(T)). \quad (29)$$

From (28) we have

$$e^{\beta^2 T} E(X^*(T)) = \bar{x}_0 + F(e^{\beta^2 T} - 1).$$

Then, applying (16) and (29), yields

$$e^{\beta^2 T} E(\bar{X}(T)) = E(\bar{X}(T)) - \frac{e^{\beta^2 T} - 1}{2\alpha} + F(e^{\beta^2 T} - 1).$$

Collecting terms and dividing by $e^{\beta^2 T} - 1 > 0$, we have

$$E(\bar{X}(T)) = F - \frac{1}{2\alpha}. \quad (30)$$

We now have:

$$\begin{aligned} y_{tb}(t, x) &= -\frac{\lambda - r}{\sigma^2 x} (x - G(t)) \\ &= -\frac{\lambda - r}{\sigma^2 x} \left\{ x - \left[F e^{-r(T-t)} - \frac{c}{r} (1 - e^{-r(T-t)}) \right] \right\} \\ &= -\frac{\lambda - r}{\sigma^2 x} \left\{ x - \left[\left(F - \frac{1}{2\alpha} \right) e^{-r(T-t)} - \frac{c}{r} (1 - e^{-r(T-t)}) + \frac{e^{-r(T-t)}}{2\alpha} \right] \right\} \\ &= \bar{y}(t, x), \end{aligned} \quad (31)$$

where in the last equality we have used (30) and (18). It is then clear that, since $y_{tb}(t, x)$ is a particular case of mean-variance investment strategy, it must lead to an optimal portfolio that is

mean-variance efficient.

ii) Consider a point $(Var(\bar{X}(T)), E(\bar{X}(T)))$ on the efficient frontier. Using (16) it is possible to find the corresponding $\bar{\alpha}$ which in turn defines the target via (30):

$$\bar{F} = E(\bar{X}(T)) + \frac{1}{2\bar{\alpha}}.$$

It is then obvious that the point $(Var(\bar{X}(T)), E(\bar{X}(T)))$ chosen on the efficient frontier can be found by solving the target-based optimization problem with target equal to \bar{F} . \square

2.4 Some comments on the target-based mean-variance efficient approach

In this section, we make some considerations on the advantages of the target-based approach for the portfolio selection of a DC pension scheme. Theorem 4 shows that every solution to a target-based optimization problem corresponds to a point on the efficient frontier, and each point of the efficient frontier can be found by solving a target-based optimization problem. The one-to-one correspondence between points of the efficient frontier and target-based optimization problems is given by the following relationship between the parameter α of the M-V approach and the value of final target of the T-B approach:

$$\alpha = \frac{e^{\beta^2 T}}{2(F - \bar{x}_0)}, \quad (32)$$

where we have used (28) and (30).

Remark 5 *Expression (32) has a relevant practical implication. In fact, it allows the scheme's member to identify her own risk aversion parameter α , hence her corresponding point on the efficient frontier, just by selecting a final target F to be reached³. This property is important for applicative purposes of the model by financial advisors of DC pension funds. Indeed, relationship (32) could be the starting point for the construction of a flexible decision-making tool for members of DC schemes.*

The fact that the target-based approach is a particular case of the mean-variance approach should put an end to the criticism of the quadratic utility function, that penalizes deviations above the target as well as deviations below it. The intuitive motivation for supporting such a utility function in DC schemes: "The choice of trying to achieve a target and no more than this has the effect of a natural limitation on the overall level of risk for the portfolio: once the target is reached, there is no reason for further exposure to risk and therefore any surplus becomes undesirable" finds here full justification in a rigorous setting. Moreover, it can be shown rigorously (see Di Giacinto, Federico, Gozzi and Vigna (2009)) that in the region of interest (i.e. for $F > \bar{x}_0$) the optimal policy found with the quadratic loss function is identical to the optimal policy found with the alternative – and maybe more appealing for financial advisors – loss function

$$\begin{cases} (F - X(T))^2 & \text{if } F > X(T) \\ 0 & \text{if } F \leq X(T). \end{cases}$$

Furthermore, we would like to point out that the T-B approach is very easy to understand for the scheme's member, immediate to implement and quite flexible to allow for a variety of needs and

³The link between quadratic utility function and M-V approach was mentioned by Bielecky et al. (2005). They noticed, however, that the portfolio's expected return would be unclear to determine a priori. In contrast, here we provide the exact expected return and variance of the optimal portfolio via optimization of the quadratic loss function. Differently from them, we are able to determine completely the exact point on the efficient frontier of portfolios.

preferences. In fact, the choice of a final target to be achieved at retirement is by no means much easier to make than the choice of a generic coefficient of risk aversion relative to some abstract utility function (see also Remark 5).

Last but not least, the property that the T-B investment strategies are M-V efficient should make this approach appealing also to pension fund investment managers, whose performance is still mainly based on M-V criteria.

A final remark about an intrinsic feature of the optimal efficient investment strategies. From (22) we can see that another direct consequence of the positivity of $Z(t)$ is the fact that under the target-based approach the amount invested in the risky asset under optimal control is always positive. Obviously, this is the case also for the mean-variance approach. This leads us to the formulation of the following corollary.

Corollary 6 *Consider the financial market and the wealth equation as in section 2.1. Consider the efficient frontier of feasible portfolios, as outlined in section 2.2. Then, the optimal amount invested in the risky asset at any time $0 \leq t < T$ is strictly positive.*

Proof. This follows from (31), (25), (26) and (27). □

This is a desirable property, given that the constrained portfolio problem has not been solved yet for the target-based approach. In fact, this natural feature allows to reduce the bilateral constrained portfolio problem in the no-borrowing constraint problem, given that the no-short selling property comes with no cost for the nature of the problem. Solving the no-short selling constrained problem with the target-based approach in the decumulation phase of a defined contribution pension scheme is subject of ongoing research.

3 Expected Utility approach versus Mean Variance approach

In this section, we give guidelines for comparison between an optimal efficient portfolio found via the M-V approach and an optimal portfolio found via the more general expected utility (EU) maximization approach. Let us point out that, as shown in section 2, the M-V approach is a particular case of EU approach, selecting a quadratic loss function. Thus, the comparison is between final optimal wealth found with different utility functions. This comparison is interesting from a theoretical point of view, because in stochastic control problems it is typically quite difficult to determine the distance of a sub-optimal solution from the optimal one. On the contrary, in this case, due to the nature of the control problem, there is a natural way to compare optimal solutions to different problems.

The individual's aim is now finding the optimal investment strategy over time that maximizes the expected value of final wealth. She then wants to solve

$$\begin{aligned} & \text{Maximize} \quad (J(y(\cdot))) \equiv E[U(X(T))], \\ & \text{subject to} \quad \begin{cases} y(\cdot) & \text{admissible} \\ X(\cdot), y(\cdot) & \text{satisfy (1).} \end{cases} \end{aligned} \tag{33}$$

Problem (33) is a standard optimization problem that can be dealt with via classical control theory. We refer the interested reader to classical texts such as Yong and Zhou (1999), Øksendal (1998),

Björk (1998). In section 2, we have shown that a member of a defined contribution pension scheme wanting to solve the following mean-variance problem

$$\begin{aligned} & \text{Minimize } (J_1(y(\cdot)), J_2(y(\cdot))) \equiv (-E(X(T)), \text{Var}(X(T))) \\ & \text{subject to } \begin{cases} y(\cdot) \text{ admissible} \\ X(\cdot), y(\cdot) \text{ satisfy (1),} \end{cases} \end{aligned}$$

where $\alpha > 0$ measures her risk aversion, should invest optimally in such a way as to obtain a final fund, $\bar{X}(T)$ that has the following mean:

$$E(\bar{X}(T)) = \bar{x}_0 + \frac{e^{\beta^2 T} - 1}{2\alpha}, \quad (34)$$

and the following variance:

$$\text{Var}(\bar{X}(T)) = \frac{e^{\beta^2 T} - 1}{4\alpha^2}. \quad (35)$$

In other words, for this problem there exists no portfolio that has a final mean equal to (34) with a variance strictly lower than (35). Equivalently, there exists no portfolio that has a final variance equal to (35) with a mean strictly greater than (34).

Therefore, it is obvious that if one derives the expectation and the variance of the final wealth under optimal control associated to the problem of maximization of $E(U(X(T)))$, $E(X_U^*(T))$ and $\text{Var}(X_U^*(T))$ and sets

$$E(X_U^*(T)) = E(\bar{X}(T)),$$

then, necessarily:

$$\Rightarrow \text{Var}(X_U^*(T)) \geq \text{Var}(\bar{X}(T)). \quad (36)$$

Alternatively, if one sets

$$\text{Var}(X_U^*(T)) = \text{Var}(\bar{X}(T)),$$

then, necessarily:

$$\Rightarrow E(X_U^*(T)) \leq E(\bar{X}(T)). \quad (37)$$

The obvious inequalities (36) and (37) express a result that since long has been known in the literature, about the comparison between the two leading approaches for portfolio selection, mean-variance and expected utility. Regarding this comparison, it is well-known that in the single-period framework the mean-variance approach and expected utility optimization coincide if either the utility function is quadratic or the asset returns are normal. Furthermore, in the continuous-time framework when prices are log-normal there is consistency between optimal choices and mean-variance efficiency at instantaneous level (see Merton (1971) and also Campbell and Viceira (2002)). However, this does not imply that an optimal policy should *remain* efficient also after two consecutive instants or, more in general, on a time interval greater than the instantaneous one. In fact, in general it does not. In previous financial literature, the lack of efficiency of optimal policies in continuous-time was noted for instance by some empirical works that compare mean-variance efficient portfolios with expected utility optimal portfolios and find that there are indeed differences between those portfolios. Among these, Hakansson (1971), Grauer (1981) and Grauer and Hakansson (1993). Hakansson (1971) compared mean-variance portfolios with the growth optimal portfolios (derived via logarithmic utility function) and found that the characteristics of the two portfolios are different. Grauer (1981) compared empirically the portfolio composition for the growth optimal model and the mean-variance model under the assumption that asset returns are approximately normally

distributed and found that there are significant differences both in the mix of risky asset and in the assets chosen. Grauer and Hakansson (1993) found that, despite the established consensus that portfolios chosen on the basis of the mean-variance approach can closely approximate portfolios chosen by expected utility maximization, with annual revisions the mean-variance model does not approximate the power utility portfolio. Related work on the comparison between M-V and EU approach can be found in Zhou (2003). The impact of the time horizon on the asset allocation has been investigated also by Jurek and Viceira (2006).

Up to our knowledge, there is no article that formalizes the extent of inefficiency. Nor the dependence of inefficiency on the parameters of the model.⁴ In a very natural way, either the difference

$$\text{Var}(X_U^*(T)) - \text{Var}(\bar{X}(T)) \geq 0$$

or the difference

$$E(\bar{X}(T)) - E(X_U^*(T)) \geq 0$$

quantify the degree of mean-variance inefficiency of the utility function U .

We therefore define the Variance Inefficiency associated to the utility function U and the time horizon T as

$$VI(X_U^*(T)) := \text{Var}(X_U^*(T)) - \text{Var}(\bar{X}(T)), \quad (38)$$

whenever $E(\bar{X}(T)) = E(X_U^*(T))$ and the Mean Inefficiency associated to the utility function U and the time horizon T as

$$MI(X_U^*(T)) := E(\bar{X}(T)) - E(X_U^*(T)),$$

whenever $\text{Var}(X_U^*(T)) = \text{Var}(\bar{X}(T))$. These two inefficiency measures are focused on the different variances and on the different expectations of final portfolios, respectively. In particular, VI specifies how much additional risk an EU maximizer (with utility function U) has to bear if she aims to the same expected final wealth. The MI indicates what is the loss in expected final wealth if she wants to keep the same level of risk.

Two issues may be of interest for a generic utility function U :

1. Are there special cases, where $VI(X_U^*(T)) = MI(X_U^*(T)) = 0$? In these cases, the inefficiency is null and the strategy that is optimal under EU with the function U turns out to be also M-V efficient.
2. What is the dependence of the inefficiency on the relevant parameters of the problem, namely the risk aversion of the member, the Sharpe ratio β , the time horizon T , the initial wealth x_0 and the contribution rate c ?

While the answers are obvious in the case of a quadratic loss or utility function, it seems a difficult task to answer these questions for a general utility function U . However, it is possible to give answers whenever the form of the utility function is selected. In the next section, we will consider the most popular utility function used for portfolio selection in the economic and financial literature, i.e. those that exhibit constant absolute risk aversion (CARA), that is the exponential utility function, and those that exhibit constant relative risk aversion (CRRA), that is the logarithmic and the power utility functions. These classes of utility functions are also those analyzed by Hakansson (1971), Grauer (1981) and Grauer and Hakansson (1993).

⁴In related work, Bucciol and Miniaci (2011) make use of the mean-variance inefficiency of an observed portfolio. They find the risk aversion coefficient α that minimizes the distance between the certainty equivalent return generated by the observed portfolio and that corresponding to the optimal efficient portfolio. However, they do not consider an expected utility framework with a generic utility function U , and they do not measure the mean-variance inefficiency of the U -optimal portfolio.

4 Analysis of mean-variance inefficiency for CARA and CRRA utility function

In this section, we calculate the variance inefficiency VI for the CARA and the CRRA utility functions and give answers to the two questions arisen above, i.e. what are the degenerate cases when the inefficiency is null and what is the dependence of the inefficiency on the model's parameters. In particular, we prove that in all cases the inefficiency is null when the risk aversion is infinite or either the time horizon or the Sharpe ratio is null. We also prove the less trivial result that, in all cases, the inefficiency is decreasing with the risk aversion coefficient and it is increasing with the time horizon T and the Sharpe ratio β . Although this is an intuitive result that lends itself to easy interpretation, the proof in the power case is quite technical. Indeed, Lemma 11 and Theorem 14 are the core of the paper from the mathematical point of view. The dependence on the initial wealth x_0 and on the contribution rate c is different depending on the class of utility function chosen: with CRRA functions VI is increasing in x_0 and c , with the CARA class VI is independent of them. We have performed the analysis focusing on the variance inefficiency VI, but it is clear that all the results hold also for the mean inefficiency MI. In what follows, we will treat the three cases - exponential, logarithmic, power utility functions - separately.

4.1 CARA: Exponential utility function

Consider the exponential utility function

$$U(x) = -\frac{1}{k}e^{-kx},$$

with (constant) Arrow-Pratt coefficient of absolute risk aversion equal to

$$ARA(x) = -\frac{U''(x)}{U'(x)} = k > 0.$$

It can be shown (see Vigna (2009) and also the Appendix) that the expected final wealth is:

$$E(X^*(T)) = \left(x_0 + \frac{c}{r}\right) e^{rT} - \frac{c}{r} + \frac{\beta^2 T}{k} = \bar{x}_0 + \frac{\beta^2 T}{k}, \quad (39)$$

and the variance of the final fund is

$$Var(X^*(T)) = E((X^*(T))^2) - E^2(X^*(T)) = \frac{\beta^2 T}{k^2}.$$

By equating the expected final funds in (39) and in (34), we find that $E(X_U^*(T)) = E(\bar{X}(T))$ if and only if

$$\frac{\beta^2 T}{k} = \frac{e^{\beta^2 T} - 1}{2\alpha}. \quad (40)$$

Therefore, using (40), after some algebra we find that the Variance Inefficiency is

$$VI(X^*(T)) = \frac{(e^{\beta^2 T} - 1)}{2\alpha k} \left(1 - \frac{k}{2\alpha}\right) = \frac{\beta^2 T}{k^2} \left(1 - \frac{\beta^2 T}{e^{\beta^2 T} - 1}\right). \quad (41)$$

Looking at the form of the VI, it is straightforward to show the expected result that the optimal portfolio found with EU with the exponential utility function is (strictly) not efficient.

Proposition 7 *In the exponential case, if $\beta^2 T > 0$ and $k < +\infty$, then $VI(X^*(T)) > 0$.*

Proof. The proof is obvious, considering that $x < e^x - 1$ for $x \neq 0$. □

Moreover, let us make some comments on the extreme cases in which the two portfolios coincide and the inefficiency (41) is null. It is rather obvious that for $k \rightarrow +\infty$ the optimal portfolio is the riskless one, with mean \bar{x}_0 and zero variance, since the investor has infinite risk aversion. At the same time, due to (40), also $\alpha \rightarrow +\infty$ and the efficient portfolio is the riskless one. Similarly, it is obvious that the difference in (41) is null also in the case $e^{\beta^2 T} = 1$. In fact, this is possible if either $\beta = 0$ or $T = 0$. In both cases, we have that the optimal portfolio is invested entirely in the riskless asset and the final deterministic portfolio at time $T \geq 0$ is \bar{x}_0 .

As a consequence, the following theorem holds.

Theorem 8 *Assume that the financial market and the wealth equation are as described in section 2.1. Assume that the portfolio selection problem is solved via maximization of the expected utility of final wealth at time T , with preferences described by the utility function $U(x) = -\frac{1}{k}e^{-kx}$. Then, the Variance Inefficiency (38) is given by (41) and:*

- i) is null if and only if either the Sharpe ratio β is null, or the time horizon T is null, or the individual has infinite absolute risk aversion;*
- ii) is a decreasing function of the absolute risk aversion coefficient $k > 0$;*
- iii) is an increasing function both of the Sharpe ratio β and the time horizon T ;*
- iv) does not depend on the initial fund x_0 and on the contribution rate c .*

Proof. The claim i) comes from the discussion above. The claims ii), iii) and iv) are obvious, given (41). □

4.2 CRRA: Logarithmic utility function

Consider the logarithmic utility function

$$U(x) = \ln x.$$

The (constant) Arrow-Pratt coefficient of relative risk aversion is

$$RRA(x) = -\frac{U''(x)}{U'(x)}x = 1.$$

It can be shown (see Vigna (2009) and also the Appendix) that the expected final wealth is

$$E(X^*(T)) = e^{AT} \left(x_0 + \frac{c}{r}(1 - e^{-rT}) \right) = \bar{x}_0 e^{\beta^2 T}, \quad (42)$$

and the variance of the final fund is

$$Var(X^*(T)) = (e^{KT} - e^{2AT}) \left(x_0 + \frac{c}{r}(1 - e^{-rT}) \right)^2 = (E(X^*(T)))^2 (e^{\beta^2 T} - 1),$$

where

$$A = r + \beta^2, \quad K = 2r + 3\beta^2. \quad (43)$$

By equating the expected final funds in (42) and in (34), we find that $E(X_U^*(T)) = E(\bar{X}(T))$ if and only if

$$e^{\beta^2 T} - 1 = \frac{e^{\beta^2 T} - 1}{2\alpha\bar{x}_0}, \quad (44)$$

which happens if and only if

$$\alpha = \frac{1}{2\bar{x}_0}.$$

Therefore, using (44), after some algebra we find that the Variance Inefficiency is

$$VI(X^*(T)) = \bar{x}_0^2 (e^{\beta^2 T} - 1)^2 (e^{\beta^2 T} + 1). \quad (45)$$

Looking at the form of the VI, it is straightforward to show the expected result that the optimal portfolio found with EU with the logarithmic utility function is (strictly) not efficient.

Proposition 9 *In the logarithmic case, if $\beta^2 T > 0$, then $VI(X^*(T)) > 0$.*

Proof. The proof is obvious. □

As before, given that $\bar{x}_0 > 0$ for the problem not to be trivial, the difference in (45) is null if and only if $e^{\beta^2 T} = 1$. As observed earlier, this is possible if either $\beta = 0$ or $T = 0$. In both cases, we have that the optimal portfolio is invested entirely in the riskless asset and the final portfolio at time $T \geq 0$ is \bar{x}_0 .

As a consequence, the following theorem holds.

Theorem 10 *Assume that the financial market and the wealth equation are as described in section 2.1. Assume that the portfolio selection problem is solved via maximization of the expected utility of final wealth at time T , with preferences described by the utility function $U(x) = \ln(x)$. Then, the Variance Inefficiency (38) is given by (45) and is:*

- i) null if and only if either the Sharpe ratio β is null, or the time horizon T is null;*
- ii) an increasing function both of the Sharpe ratio β and the time horizon T ;*
- iii) an increasing function of the initial fund $x_0 \geq 0$ and of the contribution rate $c \geq 0$.*

Proof. The claim i) comes from the discussion above. The claims ii) and iii) are obvious, given (45).

4.3 CRRA: Power utility function

Consider the power utility function

$$U(x) = \frac{x^\gamma}{\gamma},$$

with $\gamma < 1$ and (constant) Arrow-Pratt coefficient of relative risk aversion equal to

$$RRA(x) = -\frac{U''(x)}{U'(x)}x = 1 - \gamma.$$

It can be shown (see Vigna (2009) and also the Appendix) that the expected final wealth is

$$E(X^*(T)) = e^{AT} \left(x_0 + \frac{c}{r} (1 - e^{-rT}) \right) = \bar{x}_0 e^{\frac{\beta^2 T}{1-\gamma}}, \quad (46)$$

and the variance of the final fund is

$$\text{Var}(X^*(T)) = (e^{KT} - e^{2AT})(x_0 + \frac{c}{r}(1 - e^{-rT}))^2 = (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1)(E(X^*(T)))^2,$$

where A and K are given by (43). By equating the expected final funds in (46) and in (34), we find that $E(X_U^*(T)) = E(\bar{X}(T))$ if and only if

$$e^{\frac{\beta^2 T}{1-\gamma}} - 1 = \frac{e^{\beta^2 T} - 1}{2\alpha\bar{x}_0}. \quad (47)$$

It is clear that for $\beta^2 T = 0$ we have $\text{Var}(X^*(T)) = \text{Var}(\bar{X}(T)) = 0$, so that the Variance Inefficiency is null, as in previous cases. As previously, this is possible if either $\beta = 0$ or $T = 0$. In both cases, we have that the optimal portfolio is invested entirely in the riskless asset and the final deterministic portfolio at time $T \geq 0$ is \bar{x}_0 . One can also see that for $\gamma \rightarrow -\infty$ the optimal portfolio is the riskless one, with mean \bar{x}_0 and zero variance, since the investor has infinite risk aversion. At the same time, due to (47) also the efficient portfolio will be the riskless one. Therefore, also in this case the VI is null.

Let $\beta^2 T > 0$ and $-\infty < \gamma < 1$. Using (47), after some algebra we find that the Variance Inefficiency is:

$$VI(X^*(T)) = \frac{\bar{x}_0^2}{e^{\beta^2 T} - 1} \left(e^{\frac{2\beta^2 T}{(1-\gamma)}} (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1)(e^{\beta^2 T} - 1) - (e^{\frac{\beta^2 T}{1-\gamma}} - 1)^2 \right). \quad (48)$$

Since $VI \geq 0$ by definition, it is clear from (48) that VI is an increasing function of the initial wealth x_0 and the contribution rate c . However, assessing the dependence of the inefficiency on the other parameters of the model, i.e. $1 - \gamma$, β and T , is quite a difficult task. To this aim, we perform the following change of variables:

$$a := \frac{1}{1-\gamma} \quad b := e^{\beta^2 T}. \quad (49)$$

The inefficiency becomes a function of a and b :

$$VI(X^*(T)) = VI(a, b) = \frac{\bar{x}_0^2}{b-1} (b^{a^2+2a+1} - b^{a^2+2a} - b^{2a+1} + 2b^a - 1), \quad (50)$$

with $a \in (0, +\infty)$ and $b \in (1, +\infty)$. In order to prove that the VI is decreasing in the risk aversion coefficient $1 - \gamma$, we need to prove that

$$\frac{\partial VI}{\partial a} > 0.$$

Similarly, in order to prove that the VI is increasing in time and Sharpe ratio of the risky asset, we need to prove that

$$\frac{\partial VI}{\partial b} > 0. \quad (51)$$

The first result is a corollary of the following lemma.

Lemma 11 *Let the function*

$$w : [0, +\infty) \rightarrow [0, +\infty)$$

be given by

$$w(a) = b^{a^2+2a+1} - b^{a^2+2a} - b^{2a+1} + 2b^a - 1 \quad (52)$$

where the parameter $b \in (1, +\infty)$. Then,

$$\frac{dw}{da} > 0. \quad (53)$$

Proof. Claim (53) is equivalent to show that

$$f'_b(a) > g'_b(a) \quad \forall a \in [0, +\infty) \quad (54)$$

with

$$f_b(a) := b^{a^2+2a+1} + 2b^a,$$

and

$$g_b(a) := b^{a^2+2a} + b^{2a+1} + 1.$$

We have

$$\begin{aligned} f'_b(a) &= (2b^a + (2a + 2)b^{a^2+2a+1}) \log b \\ f''_b(a) &= (2b^a + (2a + 2)^2 b^{a^2+2a+1})(\log b)^2 + 2b^{a^2+2a+1} \log b, \end{aligned}$$

and

$$\begin{aligned} g'_b(a) &= (2b^{2a+1} + (2a + 2)b^{a^2+2a}) \log b \\ g''_b(a) &= (4b^{2a+1} + (2a + 2)^2 b^{a^2+2a})(\log b)^2 + 2b^{a^2+2a} \log b. \end{aligned}$$

Then,

$$\lim_{a \rightarrow 0^+} f'_b(a) = \lim_{a \rightarrow 0^+} g'_b(a) = (2 + 2b) \log b.$$

However,

$$\begin{aligned} \lim_{a \rightarrow 0^+} f''_b(a) &= (4b + 2)(\log b)^2 + 2b \log b \\ \lim_{a \rightarrow 0^+} g''_b(a) &= (4 + 4b)(\log b)^2 + 2 \log b \end{aligned}$$

so that, since $b > 1$, we have:

$$f''_b(0) - g''_b(0) = 2 \log b(b - 1 - \log b) > 0.$$

Since $f'_b(0) = g'_b(0)$ and $f''_b(0) > g''_b(0)$, if we show that $f''_b(a) > g''_b(a)$ for all $a \in [0, +\infty)$ the claim (54) is proven. We have:

$$\begin{aligned} f''_b(a) - g''_b(a) &= (\log b)^2(2b^a + (2a + 2)^2 b^{a^2+2a+1} - 4b^{2a+1} + (2a + 2)^2 b^{a^2+2a}) + 2 \log b(b^{a^2+2a+1} - b^{a^2+2a}) \\ &= 2 \log b[(2(a + 1)^2(b^{a^2+2a+1} - b^{a^2+2a}) + b^a - 2b^{2a+1}) \log b + (b^{a^2+2a+1} - b^{a^2+2a})]. \end{aligned}$$

We have

$$f''_b(a) > g''_b(a) \quad (55)$$

if and only if

$$(2(a + 1)^2(b^{a^2+2a+1} - b^{a^2+2a}) + b^a - 2b^{2a+1}) \log b + (b^{a^2+2a+1} - b^{a^2+2a}) > 0$$

that is true if and only if

$$(b^{a^2+2a+1} - b^{a^2+2a})(1 + 2 \log b(a + 1)^2) > (2b^{2a+1} - b^a) \log b. \quad (56)$$

In turn, (56) is equivalent to

$$h(a) > k(a) \quad (57)$$

for $a \in (0, +\infty)$ with

$$h(a) := (b^{a^2+2a+1} - b^{a^2+2a})(1 + 2 \log b(a + 1)^2)$$

and

$$k(a) := (2b^{2a+1} - b^a) \log b.$$

It is easy to see that

$$h(0) - k(0) = b - 1 - \log b > 0.$$

It is also possible to show that $h'(a) > k'(a)$. In fact,

$$h'(a) = (a + 1) \log b (b^{a^2+2a+1} - b^{a^2+2a}) (6 + 4 \log b (a + 1)^2)$$

and

$$k'(a) = (\log b)^2 (4b^{2a+1} - b^a).$$

Therefore, using the fact that $b - 1 > \log b$, we have

$$\begin{aligned} h'(a) - k'(a) &= (a + 1) \log b (b^{a^2+2a+1} - b^{a^2+2a}) (6 + 4 \log b (a + 1)^2) - (\log b)^2 (4b^{2a+1} - b^a) \\ &= 4(\log b)^2 (b^{a^2+2a+1} - b^{2a+1}) + a \log b (6 + 4 \log b (a + 1)^2) (b^{a^2+2a+1} - b^{a^2+2a}) + \\ &\quad + \log b (6 + 4 \log b (a^2 + 2a)) b^{a^2+2a} (b - 1) - 4(\log b)^2 b^{a^2+2a} + (\log b)^2 b^a \\ &> 4(\log b)^2 (b^{a^2+2a+1} - b^{2a+1}) + a \log b (6 + 4 \log b (a + 1)^2) (b^{a^2+2a+1} - b^{a^2+2a}) + \\ &\quad + 2(\log b)^2 b^{a^2+2a} + 4(\log b)^3 (a^2 + 2a) b^{a^2+2a} + (\log b)^2 b^a > 0. \end{aligned}$$

Since $h(0) > k(0)$ and $h'(a) > k'(a)$ for all $a > 0$, (57) holds. This in turn implies (55), that implies (54). \square

Corollary 12 *Let $VI(a, b)$ be the function defined in (50). Then, $\frac{\partial VI}{\partial a} > 0$ for all $a > 0$.*

Proof. This is obvious by Lemma 11, observing from (50) and (52) that $VI(a, b) = \frac{\bar{x}_0^2}{b-1} w(a)$. \square

The strict positivity of VI in the power case comes now as a corollary.

Corollary 13 *In the power case, if $\beta^2 T > 0$ and $\gamma < 1$, then $VI(X^*(T)) > 0$.*

Proof. Due to (49) and (50), it is enough to show that $VI(a, b) > 0$ for all $a \in (0, +\infty)$ and $b \in (1, +\infty)$. Observe that $VI(0, b) = 0$ and $\frac{\partial VI}{\partial a} > 0$ for $a > 0$. Hence, the claim. \square

The claim (51) is proven by the following Theorem.

Theorem 14 *Let $VI(a, b)$ be the function defined in (50). Then, $\frac{\partial VI}{\partial b} > 0$ for all $b > 1$.*

Proof. The proof is by contradiction. For notational convenience, for fixed $a > 0$, let us call $\frac{\partial VI(b)}{\partial b} = f_a(b)$. The steps of the proof are the following:

1. Prove that $\lim_{b \rightarrow 1^+} f_a(b) = 0$.
2. Prove that if there exists $b_0 > 1$ s.t. $f_a(b_0) \leq 0 \Rightarrow VI_a(b_0) := VI(a, b_0) \leq 0$.
3. This ends the proof, because we know by Corollary 13 that $VI_a(b) > 0$ for all $b > 1$.

By differentiating (50) with respect to b one gets the function

$$f_a(b) = \frac{\bar{x}_0^2}{(b-1)^2} \left((a^2 + 2a) b^{a^2+2a-1} (b-1)^2 + 2ab^{2a} (1-b) + (b-1) 2ab^{a-1} + (b^a - 1)^2 \right).$$

It is straightforward to see that

$$\lim_{b \rightarrow 1^+} f_a(b) = \bar{x}_0^2 \left(\lim_{b \rightarrow 1^+} (a^2 + 2a) b^{a^2+2a-1} + 2a \lim_{b \rightarrow 1^+} \frac{b^{a-1} - b^{2a}}{b-1} + \left(\lim_{b \rightarrow 1^+} \frac{b^a - 1}{b-1} \right)^2 \right) = 0.$$

Now assume that there exists $b > 1$ s.t. $f_a(b) \leq 0$. We have

$$\begin{aligned}
f_a(b) \leq 0 &\iff 2ab^{2a}(b-1) \geq (a^2 + 2a)b^{a^2+2a-1}(b-1)^2 + (b-1)2ab^{a-1} + (b^a - 1)^2 \\
&\iff b^{2a} \geq \frac{(a^2+2a)}{2a}b^{a^2+2a-1}(b-1) + b^{a-1} + \frac{(b^a-1)^2}{2a(b-1)} \\
&\iff -\frac{(b-1)}{x_0^2}VI_a(b) \geq b \left(\frac{(a^2+2a)}{2a}b^{a^2+2a-1}(b-1) + b^{a-1} + \frac{(b^a-1)^2}{2a(b-1)} \right) - b^{a^2+2a+1} + b^{a^2+2a} - 2b^a + 1 \\
&\iff -\frac{(b-1)}{x_0^2}VI_a(b) \geq \frac{a}{2}b^{a^2+2a}(b-1) - (b^a - 1) + \frac{b(b^a-1)^2}{2a(b-1)} \\
&\iff -\frac{2a(b-1)^2}{x_0^2}VI_a(b) \geq a^2b^{a^2+2a}(b-1)^2 - 2a(b-1)(b^a - 1) + b(b^a - 1)^2 \\
&\iff -\frac{2a(b-1)^2}{x_0^2}VI_a(b) > a^2(b-1)^2 - 2a(b-1)(b^a - 1) + (b^a - 1)^2 \geq 0.
\end{aligned}$$

Thus, we have proven that

$$f_a(b) \leq 0 \Rightarrow VI_a(b) < 0,$$

that is a contradiction.

Therefore, it must be $f_a(b) = \frac{\partial VI(b)}{\partial b} > 0$ for all $b > 1$. \square

We have proven the following Theorem:

Theorem 15 *Assume that the financial market and the wealth equation are as described in section 2.1. Assume that the portfolio selection problem is solved via maximization of the expected utility of final wealth at time T , with preferences described by the utility function $U(x) = \frac{x^\gamma}{\gamma}$. Then, the Variance Inefficiency (38) is given by (48) and is:*

- i) null if and only if either the Sharpe ratio β is null, or the time horizon T is null, or the individual has infinite relative risk aversion;*
- ii) a decreasing function of the relative risk aversion coefficient $1 - \gamma > 0$;*
- iii) an increasing function both of the Sharpe ratio β and the time horizon T ;*
- iv) an increasing function of the initial fund $x_0 \geq 0$ and of the contribution rate $c \geq 0$.*

Proof. Claims i) and iv) come from the discussion at the beginning of the section. Claim ii) comes from Corollary 12, claim iii) comes from Theorem 14. \square

4.4 The special case $c = 0$: the usual portfolio selection problem

It is rather clear from the previous analysis, that the inequalities still hold when $c = 0$, provided that $x_0 > 0$. We focus in particular on Propositions 7, and 9 and Corollary 13. Therefore, we find that in the usual portfolio selection analysis in continuous-time, in a standard Black & Scholes financial market the EU maximization criterion with CARA and CRRA utility functions leads to an optimal portfolio that is not mean-variance efficient. We can summarize this result in the following corollary.

Corollary 16 *Assume that an investor wants to invest a wealth of $x_0 > 0$ for the time horizon $T > 0$ in a financial market as in section 2.1 and wealth equation (1) with $c = 0$. Assume that she maximizes expected utility of final wealth at time T . Then, the couple $(\text{Var}(X_U^*(T)), E(X_U^*(T)))$ associated to the final wealth under optimal control $X_U^*(T)$ is not mean-variance efficient in the following cases:*

- i) $U(x) = -\frac{1}{k}e^{-kx}$;*
- ii) $U(x) = \ln x$;*
- iii) $U(x) = \frac{x^\gamma}{\gamma}$.*

Proof. The proof is obvious, by setting $c = 0$ in the proofs of Propositions 7 and 9 and Corollary 13 and observing that all strict inequalities still hold. \square

Corollary 16 gives a theoretical foundation for the empirical works by Hakansson (1971), Grauer (1981) and Grauer and Hakansson (1993) on the lack of efficiency of optimal policies in continuous-time. In addition, the inefficiency measures VI and MI can be considered a useful and practical tool for the valuation of the extent of inefficiency of optimal portfolios, in important contexts such as pension funds and also others.

Remark 17 (Parallel with time-consistent formulation of the mean-variance problem)

It is worth to notice an interesting relationship with the time-consistent version of the mean-variance portfolio selection problem, developed in Basak and Chabakauri (2010) and Björk and Murgoci (2010). It is not difficult to see that the expressions for the optimal time-consistent strategy in the simplest formulation of the mean-variance problem of both papers (i.e. equation (41) of Basak and Chabakauri (2010) and Proposition 7.1 of Björk and Murgoci (2010)) are identical to the optimal investment strategy obtained via maximization of EU with the exponential utility function (60). This link between the dynamic mean-variance optimal strategy and the optimal investment policy that one would obtain with a CARA utility function has been noted by Basak and Chabakauri (2010) in their Remark 1 (Recovering time-consistent objective function). However, their main conclusion on this was that the equality of CARA-type strategies and dynamic mean-variance strategies generalizes the well-known equivalence of mean variance and CARA optimization in a one-period setting. It is clear from the analysis of this paper (and in particular from Corollary 16) that this generalization in fact does not hold. As a consequence, it is rather puzzling that a mean-variance optimizer operating in a complete market who wants to be time-consistent modifies her objective function in the spirit of Basak and Chabakauri (2010) and Björk and Murgoci (2010) and, as a result, ends up to behave as if she was CARA optimizer, implying behaving in a quite mean-variance inefficient way. From the practical point of view, the actual distance of the wealth obtained with CARA preferences from the mean-variance efficient frontier turns out to be relevant with not particularly long time horizons, as will be shown in Section 5 below. Therefore, in such a simple market, it is difficult to understand why in practice a mean-variance optimizer should be tempted to deviate from the pre-commitment policy.

5 Numerical application

5.1 General framework

In this section, with some numerical investigations we intend to analyze the extent of inefficiency of optimal portfolios for DC pension schemes whenever CARA and CRRA utility functions are used to solve the portfolio selection problem. We will do this by comparing optimal inefficient portfolios with the corresponding mean-variance efficient one. Theorems 8, 10 and 15 show that the inefficiency decreases with the risk aversion and increases with the time horizon and the Sharpe ratio. Therefore, in this section, we illustrate the extent of inefficiency when the risk aversion, the time horizon and the Sharpe ratio change. The parameters that remain constant throughout the examples are $r = 0.03$, $\lambda = 0.08$, $c = 0.1$, $x_0 = 1$. The value of the volatility σ will take values between 0.1 and 0.25, to limit the Sharpe ratio between 0.2 and 0.5. The time duration will be chosen to vary between one year and forty years, to allow for all possible entry ages (one-year duration – that is not typical for pension funds – has been selected in order to allow comparisons with the common portfolio selection problem). The choice of the risk aversion parameters is more delicate and less

obvious, and will be treated in the appropriate section (Section 5.2).

Section 5.2 reports results when the risk aversion changes, Section 5.3 those when the time horizon changes, and Section 5.4 those when the Sharpe ratio changes. In Section 5.5 we have carried out Monte Carlo simulations to illustrate numerically the impact of the Variance Inefficiency, in terms of distribution of final wealth. For the reader's convenience, in Section 5.2 we have also plotted the efficient frontier and the optimal portfolios in the standard deviation-mean plan. Similar figures could appear also in Sections 5.3 and 5.4, but we have omitted them, in order to limit the length of the paper.

In each situation, we focus on both the Variance Inefficiency and the Mean Inefficiency. However, for each of them we will follow two approaches, depending on whether the inefficiency is measured in absolute or in relative terms. These approaches are introduced in the following.

Focus on VI

In order to measure the Variance Inefficiency, we present two alternatives: the VI in absolute terms and the VI in relative terms. They are defined as follows:

$$\text{Absolute VI} := VI(X^*(T)) = \text{Var}(X^*(T)) - \text{Var}(\bar{X}(T)),$$

$$\text{Relative VI} := \frac{VI(X^*(T))}{\text{Var}(\bar{X}(T))} = \frac{\text{Var}(X^*(T))}{\text{Var}(\bar{X}(T))} - 1.$$

The relative VI has been introduced at this stage only, because it may help presenting and understanding results more than the absolute VI. In fact, the relative VI measures the percentage increase in the variance of final wealth when moving away from the efficient frontier with some EU-optimal policy, by keeping the same expected final wealth.

Remark 18 *Interestingly, we notice that in the exponential case, while the absolute VI varies with the risk aversion k , the relative VI does not. In fact:*

$$\frac{\text{Var}(X^*(T))}{\text{Var}(\bar{X}(T))} = \frac{\beta^2 T}{k^2} \cdot \frac{4\alpha^2}{e^{\beta^2 T} - 1} = \frac{\beta^2 T}{e^{\beta^2 T} - 1} \cdot \frac{4\alpha^2}{k^2} = \frac{\beta^2 T}{e^{\beta^2 T} - 1} \cdot \left(\frac{e^{\beta^2 T} - 1}{\beta^2 T} \right)^2 = \frac{e^{\beta^2 T} - 1}{\beta^2 T},$$

where we have used (40). We notice that the relative VI does, instead, depend on both time and Sharpe ratio. In particular, expectedly, it is increasing in both of them⁵.

Focus on MI

As before, in order to measure the Mean Inefficiency, we present two alternatives: the MI in absolute terms and the MI in relative terms. They are defined as follows:

$$\text{Absolute MI} := MI(X^*(T)) = E(\bar{X}(T)) - E(X^*(T)),$$

⁵This comes easily from the fact that $e^{-x} > 1 - x$ for $x \neq 0$.

$$\text{Relative MI} := \frac{MI(X^*(T))}{E(\bar{X}(T))} = 1 - \frac{E(X^*(T))}{E(\bar{X}(T))}.$$

Similarly to before, the relative MI has the advantage of measuring the inefficiency in relative terms. In fact, it consists of the percentage drop in expected value of final wealth, when moving away from the efficient frontier with some EU-optimal policy, by keeping the same level of risk.

Differently from before, the relative MI depends on the risk aversion also in the exponential case.

5.2 Changing the risk aversion

In this section, we estimate the inefficiency when the risk aversion changes. We choose typical values for the time duration and the Sharpe ratio, namely, $\beta = 0.33$ (implied by $\sigma = 0.15$), and $T = 20$. Therefore, the fund achievable under the riskless strategy is $\bar{x}_0 = 4.56$. It is far beyond the scope of this paper to discuss the choice of appropriate values for the parameters of absolute and relative risk aversion for the exponential and the power utility function. However, we notice that while there seems to be overall agreement across the literature regarding typical values of the RRA coefficient, this is not the case for the choice of the ARA coefficient. In addition, there seems to be little evidence of constant absolute risk aversion displayed by investors (see for instance, Guiso and Paiella (2008)). The value of $ARA = 20$ used by Battocchio and Menoncin (2004) is not appropriate in this context, because it would imply an α value of around 37, with implied final target $F = 4.67$, too much close to the basic value achievable with the riskless strategy, $\bar{x}_0 = 4.56$. Therefore, such high values of k , used also elsewhere in the literature (see for instance Jorion (1985)) have to be considered too high in this model with this time horizon. On the other hand, Guiso and Paiella (2008) suggest that the average absolute risk aversion should range around 0.02, a too low value for this context, implying a final target of $F = 129$, clearly unreasonable. We have then decided to test different levels of risk aversion for the power case, as in many previous works of this kind.

We will be considering $RRA=1$ (logarithmic utility), $RRA=2$ and $RRA=5$. In each case, we will find the corresponding parameter α of the M-V approach and then the corresponding k -value of the exponential model. The choice of $RRA = 2$ is motivated by the evident consensus in the literature regarding constant relative risk aversion coefficient of about 2. See, for instance Schlechter (2007), who sets a minimum bound of around 1.92 with no savings, and of 2.42 in the presence of savings. More specifically, regarding active members of pension schemes, Canessa and Dorich (2008) in a recent survey reported an overall average of relative risk aversion of about 1.81, depending on the age of the group under investigation. In particular, the RRA coefficient of the group under study varies between 1.59 and 1.88 for younger members, and between 2.21 and 2.25 for older ones. The choice of $RRA = 5$, motivated by the importance of showing results relative to higher risk aversion, is in line with similar choices for DC pension plans members (see Cairns et al. (2006)) and is consistent with the choice of the final target operated by Højgaard and Vigna (2007). Not least, $RRA=5$ gives an expected final fund very similar to that empirical found by application of the lifestyle strategy (an investment strategy largely adopted in DC plans) and therefore allows consistent comparisons in Section 5.5.

We have then the following three cases:

- low risk aversion: $RRA = 1$, that is the logarithmic utility function;
- medium risk aversion: $RRA = 2$;
- high risk aversion: $RRA = 5$.

Remark 19 We could have decided to base our analysis focusing on the M-V approach, by fixing a priori appropriate values of F , and finding a posteriori the corresponding values of $1 - \gamma$ and k . This would have been a sensible choice. However, most of the literature on portfolio selection uses the CRRA class of utility functions. Therefore, in order to facilitate comparisons with other works, we have focused on the EU approach with the CRRA class. This choice inevitably brings some drawbacks when the risk aversion displayed results to be too low and not consistent with likely choices of pension fund members. In particular, when the focus is on the MI, the values of RRA considered lead to F -values of the final target to be remarkably high. In this case a better choice of the RRA would probably be 8 or 10. In calculations not reported here, we have calculated absolute and relative MI with these higher values of RRA, and, obviously, the results found turn out to be slightly milder than those reported here. Nevertheless, from the qualitative point of view, all the conclusions still hold.

Focusing on the Variance Inefficiency

We here focus on the Variance Inefficiency. By using the relationship that links α and RRA when the expected values of final wealth $E(X^*(T))$ and $E(\bar{X}(T))$ are equated, we find the α -values corresponding to the different RRA values, then we find the corresponding values of the target F and finally the corresponding k -values in the exponential model. These results are reported in Table 1, that reports also the mean $E(\bar{X}(T))$ and the standard deviation $\sigma(\bar{X}(T))$ of the efficient portfolio.

RRA	α	F	ARA	MV efficient	MV efficient
$1 - \gamma$			k	$E(\bar{X}(T))$	$\sigma(\bar{X}(T))$
1	0.11	46.66	0.06	42.1	13.09
2	0.44	14.99	0.24	13.86	3.24
5	1.61	7.43	0.87	7.12	0.89

Table 1: Parameters values and M-V efficient portfolio with different RRA, when focus is on VI.

Table 2 reports for each risk profile the standard deviation of the final wealth in the MV, in the power and exponential cases, and the corresponding VI measures, both in absolute and in relative terms.

RRA	MV efficient	Power	Exponential	Power	Exponential	Power	Exponential
$1 - \gamma$	$\sigma(\bar{X}(T))$	$\sigma(X^*(T))$	$\sigma(X^*(T))$	Absolute VI	Absolute VI	Relative VI	Relative VI
1	13.09	120.76	25.18	14413.19	462.87	8415%	270%
2	3.24	11.94	6.23	132.2	28.39	1258%	270%
5	0.89	2.16	1.71	3.91	2.14	494%	270%

Table 2: Variance Inefficiency for different RRA values, when $T=20$ and $\beta = 0.33$.

As expected, the absolute VI decreases when the risk aversion increases. It is evident the great extent of inefficiency especially in the power case when the risk aversion is too low. Namely, the relative VI in the logarithmic case is 8415%. As shown in Remark 18, while the absolute VI in the exponential case decreases when the risk aversion increases, the relative VI remains constant and equal to 270%. A more effective interpretation of these figures will be provided in Section 5.5. We also observe the interesting feature that in each scenario the inefficiency produced by the

exponential utility function is lower than that of the power utility. In order to give a clearer picture of the comparison between portfolios, Figures 1, 2 and 3 report in the standard deviation/mean diagram the efficient frontier and the optimal portfolios in the cases RRA=1, 2, 5, respectively.

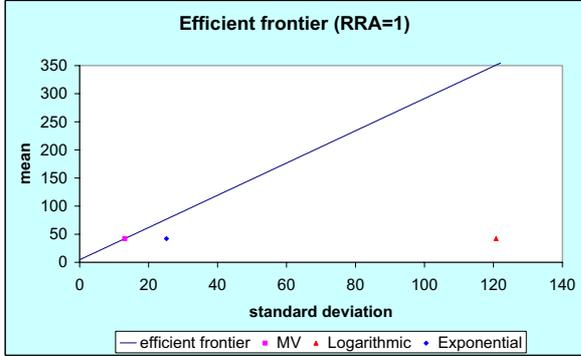


Figure 1.

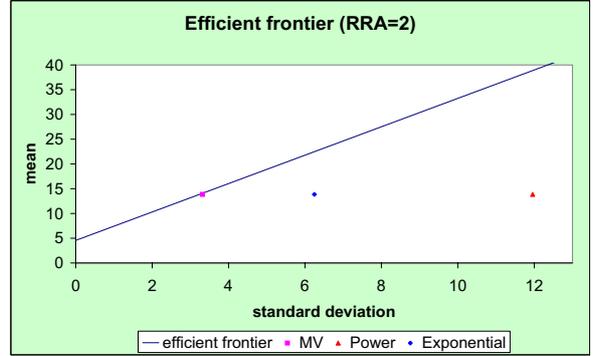


Figure 2.

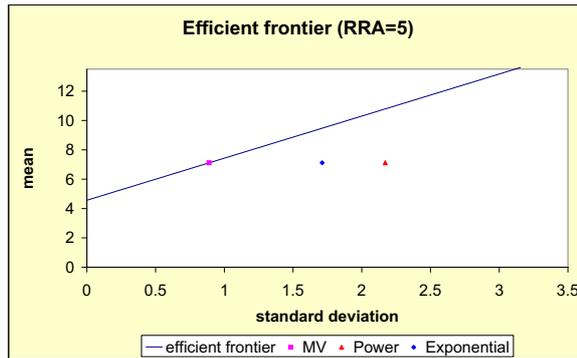


Figure 3.

The inefficiency for the logarithmic utility function is evident. This can be explained by observing that the inefficiency for the logarithmic utility function (45) is cubic in $e^{\beta^2 T}$, whereas it is quadratic in $e^{\beta^2 T}$ for the exponential case. Thus, with a high value of $\beta^2 T$ the inefficiency of the logarithmic function becomes more evident. This suggests that the logarithmic utility function is not appropriate for long time horizons or for high Sharpe ratios. As noted already, the exponential utility function is less inefficient than the power utility function.

Focusing on the Mean Inefficiency

We here focus on the Mean Inefficiency. By using the relationship that links α and RRA when the variances of final wealth $Var(X^*(T))$ and $Var(\bar{X}(T))$ are equated (not reported in the paper, but easily derivable by straight application of the definitions), we find the α -values corresponding to the different RRA values, then we find the corresponding values of the target F and finally the corresponding k -values in the exponential model. All the results relative to the three scenarios are

reported in Table 3, that reports also the mean $E(\bar{X}(T))$ and the standard deviation $\sigma(\bar{X}(T))$ of the efficient portfolio.

RRA $1 - \gamma$	α	F	ARA k	MV efficient $E(\bar{X}(T))$	MV efficient $\sigma(\bar{X}(T))$
1	0.01	393.07	0.01	350.97	120.76
2	0.12	42.99	0.12	38.82	11.94
5	0.66	11.54	0.68	10.78	2.16

Table 3: Parameters values and M-V efficient portfolio with different RRA, when focus is on MI.

Table 4 reports for each risk profile the expected value of the final wealth in the MV, in the power and exponential cases, and the corresponding MI measures, both in absolute and in relative terms.

RRA $1 - \gamma$	MV efficient $E(\bar{X}(T))$	Power $E(X^*(T))$	Exponential $E(X^*(T))$	Power Absolute MI	Exponential Absolute MI	Power Relative MI	Exponential Relative MI
1	350.97	42.1	184.59	308.86	166.38	88%	47%
2	38.82	13.85	22.37	24.96	16.45	64%	42%
5	10.78	7.11	7.79	3.66	2.98	34%	28%

Table 4: Mean Inefficiency for different RRA values, when $T=20$ and $\beta = 0.33$.

We observe that the drop in expected final wealth when the risk is kept fixed is quite high with low risk aversion and power utility function: it amounts to 88% for the logarithmic utility function and to 64% for power utility function with $RRA=2$. The relative MI for the exponential function ranges between 47% with low risk aversion to 28% with high risk aversion.

The results are probably more immediate to interpret in absolute terms. For instance, when $RRA=2$, with the same level of risk the distribution of final wealth for a MV-efficient optimizer is spread around the mean value of 39, while for the power EU-optimizer it is spread around the mean value of 14. In a more realistic setting with higher risk aversion ($RRA=5$), with the same level of risk, the distribution of final wealth for a MV-efficient optimizer is spread around the mean value of about 11, while for the power EU-optimizer it is spread around the mean value of approximately 7 and for the exponential EU-optimizer it is spread around the mean value of about 8. These results would be of clear interpretation for every pension fund member.

As before, Figures 4, 5, and 6 plot in the standard deviation/mean diagram the efficient frontier and the optimal portfolios in the cases $RRA = 1, 2, \text{ and } 5$, respectively.

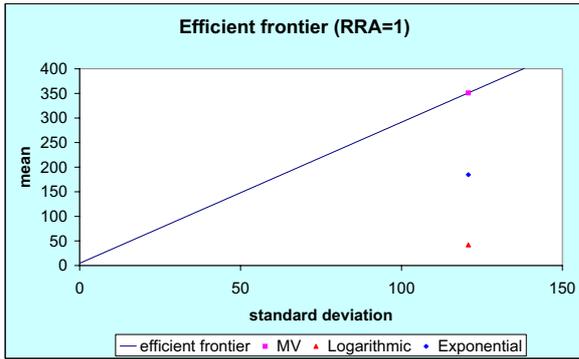


Figure 4.

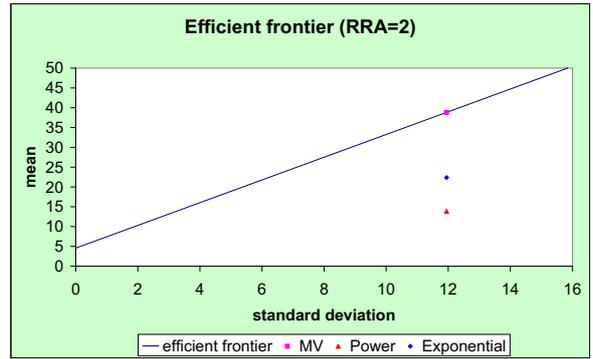


Figure 5.

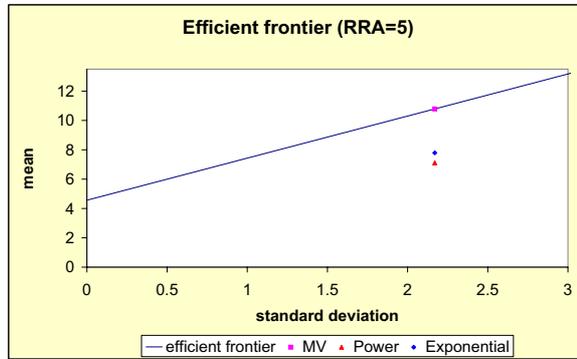


Figure 6.

As in the comparison with the VI, the optimal portfolios get closer to the efficient frontier when the risk aversion increases, that is an obvious result. As previously, the exponential portfolio performs less inefficiently than the power one. In the case of high risk aversion (RRA=5) the difference between the power and the exponential portfolios is quite small (and it is even smaller with RRA=10), while with low risk aversion there is a remarkable difference between the two.

The main conclusion that can be gathered from this section is that in realistic situations for decision making in a DC fund – pictured in Figure 6 – the loss in relative terms of expected final wealth if one wants to keep the same level of risk ranges between 27% and 34%. This result would be likely to discourage pension fund’s members to choose optimal not-efficient strategies driven by power or exponential utility function.

5.3 Changing the time horizon

In this section, we investigate the extent of the mean-variance inefficiency with different time horizons, by selecting $\beta = 0.33$ and RRA=5. In order to limit the length of the paper, in this section we report only the tables of VI and MI, disregarding the plot of the efficient frontier and the optimal portfolios in the mean-standard deviation plan. We calculate both VI and MI with time durations $T = 1, 2, 5, 10, 15, 20, 30$ and 40. Table 5 and 6 report absolute and relative MI and VI, respectively.

T	$E(\bar{X}(T))$	MV efficient $\sigma(\bar{X}(T))$	Power $\sigma(X^*(T))$	Exponential $\sigma(X^*(T))$	Power Absolute VI	Exponential Absolute VI	Power Relative VI	Exponential Relative VI
1	1.15	0.074	0.077	0.076	0.0004	0.0003	8%	5%
2	1.32	0.11	0.12	0.12	0.002	0.001	17%	11%
5	1.9	0.23	0.28	0.26	0.02	0.01	51%	33%
10	3.14	0.43	0.66	0.59	0.25	0.16	133%	83%
15	4.83	0.66	1.26	1.06	1.17	0.68	268%	157%
20	7.11	0.89	2.16	1.71	3.91	2.14	494%	270%
30	14.26	1.33	5.38	3.8	27.24	12.67	1528%	710%
40	26.88	1.72	11.85	7.51	137.67	53.43	4621%	1793%

Table 5: Variance Inefficiency for different durations T , when RRA=5 and $\beta = 0.33$.

T	$\sigma(\bar{X}(T))$	MV efficient $E(\bar{X}(T))$	Power $E(X^*(T))$	Exponential $E(X^*(T))$	Power Absolute MI	Exponential Absolute MI	Power Relative MI	Exponential Relative MI
1	0.07	1.1584	1.1574	1.1577	0.001	0.0007	0.09%	0.06%
2	0.12	1.33	1.325	1.327	0.004	0.003	0.36%	0.25%
5	0.28	1.94	1.9	1.91	0.04	0.03	2.34%	1.7%
10	0.66	3.47	3.14	3.22	0.33	0.25	9.5%	7.2%
15	1.26	6.09	4.83	5.1	1.25	0.99	21%	16%
20	2.16	10.78	7.11	7.79	3.66	2.98	34%	28%
30	5.38	35.33	14.26	17.16	21.07	18.17	60%	51%
40	11.85	119.84	26.88	36.05	92.96	83.79	78%	70%

Table 6: Mean Inefficiency for different durations T , when RRA=5 and $\beta = 0.33$.

Tables 5 and 6 show the expected – and maybe relieving – result that with short time durations the extent of inefficiency is quite small. In fact, with $T = 1, 2$ both absolute and relative VI and MI take very small values, and from the practical point of view the inefficiency can be neglected. This allows us to say that for the usual one-year time horizon portfolio selection, the EU-optimal policies with CARA and CRRA utility functions are a good approximation of the MV-efficient strategy. However, the scenario changes significantly when the time duration increases. Namely, for $T = 15, 20, 30$, that are typical time horizons for pension funds, the relative MI ranges between 16% and 60%. For longer time duration, e.g. $T = 40$, appropriate for young workers, the relative MI amounts to about 70% -80%. The relative VI values are even much higher than those of MI. To give a better idea, when the time horizon is 30 years the final wealth of the M-V optimizer is spread around the mean value of 35, while that of the power EU-optimizer is spread around the mean value of 14 and that of the exponential EU-optimizer is spread around the mean value of 17. When the time horizon is 15 years the final wealth of the M-V optimizer is spread around the mean value of 6, while that of both the power and the exponential EU-optimizer is spread around the mean value of about 5. Thus, we believe that the impact of mean-variance inefficiency is remarkable with long time durations and should be taken into serious consideration by pension fund investment managers, when deciding the appropriate portfolio selection rule.

5.4 Changing the Sharpe ratio

In this section, we investigate the extent of the mean-variance inefficiency with different Sharpe ratio, by selecting $T = 20$ and RRA=5. As before, in order to limit the length of the paper, in this section we report only the tables of VI and MI, disregarding the plot of the efficient frontier and the optimal portfolios in the mean-standard deviation plan. We calculate both VI and MI with Sharpe

ratio equal to $\beta = 0.2, 0.33, 0.4, 0.5$ (corresponding to $\sigma = 0.25, 0.15, 0.125, 0.1$ respectively). Tables 7 and 8 report absolute and relative MI and VI, respectively.

β	$E(\bar{X}(T))$	MV efficient $\sigma(\bar{X}(T))$	Power $\sigma(X^*(T))$	Exponential $\sigma(X^*(T))$	Power Absolute VI	Exponential Absolute VI	Power Relative VI	Exponential Relative VI
0.2	5.35	0.715	0.965	0.885	0.4208	0.272	82%	53%
0.33	7.11	0.89	2.16	1.71	3.914	2.141	494%	270%
0.4	8.65	0.84	3.19	2.28	9.51	4.51	1338%	635%
0.5	12.4	0.64	5.83	3.5	33.63	11.87	8068%	2848%

Table 7: Variance Inefficiency for different Sharpe ratio β , when $RRA=5$ and $T = 20$.

β	$\sigma(\bar{X}(T))$	MV efficient $E(\bar{X}(T))$	Power $E(X^*(T))$	Exponential $E(X^*(T))$	Power Absolute MI	Exponential Absolute MI	Power Relative MI	Exponential Relative MI
0.2	0.96	5.6313	5.3541	5.426	0.2771	0.2052	4.92%	3.64%
0.33	2.16	10.785	7.115	7.796	3.669	2.989	34.02%	27.71%
0.4	3.19	20.07	8.65	10.28	11.42	9.79	56.89%	48.77%
0.5	5.83	75.41	12.4	17.61	63.01	57.8	83.55%	76.64%

Table 8: Mean Inefficiency for different Sharpe ratio β , when $RRA=5$ and $T = 20$.

Tables 7 and 8 do not need many comments. When the Sharpe ratio is very low, i.e. with poor performances of the financial markets, the impact of mean inefficiency is quite small, ranging around 4%. However, with medium-high values of β , i.e. $\beta = 0.4$, i.e. in the presence of favourable market conditions, the relative MI amounts around a significant 49-57%, depending on the utility function chosen. This has to be explained as follows. A poor (good) performance of the risky asset produces on the optimal investment strategy the same effect produced by high (low) risk aversion, i.e. leads to less (more) aggressive strategies, that imply lower (higher) inefficiency.

5.5 Numerical simulations to understand the impact of VI

While the absolute and the relative MI are easy to understand, as the practical consequences of loss of expected final wealth are quite immediate for every lay person, the deep understanding of the increase in variance of final wealth is much less obvious. In other words, Figures 1, 2 and 3 or the values of the VI shown in Tables 2, 5 and 7 illustrate the extent of variance inefficiency and comparison between different portfolios, but may be of difficult understanding. Therefore, it may be desirable to provide some more useful insight about the practical consequences of inefficiency by deriving in a simulations framework the distribution of the final fund. To this aim, we have carried out Monte Carlo simulations for the risky asset, and have seen how the variance inefficiency translates into distribution of final wealth. For illustrative purposes, we will also report results for the lifestyle strategy (see e.g. Cairns et al. (2006)), widely used by DC pension plans in UK. In the lifestyle strategy the fund is invested fully in the risky asset until 10 years prior to retirement, and then is gradually switched into the riskless asset by switching 10% of the portfolio from risky to riskless asset each year.

In this example, we focus on the basic scenario characterized by $RRA = 5$, $\beta = 0.33$ and $T = 20$, that imply absolute VI equal to 3.91 in the power case and 2.14 in the exponential case, and relative VI equal to 494% in the power case and 270% in the exponential case. We have carried out 1000 Monte Carlo simulations and applied the optimal policies derived via the mean-variance approach

and via the EU approach with power and exponential utility functions, plus the lifestyle strategy. For consistent comparisons, for each of the four investment strategies tested we have created the same 1000 scenarios, by applying in each case the same stream of pseudo random numbers.

As in Højgaard and Vigna (2007), we see that all optimal investment strategies tend to apply a remarkable amount of borrowing for small values of x . Since borrowing is likely to be ruled out by the scheme itself or by the legislation, we introduce applicable suboptimal strategies which are cut off at 0 or 1 if the optimal strategy goes beyond the interval $[0, 1]$. For this reason, in the tables and figures that follow we will name each strategy adding the word "cut"⁶. Clearly, the lifestyle strategy does not need this cutting procedure. It must be said that imposing restriction on the controls would change substantially the formulation of the problem and would make it very difficult to tackle mathematically. Up to our knowledge, the only work where an optimization problem with constraints has been thoroughly treated in the accumulation phase of a DC scheme is Di Giacinto et al. (2010).

Table 9 reports for the four strategies considered some percentiles of the distribution of the final wealth, its mean and standard deviation, the probability of reaching the target and the mean shortfall, defined as the mean of the deviation of the fund from the target, given that the target is not reached. We remind that in the T-B approach with optimal policies the target can be approached very closely but cannot be reached. This explain the observed null probability of reaching the target with the M-V cut strategy. We also recall that the target in this case is 7.43. Figure 7 plots the suboptimal portfolios for the four strategies considered together with the efficient frontier.

Final wealth	MV cut	Power cut	Exponential cut	Lifestyle
5th perc.	3.65	4.05	4.05	3.8
25th perc.	6.36	5.28	5.6	5.13
50th perc.	7.1	6.45	6.71	6.61
75th perc.	7.32	7.93	7.88	8.72
95th perc.	7.4	10.63	9.57	13.57
Mean	6.54	6.78	6.77	7.32
Standard deviation	1.22	2.05	1.68	3.06
Prob reaching target	0	0.31	0.34	0.45
Mean shortfall	0.88	1.76	1.61	1.7

Table 9: Target $=F = 7.43$.

⁶It must be mentioned that suboptimal policies of the same type were applied by Gerrard et al. (2006) in the decumulation phase of a DC scheme, and proved to be satisfactory: with respect to the unrestricted case, the effect on the final results turned out to be negligible and the controls resulted to be more stable over time.

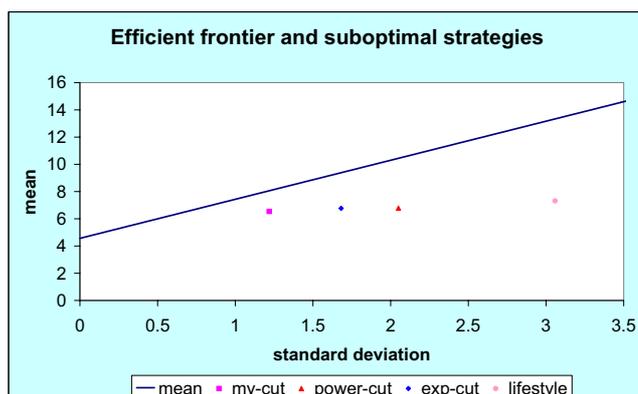


Figure 7.

A few comments can be gathered from Table 9 and Figure 7. Maybe the most important result is evident from Figure 7: the strategy which is most close to the efficient frontier is the mv-cut, followed by the exponential-cut, followed by the power-cut and then by the largely inefficient lifestyle strategy. In particular, for being efficient the lifestyle strategy should provide either a standard deviation of about 0.96 (rather than 3.06) with same level of mean, or a mean of 13.34 (rather than 7.32) with the same level of standard deviation.

The mv-cut, power-cut and exponential-cut provide, as expected, almost the same mean, but the mv-cut has a standard deviation much lower than that of the other two strategies. This can be found also by inspection of the percentiles of final wealth: in the mv-cut strategy in 75% of the scenarios the final wealth lies between 6.36 and 7.423 (that is the maximum value, not reported in Table 9). Considering that the target is 7.426, we find this is a satisfactory result.

The much lower dispersion of the mv-cut has as a direct consequence also on the mean shortfall value: the target is never reached, but the average distance from it is rather small, namely 0.88 which is 12% of the target. This is not the case for the power-cut and the exponential-cut strategies: in the former (latter) case the target is not reached in 69% (66%) of the cases with a mean shortfall of 1.76 (1.61), that amounts to 24% (22%) of the target.

As a final comment, we would like to add that it is certainly true that the higher dispersion of the exponential-cut and power-cut with respect to the mv-cut strategy means also a longer right tail of the distribution of final wealth, implying possibility of exceeding the target in about 30% of the cases. However, we believe that most active members of a pension scheme would not be willing to accept a significantly higher reduction in targeted wealth in exchange of having the chance of exceeding the targeted wealth in 30% of the cases.

6 Concluding remarks and further research

In this paper, we have investigated on different aspects of the mean-variance optimization approach and on its suitability in the accumulation phase of a defined contribution pension scheme.

First of all, we have solved the mean-variance portfolio selection problem in a DC plan via the

embedding technique introduced by Zhou and Li (2000). We have shown that the M-V approach is equivalent to a target-based approach, driven by the achievement of a desired target via minimization of a quadratic loss function. We have supported the target-based approach for portfolio selection in pension funds for several advantages.

Firstly, it transforms the difficult problem of selecting the individual's risk aversion coefficient of a generic utility function into the easiest task of choosing an appropriate target. In fact, it is evidently difficult for an agent to specify her own utility function and the corresponding risk aversion parameter. On the contrary, it is relatively easy to reason in terms of targets to reach. This was observed also by Kahneman and Tversky (1979) in their classical paper on Prospect Theory and more recently by Bordley and Li Calzi (2000).

Secondly, it is quite intuitive and largely adaptable to the member's needs and preferences, due to the flexibility in choosing the target – typically linked also to initial wealth and market conditions. Thirdly, it is efficient in the mean-variance sense. This makes this approach appealing for both the member and the investment manager. Indeed, for most individuals it is rather immediate to understand the mean-variance criterion. It is enough to show them two distributions of final wealth with same mean and different variances: in the context of pension funds, most workers would probably choose the distribution with lower variance. Alternatively, if the financial advisor shows two wealth distribution with same variance and different mean, the member would probably choose the distribution with higher mean. Moreover, the mean-variance criterion is still the most used tool to value and compare investment funds performances. It is evidently appreciable if member and investment manager pursue the same goal.

Then, we have addressed the issue of comparison between an optimal portfolio derived with the expected utility approach with a generic utility function and the corresponding efficient portfolio. We have defined the M-V inefficiency through the natural notions of Variance Inefficiency and Mean Inefficiency. In the special cases of CARA and CRRA utility functions, we have proven the intuitive but not trivial results that the inefficiency decreases with the risk aversion, and increases with the time horizon and the Sharpe ratio of the risky asset. The dependence of the inefficiency on initial wealth and contribution rate is different for the CARA and the CRRA classes, independent in the former case, increasing in the latter. As a corollary, we have proven the expected and known result that the CARA and the CRRA utility functions produce optimal portfolios that are inefficient in the mean-variance setting.

Finally, we have presented a numerical application aimed at showing the extent of inefficiency in DC pension schemes. We have defined the variance and the mean inefficiency also in relative terms. In particular, the relative VI measures the percentage increase of variance when moving away from the efficient frontier, keeping the same expected final wealth; the relative MI indicates the percentage drop in expected final wealth when the level of risk remains the same. We have calculated absolute and relative VI and MI in a number of cases, and tested their sensitivity to changes in the risk aversion, the time horizon and the Sharpe ratio. The most interesting results that we have found are those related to the dependence of the inefficiency on the time horizon. We have found out that with short time durations (i.e. up to five years) the inefficiency is quite small and both the CARA and the CRRA optimal portfolios can be considered good approximations of the mean-variance efficient portfolio. This can be seen an extension of Merton (1971) classical result, stating that at instantaneous level EU-optimality and M-V efficiency coincide. Furthermore, this seems also a relieving result, for most of the financial literature on portfolio selection makes use of these two classes of utility functions. However, when the time horizon increases, e.g. for durations longer than 15 years, the inefficiency increases remarkably and makes results likely to be unacceptable from the member's point of view. Indeed, with time durations of 15-30 years, quite typical

for pension funds, the relative MI ranges between 15% and 60%. This significant loss of expected value of final wealth is likely to be highly undesirable for pension fund members. Similarly, when the inefficiency is measured with the variance inefficiency, for durations longer than 15 years the increase in standard deviation of final wealth ranges between about 150% and as much as 1500%. This means that the member should accept in most cases a significantly higher negative deviation from the target, in exchange of having the chance of exceeding it in a few cases. Considering that the targeted wealth is future retirement income, we may expect the average prudential pension fund member not to be willing to accept these conditions. Concluding, the practical impact of inefficiency with long time horizons illustrated by these numerical results provides another element to support the T-B approach in DC pension schemes.

This work leaves ample scope for further research. The two questions arisen in Section 3 could be answered for other classes of utility functions, still within the more general HARA class. We could consider a model with time-dependent drift and volatility. At a much more difficult level, stochastic volatility could be included in the model. The extension to the multi-period discrete time framework is also appealing. Finally, the inclusion of a stochastic interest rate in the financial market is also worth exploring. Namely, a financial market that includes bond assets is crucial in a long time horizon context such as pension funds. In addition, this extension would be in line with the most advanced models for portfolio allocation in pension funds (see, for instance, Battocchio and Menoncin (2004), Boulier et al. (2001), Cairns et al. (2006), Deelstra et al. (2003), Gao (2008)). Therefore, this challenging task is in the agenda for future research.

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Appendix

A The efficient frontier

To find the efficient frontier of portfolios let us introduce the following notation:

$$\begin{aligned} y_0 &\equiv x_0 + \frac{c}{r} & \theta &\equiv 1 - e^{-\beta^2 T} \\ \rho &\equiv e^{-(\beta^2 - r)T} & \phi &\equiv e^{-(\beta^2 - 2r)T}. \end{aligned} \quad (58)$$

From (12) and (13), we have:

$$E(\bar{X}(T)) = y_0 \rho - \frac{c}{r}(1 - \theta) + \delta \theta, \quad (59)$$

and

$$E(\bar{X}^2(T)) = y_0^2 \phi - 2\frac{c}{r}y_0 \rho + \frac{c^2}{r^2}(1 - \theta) + \delta^2 \theta.$$

Therefore

$$\begin{aligned} Var(\bar{X}(T)) &= E(\bar{X}^2(T)) - E(\bar{X}(T))^2 = y_0^2 \phi - 2\frac{c}{r}y_0 \rho + \frac{c^2}{r^2}(1 - \theta) + \delta^2 \theta - \\ &- y_0^2 \rho^2 - \frac{c^2}{r^2}(1 - \theta)^2 - \delta^2 \theta^2 + 2y_0 \rho \frac{c}{r}(1 - \theta) - 2y_0 \rho \delta \theta + 2\frac{c}{r}(1 - \theta)\delta \theta. \end{aligned}$$

After a few passages and noticing that

$$\phi - \rho^2 = \phi \theta,$$

we have

$$Var(\bar{X}(T)) = y_0^2 \theta \phi + \theta(1 - \theta) \left(\delta + \frac{c}{r} \right)^2 - 2y_0 \rho \theta \left(\delta + \frac{c}{r} \right).$$

From (59), we have

$$\theta \left(\delta + \frac{c}{r} \right) = E(\bar{X}(T)) - y_0 \rho + \frac{c}{r}.$$

Therefore

$$\begin{aligned} Var(\bar{X}(T)) &= y_0^2 \theta \phi + \theta(1-\theta) \frac{(E(\bar{X}(T)) - y_0 \rho + \frac{c}{r})^2}{\theta^2} - 2y_0 \rho \left(E(\bar{X}(T)) - y_0 \rho + \frac{c}{r} \right) \\ &= \frac{1-\theta}{\theta} \left[\frac{y_0^2 \phi \theta^2}{1-\theta} + \left(E(\bar{X}(T)) - y_0 \rho + \frac{c}{r} \right)^2 - \frac{2y_0 \rho \theta}{1-\theta} \left(E(\bar{X}(T)) - y_0 \rho + \frac{c}{r} \right) \right] \\ &= \frac{1-\theta}{\theta} \left[\frac{\phi \theta^2 + \rho^2 + \rho^2 \theta}{1-\theta} y_0^2 + 2E(\bar{X}(T)) \frac{c}{r} + \frac{c^2}{r^2} + E(\bar{X}(T))^2 - 2y_0 \frac{\rho}{1-\theta} \left(E(\bar{X}(T)) + \frac{c}{r} \right) \right]. \end{aligned}$$

Now notice that

$$\frac{\phi \theta^2 + \rho^2 + \rho^2 \theta}{1-\theta} = e^{2rT} \quad \text{and} \quad \frac{\rho}{1-\theta} = e^{rT}.$$

So that

$$\begin{aligned} Var(\bar{X}(T)) &= \frac{1-\theta}{\theta} \left[y_0^2 e^{2rT} + \left(E(\bar{X}(T)) + \frac{c}{r} \right)^2 - 2y_0 e^{rT} \left(E(\bar{X}(T)) + \frac{c}{r} \right) \right] \\ &= \frac{1-\theta}{\theta} \left[\left(E(\bar{X}(T)) + \frac{c}{r} \right) - y_0 e^{rT} \right]^2 \\ &= \frac{e^{-\beta^2 T}}{1-e^{-\beta^2 T}} \left[E(\bar{X}(T)) - \left(x_0 e^{rT} + c \frac{e^{rT}-1}{r} \right) \right]^2 \\ &= \frac{e^{-\beta^2 T}}{1-e^{-\beta^2 T}} \left[E(\bar{X}(T)) - \bar{x}_0 \right]^2 \\ &= \frac{e^{-\beta^2 T}}{1-e^{-\beta^2 T}} \left(\frac{e^{\beta^2 T}-1}{2\alpha} \right)^2 = \frac{e^{\beta^2 T}-1}{4\alpha^2}, \end{aligned}$$

where in the forth last equality we have used (58).

B Derivation of expected values and variances with EU approach

B.1 Exponential utility function

The value function of this problem is (see for instance Devolder et al. (2003)):

$$V(t, x) = -\frac{1}{k} \exp[-k(a(t) + b(t)(x - f(t)))],$$

with

$$\begin{aligned} a(t) &= \frac{\beta^2}{2k}(T-t), \\ b(t) &= e^{r(T-t)}, \\ f(t) &= \frac{c}{r}(e^{-r(T-t)} - 1). \end{aligned}$$

The optimal amount invested in the risky asset at time t if the wealth is x is:

$$xy^*(t, x) = \frac{\beta}{\sigma k} e^{-r(T-t)}. \quad (60)$$

The evolution of the fund under optimal control $X^*(t)$ is given by:

$$dX^*(t) = \left[\frac{\beta^2}{k} e^{-r(T-t)} + xr + c \right] dt + \frac{\beta}{k} e^{-r(T-t)} dW(t). \quad (61)$$

By application of Ito's lemma to (61) the evolution of its square, $(X^*(t))^2$ is given by

$$d(X^*(t))^2 = \left[2r(X^*(t))^2 + \left(2c + \frac{2\beta^2}{k} e^{-r(T-t)} \right) X^*(t) + \frac{\beta^2}{k^2} e^{-2r(T-t)} \right] dt + \frac{2\beta X^*(t)}{k} e^{-r(T-t)} dW(t). \quad (62)$$

If we take expectations on lhs and rhs in both (61) and (62) we have the following ODEs:

$$dE(X^*(t)) = \left[rE(X^*(t)) + \left(c + \frac{\beta^2}{k} e^{-r(T-t)} \right) \right] dt, \quad (63)$$

$$dE((X^*(t))^2) = \left[2rE((X^*(t))^2) + \left(2c + \frac{2\beta^2}{k} e^{-r(T-t)} \right) E(X^*(t)) + \frac{\beta^2}{k^2} e^{-2r(T-t)} \right] dt. \quad (64)$$

Solving (63) and (64) with initial conditions $E(X^*(0)) = x_0$ and $E((X^*(0))^2) = x_0^2$, gives us:

$$E(X^*(t)) = x_0 e^{rt} + \frac{c}{r} (e^{rt} - 1) + \frac{\beta^2 t}{k} e^{-r(T-t)},$$

and

$$E((X^*(t))^2) = x_0^2 e^{2rt} + \frac{c^2}{r^2} (1 - e^{2rt}) + \frac{2c}{r} \left(x_0 + \frac{c}{r} - \frac{\beta^2}{kr} e^{-rT} \right) e^{rt} (e^{rt} - 1) + \frac{2\beta^2}{k} \left(\frac{c}{r^2} + \left(x_0 + \frac{c}{r} \right) t \right) e^{2rt-rT} - \frac{2\beta^2 c}{kr^2} (1 + rt) e^{-r(T-t)} + \frac{\beta^2 t}{k^2} (1 + \beta^2 t) e^{-2r(T-t)}.$$

Therefore, at retirement $t = T$ we have:

$$E(X^*(T)) = \left(x_0 + \frac{c}{r} \right) e^{rT} - \frac{c}{r} + \frac{\beta^2 T}{k} = \bar{x}_0 + \frac{\beta^2 T}{k},$$

and

$$E((X^*(T))^2) = e^{2rT} \left(x_0 + \frac{c}{r} \right)^2 - \frac{2c}{r} \left(x_0 + \frac{c}{r} \right) e^{rT} + \frac{c^2}{r^2} + \frac{2\beta^2 T}{k} \left(x_0 + \frac{c}{r} \right) e^{rT} + \frac{\beta^2 T}{k^2} - \frac{2c\beta^2 T}{kr} + \frac{(\beta^2 T)^2}{k^2}.$$

Thus, the variance of the final fund is

$$Var(X^*(T)) = E((X^*(T))^2) - E^2(X^*(T)) = \frac{\beta^2 T}{k^2}.$$

B.2 Logarithmic utility function

The value function is:

$$V(t, x) = \ln(b(t)) + \ln(x + a(t)),$$

with

$$a(t) = \frac{c}{r} (1 - e^{-r(T-t)}),$$

$$b(t) = e^{(r + \frac{\beta^2}{2})(T-t)}.$$

The optimal amount invested in the risky asset is:

$$xy^*(t, x) = \frac{\beta}{\sigma} \left(x + \frac{c}{r} (1 - e^{-r(T-t)}) \right).$$

The evolution of the fund under optimal control $X^*(t)$ is given by:

$$dX^*(t) = \left[(r + \beta^2) X^*(t) + \left(c + \frac{c\beta^2}{r} (1 - e^{-r(T-t)}) \right) \right] dt + \beta \left(X^*(t) + \frac{c}{r} (1 - e^{-r(T-t)}) \right) dW(t). \quad (65)$$

By application of Ito's lemma to (65) the evolution of its square, $(X^*(t))^2$ is given by

$$d(X^*(t))^2 = \left[(2r + 3\beta^2) (X^*(t))^2 + \left(2c + \frac{4c\beta^2(1-e^{-r(T-t)})}{r} \right) X^*(t) + \frac{\beta^2 c^2 (1-e^{-r(T-t)})^2}{r^2} \right] dt + 2\beta X^*(t) \left(X^*(t) + \frac{c(1-e^{-r(T-t)})}{r} \right) dW(t). \quad (66)$$

If we take expectations on lhs and rhs in both (65) and (66) we have the following ODEs:

$$dE(X^*(t)) = \left[(r + \beta^2) E(X^*(t)) + \left(c + \frac{c\beta^2(1-e^{-r(T-t)})}{r} \right) \right] dt, \quad (67)$$

$$dE((X^*(t))^2) = \left[(2r + 3\beta^2) E((X^*(t))^2) + \left(2c + \frac{4c\beta^2(1-e^{-r(T-t)})}{r} \right) E(X^*(t)) + \frac{\beta^2 c^2 (1-e^{-r(T-t)})^2}{r^2} \right] dt. \quad (68)$$

Solving (67) and (68) with initial conditions $E(X^*(0)) = x_0$ and $E((X^*(0))^2) = x_0^2$, gives us:

$$E(X^*(t)) = x_0 e^{At} + \frac{c}{r} (e^{At} - 1 + e^{-r(T-t)} - e^{-rT+At}),$$

with

$$A = r + \beta^2,$$

and

$$E((X^*(t))^2) = x_0^2 e^{Kt} + \frac{c^2}{r^2} [(e^{-2r(T-t)} - e^{-2rT+Kt}) - 2(e^{-r(T-t)} - e^{-rT+Kt}) + 2(x_0 \frac{r}{c} + 1 - e^{-rT})(e^{At} - e^{Kt} - e^{-r(T-t)+At} + e^{-rT+Kt}) + 1 - e^{Kt}],$$

with

$$K = 2r + 3\beta^2.$$

Therefore, at retirement $t = T$ we have:

$$E(X^*(T)) = e^{AT} (x_0 + \frac{c}{r} (1 - e^{-rT})) = \bar{x}_0 e^{\beta^2 T},$$

and

$$E((X^*(T))^2) = e^{KT} (x_0 + \frac{c}{r} (1 - e^{-rT}))^2.$$

Thus, the variance of the final fund is

$$Var(X^*(T)) = (e^{KT} - e^{2AT}) (x_0 + \frac{c}{r} (1 - e^{-rT}))^2 = (E(X^*(T)))^2 (e^{\beta^2 T} - 1).$$

B.3 Power utility function

The value function is (see for instance Devolder et al. (2003)):

$$V(t, x) = b(t) \frac{(x - a(t))^\gamma}{\gamma},$$

with

$$a(t) = -\frac{c}{r} (1 - e^{-r(T-t)}),$$

$$b(t) = \exp \left[\gamma \left(r + \frac{\beta^2}{2(1-\gamma)} \right) (T-t) \right].$$

The optimal amount invested in the risky asset at time t if the wealth is x is:

$$xy^*(t, x) = \frac{\beta}{\sigma(1-\gamma)} \left(x + \frac{c(1-e^{-r(T-t)})}{r} \right).$$

The evolution of the fund under optimal control $X^*(t)$ is given by:

$$dX^*(t) = \left[\left(r + \frac{\beta^2}{1-\gamma} \right) X^*(t) + \left(c + \frac{c\beta^2(1-e^{-r(T-t)})}{r(1-\gamma)} \right) \right] dt + \frac{\beta}{1-\gamma} \left(X^*(t) + \frac{c(1-e^{-r(T-t)})}{r} \right) dW(t). \quad (69)$$

By application of Ito's lemma to (69) the evolution of its square, $(X^*(t))^2$ is given by

$$d(X^*(t))^2 = \left[\left(2r + \frac{2\beta^2}{1-\gamma} + \frac{\beta^2}{(1-\gamma)^2} \right) (X^*(t))^2 + \left(2c + \frac{2c\beta^2(1-e^{-r(T-t)})(2-\gamma)}{r(1-\gamma)^2} \right) X^*(t) + \frac{\beta^2 c^2 (1-e^{-r(T-t)})^2}{r^2 (1-\gamma)^2} \right] dt + \frac{2\beta X^*(t)}{1-\gamma} \left(X^*(t) + \frac{c(1-e^{-r(T-t)})}{r} \right) dW(t). \quad (70)$$

If we take expectations on lhs and rhs in both (69) and (70) we have the following ODEs:

$$dE(X^*(t)) = \left[\left(r + \frac{\beta^2}{1-\gamma} \right) E(X^*(t)) + \left(c + \frac{c\beta^2(1-e^{-r(T-t)})}{r(1-\gamma)} \right) \right] dt, \quad (71)$$

$$dE((X^*(t))^2) = \left[\left(2r + \frac{2\beta^2}{1-\gamma} + \frac{\beta^2}{(1-\gamma)^2} \right) E((X^*(t))^2) + \left(2c + \frac{2c\beta^2(1-e^{-r(T-t)})(2-\gamma)}{r(1-\gamma)^2} \right) E(X^*(t)) + \frac{\beta^2 c^2 (1-e^{-r(T-t)})^2}{r^2 (1-\gamma)^2} \right] dt. \quad (72)$$

Solving (71) and (72) with initial conditions $E(X^*(0)) = x_0$ and $E((X^*(0))^2) = x_0^2$, gives us:

$$E(X^*(t)) = x_0 e^{At} + \frac{c}{r} (e^{At} - 1 + e^{-r(T-t)} - e^{-rT+At}),$$

with

$$A = r + \frac{\beta^2}{1-\gamma},$$

and

$$E((X^*(t))^2) = x_0^2 e^{Kt} + \frac{c^2}{r^2} [(e^{-2r(T-t)} - e^{-2rT+Kt}) - 2(e^{-r(T-t)} - e^{-rT+Kt}) + 2(x_0 \frac{r}{c} + 1 - e^{-rT})(e^{At} - e^{Kt} - e^{-r(T-t)+At} + e^{-rT+Kt}) + 1 - e^{Kt}],$$

with

$$K = 2r - \frac{\beta^2(2\gamma-3)}{(1-\gamma)^2}.$$

Therefore, at retirement $t = T$ we have:

$$E(X^*(T)) = e^{AT} \left(x_0 + \frac{c}{r} (1 - e^{-rT}) \right) = \bar{x}_0 e^{\frac{\beta^2 T}{1-\gamma}},$$

and

$$E((X^*(T))^2) = e^{KT} \left(x_0 + \frac{c}{r} (1 - e^{-rT}) \right)^2.$$

Thus, the variance of the final fund is

$$\text{Var}(X^*(T)) = (e^{KT} - e^{2AT})\left(x_0 + \frac{C}{r}(1 - e^{-rT})\right)^2 = \left(e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1\right)(E(X^*(T)))^2.$$

It is worth noticing that, apart from the value function, the results for the logarithmic utility can be found by setting $\gamma = 0$ in the power case, that is an expected result.