Collegio Carlo Alberto

Interest Rate Rules, Endogenous Cycles, and Chaotic Dynamics in Open Economies

Marco Airaudò

Luis-Felipe Zanna

No. 171

December 2010

Carlo Alberto Notebooks

www.carloalberto.org/working_papers

© 2010 by Marco Airaudò and Luis-Felipe Zanna. Any opinions expressed here are those of the authors and not those of the Collegio Carlo Alberto.

Interest Rate Rules, Endogenous Cycles, and Chaotic Dynamics in Open Economies *

Marco Airaudo[†] Luis-Felipe Zanna[‡]

This Draft: December 10th, 2010

Abstract

In this paper we present an extensive analysis of the consequences for global equilibrium determinacy in flexible-price open economies of implementing active interest rate rules, i.e., monetary rules where the nominal interest rate responds more than proportionally to changes in inflation. We show that conditions under which these rules generate aggregate instability by inducing liquidity traps, endogenous cycles, and chaotic dynamics depend on particular characteristics of open economies, including the degree of trade openness and the degree of exchange rate pass-through into import prices. For instance, in our model, we find that a rule that responds to expected future inflation is more prone to induce endogenous cyclical and chaotic dynamics the more open the economy and the higher the degree of exchange rate pass-through.

Keywords: Small Open Economy, Interest Rate Rules, Taylor Rules, Multiple Equilibria, Chaos and Endogenous Fluctuations.

JEL Classifications: E32, E52, F41

*The authors would like to thank the following people for comments and suggestions: Gian-Italo Bischi, David Bowman, Chris Erceg, Jon Faust, Laura Gardini, Dale Henderson, and the seminar participants at the Federal Reserve Board, LUISS, the University of Urbino, the Society for Non-Linear Dynamics and Econometrics and the Latin American Meeting of the Econometric Society. Part of this work was completed while Airaudo was Junior Fellow and then Visiting Fellow at Collegio Carlo Alberto. He is grateful to the Collegio for all the hospitality and research support. The views expressed in this paper are solely the responsibility of the authors and should not be interpreted as reflecting those of the IMF.

[†]Corresponding Author. Department of Economics, LeBow College of Business, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104, USA. Email: marco.airaudo@drexel.edu, Phone: +1-215-8956982.

[‡]International Monetary Fund, 700 19th Street, N.W., Washington, D.C., 20431; email: fzanna@imf.org.

1 Introduction

The recent financial crisis has revived concerns in many countries about the comeback of liquidity traps, where monetary policy ceases to be effective. With falling asset and housing prices, and the mounting fear of a credit crunch as well as deflation, central banks across the world have been forced to cut interest rates to unprecedented low levels, rapidly approaching the zero lower bound. However, even in the absence of a crisis, liquidity traps can be also driven by people's arbitrary revisions in expectations, as argued by Benhabib, Schmitt-Grohé, and Uribe (2001b, 2002a,b). Given the zero lower bound for nominal interest rates, if monetary policy takes the form of an *active* Taylor-type interest rate rule, whereby the nominal interest rate reacts more than proportionally with respect to inflation, then nominal interest rates may decline towards zero, simply because expectations of a deflationary spiral become self-validated in a Rational Expectations Equilibrium. In fact, the liquidity trap equilibrium is only one of the varieties of equilibria, among self-fulfilling endogenous cycles and chaotic dynamics, that can be induced by an active rule.

Most of the literature about interest rate rules and self-fulfilling endogenous cyclical and chaotic dynamics, including liquidity traps, has been developed in the context of a *global* equilibrium analysis of closed economy models.¹ In this paper, we take one step further and assess how the existence of these self-fulfilling fluctuations may be affected by opening the economy to international trade and the presence of complete international capital markets. Interestingly, our global equilibrium analysis shows that the presence of these endogenous fluctuations is related to two prominent features of open economies, namely the degree of trade openness and the degree of exchange rate pass-through into import prices.

We find that in a traditional flexible-price Small Open Economy (SOE) model with traded and non-traded goods and imperfect exchange pass-through, an active interest rate rule that responds to expected future CPI inflation is more prone to induce endogenous cyclical and chaotic dynamics the more open the economy and the higher the exchange rate pass-through. As in closed economy models, the global equilibrium analysis unveils the existence of two steady states: a high steady state, frequently associated with the inflation target of the monetary authority, and a low steady state, which becomes the unintended target or state associated with liquidity traps. In this context, and depending on open economy features, we find that if consumption and money are Edgeworth complements in utility, these cyclical and chaotic dynamics occur around the liquidity trap steady state. If instead they are substitutes, these dynamics appear around the high steady state. Key to these results is the elasticity of the real exchange rate with respect to the nominal interest rate, set by the rule. This elasticity is *qualitatively* affected by the complementarity/substitutability between consumption and money, and *quantitatively* affected by the degree of trade openness.

Our results are relevant for monetary policy analysis. They suggest that in order to avoid destabilizing endogenous cycles and chaos in *open* economies, the design and implementation of interest rate rules should

¹In contrast to a local equilibrium analysis, a global equilibrium analysis does not rely on local approximations of a model. See for instance, Alstadheim and Henderson, (2006), Benhabib et al. (2001b, 2002a,b), and Eusepi (2007), among others.

take into account not only the interest response coefficient to inflation, but also structural characteristics of the economies, such as the degrees of openness and exchange rate pass-through. This is important given that countries differ in terms of monetary policy responses to inflation and structural characteristics. For instance, Clarida, Galí, and Gertler (1998) and Lubik and Schorfheide (2007), among others, provide empirical evidence of some small open economies following active interest rate rules with different responses to inflation; while Campa and Goldberg (2006) and Frankel, Parsley, and Wei (2005) suggest that the degrees of openness and exchange rate pass-through into import prices vary across economies and have changed over time.

Our work contributes to the revival of theoretical and empirical literature about the macroeconomic consequences of implementing Taylor-type interest rate rules in SOE models.² Previous works have tried to identify conditions under which these rules may lead to multiple *local* equilibria, as they rely on local approximations of the models.³ For instance, De Fiore and Liu (2005), Llosa and Tuesta (2008), and Zanna (2003), among others, discuss the importance of the degree of openness of the economy for local determinacy. Zanna (2003) also discusses the importance of the degree of the exchange rate pass-through. By restricting the attention to the local dynamics, disregarding the zero lower bound, the open economy literature has gained in tractability but has also overlooked a possibly wider set of equilibrium dynamics. In this regard, and to the best of our knowledge, our work is the first attempt in the open economy literature to understand how interest rate rules may lead to global endogenous fluctuations.

In addition, our paper contributes to the findings of the closed economy literature by pointing out that the existence of cyclical and chaotic dynamics under interest rate rules is to some extent robust to specific ways in which money is introduced in a model and to high relative risk aversion coefficients. Benhabib et al. (2002a) highlight the role of money in the production function for the occurrence of these dynamics and find that chaos always occurs for sufficiently low coefficients of risk aversion.⁴ Our results suggest that these dynamics occur even if money is introduced in the utility function and that in fact the range of risk aversion coefficients consistent with chaotic dynamics is larger for more open economies.

The remainder of this paper is organized as follows. In Section 2, we present a flexible-price model with its main assumptions. We define the open economy equilibrium and derive some basic steady state results. In Section 3, we pursue local and global equilibrium analyses for an interest rate rule that responds to expected future CPI inflation, focusing on the role played by the degree of openness. In Section 4 we investigate how the degree of exchange rate pass-through can affect the global equilibrium analysis under this rule. In Section 5, we pursue a sensitivity analysis to gauge the robustness of our results with respect to rules that respond to current and past CPI inflation, the presence of incomplete markets, and different timings of real

²See Ball (1999), Clarida et al. (1998), Galí and Monacelli (2008), and Svensson (2000), among others.

³For two country models see Batini, Levine, and Pearlman (2004), Benigno and Benigno (2008) and Leith and Wren-Lewis (2009), among others.

⁴See also Michener and Ravikumar (1998) that proves the existence of cycles and chaos for cash-in-advance economies under money-growth rules.

money balances in liquidity services. Finally, Section 6 concludes.

2 A Flexible-Price Model

2.1 The Household-Firm Unit

Consider a Small Open Economy (SOE) populated by a large number of identical and infinitely lived household-firm units. Each unit derives utility from consumption (c_t), real money balances (m_t^d), and not working ($1 - h_t^T - h_t^N$) according to

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{[(c_t)^\gamma (m_t^d)^{1-\gamma}]^{1-\sigma} - 1}{1-\sigma} + \psi(1 - h_t^T - h_t^N) \right\} \quad (1)$$

$$c_t = (c_t^T)^\alpha (c_t^N)^{(1-\alpha)}, \quad (2)$$

where $\beta, \gamma \in (0, 1)$, and $\psi, \sigma > 0$ but $\sigma \neq 1$; E_0 is the expectations operator conditional on the set of information available at time 0; c_t^T and c_t^N denote the consumption of traded and non-traded goods; $m_t^d = \frac{M_t^d}{p_t}$ are real money balances defined in terms of the Consumer Price Index (CPI) p_t ; h_t^T and h_t^N stand for labor supplied to the production of traded and non-traded goods respectively, and $\alpha \in (0, 1)$ is the share of traded goods in the consumption aggregator (2). We interpret this share as a measure of the degree of trade openness of the economy. As α goes to zero, domestic agents do not value internationally traded goods and the economy is basically closed. Whereas if α goes to one, non-traded goods are negligible in consumption and the economy becomes completely open.

The functional forms in (1) are general enough to derive the main results. They allow us to show *analytically* how cyclical dynamics induced by an interest rate rule depend on the degree of openness α .⁵ To keep our analysis as general as possible, we do not place *a priori* assumptions on the sign of the cross-partial derivative of utility with respect to consumption and real money balances, U_{cm} . By defining $U = \frac{(c_t^\gamma m_t^{1-\gamma})^{1-\sigma} - 1}{1-\sigma}$ we obtain that the sign of U_{cm} satisfies $\text{sign}\{U_{cm}\} = \text{sign}\{(1-\sigma)\}$.⁶ This allows us to distinguish between the following cases: (i) the case of Edgeworth substitutes for which $U_{cm} < 0$, when $\sigma > 1$; and the case of complements for which $U_{cm} > 0$, when $\sigma < 1$.⁷ Moreover, given that $\gamma \in (0, 1)$ and

⁵We could consider a CES function for aggregate consumption (2) to emphasize the fact that the intratemporal elasticity of substitution between the two types of consumption goods can be different from one. However, this would not affect our conclusions, but simply prevent us from obtaining analytical results.

⁶Both Benhabib et al. (2001a) and Matsuyama (1990) show that the sign of this derivative has important consequences for the existence of local or global sunspot equilibria in closed economy monetary models. From a theoretical point of view, a negative partial derivative can be motivated on the following grounds: a) it is observationally equivalent to a model with real balances entering into the production function (Benhabib et al., 2001a); or b) the resources saved by holding real money balances are imperfect substitutes of the consumption good (Matsuyama, 1990). Castelnovo (2008) reviews the empirical literature on this and provides further evidence on the non-separability between consumption and real money balances in utility.

⁷The case of $\sigma = 1$ corresponds to the case of separability between consumption and money in the utility function. It implies

that the coefficient of relative risk aversion (CRRA) can be expressed as $\tilde{\sigma} \equiv -\frac{U_{ccc}}{U_c} = 1 - \gamma(1 - \sigma)$, then $\sigma \gtrless 1$ implies $\tilde{\sigma} \gtrless 1$. As a result of this, we will refer to σ as the “risk aversion parameter.”

The representative unit produces traded and non-traded goods by employing labor and the technologies

$$y_t^T = z_t^T (h_t^T)^{\theta_T} \quad \text{and} \quad y_t^N = z_t^N (h_t^N)^{\theta_N}, \quad (3)$$

where $\theta_T, \theta_N \in (0, 1)$ and z_t^T and z_t^N are productivity shocks following stationary AR(1) stochastic processes. We assume that these shocks are the sole source of fundamental uncertainty.

The Law of One Price (LOP) holds for traded goods and we normalize the foreign price of the traded good, P_t^{Tw} , to one.⁸ Hence $P_t^T = \mathcal{E}_t$, where P_t^T is the domestic currency price of traded goods and $\mathcal{E}_t$ is the nominal exchange rate. This and equation (2) imply the following expression for the CPI:

$$p_t \equiv \frac{(\mathcal{E}_t)^\alpha (P_t^N)^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}}, \quad (4)$$

which can be used to derive the gross CPI-inflation rate

$$\pi_t \equiv \frac{p_t}{p_{t-1}} = \epsilon_t^\alpha (\pi_t^N)^{(1-\alpha)}. \quad (5)$$

As can be seen, the CPI inflation is just a weighted average of the (gross) nominal devaluation rate or traded goods inflation, $\epsilon_t \equiv \frac{\mathcal{E}_t}{\mathcal{E}_{t-1}}$, and the (gross) non-traded goods inflation, $\pi_t^N \equiv \frac{P_t^N}{P_{t-1}^N}$, with weights related to the degree of openness, α . The real exchange rate, in turn, is defined as the ratio of the price of traded goods and the price of non-traded goods

$$e_t \equiv \frac{\mathcal{E}_t}{P_t^N}, \quad (6)$$

and evolves according to

$$\frac{e_t}{e_{t-1}} = \frac{\epsilon_t}{\pi_t^N}. \quad (7)$$

We assume that the household-firm units have access to a complete set of internationally traded claims, as has become very common in the open economy literature such as Clarida, Galí, and Gertler (2001) and Galí and Monacelli (2008), among others. In each period $t \geq 0$, the agents can purchase two types of financial assets: fiat money M_t^d and nominal state contingent claims, D_{t+1} . The latter pay one unit of foreign currency for a specific realization of the fundamental shocks in $t + 1$.⁹

no distortionary effects of transaction money demand. It can be easily shown that no equilibrium cycles occur in this case.

⁸Below we relax this assumption by introducing non-traded distribution costs. This allow us to model an imperfect degree of exchange rate pass-through into import prices.

⁹From the technical point of view, complete markets serve the purpose of ruling out the unit root problem of the small open economy (see Schmitt-Grohé and Uribe, 2003). This allows us to derive meaningful local determinacy of equilibrium results and compare them to the ones derived under our global equilibrium analysis. But note that our global results still hold if instead we assume incomplete markets, as we show below.

Under complete markets the representative agent's flow constraint can be written as

$$M_t^d + E_t Q_{t,t+1} D_{t+1} \leq W_t + \mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t \tau_t - \mathcal{E}_t c_t^T - P_t^N c_t^N, \quad (8)$$

where $E_t Q_{t,t+1} D_{t+1}$ denotes the cost of all contingent claims bought at the beginning of period t and $Q_{t,t+1}$ refers to the period- t price of a claim to one unit of currency delivered in a particular state of period $t+1$, divided by the probability of occurrence of that state and conditional on information available in period t . Constraint (8) says that the total end-of-period nominal value of the financial assets can be worth no more than the value of the financial wealth brought into the period, $W_t = M_{t-1}^d + D_t$, plus non-financial income net of the value of taxes, $\mathcal{E}_t \tau_t$, and the value of consumption spending. Using the fact that the period- t price of a claim that pays one unit of currency in every state in period $t+1$ is equal to the inverse of the risk-free gross nominal interest rate, that is $E_t Q_{t,t+1} = \frac{1}{R_t}$, we can write constraint (8) as the following period-by-period constraint:

$$E_t Q_{t,t+1} W_{t+1} \leq W_t + \mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t \tau_t - \frac{R_t - 1}{R_t} M_t^d - \mathcal{E}_t c_t^T - P_t^N c_t^N. \quad (9)$$

The representative unit is also subject to a Non-Ponzi game condition

$$\lim_{j \rightarrow \infty} E_t q_{t+j} W_{t+j} \geq 0 \quad (10)$$

at all dates and under all contingencies, where q_t represents the period-zero price of one unit of currency to be delivered in a particular state of period t divided by the probability of occurrence of that state and given information available at time 0. It satisfies $q_t = Q_{0,1} Q_{1,2} \dots Q_{t-1,t}$ with $q_0 \equiv 1$.

The problem of the representative household-firm unit reduces then to choosing the sequences $\{c_t^T, c_t^N, h_t^T, h_t^N, M_t^d, W_{t+1}\}_{t=0}^{\infty}$ in order to maximize (1) subject to (2), (3), (9), and (10), given W_0 and the time paths of $R_t, \mathcal{E}_t, P_t^N, Q_{t,t+1}$, and τ_t . Since the utility function specified in (1) implies that the preferences of the agent display non-satiation, both constraints (9) and (10) hold with equality. The first order conditions correspond to these constraints and

$$\alpha \gamma (c_t^T)^{\alpha \gamma (1-\sigma)-1} (c_t^N)^{(1-\alpha) \gamma (1-\sigma)} (m_t^d)^{(1-\gamma)(1-\sigma)} = \lambda_t \quad (11)$$

$$\frac{\alpha c_t^N}{(1-\alpha) c_t^T} = e_t \quad (12)$$

$$\frac{\lambda_t}{e_t} \theta_N (h_t^N)^{(\theta_N-1)} = \psi = \lambda_t \theta_T (h_t^T)^{(\theta_T-1)} \quad (13)$$

$$m_t^d = \left(\frac{1-\gamma}{\gamma} \right) \left(\frac{\alpha}{1-\alpha} \right)^{\alpha} \left(\frac{R_t}{R_t-1} \right) c_t^N e_t^{-\alpha} \quad (14)$$

$$\frac{\lambda_t}{\mathcal{E}_t} Q_{t,t+1} = \beta \frac{\lambda_{t+1}}{\mathcal{E}_{t+1}} \quad (15)$$

where $\frac{\lambda_t}{\mathcal{E}_t}$ is the Lagrange multiplier of the budget constraint.

The interpretation of conditions (11)-(15) is straightforward. Equation (11) is the intertemporal envelope condition that makes the marginal utility of consumption of traded goods equal to the marginal utility of wealth measure in terms of traded goods (λ_t). Condition (12) implies that the marginal rate of substitution between traded and non-traded goods must be equal to the real exchange rate, while condition (13) equalizes the value of the marginal products of labor in both sectors. Equation (14) represents the demand for real money balances, and condition (15) describes a standard pricing equation for one-step-ahead nominal contingent claims for each period t and for each possible state of nature.

2.2 The Government

The government issues money, M_t^s , and a one period risk-free domestic bond, B_t^s , which pays a gross risk-free nominal interest rate R_t . It cannot issue or hold state contingent claims. It also levies taxes, pays interest on its debt, and receives revenues from seigniorage. The government follows a generic Ricardian policy by choosing the path of the taxes to satisfy its intertemporal budget constraint in conjunction with a transversality condition at all times. As a result of this policy, we can ignore the budget constraint of the government in the analysis to follow.

Monetary policy, on the other hand, is described as a forward-looking interest rate rule, which sets the nominal interest rate, R_t , as a continuous and increasing function of the deviation of the expected future CPI inflation rate, $E_t \pi_{t+1}$, from a target, π^* .¹⁰ For analytical purposes, we use the following functional form:

$$R_t = \rho(E_t \pi_{t+1}) \equiv 1 + (R^* - 1) \left(\frac{E_t \pi_{t+1}}{\pi^*} \right)^{\frac{A}{R^* - 1}}, \quad (16)$$

where $R^* = \frac{\pi^*}{\beta}$ corresponds to the interest rate target, and assume that R_t always satisfies the zero bound on the nominal interest rate, i.e., $R_t > 1$. In addition, we focus on active rules. That is, rules that at the inflation target respond aggressively to inflation, implying that the elasticity to inflation $\xi \equiv \frac{\rho'(\pi^*)\pi^*}{\rho(\pi^*)} = \frac{A}{R^*}$ is strictly bigger than 1.

Assumption 0: *The rule is active: $\xi = \frac{A}{R^*} > 1$.*

2.3 International Capital Markets

Besides complete markets, we assume free international capital mobility. Then the no-arbitrage condition $Q_{t,t+1}^w = Q_{t,t+1} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}$ holds, where $Q_{t,t+1}^w$ refers to the period- t foreign currency price of a claim to one unit

¹⁰In the sensitivity analysis presented in Section 5 we also study contemporaneous and backward-looking rules.

of foreign currency delivered in a particular state of period $t + 1$ divided by the probability of occurrence of that state and conditional on information available in period t .

Furthermore, a condition similar to (15) must hold for the representative agent in the Rest of The World (ROW). That is, $\frac{\lambda_t^w}{P_t^{Tw}} Q_{t,t+1}^w = \frac{\lambda_{t+1}^w}{P_{t+1}^{Tw}} \beta^w$ where λ_t^w represents the marginal utility of nominal wealth in the ROW and β^w denotes the subjective discount rate of the ROW. Combining this with condition (15) and assumptions $P_t^{Tw} = 1$, $\beta^w = \beta$, and $P_t^T = \mathcal{E}_t$ yields $\frac{\lambda_{t+1}}{\lambda_t} = \frac{\lambda_{t+1}^w}{\lambda_t^w}$, which holds at all dates and under all contingencies. This condition implies that the domestic marginal utility of wealth is proportional to its foreign counterpart: $\lambda_t = \Lambda \lambda_t^w$, where Λ is a constant that determines the wealth difference between the SOE and the ROW. As in Schmitt-Grohé and Uribe (2003), we assume that λ_t^w is constant overtime implying that λ_t is also constant

$$\lambda_t = \lambda = \Lambda \lambda^w. \quad (17)$$

This allows us to write condition (15) as $Q_{t,t+1} = \frac{\mathcal{E}_t}{\mathcal{E}_{t+1}} \beta = \frac{\beta}{\epsilon_{t+1}}$, which together with $E_t Q_{t,t+1} = \frac{1}{R_t}$ can be used to derive the following expression, reminiscent of an uncovered interest parity condition:

$$R_t = \frac{1}{\beta E_t \frac{1}{\epsilon_{t+1}}}. \quad (18)$$

2.4 The Definition of Equilibrium

We will focus on perfect foresight equilibria, where agents in the economy forecast correctly all the anticipated variables. Therefore for any variable x_t , we have that $E_t x_{t+j} = x_{t+j}$ for $j \geq 0$, implying that we can drop the expectation operator in the previous equations. For instance, condition (18) becomes

$$R_t = \frac{\epsilon_{t+1}}{\beta}, \quad (19)$$

which corresponds to the typical uncovered interest parity condition as long as $\frac{1}{\beta}$ represents the foreign international interest rate.¹¹

To provide a definition of the equilibrium dynamics, we derive a reduced non-linear form of the model. By combining conditions (11)-(14) with (17), and the market clearing conditions for money and the non-traded good ($M_t^d = M_t^s = M_t$ and $y_t^N = (h_t^N)^{\theta_N} = c_t^N$), we can express the real exchange rate e_t as a function of the nominal interest rate R_t :

$$e_t \equiv e(R_t) = \kappa \left(\frac{R_t}{R_t - 1} \right)^\nu, \quad (20)$$

where κ and ν are constants that depend on structural parameters. In particular $\nu = \frac{(\sigma-1)(1-\gamma)(1-\theta_N)}{\sigma[\theta_N + \alpha(1-\theta_N)] + (1-\alpha)(1-\theta_N)}$.

¹¹ This holds by the previous analysis since $E_t Q_{t,t+1}^w = \frac{1}{R_t^w} = \beta$.

For the discussion below, it will prove useful to notice that

$$e'(R_t) > 0 \text{ for } \sigma < 1 \text{ and } e'(R_t) < 0 \text{ for } \sigma > 1. \quad (21)$$

To see why, consider the case of $\sigma < 1$ and, without loss of generality, assume that the economy has a fixed endowment of non-traded goods, such that $y_t^N = c_t^N = \bar{y}^N$. This together with (12), (14) and (17) allow us to write condition (11) as follows:

$$U_{c^T} (c^T(e_t, \bar{y}^N), \bar{y}^N, m^d(\bar{y}^N, e_t, R_t)) = \lambda, \quad (22)$$

implying that the marginal utility from traded-goods consumption must be constant.¹² By the definition of c_t in (2) and the fact that $\text{sign}\{U_{cm}\} = \text{sign}\{(1 - \sigma)\}$, the cross-derivative $U_{c^T m}$ is positive. Now suppose the nominal interest rate R_t shifts upwards. By the money demand equation (14), m_t^d decreases, pushing down the left hand side of (22). To restore the equilibrium a real exchange rate depreciation is necessary, since a higher e_t would imply lower traded-good consumption, and, by concavity, push up marginal utility. Hence, when $\sigma < 1$, e_t and R_t are positively related. By a similar argument, it is possible to motivate why $e'(R_t) < 0$ when $\sigma > 1$.¹³

Furthermore, and in order to derive the reduced non-linear form of the model, we can combine equations (5), (7), and (19), to express the (expected) real depreciation $\frac{e_{t+1}}{e_t}$ in terms of the ex-post real interest rate $\frac{R_t}{\pi_{t+1}}$:

$$\left(\frac{e_{t+1}}{e_t}\right)^{1-\alpha} = \beta \frac{R_t}{\pi_{t+1}}. \quad (23)$$

In turn, this equation can be used with the rule (16), the definition $R^* = \frac{\pi^*}{\beta}$, and real exchange rate equation (20) to derive the following expression:

$$\left(\frac{R_{t+1}}{R_{t+1} - 1}\right)^\chi = \frac{R_t}{R^*} \left(\frac{R^* - 1}{R_t - 1}\right)^{\frac{R^* - 1}{A}} \left(\frac{R_t}{R_t - 1}\right)^\chi, \quad (24)$$

where

$$\chi = \frac{(\sigma - 1)(1 - \alpha)(1 - \gamma)(1 - \theta_N)}{\sigma[\theta_N + \alpha(1 - \theta_N)] + (1 - \alpha)(1 - \theta_N)}. \quad (25)$$

Equation (24) is a first order difference equation that represents the reduced non-linear form of the model, which will be used to pursue the local and global determinacy of equilibrium analyses. The type of equilibrium we will be studying corresponds to a *Perfect Foresight Equilibrium*, as stated in the following definition.

Definition 1 *Given the target R^* and the initial condition R_0 , a Perfect Foresight Equilibrium (PFE), under*

¹²Under incomplete markets, this marginal utility can be constant as well, provided we assume that the subjective discount rate is the inverse of the world real interest rate.

¹³If $\sigma = 1$, movements in R_t do not affect e_t , and the real exchange rate is constant along the equilibrium path.

a forward-looking interest rate rule, is a deterministic process $\{R_t\}_{t=0}^{\infty}$, with $R_t > 1$ for any t , satisfying equation (24).

Although this definition is stated exclusively in terms of the nominal interest rate (R_t), it must be clear that multiple PFE solutions to (24) imply *real* local and/or global indeterminacy of all the endogenous variables.¹⁴ In our model, the indeterminacy of R_t implies real indeterminacy, as a result of the non-separability in the utility function between money and consumption. In fact, for any given R_t , equations (5), (7), (11)-(14), (17), (19), and the market clearing conditions for money and the non-traded good, can be used to obtain all the remaining real endogenous variables.

To pursue the equilibrium analysis we first identify the steady states of the economy. From equation (5), (7), and (23), we obtain that at the steady-state, $\pi^{Nss} = \epsilon^{ss} = \pi^{ss}$, and $R^{ss} = \frac{\pi^{ss}}{\beta}$. Using these and the rule (16) we have that

$$(R^* - 1)^{\frac{R^*-1}{A}} R^{ss} = R^* (R^{ss} - 1)^{\frac{R^*-1}{A}}. \quad (26)$$

Clearly $R^{ss} = R^* > 1$ is a solution to (26), and therefore a feasible steady state. But if the rule is active at R^* , that is if $\xi = \frac{A}{R^*} > 1$, then another lower steady state $R^L \in (1, R^*)$ exists. At this state, the elasticity of the rule to inflation satisfies $\xi = \frac{A}{R^L} < 1$. We will call this state the passive steady state. The following proposition formalizes the existence of the passive steady state R^L .

Proposition 1 *If $\frac{A}{R^*} > 1$ (an active rule) and $R^{ss} > 1$ (the zero lower bound) then there exists a solution $R^{ss} = R^L \in (1, R^*)$ solving (26) besides the solution $R^{ss} = R^*$.*

Proof. The proof is available from the authors upon request. ■

The existence of two steady states plays a crucial role in the derivation of our results. As in Benhabib et al. (2002a), it is independent of the non-policy structural parameters implying that no fold bifurcation (appearance or disappearance of steady states) occurs because of changes in these parameters. What distinguishes our model from theirs are the equilibrium dynamics *off* the two steady states. This distinction stems mainly from the following two features. First, by introducing traded and non-traded goods we present an economy with two goods that are fundamentally different in terms of the degree of openness to international trade. As we will see below this degree, measured by α , will influence the equilibrium dynamics. Second, by considering money in the non-separable utility function, we are able to study how the existence of cyclical dynamics depend on the risk aversion coefficient σ , which is in turn related to whether money and consumption are either Edgeworth complements ($\sigma < 1$) or substitutes ($\sigma > 1$).

In the analysis to follow we will keep the other structural parameters (β , γ and θ_N) and the policy parameters (A and R^*) constant. This will allow us to compare the dynamics of economies that implement

¹⁴By indeterminacy we refer to a situation where one or more real variables are not pinned down by the model. We use the following terms interchangeably: (i) “indeterminacy” and “multiple equilibria”, and (ii) “determinacy” and a “unique equilibrium.”

the same monetary rule but differ in the degree of openness α and the risk aversion parameter σ . We will proceed in two steps. First, we will analyze how these dynamics are affected by the composite parameter χ defined in (25). Second, by taking into account the dependence of χ on both α and σ , we will unveil the effect of the degree of openness and the risk aversion parameter on the existence of local and global dynamics (cycles and chaos). In this sense, we will regard χ as a function of α and σ , $\chi(\alpha, \sigma)$. For this second step, we will use and refer to (25), and the Lemmata 4 and 5 in the Appendix. In turn, these Lemmata and subsequent propositions will use the following definitions

$$\chi_{\max} \equiv \frac{(1-\gamma)(1-\theta_N)}{\theta_N} \in (0, +\infty) \quad \chi_{\min} \equiv -(1-\gamma) \in (-1, 0) \quad (27)$$

$$\mu(\sigma) = \frac{(\sigma-1)(1-\gamma)(1-\theta_N)}{\sigma\theta_N + (1-\theta_N)} \quad (28)$$

where $\chi_{\max}$ and $\chi_{\min}$ are considered scalars and $\mu(\sigma)$ is considered a function of σ .

Definition 2 Using (27) and (28) define the scalars $\sigma^i \equiv \frac{1-\frac{\Upsilon^i}{\chi_{\min}}}{1-\frac{\Upsilon^i}{\chi_{\max}}}$ and the functions $\alpha^i(\sigma) \equiv \frac{1-\frac{\Upsilon^i}{\mu(\sigma)}}{1-\frac{\Upsilon^i}{\chi_{\min}}}$ for $i = w, k, f, d$, where $\Upsilon^w \equiv R^L \left(1 - \frac{R^*-1}{A}\right) - 1 < 0$, $\Upsilon^k \equiv \left(1 - \frac{R^*}{A}\right) (R^* - 1) > 0$, $\Upsilon^f \equiv \frac{\Upsilon^w}{2}$, $\Upsilon^d \equiv \frac{\Upsilon^k}{2}$, and the functions $\alpha^i(\sigma)$ are characterized in Lemma 4 when $\sigma > 1$ and in Lemma 5 when $\sigma \in (0, 1)$.

3 Equilibrium Dynamics under Forward-Looking Rules

Our study of rules that respond to expected future inflation, or forward-looking rules, is motivated by the evidence provided by Clarida et al. (1998). To derive analytical results for both local and global equilibrium analyses, we will assume that the parameters γ , θ_N , A , and R^* satisfy the following:

Assumption 1: $\chi_{\max} > \frac{1}{2}(R^* - 1) \left(1 - \frac{R^*}{A}\right)$.

Assumption 2: $\chi_{\min} < \frac{1-R^*}{A}$.

Assumption 3: $R^* - 1 > A(R^L - 1)$.

We briefly explain these assumptions. Assumption 1 is necessary and sufficient for the existence of ranges of α and σ where local indeterminacy occurs, as well as for the existence of a flip bifurcation in the case of $\sigma > 1$. Assumption 2 allows for monotonic liquidity traps for low values of σ . It could be dropped without affecting the possibility of cycles around the passive steady state. Finally Assumption 3, which basically requires enough separation between the two steady states, is useful in proving the existence of a flip bifurcation in the case of $\sigma < 1$. The calibration exercise that we present below suggests that these assumptions are not unrealistic.¹⁵

¹⁵For given monetary policy parameters and for given θ_N , both Assumptions 1 and 2 imply a minimum share of real balances in utility. We can in fact rewrite Assumption 1 as $(1-\gamma) > \frac{1}{2} \frac{\theta_N}{(1-\theta_N)} \left(\frac{R^*-1}{A}\right) (A - R^*)$. For the calibration used in Table 1 below, the right hand side of this inequality is about 0.006, while we set $1-\gamma = 0.03$ consistently with the literature. Similarly, Assumption 2 can be written as $1-\gamma > \frac{R^*-1}{A}$, with the right hand side equal to 0.007 in our calibration.

3.1 The Local Determinacy of Equilibrium Analysis

We start with the local determinacy analysis. That is, we focus on PFE which remain bounded within a small neighborhood of the targeted high steady state. To do this, we log-linearize equation (24) around R^* , yielding

$$\hat{R}_{t+1} = \left(1 + \frac{\frac{R^*}{A} - 1}{\frac{\chi}{R^* - 1}}\right) \hat{R}_t. \quad (29)$$

Since R_t is a non-predetermined variable, studying local determinacy is equivalent to finding conditions that make the linear difference equation (29) explosive. The next Lemma shows that these conditions depend on χ .

Lemma 1 *Define $\Upsilon^d \equiv \frac{1}{2}(R^* - 1) \left(1 - \frac{R^*}{A}\right) > 0$ and consider $\chi \in \mathbb{R}$. Suppose the government follows an active forward-looking rule then: 1) the equilibrium is locally unique if $\chi < \Upsilon^d$; 2) there exist locally multiple equilibria if $\chi > \Upsilon^d$.*

Proof. See the Appendix. ■

These simple determinacy of equilibrium conditions for χ can be reinterpreted in terms of the degree of openness α and the risk aversion parameter σ in the following Proposition.

Proposition 2 *Consider σ^d and $\alpha^d(\sigma)$ from Definition 2, where $\sigma^d > 1$ and $\alpha^d : (1, +\infty) \rightarrow (-\infty, 1)$. Suppose that the government follows an active forward-looking rule.*

1. *There exists locally a unique equilibrium,*

- (a) *if consumption and money are Edgeworth complements, i.e. $\sigma \in (0, 1)$, and for any degree of openness, i.e. $\alpha \in (0, 1)$;*
- (b) *if consumption and money are Edgeworth substitutes, i.e. $\sigma > 1$, and the economy is sufficiently open satisfying $\alpha > \alpha_{\min}^d$ where $\alpha_{\min}^d \equiv \max\{0, \alpha^d(\sigma)\}$ is positive and strictly increasing for $\sigma > \sigma^d$, but constant and equal to zero for any $\sigma \in (1, \sigma^d]$.*

2. *There exist locally multiple equilibria if consumption and money are Edgeworth substitutes satisfying $\sigma > \sigma^d$, and the economy is sufficiently closed satisfying $\alpha \in (0, \alpha^d(\sigma))$.*

Proof. See the Appendix. ■

The results of this proposition show the importance of α and σ in the local characterization of the equilibrium. In a nutshell, active forward-looking rules guarantee local determinacy in the following cases: when regardless of the degree of openness, the risk aversion parameter σ is sufficiently low; and when the economy is sufficiently open for high values of σ .¹⁶ Local determinacy, however, does not guarantee global equilibrium determinacy, as we show below.

¹⁶In fact, this result is more general since a quick inspection of Proposition 2 and Lemma 1 suggest that if the economy is very open, an active rule leads to a unique equilibrium regardless of the values of the other structural parameters.

3.2 The Global Determinacy of Equilibrium Analysis

For the global analysis, we rewrite equation (24) as the forward mapping $R_{t+1} = f(R_t)$ where

$$f(R_t) \equiv \frac{1}{1 - J(R_t)^{\frac{1}{\chi}}} \quad (30)$$

and

$$J(R_t) \equiv \frac{R^*}{(R^* - 1)^{\frac{R^* - 1}{A}}} \frac{(R_t - 1)^{\chi + \frac{R^* - 1}{A}}}{R_t^{1 + \chi}}. \quad (31)$$

This analysis corresponds to studying the global PFE dynamics that satisfy $R_{t+1} = f(R_t)$ for a given initial condition $R_0 > 1$ and subject to the zero-lower-bound condition $f^n(R_0) > 1$ for any $n \geq 1$. A PFE is said to be globally indeterminate, if there exist multiple initial conditions R_0 for which the non-linear mapping (30) satisfies the zero lower bound at any point in time. Global indeterminacy will take the form of cyclical and chaotic dynamics conforming to the following definitions.

Definition 3 *Period-n cycle.* A value “ R ” is a point of a period- n cycle if it is a fixed point of the n -th iterate of the mapping $f(\cdot)$, i.e. $R = f^n(R)$, but not a fixed point of an iterate of any lower order. If “ R ” is such, we call the sequence $\{R, f(R), f^2(R), \dots, f^{n-1}(R)\}$ a period- n cycle.

Definition 4 *Topological chaos.* The mapping $f(\cdot)$ is topologically chaotic if there exists a set “ S ” of uncountable many initial points, belonging to its domain, such that no orbit that starts in “ S ” will converge to one another or to any existing period orbit.

The global analysis requires a full characterization of $f(\cdot)$ in (30) around its stationary solutions over its entire domain. Lemma 6 in the Appendix investigates the properties of the mapping $f(\cdot)$ showing that they depend critically on χ . Here we only provide a big picture of the analysis. First of all, the Lemma specifies conditions under which $f(\cdot)$ satisfies the zero-lower-bound requirement. Second, it makes use of the following conditions

$$\text{sign}\{f'(R_t)\} = \text{sign}\left\{\frac{J'(R_t)}{\chi}\right\} \quad \text{and} \quad \text{sign}\{J'(R_t)\} = \text{sign}\left\{\frac{1 + \chi}{1 - \frac{R^* - 1}{A}} - R_t\right\}, \quad (32)$$

which imply that for $\chi \neq 0$ then $R^J \equiv \frac{1 + \chi}{1 - \frac{R^* - 1}{A}}$ is a critical point of $f(\cdot)$ as long as $R^J > 1$. With these conditions, the Lemma shows that the mapping $R_{t+1} = f(R_t)$ is always single-peaked for $\chi > 0$, whereas for $\chi < 0$ it is single-troughed only if $\chi + \frac{R^* - 1}{A} > 0$.

Figure 1 displays a graphical representation of the cases where the equilibrium mapping $R_{t+1} = f(R_t)$ has a critical point between the two steady states. The right hand side of the figure considers the case of $\chi < 0$ and $\chi + \frac{R^* - 1}{A} > 0$, while the left hand side presents the case of $\chi > 0$. In the left one, $f(\cdot)$ always satisfies the equilibrium conditions for any $R_t \in (1, +\infty)$ and crosses the 45° line twice at the steady states

Figure 1: This figure shows the mapping $R_{t+1} = f(R_t)$ for (i) $\chi > 0$, and (ii) $\chi < 0$ but $\chi + \frac{R^*-1}{A} > 0$. R_t denotes the nominal interest rate. The diagonal dotted line corresponds to the 45° line.

R^L and R^* . Furthermore, $\lim_{R_t \rightarrow 1} f(R_t) = \lim_{R_t \rightarrow +\infty} f(R_t) = 0$ and there is a maximum at $R^J \in (R^L, R^*)$. In the right hand side, all equilibrium conditions are satisfied only within a subset $(R^l, R^u) \subset (1, +\infty)$ defined in Lemma 6. Within that set, $f(\cdot)$ crosses the 45° line at R^L and R^* , as in the previous case, but now $R^J \in (R^L, R^*)$ is a minimum and $\lim_{R_t \rightarrow R^l+} f(R_t) = \lim_{R_t \rightarrow R^u-} f(R_t) = +\infty$. These cases are interesting, as they imply a negative derivative of $f(\cdot)$ at either one of the two steady states, which in turn is a necessary condition for the existence of cyclical dynamics in continuously differentiable maps.¹⁷

Lemma 6 and Figure 1 suggest that the sign of χ determines whether cycles may appear around either the active or the passive steady state. Hence we proceed to look for flip bifurcation thresholds for χ i.e. critical values of χ that determine a change in the stability properties of the steady state where the map $f(\cdot)$ is negatively sloped. If the steady state is stable, any equilibrium orbit that starts in a map invariant set centered around this state will asymptotically converge to the steady state itself, monotonically or spirally. Thus endogenous equilibrium cycles are impossible. On the contrary, if the steady state is unstable, such orbit will keep oscillating within the map invariant set and either will converge to a stable n -period cycle, or not converge at all, displaying aperiodic but bounded dynamics (chaotic equilibrium paths). We first consider the case of $\chi \in \left(R^L \left(1 - \frac{R^*-1}{A}\right) - 1, 0\right)$ and show that endogenous cyclical dynamics of period 2 can occur around the passive steady state.

Lemma 2 Let $\Upsilon^w \equiv R^L \left(1 - \frac{R^*-1}{A}\right) - 1$, and define the points $\underline{R} \equiv f(R^J)$ and $\tilde{R} \equiv f^{-1}(R^*)$. Consider $\chi \in (\Upsilon^w, 0)$ and assume that $f_{\min} \equiv f(R^J) \geq \tilde{R}$.¹⁸ Then:

¹⁷See Lorenz (1993).

¹⁸This assumption rules out explosive paths for initial conditions between the two steady states. It is similar in flavor to the one used by Boldrin, Nishimura, Shigoka, and Yana (2001) and Matsuyama (1991). If $f_{\min} = f(R^J) < \tilde{R}$ there would not be a

1. $R^L \in (\underline{R}, R^J)$ and $R^L > \tilde{R}$;
2. the set $[\underline{R}, R^*]$ is invariant under the mapping $f(\cdot)$ and attractive for any $R_t \in [\tilde{R}, \underline{R}]$ where $\tilde{R} < \underline{R}$;
3. period-2 cycles within $[\underline{R}, R^*]$ and centered around the passive steady state occur when $\chi \in (\frac{\Upsilon^w}{2}, 0)$.

Proof. See the Appendix. ■

Instead, for the case of $\chi \in (0, (1 - \frac{R^*}{A})(R^* - 1))$, endogenous cyclical dynamics of period 2 exist around the active steady state.

Lemma 3 Let $\Upsilon^k \equiv (1 - \frac{R^*}{A})(R^* - 1)$, and define the points $\bar{R} \equiv f(R^J)$ and $\tilde{R} \equiv f^{-1}(R^*)$. Consider $\chi \in (0, \Upsilon^k)$ and assume $f_{\max} \equiv f(R^J) \leq \tilde{R}$. Then:

1. $R^* \in (R^J, \bar{R})$ and $\tilde{R} > R^L$;
2. the set $[R^L, \bar{R}]$ is invariant under the mapping $f(\cdot)$ and attractive for any $R_t \in (\bar{R}, \tilde{R}]$ where $\bar{R} < \tilde{R}$;
3. period-2 cycles within $[R^L, \bar{R}]$ and centered around the active steady state occur when $\chi \in (0, \frac{1}{2}\Upsilon^k)$.

Proof. See Appendix. ■

We proceed translating the conditions for endogenous cycles derived in terms of χ into conditions described in terms of the degree of openness α and the risk aversion parameter σ . The following proposition states that either at sufficiently low or high risk aversion coefficients (σ), forward looking rules are more prone to induce endogenous cyclical dynamics the more open the economy; while for σ sufficiently close to 1, forward looking rules lead to those dynamics regardless of the degree of openness.¹⁹

Proposition 3 Suppose that the government follows an active forward-looking rule.

1. Consider σ^f and $\alpha^f(\sigma)$ from Definition 2, where $\sigma^f \in (0, 1)$ and $\alpha^f : (0, 1) \rightarrow (-\infty, 1)$, and assume that consumption and money are Edgeworth complements, i.e. $\sigma \in (0, 1)$. Then period-2 equilibrium cycles exist around the passive steady state if the economy is sufficiently open satisfying $\alpha > \alpha_{\min}^f$, where $\alpha_{\min}^f \equiv \max\{0, \alpha^f(\sigma)\}$ is positive and strictly decreasing for $\sigma \in (0, \sigma^f)$, but constant and equal to zero for any $\sigma \in [\sigma^f, 1)$.
2. Consider σ^d and $\alpha^d(\sigma)$ from Definition 2, where $\sigma^d > 1$ and $\alpha^d : (1, +\infty) \rightarrow (-\infty, 1)$, and assume that consumption and money are Edgeworth substitutes, i.e. $\sigma > 1$. Then period-2 equilibrium cycles exist around the active steady state if the economy is sufficiently open satisfying $\alpha > \alpha_{\min}^d$, where $\alpha_{\min}^d \equiv \max\{0, \alpha^d(\sigma)\}$ is positive and strictly increasing for $\sigma > \sigma^d$, but constant and equal to zero for any $\sigma \in (1, \sigma^d]$.

non-trivial mapping-invariant set. This case displays a different type of multiplicity. It can be shown that there exists a set of points within the set $[\underline{R}, R^*]$ that leave such a set after a finite number of iterations, and settle to an exploding path diverging from the active steady state (see Matsuyama, 1991).

¹⁹As mentioned before, the discontinuity of the model with respect to $\sigma = 1$ corresponds to a utility function that is separable in money and consumption. In this case it is possible to show that no cyclical dynamics can occur.

Proof. See the Appendix. ■

Note that from point 2 of this Proposition, it is possible to observe that the sufficient condition for the existence of period-2 cycles when $\sigma > 1$ is the same as the condition from Proposition 2 that guarantees a unique local equilibrium. This is a clear example of why local analysis can be misleading. By log-linearizing around the steady state, local analysis implicitly assumes that any path starting arbitrarily close to it and diverging cannot be part of an equilibrium, since it will eventually explode and thus violate some transversality condition. This is not the case here, as the global analysis proves that the true non-linear map features a bounded map-invariant and attractive set around the active steady state. It is then possible to have equilibrium paths that starting arbitrarily close to the target steady state will converge to a stable deterministic cycle.²⁰ These are liquidity traps that do not converge to the passive steady state. Instead, they converge non-monotonically to a cycle around this state.

Given the functional form of $f(\cdot)$ in (30) it is very difficult to derive analytical conditions for α and σ under which forward-looking rules induce either cycles of period higher than 2 or chaotic dynamics. Therefore in order to shed some light on the role of both α and σ in delivering these dynamics, as well as to find some empirical confirmation of our analytical results, we pursue a simple parametrization-simulation exercise.

Table 1: Parametrization

θ_N	β	π^*	R^*	$1 - \gamma$	$\frac{A}{R^*}$
0.56	0.99	$1.031^{\frac{1}{4}}$	$1.072^{\frac{1}{4}}$	0.03	2.24

Table 1 summarizes the parametrization. We set the time unit to be a quarter and try to use Canada as the representative economy for the parametrization. In the simulations, we vary α over the continuum $(0, 1)$ while σ , which determines whether consumption and real money balances are either Edgeworth substitutes or complements, is selected among $\{0.79, 1.51, 2.03, 2.55\}$. In line with empirical evidence we set $1 - \gamma$ equal to 0.03.²¹ This and the values of σ correspond to values of the CRRA $\tilde{\sigma} = 1 - \gamma(1 - \sigma)$ in the set $\{0.8, 1.5, 2, 2.5\}$. From Mendoza (1995) we borrow the labor income share for the non-traded sector $\theta_N = 0.56$. The steady-state inflation and nominal interest rate are calculated as the averages of the CPI inflation and the Central Bank discount rate for Canada, during the last two decades. This yields $\pi^* = 1.031^{\frac{1}{4}}$ and $R^* = 1.072^{\frac{1}{4}}$, implying $\beta = 0.99$. For illustrative purposes we use an elasticity $\frac{A}{R^*}$ of 2.25, which satisfies the Taylor principle and is not unrealistic.²²

²⁰More recently, Eusepi (2005, 2007) and Evans, Guse, and Honkapohja (2008) have proved that both cyclical equilibria and liquidity traps are stable under adaptive learning.

²¹For the US, estimates of this parameter vary from 0.0146 to 0.039 depending on the specification of the utility function and method of estimation. See Holman (1998), among others.

²²See for instance Boivin (2006) and Cogley and Sargent (2005).

Figure 2: These orbit-bifurcation diagrams show the set of limit points for the nominal interest rate R_t , as a function of the degree of openness α , under the coefficients of relative risk aversion (CRRA) $\tilde{\sigma} = 0.8$ and $\tilde{\sigma} = 2$.

Using this parametrization, Figure 2 presents the orbit-bifurcation diagrams for the degree of openness α under both cases: complementarity ($\tilde{\sigma} = 0.8$) and substitutability ($\tilde{\sigma} = 2$) between money and consumption. With $\alpha \in (0, 1)$ on the horizontal axis and $R_t \in (1, 1.5)$ on the vertical axis, the solid lines in the diagram show the stable solutions of period n of the mapping $R_{t+1} = f(R_t)$ in (30). The Figure supports our analytical findings. First, under complementarity, cyclical and chaotic dynamics, including liquidity traps, occur around the passive steady state; whereas under substitutability, these dynamics appear around the active steady state. Second, the rule is more prone to induce these dynamics the more open the economy. To see these, note that both sides of the figure show that for low α 's, the mapping always settles on the passive steady state, R^L , when $\tilde{\sigma} = 0.8$, and on the active steady state, R^* , when $\tilde{\sigma} = 2$. Once α reaches some threshold, a stable period-2 cycle appears, as indicated by the first split into two branches. As we continue increasing α both branches split and a cascade of further period doubling occurs, yielding cycles of periods 4, 8, 16, and so on. Finally for sufficiently high values of α , the rule produces aperiodic chaotic dynamics, i.e. the attractor of the mapping changes from a finite to an infinite set of points.

Further evidence on openness and chaos is presented in Figure 3, where the Lyapunov exponent is plotted as a function of the degree of openness. A positive exponent measures the degree of exponential divergence between two series generated by the same map and starting from nearby, but not equal, initial conditions. This is a characterizing feature of chaotic systems. Consistently with Figure 2, there is evidence of chaotic

Forward-Looking Rules

Figure 3: The figure shows the Lyapunov coefficients as a function of the degree of openness α , under the coefficients of relative risk aversion (CRRA) $\tilde{\sigma} = 0.8$ and $\tilde{\sigma} = 2$.

Figure 4: This figure shows a comparison of the results from the local and global equilibrium analyses for an active forward-looking rule, as the degree of openness α and the coefficient of risk aversion σ vary. "M" stands for multiple local equilibria and "U" stands for a unique local equilibrium.

dynamics for $\alpha > 0.38$, when $\tilde{\sigma} = 0.8$, and for $\alpha > 0.5$, when $\tilde{\sigma} = 2$.²³

To summarize and compare the results from the local and the global determinacy of equilibrium analyses we construct Figure 4. It shows the combinations of the degree of openness and the risk aversion parameter, α and σ , for which there is local and/or global (in)determinacy. We plot the following two threshold frontiers: the flip bifurcation frontier $\alpha^f(\sigma)$ for period-2 cycles around the passive steady state and the frontier $\alpha^d(\sigma)$ for both local determinacy and period-2 cycles around the active steady state. Regions featuring a locally unique equilibrium are labeled by “U”, while those featuring locally multiple equilibria are labeled by “M”. Clearly, “U” appears everywhere but below the curve $\alpha^d(\sigma)$ implying that local determinacy occurs for a wide range of α and σ combinations. In fact note, once more, how local determinacy coexists with global indeterminacy. In addition, note that Figure 4 does not show the regions for liquidity traps. However, the previous analysis should make clear that as long as $\sigma < 1$, standard liquidity traps that converge to the passive steady state appear for combinations of σ and α below the frontier $\alpha^f(\sigma)$; whereas liquidity traps that converge to a cycle around the passive steady state may occur for combinations of σ and α above the frontier $\alpha^f(\sigma)$.

Before we continue our analysis, it is worth underscoring some important differences between our results and those from Benhabib et al. (2002a). First, our results, which are derived in a money-in-the-utility-function set-up, suggest that it is not necessary to assume a productive role for money to obtain cyclical and chaotic equilibria. Second, if consumption and money are complements then it is possible to have liquidity traps as shown by Benhabib et al., but our traps may be *non-monotonic* and converge to a cycle around an extremely low interest rate steady state. On the other hand, if consumption and money are substitutes then cyclical and chaotic dynamics occur only around the active steady state. Although this case is reminiscent of the one in Benhabib et al. (2002a), it also presents a subtle difference. In their closed economy model period-3 cycles and chaos always occur only for sufficiently low σ , while our results show that they can basically appear for any $\sigma > 1$ provided that there is enough degree of openness in the economy. In this sense, open economies are more prone to display these cyclical and aperiodic dynamics.

3.2.1 An Intuition of the Results

To get some intuition for our results, consider the case of $\sigma < \sigma^f$. From Figures 2 and 4 we know that, provided the economy is sufficiently open, period-2 cycles as well as higher-order cycles and chaotic dynamics occur around the passive steady state. If this were not the case, for any initial condition, the equilibrium sequence of interest rates would converge monotonically to the passive steady state (standard liquidity traps). From a technical point of view, this implies the existence of a threshold for the degree of openness $\alpha^f(\sigma)$ above which the passive steady state loses stability, i.e. a flip bifurcation point. Since the non-linear map (30) is always negatively sloped at R^L , this requires $f'(R^L) < -1$; which holds if $\chi > \frac{1}{2} \left[\left(1 - \frac{R^*-1}{A} \right) R^L - 1 \right]$.

²³Figure 3 was constructed using the *E&F Chaos* software package by Diks, Hommes, Panchenko, and Van der Weide (2008).

When α crosses $\alpha^f(\sigma)$, the *passive* steady state goes from being dynamically stable to dynamically unstable. This stability switch is a necessary condition for the existence of cycles and has to do with the elasticity of the real exchange rate to the interest rate factor $\frac{R_t}{R_t-1}$, which according to equation (20) is given by $\nu \equiv \frac{(\sigma-1)(1-\gamma)(1-\theta_N)}{\sigma[\theta_N+\alpha(1-\theta_N)]+(1-\alpha)(1-\theta_N)}$. Note that for fixed $\sigma < 1$, ν is always negative but decreasing in absolute value as α increases towards 1. That is, the more open the economy, the smaller the impact of the nominal interest rate on the real exchange rate. Then, by the rule (16) and equation (20), we can write the dynamic equilibrium condition (23) as follows:

$$[e(\rho(\pi_{t+2}))]^{1-\alpha} = [e(\rho(\pi_{t+1}))]^{1-\alpha} \beta \frac{\rho(\pi_{t+1})}{\pi_{t+1}}, \quad (33)$$

which has to be satisfied by any sequence of inflation rates in order to be a PFE.

We now use this equation to build an intuition based on sequences of inflation rates. Suppose the economy is at the passive steady state $\pi^L \equiv \beta R^L$, where the rule responds less than proportionally to inflation. That is, it is passive. Assume that suddenly the private sector expects a higher inflation in the future. By the policy rule, $\rho(\pi_{t+1})$ increases, but since the rule is passive, the real interest rate $\frac{\rho(\pi_{t+1})}{\pi_{t+1}}$ declines. If α is high, the elasticity ν is small and the increase in $\rho(\pi_{t+1})$ has a small effect on e_t and even less on $e_t^{1-\alpha}$. As a consequence, the right hand side of (33) decreases. But because of the small elasticity, to have a lower left hand side we need a substantial drop in π_{t+2} , i.e. $|\pi_{t+2}| > |\pi_{t+1}|$. Hence, when α is high, the passive steady state is dynamically unstable and the equilibrium series of inflation rates starts diverging from it.

Convergence to a cycle can be explained as follows. Suppose $\pi_{t+1} = \pi^h > \pi^L$, that is, agents expect an inflation level above the passive steady state π^L . Since the rule is passive we have that $\frac{\rho(\pi^h)}{\pi^h} < \frac{\rho(\pi^L)}{\pi^L}$; and because of the high openness (hence, a low ν), the term $[e(\rho(\pi))]^{1-\alpha}$ is only slightly affected by changes in inflation, implying that the right hand side of (33) is then lower than at the steady state. But then, by (33), next period inflation must be below the steady state, i.e. $\pi_{t+2} = \pi^l < \pi^L$. Following the same argument, in the next period it must be that $\frac{\rho(\pi^l)}{\pi^l} > \frac{\rho(\pi^L)}{\pi^L}$, pushing the right hand side of (33) above the steady state, thus requiring next period inflation to be at $\pi^h > \pi^L$, and so on.

3.2.2 The Importance of the Elasticity of the Rule to Inflation

So far we have studied the non-linear dynamics properties of the PFE with respect to the CRRA $\tilde{\sigma}$ and the degree of openness α , keeping fixed the elasticity of the interest rate rule to CPI inflation at the active steady state, $\xi \equiv \frac{A}{R^*}$. This, however, does *not* mean that the elasticity of the rule is irrelevant for the existence of cyclical and chaotic dynamics. The policy response is also crucial. To see this, Figure 5 displays the orbit-bifurcation diagrams for the cases of (i) $\tilde{\sigma} = 0.8$ and $\alpha = 0.4$ and (ii) $\tilde{\sigma} = 2$ and $\alpha = 0.55$, with the remaining parameters as in Table 1.²⁴ Clearly, for case (i) cyclical and chaotic dynamics occur around the passive steady state only for intermediate elasticities of the policy rule, i.e. $\xi \in (1.43, 3.9)$. Outside this

²⁴Figure 5 and 6 were constructed using the *E8F Chaos* software package by Diks et al. (2008).

Orbit-Bifurcation Diagrams

Forward-Looking Rules

Figure 5: These orbit-bifurcation diagrams show the set of limit points for the nominal interest rate R_t , as a function of the elasticity of the rule with respect to inflation, under the following two cases: (i) $\tilde{\sigma} = 0.8$ and $\alpha = 0.4$ and (ii) $\tilde{\sigma} = 2$ and $\alpha = 0.55$.

range, the economy converges to the passive steady state. On the other hand for case (ii), cycles and chaos exist when ξ takes values between 1.65 and 2.88.²⁵

Furthermore, it is possible to use basins of attractions to study quantitatively the importance of the interaction between the degree of openness α and the elasticity ξ for our results. For $\tilde{\sigma} = 0.8$ and $\tilde{\sigma} = 2$, Figure 6 shows the nature of the attractor of converging equilibrium trajectories, for different combinations of ξ (vertical axis) and α (horizontal axis). In the left hand side of Figure 6, when $\tilde{\sigma} = 0.8$, the light grey area corresponds to trajectories converging to the passive steady state, including liquidity traps. Moving rightward, the darker grey areas correspond to period-2 and then period-4 cycles, while the dotted area represent period-3 as well as higher order cycles and chaos. For combinations of α and ξ within the white area, the economy diverges. The areas of the right hand side of the figure, when $\tilde{\sigma} = 2$, have the same interpretation, with two caveats: first, the light grey area associated with low α 's (and below the dark grey area) corresponds to trajectories converging to the active steady state; while the light grey area related to high α 's (and above the dark grey area) represents those combinations of α and ξ that induce trajectories violating the restriction $R_t > 1$ for any t .

Table 2 underscores the interaction between monetary policy and structural characteristics in determining the existence of endogenous cyclical and aperiodic dynamics. It shows parametrizations of structural

²⁵ For $\xi > 2.88$, we find that the equilibrium trajectories converge to $R = 1$, thus violating our zero-lower-bound restriction. This result is consistent with the following facts. First, $R = 1$ is another steady state of our non-linear difference equation. Second, referring to the left panel of Figure 1 and recalling the results of Lemma 3, it can be shown that for a sufficiently aggressive rule the set $[R^L, \bar{R}]$ is not invariant under mapping f . That is, for any initial condition within such set, the equilibrium trajectory would eventually move below R^L (which is dynamically unstable), and from there to the zero lower bound.

Figure 6: Basis of attractions for different combinations of the degree of openness α and the elasticity of the rule to inflation ξ , under $\tilde{\sigma} = 0.8$ and $\tilde{\sigma} = 2$. See the main text for the interpretation of the colored areas.

characteristics for a group of countries which are explicitly pursuing an inflation targeting policy, as well as the thresholds for elasticities ξ above which endogenous cycles and chaotic dynamics may occur. Note that these thresholds may take realistic values, especially for Norway and Sweden. Interestingly, Bask (2002) and Gogas and Serletis (2000) provide evidence of chaotic dynamics for either the nominal or the real exchange rates for the Scandinavian countries.

Table 2: Responsiveness to Inflation, Cycles, and Chaos

Country	Parametrization				2-Period Cycles 1/	Chaos 1/
Norway	$\sigma = 1.58$	$\alpha = 0.34$	$\pi^* = 0.025$	$\theta_N = 0.57$	$\xi > 1.75$	$\xi > 2.26$
Sweden	$\sigma = 1.58$	$\alpha = 0.26$	$\pi^* = 0.02$	$\theta_N = 0.65$	$\xi > 1.7$	$\xi > 2.17$
United Kingdom	$\sigma = 1.59$	$\alpha = 0.34$	$\pi^* = 0.02$	$\theta_N = 0.45$	$\xi > 2.61$	$\xi > 5.47$
Australia	$\sigma = 1.58$	$\alpha = 0.31$	$\pi^* = 0.025$	$\theta_N = 0.48$	$\xi > 2.26$	$\xi > 3.80$
New Zealand	$\sigma = 1.58$	$\alpha = 0.38$	$\pi^* = 0.015$	$\theta_N = 0.56$	$\xi > 1.94$	$\xi > 2.75$

Sources: Ogaki et al. (1996) for σ ; Petursson (2004) for π^* ; Bentolilla and Saint Paul (2003) for θ_N ; and Campa and Goldberg (2006) for α .

1/ The thresholds for ξ are for a forward-looking rule and assume $1 - \gamma = 0.03$ for all countries.

4 Distribution Costs and Imperfect Exchange Rate Pass-Through

We have assumed that the Law of One Price (LOP) holds at the consumption level for traded goods and normalized their foreign price to one. This in turn implied perfect exchange rate pass-through into import prices. In this Section we relax this assumption by introducing distribution costs. As in Burnstein, Neves, and Rebelo (2003), we assume that the traded good needs to be combined with some non-traded distribution services before it is consumed.²⁶ To consume one unit of the traded good, it is required η units of the non-traded good. Let $\tilde{P}_t^T$ and P_t^T be the prices in the domestic currency that producers of traded goods receive and that consumers pay, respectively. Then the consumer price of the traded good is

$$P_t^T = \tilde{P}_t^T + \eta P_t^N. \quad (34)$$

To simplify the analysis we assume that the LOP holds for traded goods at the production level and normalize the foreign price of the traded good to one ($\tilde{P}_t^{Tw} = 1$). Thus $\tilde{P}_t^T = \mathcal{E}_t \tilde{P}_t^{Tw} = \mathcal{E}_t$.

The presence of distribution services leads to imperfect exchange rate pass-through into import prices.²⁷ To see this we combine (34), $\tilde{P}_t^T = \mathcal{E}_t$ and $e_t = \mathcal{E}_t / P_t^N$ to obtain

$$\pi_t^T = \left(\frac{e_{t-1}}{e_{t-1} + \eta} \right) \epsilon_t + \left(\frac{\eta}{e_{t-1} + \eta} \right) \pi_t^N$$

where $\pi_t^T \equiv \frac{P_t^T}{P_{t-1}^T}$, $\epsilon_t \equiv \frac{\mathcal{E}_t}{\mathcal{E}_{t-1}}$ and $\pi_t^N \equiv \frac{P_t^N}{P_{t-1}^N}$ correspond to the gross inflation of import prices, nominal depreciation, and non-traded inflation, respectively. Clearly if $\eta = 0$ then there is perfect pass-through of the nominal depreciation rate into the inflation of import prices and it is measured by $\frac{\partial \pi_t^T}{\partial \epsilon_t} = 1$. This is the case already studied. But if $\eta > 0$ then we obtain imperfect exchange rate pass-through measured by $\frac{\partial \pi_t^T}{\partial \epsilon_t} = \left(\frac{e_{t-1}}{e_{t-1} + \eta} \right) \in (0, 1)$. As the parameter of distribution costs, η , increases then the degree of exchange rate pass-through decreases.

To pursue the determinacy of equilibrium analysis we obtain a reduced non-linear form of the model that describes the dynamics of this economy. In contrast to the case of perfect exchange rate pass-through, it is not possible to derive explicitly a difference equation similar to (24) that depends exclusively on the nominal interest rate R_t . Under imperfect exchange rate pass-through the dynamics are determined by the system

$$\left(\frac{e_{t+1} + \eta}{e_t + \eta} \right)^\alpha \left(\frac{e_t}{e_{t+1}} \right) = \left(\frac{R^*}{R_t} \right) \left(\frac{R_t - 1}{R^* - 1} \right)^{\frac{R^* - 1}{A}}$$

²⁶ Distribution services in open economy models have become very useful to account quantitatively for some dynamics of the real and nominal exchange rates. See for instance Burnstein, Eichenbaum, and Rebelo (2005), among others.

²⁷ This is certainly not the only approach to model imperfect exchange rate pass-through. However, there is no consensus in the theoretical literature about the best approach. For other approaches see Monacelli (2003), among others.

Figure 7: Orbit-bifurcation diagrams for the distribution costs parameter (η). R_t denotes the nominal interest rate. The diagrams show the set of limit points as a function of η , under the following two cases: (i) $\tilde{\sigma} = 0.8$ and $\alpha = 0.4$ and (ii) $\tilde{\sigma} = 2$ and $\alpha = 0.4$.

$$\left(\frac{e_t + \eta}{e_{t+1} + \eta} \right)^{[\alpha(1-\sigma)+\sigma]} \left[\frac{(1-\alpha)e_{t+1} + \eta}{(1-\alpha)e_t + \eta} \right]^\sigma \left(\frac{e_{t+1}}{e_t} \right)^{\left(1 + \frac{\sigma\theta_N}{1-\theta_N}\right)} = \left[\left(\frac{R_{t+1} - 1}{R_{t+1}} \right) \left(\frac{R_t}{R_t - 1} \right) \right]^{(1-\gamma)(1-\sigma)}.$$

Since it is not possible to derive analytical results, we rely on numerical simulations to assess the impact of the distribution costs parameter η (and therefore the degree of exchange rate pass-through) on the existence of cyclical and chaotic dynamics.

Recalling the parametrization of Table 1, we construct Figure 7 for two cases: (i) $\alpha = 0.4$ and $\tilde{\sigma} = 0.8$ (left hand side) and (ii) $\alpha = 0.4$ and $\tilde{\sigma} = 2.0$ (right hand side). The figure reveals that increasing η , or in other words decreasing the degree of exchange rate pass-through, has a non-trivial impact on the dynamics. Consider the case of $\alpha = 0.4$ and $\tilde{\sigma} = 0.8$. Starting from $\eta = 0$, as η increases, the economy moves from displaying chaotic dynamics into displaying periodic cyclical dynamics. As η continues increasing the period of cycles decrease. Beyond $\eta \approx 0.6$ cycles disappear and the only fixed point that subsists corresponds to the passive steady state. On the other hand, for $\alpha = 0.4$ and $\tilde{\sigma} = 0.8$, if $\eta = 0$ the economy presents period-2 cycles around the active steady state. However, as η increases, cycles disappear and the only attractor that subsists is the active steady state. These results can be summarized in the following Proposition.

Proposition 4 *Forward-looking rules are more prone to induce cyclical and chaotic dynamics the higher the degree of exchange rate pass-through.*

5 Sensitivity Analysis

So far we have assumed complete international financial markets, flexible prices, a forward-looking rule, and money liquidity services that satisfy the “Cash-When-I’m Done” convention.²⁸ Now we study the consequences of relaxing these assumptions. To simplify the analysis we still assume perfect exchange rate pass-through. We show that our previous global findings of Section 3 hold even if we consider incomplete financial markets. We also argue that cyclical and chaotic dynamics are less likely to occur under backward-looking rules, while they can still appear under contemporaneous rules depending on the degrees of trade openness and risk aversion. Similarly to Eusepi (2005), for the case of sticky prices, our simulations show that liquidity traps are the only type of global indeterminacy. And lastly, we prove that the alternative timing for real money balances, known as the “Cash-in-Advance” timing, affects the previously derived bifurcation thresholds, but does not preclude the existence of cyclical and chaotic dynamics.

5.1 Incomplete Markets

The assumption about complete markets is not essential for the existence of cyclical and chaotic dynamics. Assuming incomplete markets leads to the same global indeterminacy. To see this, assume that the agent is blessed with perfect foresight and has access to an international bond b_t^w and a domestic bond B_t issued by the government. The former pays a constant international interest rate, R^w , and the latter pays an interest rate, R_t . Using this and the assumptions in Subsection 2.1 we rewrite the agent’s budget constraint as

$$B_t + \mathcal{E}_t b_t^w + M_t = R^w \mathcal{E}_t b_{t-1}^w + M_{t-1} + R_{t-1} B_{t-1} + \mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t \tau_t - \mathcal{E}_t c_t^T - P_t^N c_t^N$$

The agent chooses the sequences $\{c_t^T, c_t^N, h_t^T, h_t^N, M_t^d, b_t^w, B_t\}_{t=0}^\infty$ in order to maximize (1) subject to (2) and (3), the budget constraint and corresponding transversality conditions, given M_{-1}^d, b_{-1}^w , and B_{-1} and the time paths of $R_t, \mathcal{E}_t, P_t^N, R^w$ and τ_t . The first order conditions for b_t^w and B_t correspond to

$$\lambda_t = \beta R^w \lambda_{t+1} \tag{35}$$

and

$$\lambda_t = \frac{\beta R_t \lambda_{t+1}}{\epsilon_t}, \tag{36}$$

whereas the conditions for $\{c_t^T, c_t^N, h_t^T, h_t^N, M_t^d\}$ can be written as (11)-(14).

As is common in the SOE literature we assume $\beta R^w = 1$, which implies by (35) that $\lambda_t = \lambda_{t+1}$. Using this, it is simple to show that the global dynamics are still governed by (24). To see this, use $\lambda_t = \lambda_{t+1}$ and (36) to obtain $\epsilon_t = \beta R_t$ which is identical to (19). This condition together with $\lambda_t = \lambda_{t+1}$, (5), (7), (11)-(14), and market clearing conditions can be used to derive an identical equation to (23), which in tandem with a

²⁸This convention means that the real money balances that provide utility are those left after leaving the goods market.

forward-looking rule (16) allow us to find an identical difference equation to (24). Then the previous cyclical and chaotic dynamics results under complete markets still hold under incomplete markets.

However, note that under incomplete markets cycles and chaos may occur in the accumulation of foreign bonds, $b_t^w - b_{t-1}^w$, and therefore in the current account, ca_t . This can be seen by using the current account equation

$$ca_t = (R^w - 1)b_{t-1}^w + (h_t^T)^{\theta_T} - c_t^T = b_t^w - b_{t-1}^w,$$

and recalling that global indeterminacy of the nominal interest rate in our model implies also global indeterminacy of the rest of real variables

5.2 Contemporaneous and Backward-Looking Rules

Next we explore the consequences of varying the timing of the rule. We will study contemporaneous and backward-looking rules that still respond to the CPI inflation.

5.2.1 Contemporaneous Rules

Motivated by the recent estimations of rules by Lubik and Schorfheide (2007) for the United kingdom, Canada, Australia and New Zealand, we study the determinacy of equilibrium for contemporaneous rules that respond to current CPI inflation: $R_t = 1 + (R^* - 1) \left(\frac{\pi_t}{\pi^*} \right)^{\frac{A}{R^* - 1}}$ with $R^* = \frac{\pi^*}{\beta}$ and $R_t > 1$. Under these rules the equilibrium dynamics are described by

$$\left(\frac{R_{t+1} - 1}{R_{t+1}} \right)^\chi \left(\frac{R^* - 1}{R_{t+1} - 1} \right)^{\frac{R^* - 1}{A}} = \left(\frac{R_t - 1}{R_t} \right)^\chi \frac{R^*}{R_t} \quad (37)$$

where χ defined in (25) depends on α and σ . An explicit representation for either the forward or the backward dynamics of (37) is not available in this case. Although it is feasible to derive some analytical results as before, for reasons of space, we only present some numerical simulations. These are sufficient to make the point that the degree of openness still affects the appearance of complex dynamics.²⁹

Using the parametrization of Table 1 we construct a panel in Figure 8. This figure presents some qualitative properties of the global dynamics of the model for contemporaneous rules for different degrees of openness while setting $\tilde{\sigma} = 2.5$ (substitute goods). For a given $\alpha \in \{0.01, 0.38, 0.90\}$, each column of the panel plots the map $R_{t+1} = f(R_t)$, implicitly defined in equation (37), its second iterate $R_{t+2} = f^2(R_t)$, and its third iterate $R_{t+3} = f^3(R_t)$, respectively. The straight line corresponds to the 45° line. Note that for $\alpha \in \{0.01, 0.38\}$, namely an almost closed economy and a moderately open economy, the second and third iterates, $R_{t+2} = f^2(R_t)$ and $R_{t+3} = f^3(R_t)$, have fixed points different from the steady state values R^* and R^L , implying the existence of cycles of periods 2 and 3. Then by the Sarkovskii's (1964) Theorem and the

²⁹The analytical results of this part of the paper are available upon request.

Figure 8: This graph shows the first, the second, and the third iterates of the implicit mapping $R_{t+1} = f(R_t)$ defined in (37) for different degrees of openness of the economy (α).

Li and Yorke's (1975) Theorem, $f(R_t)$ features cycles of any order, as well as aperiodic cyclical dynamics (topological chaos).³⁰ But observe that for $\alpha = 0.01$, cycles and chaos appear around the active steady state, while for $\alpha = 0.38$ they occur around the passive steady state. Finally, for very open economies ($\alpha = 0.90$), no cycles and chaotic dynamics appear at all. The monotonicity of the map $f(R_t)$ implies that standard liquidity traps are the only type of global equilibrium multiplicity in this case.³¹

When consumption and money are complements, endogenous global fluctuations do not exist. This is an important difference with respect to the results derived under forward-looking rules. But a close look at the results under substitutability reveal more differences. First, for contemporaneous rules, very open economies seem to be less prone to endogenous cycles. Second, the qualitative changes induced by a variation in the degree of openness can be even more dramatic, since they might even trigger a switch from one steady state to the other as the focus of fluctuations. The following proposition summarizes these results.

³⁰See Lorenz (1993) for a precise statement of these two theorems.

³¹It is also possible to find the exact numerical values of the α thresholds triggering a qualitative switch in dynamics. We find that period-2 cycles appear around the active steady state when $\alpha \in (0.001, 0.22)$; and period-3 cycles occur around the active steady state when $\alpha \in (0.001, 0.16)$. Pushing α up, period-2 cycles appear around the passive steady state for $\alpha \in (0.25, 0.43)$, whereas period-3 cycles (and therefore chaos) exist for $\alpha \in (0.33, 0.38)$. Finally for $\alpha > 0.39$ only liquidity traps exist.

Proposition 5 *Under active contemporaneous rules:*

1. *if consumption and money are Edgeworth complements, i.e. $\sigma \in (0, 1)$, there cannot be equilibrium cycles of any periodicity for any degree of openness $\alpha \in (0, 1)$;*
2. *if consumption and money are Edgeworth substitutes, i.e. $\sigma > 1$,*
 - (a) *there cannot be equilibrium cycles of any periodicity if the economy is sufficiently open;*
 - (b) *cyclical and chaotic dynamics occur around the passive steady state for intermediate degrees of openness;*
 - (c) *cyclical and chaotic dynamics around the active steady state occur if the economy is sufficiently closed.*

5.2.2 Backward-Looking Rules

We conclude the analysis of different timings for the rule by studying backward-looking rules defined as $R_t \equiv 1 + (R^* - 1) \left(\frac{\pi_{t-1}}{\pi^*} \right)^{\frac{A}{R^* - 1}}$ with $\xi \equiv \frac{A}{R^*} > 1$. This specification in tandem with equation $\pi_{t+1} \left(\frac{R_{t+1}}{R_{t+1} - 1} \right)^x = \left(\frac{R_t}{R_t - 1} \right)^x \beta R_t$, which can be derived by combining equations (20) and (23), conform a system of two first-order difference equations that can be used to pursue the global determinacy analysis. Since it is very difficult to derive analytical results, we rely on simulations in order to assess whether for different values of α and σ , the system presents cycles or chaos. The simulation results show that these dynamics are not present, regardless of the degree of openness α when $\sigma \in \{0.79, 1, 51, 2.03, 2.55\}$ (or equivalently when the CRRA coefficient $\tilde{\sigma} \in \{0.8, 1.5, 2, 2.5\}$). The interest rate converges to either the active or the passive steady-state. Nevertheless, this is not sufficient evidence to conclude that backward-looking rules will preclude the existence of cyclical dynamics. In fact, in Airaudo and Zanna (2009) we show that these rules may still lead to cyclical dynamics in open economies that face nominal price rigidities.

5.3 Nominal Price Rigidities

We proceed to assess whether cyclical and chaotic dynamics still arise once we introduce sticky prices. To do so, we introduce price stickiness for non-traded goods, following the specification in Benhabib et al. (2001a) of quadratic price adjustment costs in the utility function. In this sense, the representative agent j now maximizes:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\left[(c_{jt})^\gamma (m_{jt}^d)^{1-\gamma} \right]^{1-\sigma} - 1}{1-\sigma} + \psi(1 - h_{jt}^T - h_{jt}^N) - \frac{\phi}{2} \left(\frac{P_{jt}^N}{P_{jt-1}^N} - \pi^* \right)^2 \right\}$$

where c_{jt} is defined as in (2). We assume that the non-traded good is a composite good, and that, because of monopolistic competition, the household-firm unit j can choose the price of the non-traded good it supplies,

P_{jt}^N , subject to the demand constraint $y_{jt}^N \geq c_t^N \left(\frac{P_{jt}^N}{P_t^N} \right)^{-\mu}$, where $c_t^N = \int_0^1 c_{jt}^N dj$ and $\mu > 1$. The budget constraint for the representative household-firm unit j is identical to (8), apart for the subscript j . For computational tractability, we focus on the case of a contemporaneous interest rate rule, as in Section 5.2.1. Under symmetry, the equilibrium dynamics are entirely described by the following conditions:³²

$$(\pi_{t+1}^N - \pi^*) \pi_{t+1}^N = \frac{1}{\beta} (\pi_t^N - \pi^*) \pi_t^N + \frac{\lambda(\mu-1)}{\beta\phi} \frac{c_t^N}{e_t} - \frac{\psi\mu}{\theta^N \beta\phi} (c_t^N)^{\frac{1}{\theta^N}}$$

$$e_t = e_{t-1} \frac{\epsilon_t}{\pi_t^N}$$

$$\epsilon_{t+1} = \beta \left[1 + (R^* - 1) \left(\frac{\pi_t}{\pi^*} \right)^{\frac{A}{R^*-1}} \right]$$

where $c_t^N = \tilde{\kappa} e_t^\varphi \left[(R^* - 1) \left(\frac{\pi_t}{\pi^*} \right)^{\frac{A}{R^*-1}} \right]^{-\omega} \left[1 + (R^* - 1) \left(\frac{\pi_t}{\pi^*} \right)^{\frac{A}{R^*-1}} \right]^\omega$ and $\pi_t = \epsilon_t^\alpha (\pi_t^N)^{(1-\alpha)}$, with $\tilde{\kappa} > 0$, $\varphi \equiv \frac{1-\alpha(1-\sigma)}{\sigma}$ and $\omega \equiv \frac{(1-\gamma)(1-\sigma)}{\sigma}$.

First of all, it is straightforward to show that we still have two distinct steady states. Second, it is possible to show that, for any degree of openness and any elasticity $\xi > 1$, the active steady state is dynamically unstable. For the rest, we rely on numerical simulations, varying the degree of nominal price rigidity ϕ in the range [2.8, 44], which is in line with Dib's (2003) estimates. We consider different combinations of degree of openness α and the CRRA coefficient $\tilde{\sigma}$, while keeping the remaining parameters as in Table 1. Through a wide range of experiments, we have not been able to detect cycles of any periodicity, around neither of the two steady states. Under sticky prices, the only type of global indeterminacy seems to be the liquidity trap. In this sense, our results are in line with those reported by Eusepi (2005).

5.4 The Timing of Real Money Balances

Our set-up follows the “Cash-When-I’m-Done” (CWID) timing for real money balances, traditionally adopted in the literature. However, Carlstrom and Fuerst (2001) have argued that this timing is counterintuitive and promoted the use of the “Cash-In-Advance” (CIA) timing, where the real money balances entering the agent’s utility are those left after leaving the bond market, but before entering the goods market. In this Subsection, we construct a simple example showing that the results presented in Section 3 for active forward-looking rules are not driven by the CWID timing.

We introduce the CIA following Appendix A of Woodford (2003), while keeping the rest of the features of the model. As before, it is possible to obtain a non-linear reduced form of the model

$$\left(\frac{R_{t+1} - 1}{R_{t+1}^\Psi} \right)^\chi = \frac{R^*}{(R^* - 1)^{\frac{R^*-1}{A}}} \frac{(R_t - 1)^{\chi + \frac{R^*-1}{A}}}{R_t^{1+\Psi\chi}}, \quad (38)$$

³²The derivation of these conditions is available from the authors upon request.

which describes the equilibrium dynamics under forward-looking rules. The difference between (38) and its CWID counterpart (24) is the coefficient $\Psi \equiv \frac{1-\theta_N(1-\sigma)}{(1-\sigma)(1-\gamma)(1-\theta_N)}$, which is positive for $\sigma \in (0, 1)$ and negative for $\sigma > 1$.³³ The structural parameter χ is still defined by (25). The following Proposition suggests that under the CIA timing, cycles are still possible and their existence depends on the degree of openness α . Nevertheless the CIA timing implies different bifurcation thresholds for α from the ones derived under the CWID timing.

Proposition 6 *Consider the definition of χ in (25) and define $\Upsilon^{cia} \equiv \frac{(1-\frac{R^*-1}{A})^{R^*-1}}{2R^*}$. Suppose $\Upsilon^{cia} > 0$ and that the government follows an active forward-looking rule. If consumption and money are Edgeworth substitutes, i.e. $\sigma > 1$, then there exist period-2 cycles when $\chi < \Upsilon^{cia}$, a condition that is satisfied for sufficiently open economies, i.e. $\alpha > \alpha^{cia}$.*

Proof. See the Appendix. ■

6 Conclusions

We show that the extent to which active interest rate rules can have perverse effects in a small open economies by inducing endogenous cyclical and chaotic dynamics depends on key features of these economies. In our model a rule that responds to expected future CPI-inflation is more prone to lead to these dynamics the more open the economy and the higher the degree of exchange rate pass-through. If consumption and money are Edgeworth complements these dynamics occur around an extremely low interest rate steady state resembling liquidity traps. On the other hand, if consumption and money are substitutes these dynamics appear around the interest rate target set by the monetary authority. We find that our results seem to hold under rules that respond to current inflation, different timings of real money balances in liquidity services, and differing assumptions about market completeness.

Because of tractability, our analytical results are restricted to the case of a flexible price and perfectly competitive economy. By simulations, we have not been able to detect the existence cyclical and chaotic dynamics for calibrated versions of our economy featuring flexible-price traded and sticky price non-traded goods. However, using a continuous time framework and appealing to the Hopf bifurcation theorem, in Airaudo and Zanna (2009) we show that the existence of cycles and its relationship to open economy features is robust to the introduction of nominal price rigidities.

Our theoretical results raise the question of whether there is any evidence of chaotic dynamics in economic data.³⁴ In this regard, Bask (2002) provides evidence for the nominal exchange rate of several currencies against the US Dollar, including the Swedish Krona, the Deutsche Mark, and the Yen. Similarly, Goga

³³One could think of the CWID set up as one where $\Psi = 1$.

³⁴Most of the papers that deal with this issue focus on financial variables, such as stock prices and returns. Hsieh (1991) and Barnett and Serletis (2000) provide a detailed overview of the available results, as well as the statistical procedures used to test for the existence of chaos, including the Lyapunov exponent.

and Serletis (2000) report evidence for the real exchange rates of a number of OECD countries.³⁵ These results seem to be in line with the simulation analysis by De Grauwe and Grimaldi (2006), which shows that deterministic chaotic dynamics can be as a good description of exchange rate fluctuations as more standard linear stochastic models.

We leave open some interesting questions for future research. Throughout the analysis, we have assumed the government follows a fixed monetary policy rule. Many authors, including Benhabib et al. (2002b), have argued for an explicit commitment to switch to a different monetary rule (exchange-rate-based or money-growth-based rules) to escape from or rule out liquidity traps. It would be interesting to assess whether cyclical and chaotic dynamics still occur in such regime-switching environments. We plan to pursue the analysis in a separate research project.

A Appendix

This first part of the appendix presents the statements of some Lemmata and their proofs as well as some of the proofs of the Lemmata stated in the paper. The second part presents the proofs of the main Propositions of the paper.

A.1 Lemmata and Proofs

A.1.1 Proof of Lemma 1

Proof. Let $\phi^f \equiv 1 + \frac{\frac{R^*}{A} - 1}{\frac{A}{R^*} - 1}$ be the slope of (29) and define $\Upsilon^d \equiv \frac{1}{2}(R^* - 1) \left(1 - \frac{R^*}{A}\right) > 0$. Since R_t is a non-predetermined variable, the equilibrium is locally unique if and only if $|\phi^f| > 1$, i.e. the linearized mapping (29) is explosive and therefore the target steady state is the unique bounded PFE.

If $\chi < 0$, $\frac{A}{R^*} > 1$ (Assumption 0), and $R_t > 1$ (the zero lower bound) then $\phi^f > 1$. Hence the equilibrium is locally unique. If $\chi > 0$, $\frac{A}{R^*} > 1$, and $R_t > 1$ then $\phi^f < 1$, meaning that in order to have a unique equilibrium we need $\phi^f < -1$. Simple algebra shows that $\chi < \Upsilon^d$ implies $\phi^f < -1$. Hence the map is explosive and the equilibrium is unique. On the other hand, it is simple to show that if $\chi > \Upsilon^d$ then $-1 < \phi^f < 1$. Hence the map is non-explosive, i.e. from any initial condition R_0 off the target steady state R^* , R_t will eventually converge to R^* . This continuity of PFE paths is the source of local multiple equilibria.

■

A.1.2 Proof of Lemma 2

Proof. From point 2 of Lemma 6, the assumption $\chi \in (\Upsilon^w, 0)$ implies that the mapping $f(\cdot)$ satisfies the zero-lower-bound condition only for $R_t \in (R^l, R^u) \subset (1, +\infty)$. Moreover within (R^l, R^u) , $f(\cdot)$ resembles an inverted logistic mapping with a minimum at R^J and $f'(R^L) < 0$ (see the right hand side of Figure 1). Then Point 1 follows from these properties of $f(\cdot)$ the fact that $f'(R_t) < 0$ for any $R_t \in (R^l, R^J)$.

³⁵Note that in our model, a chaotic nominal interest rate implies a chaotic real exchange rate, as the latter is a monotonic (non-linear) tranformation of the former.

To prove Point 2, take any $R_t \in [\underline{R}, R^*]$ whose first iterate is $R_{t+1} = \left[1 - J(R_t)^{\frac{1}{\chi(\alpha, \sigma)}}\right]^{-1}$. If $R_t \in [\underline{R}, R^J]$, then $R_{t+1} = f(R_t) > f(R^J) \equiv \underline{R}$ since $f'(R_t) < 0$ for any $R_t < R^J$; moreover $R_{t+1} < R^*$ if $f_{\min} \geq \tilde{R}$. Similarly, if $R_t \in (R^J, R^*]$ then $R_{t+1} = f(R_t) < f(R^*) = R^*$ since $f'(R_t) > 0$ for $R_t > R^J$, as well as $R_{t+1} = f(R_t) > f(R^J) = \underline{R}$. Hence we have shown that for any $R_t \in [\underline{R}, R^*]$, $R_{t+1} = f(R_t) \in [\underline{R}, R^*]$. Then $f : [\underline{R}, R^*] \rightarrow [\underline{R}, R^*]$. The attractive property of the set is straightforward to show and therefore omitted.

Point 3 is based on the fact that a sufficient condition for the existence of period-2 cycles is $f'(R^L) < -1$ which implies an unstable passive steady state. To show this we define an auxiliary function $g(R_t) \equiv R_t - f^2(R_t)$ such that period-2 cycles are the zeros of $g(R_t)$. The assumption $f_{\min} \equiv f(R^J) \geq \tilde{R}$ requires a distinction between the two cases: i) $f(R^J) > \tilde{R}$ and ii) $f(R^J) = \tilde{R}$.

In case i), the map-invariant set is $[\underline{R}, R^*]$. Clearly $g(R^L) = g(R^*) = 0$ and $g(\underline{R}) \leq 0$. Also, by the chain rule, $g'(R^{ss}) = 1 - [f'(R^{ss})]^2$, with R^{ss} being either one of the two steady states. But then $f'(R^*) = \frac{A}{R^*} > 1$ implies that $g'(R^*) < 0$. The sign of $g(R^J)$ is ambiguous. If $g(R^J) = 0$ then the period-2 cycle is $\{R^J, \underline{R}\}$. If $g(R^J) < 0$, by continuity of $f(R_t)$ on $[\underline{R}, R^*]$, then there exists a point $R^c \in (R^J, R^*)$ such that $g(R^c) = 0$. Since the set $[\underline{R}, R^*]$ is map invariant the second point of the period-2 cycle belongs to the same set.³⁶ If instead $g(R^J) > 0$, nothing guarantees that the function $g(\cdot)$ has zeros other than the two steady states. A sufficient condition for having another zero is that $g'(R^L) = 1 - [f'(R^L)]^2 < 0$, i.e. $f'(R^L) < -1$. If this holds, then $g(R) = 0$ at some point between R^L and R^J . From the definition of $f(\cdot)$ it is simple to show that $f'(R^L) < -1$ is equivalent to $\chi > \frac{1}{2}\Upsilon^w$ where $\Upsilon^w \equiv R^L \left(1 - \frac{R^*-1}{A}\right) - 1$. Since we are considering $\chi > \Upsilon^w$, in case i) period-2 cycles around the passive steady state occur when $\chi \in (\frac{1}{2}\Upsilon^w, 0)$. In case ii) all conditions of case i) hold, although in this case $g(R^J) < 0$ is always true. Therefore by a similar argument there exist period-2 cycles. ■

A.1.3 Proof of Lemma 3

Proof. Let $\Upsilon^k \equiv \left(1 - \frac{R^*}{A}\right)(R^* - 1)$. The restriction $\chi \in (0, \Upsilon^k)$ implies, by Point 1 of Lemma 6, that the mapping $f(\cdot)$ looks like a logistic map with a maximum at R^J , $f'(R^L) > 1$ and $f'(R^*) < 0$ (see the left hand side of Figure 1).

The proofs of Points 1 and 2 are similar to their counterparts in Lemma 2, so both are omitted. The proof of Point 3 involves searching for a flip bifurcation at the active steady state. Let's define an auxiliary function $g(R_t) \equiv R_t - f^2(R_t)$. Period 2 cycles are then zeros of $g(R_t)$. The assumption $f_{\max} \equiv f(R^J) \leq \tilde{R}$ requires a distinction between the two cases: $f(R^J) = \tilde{R}$ and $f(R^J) < \tilde{R}$. If $f(R^J) < \tilde{R}$, the set invariant under mapping $f(\cdot)$ is $[R^L, \bar{R}]$. Clearly $g(R^L) = g(R^*) = 0$, and $g'(R^L) < 0$, but $g(R^J) \geq 0$. If $g(R^J) = 0$, a period-2 cycle is, by construction, $\{R^J, \bar{R}\}$. If $g(R^J) > 0$, by continuity of the function $g(\cdot)$ there must exist a point $R^c \in (R^L, R^J)$ such that $g(R^c) = 0$. As R^c belongs to the map invariant set $[R^L, \bar{R}]$, $f^2(R^c) \in [R^L, \bar{R}]$ as well: a period-2 cycle exists. If instead $g(R^J) < 0$, by continuity, a sufficient condition for period-2 cycles is that $g'(R^*) = 1 - [f'(R^*)]^2 < 0$, i.e. $f'(R^*) < -1$ given that over these parametric ranges $f'(R^*) < 0$ always. Using the definition of $f(\cdot)$ in (30) and by simple differentiation, this is in fact equivalent to $\chi < \frac{\Upsilon^k}{2}$.

³⁶We could actually show that this second point belong to the interval $(\underline{R}, R^L)$, meaning that the equilibrium path jumps deterministically from the left to the right neighborhood of the passive steady state.

For the case of $f(R^J) = \tilde{R}$, all the properties of the previous case hold, but now $g(R^J) = R^J - R^L > 0$ always. So by similar arguments, a period-2 cycle always exists. ■

A.1.4 Lemma 4

Lemma 4 Keep γ and θ_N constant and let $\chi(\alpha, \sigma)$, $\chi_{\max}$, $\chi_{\min}$ and $\mu(\sigma)$ be defined as in (25), (27) and (28). Consider any real number $\Upsilon^i \in (0, \chi_{\max})$ and define a function $\alpha^i(\sigma) \equiv \frac{\left[1 - \frac{\Upsilon^i}{\mu(\sigma)}\right]}{\left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]}$ for $\sigma > 1$.

1. Over the domain $(1, +\infty)$, the function $\alpha^i(\sigma)$ satisfies the following properties:

- (a) it is continuously differentiable, strictly increasing and strictly concave;
- (b) $\lim_{\sigma \rightarrow 1^+} \alpha^i(\sigma) = -\infty$ and $\lim_{\sigma \rightarrow +\infty} \alpha^i(\sigma) = \left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]^{-1} \left[1 - \frac{\Upsilon^i}{\chi_{\max}}\right] \in (0, 1)$
- (c) $\alpha^i(\sigma) \geq 0$ for $\sigma \geq \sigma^i$ where $\sigma^i \equiv \frac{\left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]}{\left[1 - \frac{\Upsilon^i}{\chi_{\max}}\right]} > 1$;

2. For any given $\sigma > 1$, $\chi(\alpha, \sigma) \geq \Upsilon^i$ if and only if $\alpha \leq \alpha^i(\sigma)$.

3. For $\alpha \in (0, 1)$ and $\sigma > 1$ we have that $\chi(\alpha, \sigma) < \Upsilon^i$ if and only if $\alpha > \max\{0, \alpha^i(\sigma)\}$.

Proof. Recall the definitions of $\chi(\alpha, \sigma)$, $\chi_{\max}$, $\chi_{\min}$ and $\mu(\sigma)$ in (25), (27) and (28). Then take any real number $\Upsilon^i \in (0, \chi_{\max})$ and any $\sigma > 1$. If we solve explicitly the equation $\chi(\alpha, \sigma) = \Upsilon^i$ with respect to α , for a given σ , the solution is a function $\alpha^i(\sigma) \equiv \left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]^{-1} \left[1 - \frac{\Upsilon^i}{\mu(\sigma)}\right]$. Point 1(a) follows from the definition of $\alpha^i(\sigma)$, $\chi_{\max} \in (0, +\infty)$, $\chi_{\min} \in (-1, 0)$, $\Upsilon^i \in (0, \chi_{\max})$, and the fact that for $\sigma > 1$, the function $\mu(\sigma)$, defined in (28) satisfies the following: $\mu : (1, \infty) \rightarrow \mathbb{R}$, it is continuously differentiable, strictly increasing, and strictly concave with respect to σ and satisfies $\lim_{\sigma \rightarrow 1^+} \mu(\sigma) = 0$ and $\lim_{\sigma \rightarrow +\infty} \mu(\sigma) = \chi_{\max} \in (0, +\infty)$. Point 1(b) is proved as follows. Given that $\lim_{\sigma \rightarrow 1^+} \mu(\sigma) = 0$ and $\lim_{\sigma \rightarrow +\infty} \mu(\sigma) = \chi_{\max}$ then for $\Upsilon^i \in (0, \chi_{\max})$ and $\chi_{\min} \in (-1, 0)$ we have that $\lim_{\sigma \rightarrow 1^+} \alpha^i(\sigma) = -\infty$ and $\lim_{\sigma \rightarrow +\infty} \alpha^i(\sigma) = \left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]^{-1} \left[1 - \frac{\Upsilon^i}{\chi_{\max}}\right]$. The assumption of $\Upsilon^i \in (0, \chi_{\max})$ together with $\chi_{\min} \in (-1, 0)$ guarantee that $\left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]^{-1} \left[1 - \frac{\Upsilon^i}{\chi_{\max}}\right] \in (0, 1)$. And point 1(c) follows from solving $\mu(\sigma) = \Upsilon^i$ with respect to σ and obtaining $\sigma^i = \frac{\left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]}{\left[1 - \frac{\Upsilon^i}{\chi_{\max}}\right]}$. As $\Upsilon^i \in (0, \chi_{\max})$ and $\chi_{\min} \in (-1, 0)$, it must be that $\sigma^i > 1$. Using the definition of σ^i and points 1(a) and 1(b) then the rest of 1(c) follows.

Point 2 is based on the following. Since for $\sigma > 1$ and $\alpha \in (0, 1)$, the function $\chi : (0, 1) \times (1, \infty) \rightarrow \mathbb{R}$ is continuously differentiable, strictly decreasing with respect to α and satisfies $\chi(\alpha, \sigma) > 0$ then it follows that $\chi(\alpha, \sigma) \geq \Upsilon^i$ if and only if $\alpha \leq \alpha^i(\sigma)$. Lastly, to prove point 3 we proceed as follows. From point 2, we know that $\chi(\alpha, \sigma) < \Upsilon^i$ if and only if $\alpha > \alpha^i(\sigma)$. From point 1(c) of the same Lemma we have also that $\alpha^i(\sigma) \geq 0$ for $\sigma \geq \sigma^i$. Since $\alpha \in (0, 1)$ then $\chi(\alpha, \sigma) < \Upsilon^i$ for any $\alpha \in (0, 1)$ when $\sigma \leq \sigma^i$, as in this case $\alpha^i(\sigma) \leq 0$; while $\chi(\alpha, \sigma) < \Upsilon^i$ only for $\alpha > \alpha^i(\sigma) > 0$ if $\sigma > \sigma^i$. If we write this compactly, we have that $\chi(\alpha, \sigma) < \Upsilon^i$ if and only if $\alpha > \max\{0, \alpha^i(\sigma)\}$. ■

A.1.5 Lemma 5

Lemma 5 Keep γ and θ_N constant and let $\chi(\alpha, \sigma)$, $\chi_{\max}$, $\chi_{\min}$ and $\mu(\sigma)$ be defined as in (25), (27) and (28). Consider any real number $\Upsilon^i \in (\chi_{\min}, 0)$ and define a function $\alpha^i(\sigma) \equiv \frac{\left[1 - \frac{\Upsilon^i}{\mu(\sigma)}\right]}{\left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]}$ for $\sigma \in (0, 1)$.

1. Over the domain $(0, 1)$, the function $\alpha^i(\sigma)$ satisfies the following properties:

- (a) it is continuously differentiable, strictly decreasing and strictly concave;
- (b) $\lim_{\sigma \rightarrow 0} \alpha^i(\sigma) = 1$ and $\lim_{\sigma \rightarrow 1^-} \alpha^i(\sigma) = -\infty$;
- (c) $\alpha^i(\sigma) \geq 0$ for $\sigma \leq \sigma^i$ where $\sigma^i \equiv \frac{\left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]}{\left[1 - \frac{\Upsilon^i}{\chi_{\max}}\right]} \in (0, 1)$.

2. For any given $\sigma \in (0, 1)$, $\chi(\alpha, \sigma) \geq \Upsilon^i$ if and only if $\alpha \geq \alpha^i(\sigma)$.

3. For $\alpha \in (0, 1)$ and $\sigma \in (0, 1)$ we have that $\chi(\alpha, \sigma) > \Upsilon^i$ if and only if $\alpha > \max\{0, \alpha^i(\sigma)\}$.

Proof. The proof is omitted since it is very similar to the proof of Lemma 4. ■

A.1.6 Lemma 6

Lemma 6 Define the scalars $\Upsilon^p \equiv \frac{R^* - 1}{A} > 0$, $\Upsilon^k \equiv \left(1 - \frac{R^*}{A}\right)(R^* - 1) > 0$ and $R^J \equiv \frac{1 + \chi}{1 - \Upsilon^p}$. Recall the definition of $f(R_t)$ in (30)-(31) for $R_t \in (1, +\infty)$.

- 1. Suppose that $\chi > 0$. Then a) $f : (1, +\infty) \rightarrow (1, +\infty)$ and it is continuously differentiable; b) $f(\cdot)$ has a global maximum at R^J with $R^J > R^L$; c) $\lim_{R_t \rightarrow 1^+} f(R_t) = \lim_{R_t \rightarrow \infty} f(R_t) = 0$; d) $f'(R^L) > 1$ always, while $f'(R^*) \geq 0$ depending on $\chi \geq \Upsilon^k$.
- 2. Suppose that $\chi \in (-\Upsilon^p, 0)$. Let R^l and R^u be the solutions to $J(R_t) = 1$ with $R^l \in (1, R^L)$ and $R^u > R^*$. Then a) $f : (R^l, R^u) \rightarrow (1, +\infty)$ and it is continuously differentiable; b) within (R^l, R^u) , $f(\cdot)$ has a unique minimum at R^J ; c) $\lim_{R_t \rightarrow R^l} f(R_t) = \lim_{R_t \rightarrow R^u} f(R_t) = +\infty$; d) $f'(R^*) > 1$ always, while $f'(R^L) \geq 0$ depending on $\chi \leq \Upsilon^w$ with $\Upsilon^w \equiv R^L(1 - \Upsilon^p) - 1 \in (-\Upsilon^p, 0)$.

Proof. The proof is available from the authors upon request. ■

A.2 Proofs of the Propositions

A.2.1 Proof of Proposition 2

Proof. Recall the definition of $\chi(\alpha, \sigma)$ in (25). For $\sigma \in (0, 1)$ and $\alpha \in (0, 1)$ we have that $\chi : (0, 1) \times (0, 1) \rightarrow \mathbb{R}$. Then Point 1(a) follows from point 1 of Lemma 1 and the fact that $\chi(\alpha, \sigma) < 0$ if $\sigma \in (0, 1)$ and $\alpha \in (0, 1)$.

Next we prove point 1(b). For $\sigma \in (1, \infty)$ and $\alpha \in (0, 1)$ we have that $\chi : (0, 1) \times (1, \infty) \rightarrow \mathbb{R}$. Define $\Upsilon^d \equiv \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right)$. The proof of point 1(a) of Lemma 1 makes clear that $\chi = \Upsilon^d$ is the threshold value of χ that differentiates between active rules leading to either a unique equilibrium or multiple equilibria. Since we are interested in how the pair (α, σ) affects local determinacy, we look for values of α that, for given σ , solve $\chi(\alpha, \sigma) = \Upsilon^d$. The solution is a function $\alpha^d(\sigma) = \left[1 - \frac{\Upsilon^d}{\chi_{\min}}\right]^{-1} \left[1 - \frac{\Upsilon^d}{\mu(\sigma)}\right]$. Given $\Upsilon^d \in (0, \chi_{\max})$

(Assumption 1) then we can use Lemma 4 to deduce that $\alpha^d(\sigma)$ is equivalent to the function $\alpha^i(\sigma)$, defined in that Lemma, with $\Upsilon^i = \Upsilon^d$. Therefore, $\alpha^d(\sigma)$ inherits all the properties of $\alpha^i(\sigma)$. Using this and Point 3 of Lemma 4, we can conclude that for any $\sigma > 1$ then $\alpha > \alpha_{\min}^d$ with $\alpha_{\min}^d \equiv \max \{0, \alpha^d(\sigma)\}$. This is a necessary and sufficient condition for $\chi(\alpha, \sigma) < \Upsilon^d$ to hold, which, by Lemma 1, implies the existence of a unique local equilibrium.

To prove point 2 we follow the same steps of proving 1(b) taking into account that from point 2 of Lemma 4 we can deduce that $\sigma > \sigma^d$ and $\alpha \in (0, \alpha^d(\sigma))$ leads to $\chi(\alpha, \sigma) > \Upsilon^d$. This combined with point 2 of Lemma 1 imply multiple equilibria. ■

A.2.2 Proof of Proposition 3

Proof. The proof of Point 1 combines the results of Point 3 in Lemma 2 with Lemma 5. Define $\Upsilon^p \equiv \frac{R^*-1}{A}$, $\Upsilon^w \equiv R^L(1 - \Upsilon^p) - 1$ and $\Upsilon^f \equiv \frac{1}{2}\Upsilon^w$. From Point 3 of Lemma 2, the flip bifurcation threshold is $\chi = \Upsilon^f$. Since χ is a function of α and σ then this threshold can be expressed in the (α, σ) -space as the solution to $\chi(\alpha, \sigma) = \Upsilon^f$ with respect to α , for any arbitrary $\sigma \in (0, 1)$. The solution is a function $\alpha^f(\sigma) \equiv \left[1 - \frac{\Upsilon^f}{\chi_{\min}}\right]^{-1} \left[1 - \frac{\Upsilon^f}{\mu(\sigma)}\right]$. Observe that an active rule implies $-\Upsilon^p < R^L(1 - \Upsilon^p) - 1 = \Upsilon^w$ and that, from Assumption 3, $\frac{R^L-1}{R^L} < R^L - 1 < \Upsilon^p$. Then $R^L(1 - \Upsilon^p) - 1 < 0$, which implies that $\Upsilon^w < \Upsilon^f < 0$. Using this and $-\Upsilon^p < R^L(1 - \Upsilon^p) - 1 = \Upsilon^w$ we conclude that $\Upsilon^w \in (-\Upsilon^p, 0)$. From this and Assumption 2 we can see that $\Upsilon^f \in (\chi_{\min}, 0)$ satisfies the assumptions in Lemma 5. Hence $\alpha^f(\sigma)$ shares all the properties of the function $\alpha^i(\sigma)$ for $\Upsilon^i = \Upsilon^f$ stated in that Lemma. By Point 3 of Lemma 5, for an arbitrary $\sigma \in (0, 1)$, the flip bifurcation condition $\chi(\alpha, \sigma) > \Upsilon^f$ in Lemma 2 is therefore equivalent to $\alpha > \alpha_{\min}^f$ where $\alpha_{\min}^f \equiv \max \{0, \alpha^f(\sigma)\}$. From an application of Point 1 of Lemma 5 we can also see that such minimum degree of openness, $\alpha_{\min}^f$, is positive and decreasing in σ , for $\sigma \in (0, \sigma^f)$, but constant and equal to zero for $\sigma \in [\sigma^f, 1)$.

The proof of Point 2 is omitted since it is very similar to the proof of Point 1. But it uses the fact that $\sigma > 1$ and Point 3 of Lemma 3 together with Lemma 4 instead. ■

A.2.3 Proof of Proposition 6

Proof. We provide a sketch of the proof. Let $K(R_{t+1})$ and $J(R_t)$ be the left and the right hand sides of (38) respectively. First of all notice that for $\sigma > 1$, we have $\Psi < 0$ and $\chi > 0$. To simplify the notation let $\Gamma \equiv (1 - \Psi)\chi$. Then consider the function $K(\cdot)$. Since $\Gamma > 0$, it is simple to show that $\lim_{R_{t+1} \rightarrow 1^+} K(R_{t+1}) = 0$ and $\lim_{R_{t+1} \rightarrow +\infty} K(R_{t+1}) = +\infty$. Furthermore simple algebra shows that $K(\cdot)$ is strictly increasing over the entire domain $(1, +\infty)$. But then $K(\cdot)$ is also globally invertible such that there exist a well defined function $f(R_t) = K^{-1}(J(R_t))$ describing the forward equilibrium dynamics.

Now consider the function $J(R_t)$. Since $\chi + \frac{R^*-1}{A} > 0$ for any $\alpha \in (0, 1)$, then $\lim_{R_t \rightarrow 1^+} J(R_t) = 0$; while $\lim_{R_t \rightarrow +\infty} J(R_t) = +\infty$, if $\Gamma > \left(1 - \frac{R^*-1}{A}\right)$ and $\lim_{R_t \rightarrow +\infty} J(R_t) = 0$, if $\Gamma < \left(1 - \frac{R^*-1}{A}\right)$. Using the definition of Γ , let's consider the case of $\chi < \frac{\left(1 - \frac{R^*-1}{A}\right)}{(1-\Psi)}$ such that $\lim_{R_t \rightarrow +\infty} J(R_t) = 0$. It is possible to show that $J(\cdot)$ has a critical point at $R^J = \frac{1+\Psi\chi}{\left(1 - \frac{R^*-1}{A}\right)} > 1$.

Therefore if $\chi < \frac{\left(1 - \frac{R^*-1}{A}\right)}{(1-\Psi)}$ the function $J(\cdot)$ is single-peaked. This together with the fact that $K(\cdot)$ is strictly increasing over the entire domain $(1, +\infty)$ imply that the mapping $f(\cdot)$ looks like a logistic map.

Then if cycles exist they have to be centered around the active steady state. To show we can use Lemma 3. That is, a sufficient condition for period-2 cycles to exist is $f'(R^*) < -1$, which can be proved to be equivalent to $\chi < \frac{(1 - \frac{R^* - 1}{A})R^* - 1}{2R^*}$. Given $\sigma > 1$ it is possible to show that there exists $\alpha^{cia} \in (0, 1)$ such that $\chi = \frac{(1 - \frac{R^* - 1}{A})R^* - 1}{2R^*}$ and that for $\alpha > \alpha^{cia}$, $\chi < \frac{(1 - \frac{R^* - 1}{A})R^* - 1}{2R^*}$. ■

References

- [1] Alstadheim, R., Henderson, D., 2006. Price-Level Determinacy, Lower Bounds on the Nominal Interest Rate, and Liquidity Traps. *Contributions to Macroeconomics* 6, 1437-1437.
- [2] Airaudo, M., Zanna, L.F., 2009. Interest Rate Rules and Endogenous Cycles in Open Economies with Nominal Price Rigidities. Manuscript.
- [3] Ball, L., 1999. Policy Rules for Open Economies, in Taylor, J.B. (Ed), *Monetary Policy Rules*, National Bureau of Economic Research, 127-144.
- [4] Barnett, W., Serletis, A., 2000. Martingales, Non-linearities and Chaos. *Journal of Economic Dynamics and Control* 24, 703-724.
- [5] Bask, M., 2002. A Positive Lyapunov Exponent in Swedish Exchange Rates?. *Chaos, Solitons and Fractals* 14, 1295-1304.
- [6] Batini, N., Levine, P., Pearlman, J., 2004. Indeterminacy with Inflation-Forecast-Based Rules in a Two-Bloc Model. IFDP No. 797, Board of Governors of the Federal Reserve System.
- [7] Benhabib, J., Schmitt-Grohé, S., Uribe, M., 2001a. Monetary Policy Rules and Multiple Equilibria. *American Economic Review* 91, 167-184.
- [8] Benhabib, J., Schmitt-Grohé, S., Uribe, M., 2001b. The Perils of the Taylor Rules. *Journal of Economic Theory* 96, 40-69.
- [9] Benhabib, J., Schmitt-Grohé, S., Uribe, M., 2002a. Chaotic Interest Rate Rules. *American Economic Review* 92, 72-78.
- [10] Benhabib, J., Schmitt-Grohé, S., Uribe, M., 2002b. Avoiding Liquidity Traps. *Journal of Political Economy* 110, 535-563.
- [11] Benigno, G., Benigno, P., 2008. Exchange Rate Determination under Interest Rate Rules. *Journal of International Money and Finance* 27, 971-993.
- [12] Bentolilla, S., Saint-Paul, G., 2003. Explaining Movements in the Labor Share. *Contributions to Macroeconomics* 3, 1103-1103.
- [13] Boivin, J., 2006. Has U.S. Monetary Policy Changed? Evidence from Drifting Coefficients and Real-Time Data. *Journal of Money, Credit and Banking* 38, 1149-1174.
- [14] Boldrin, M., Nishimura, K., Shigoka, T., Yano, M., 2001. Chaotic Equilibrium Dynamics in Endogenous Growth Models. *Journal of Economic Theory* 96, 97-132.

- [15] Burnstein, A., Neves, J., Rebelo, S., 2003. Distribution Costs and Real Exchange Rate Dynamics. *Journal of Monetary Economics* 50, 1189-1214.
- [16] Burnstein, A., Eichenbaum, M., Rebelo, S., 2005. Large Devaluations and the Real Exchange Rate. *Journal of Political Economy* 113, 742-84.
- [17] Campa, J. M., Goldberg, L., 2006. Distribution Margins, Imported Inputs, and the Sensitivity of the CPI to Exchange Rates. Federal Reserve Bank of New York, Staff Reports N. 247.
- [18] Carlstrom, C., Fuerst, T., 2001. Timing and Real Indeterminacy in Monetary Models. *Journal of Monetary Economics* 47, 285-298.
- [19] Castelnuovo, E., 2008. Money and Output in a New-Keynesian Framework: Further Evidence for the US. University of Padua, Manuscript.
- [20] Clarida, J., Galí, J., Gertler, M., 1998. Monetary Policy Rules in Practice: Some International Evidence. *European Economic Review* 42, 1033-1067.
- [21] Clarida, J., Galí, J., Gertler, M., 2001. Optimal Monetary Policy in Open vs Closed Economies: An Integrated Approach. *American Economic Review* 91, 248-252.
- [22] Cogley, T., Sargent, T., 2005. Drifts and Volatilities: Monetary Policy and Outcomes in the Post WWII US. *Review of Economic Dynamics* 8, 262-302.
- [23] De Fiore, F., Zheng, L., 2005. Does Trade Openness Matters for Aggregate Instability?. *Journal of Economic Dynamics and Control* 29, 1165-1192.
- [24] De Grauwe, P., Grimaldi, M., 2006. Exchange Rate Puzzles: A Tale of Switching Attractors. *European Economic Review* 50, 1-33.
- [25] Dib, A., 2003. An Estimated Canadian DSGE Model with Nominal and Real Rigidities. *Canadian Journal of Economics* 36, 949-972.
- [26] Diks, C., Hommes, C., Panchenko, V., Van der Weide, R., 2008. E&F Chaos: A User Friendly Software Package for Non-Linear Economic Dynamics. *Computational Economics* 32, 221-244.
- [27] Eusepi, S., 2005. Comparing Forecast-Based and Backward-Looking Taylor Rules: a Global Analysis. Federal Reserve Bank of New York, Staff Reports, 198.
- [28] Eusepi, S., 2007. Learnability and Monetary Policy: A Global Perspective. *Journal of Monetary Economics* 54, 1115-1131.
- [29] Evans, G., Guse, E., Honkapohja, S., 2008. Liquidity Traps, Learning and Stagnation. *European Economic Review* 52, 1438-1463.
- [30] Frankel, J., Parsley, D., Wei, S., 2005. Slow Passthrough Around the World: A New Import for Developing Countries?. NBER Working Paper 11199.
- [31] Galí, J., Monacelli, T., 2008. Monetary Policy and Exchange Rate Volatility in a Small Open Economy. *Review of Economic Studies* 72, 707-734.

- [32] Gogas, P., Serletis, A., 2000. Purchase Power Parity, Non-linearity and Chaos. *Applied Financial Economics* 10, 615-622.
- [33] Holman, J., 1998. GMM Estimation of a Money-in-the-Utility-Function Model: The Implications of Functional Forms. *Journal of Money, Credit and Banking* 30, 679-698.
- [34] Hsieh, D., 1991. Chaos and Non-linear Dynamics: Applications to Financial Markets. *Journal of Finance* 46, 1839-1877.
- [35] Leith, C., Wren-Lewis, S., 2009. Taylor Rules in the Open Economy. *European Economic Review*, Forthcoming.
- [36] Li, T., Yorke, J., 1975). Period Three Implies Chaos. *American Mathematical Monthly* 82, 985-992.
- [37] Llosa, G., Tuesta, V., 2008. Determinacy and Learnability of Monetary Policy Rules in Small Open Economies. *Journal of Money, Credit and Banking* 40, 1033-1063.
- [38] Lorenz, H., 1993. *Nonlinear Dynamical Economics and Chaotic Motion*. Springer-Verlag, New York.
- [39] Lubik, T., Schorfheide, F., 2007. Do Central Banks Target Exchange Rates? A Structural Investigation. *Journal of Monetary Economics* 54, 1069-1087.
- [40] Matsuyama, K., 1990). Sunspot Equilibria (Rational Bubbles) in a Model of Money-in-the-Utility-Function. *Journal of Monetary Economics* 25, 137-144.
- [41] Matsuyama, K., 1991. Endogenous Price Fluctuations in an Optimizing Model of a Monetary Economy. *Econometrica* 59, 1617-1631.
- [42] Mendoza, E., 1995. The Terms of Trade, The Real Exchange rate and Economic Fluctuations. *International Economic Review* 36, 101-137.
- [43] Michener, R., Ravikumar, B., 1998. Chaotic dynamics in a cash-in-advance economy. *Journal of Economic Dynamics and Control* 22, 1117-1137.
- [44] Monacelli, T., 2005. Monetary Policy in a Low Pass-Through Environment. *Journal of Money Credit and Banking* 37, 1047-1066.
- [45] Ogaki, M., Ostry, J.D., Reinhart, C.M., 1996. Saving Behavior in Low and Middle-Income Developing Countries: A Comparison. *IMF Staff Papers* 43, 38-71.
- [46] Petursson, T., 2004. The Effects of Inflation Targeting on Macroeconomic Performance. Working Papers No. 23, Central Bank of Iceland.
- [47] Sarkovskii, A., 1964. Coexistence of Cycles of a Continuous Map of the Line Into Itself. *Ukraine Math Zhur* 16, 61-71.
- [48] Schmitt-Grohé, S., Uribe, M., 2003. Closing Small Open Economy Models. *Journal of International Economics* 61, 137-139.
- [49] Svensson, L.E.O., 2000. Open-economy Inflation Targeting. *Journal of International Economics* 50, 155-184.

- [50] Woodford, M., 2003. Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton, Princeton University Press.
- [51] Zanna, L. F., 2003. Interest Rate Rules and Multiple Equilibria in the Small Open Economy. IFDP No. 785, Board of Governors of the Federal Reserve System.