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**Abstract**

This paper studies the pricing implications of the sole ambiguity aversion, in a Lucas' tree economy where asset returns are ambiguous. Abstracting from a specific functional form, we disentangle the model-specific effect from the effect of ambiguity aversion. In addition, we allow the investor to change her tastes across time in a dynamically consistent way. Two phenomena are consistent with ambiguity aversion: portfolio inertia and price indeterminacy. We provide intuitive conditions to guarantee the existence and to characterize equilibria, showing that the relevant information to price asset is contained in a set of priors who is identifiable in any model used in applications. Lastly, we prove that ambiguity enriches the standard pricing formula by an additional stochastic discount factor and we calculate its explicit form for various models.

KEYWORDS: Asset Pricing, Knightian Uncertainty, Ambiguity Aversion, Indeterminacy

## 1 Introduction and overview of the results

Classical models of intertemporal asset pricing *a lá* Lucas [34], assume that agents evaluate streams of consumption through (recursive) expected utility

$$V_t(c) = u(c_t) + \beta E_p\left[V_{t+1}(c)\right]$$

This reflects the *rational expectation hypothesis*: each investor has a *unique* (subjective) probability which coincides with the law governing the "true" data generating process.

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However, a large amount of empirical evidence shows inconsistencies between the predictions of the rational expectation hypothesis and the observed data. Well known examples are the equity premium puzzle, the risk-free rate puzzle and the excess volatility puzzle.[1] The rational expectation hypothesis is overly restrictive for various reasons. One over all, investors may have little information concerning the process governing returns, therefore, they may be unable to single out a unique probability to evaluate them.[2]

Ellsberg's paradox proved that partial information has a no-negligible effect on the betting behavior of individuals and it challenged the validity of the expected utility model. This led to consider generalizations of (subjective) expected utility as the multiple prior model of Gilboa and Schmeidler [21], a pessimistic decision rule that evaluates a prospect taking its minimum expected utility over a set of priors. Epstein and Wang [16] used a dynamic version of MaxMin to study asset pricing in a Lucas'tree economy. They showed that extreme aversion to ambiguity may lead to market non-participation and equilibrium indeterminacy. Moreover, the assumption of ambiguity aversion affects another aspect of financial decisions, namely the reaction of beliefs to new information. Epstein and Schneider [14] and Illeditsch [29] show that ambiguity averse investors react asymmetrically: bad news have greater effect than good news, reflecting the pessimistic approach of the MaxMin criterion. As observed in Guidolin and Rinaldi [23] ambiguity[3] and ambiguity aversion offer a possible explanation to many observed phenomena of financial markets. This consideration is due to a bunch of theoretical results appeared in the literature (i.e. Epstein and Wang [16], Chen and Epstein [6], Leippold et al. [33] are some examples). However, each work uses a different intertemporal model of decision under ambiguity and hence, it is not clear *how to disentangle the model-specific effect from the effect of the sole ambiguity aversion*, in pricing assets. For example, assuming a MaxMin investor, which seems very natural in this context, has many implicit and, maybe, unwanted

[1] See respectively Mehra and Prescott [38], Weil [47] and Grossman and Shiller [22].

[2] "Even after observing 206 year of data (1802 to 2007), investors do not know the parameters of the return-generating process, especially the parameters related to the conditional expected return." Pastor and Stambaugh [41].

[3] "In dynamic, Lucas-type endowment models, ambiguity has also been found to be a potential cause for asset price indeterminacy and therefore for endogenous volatility (of a "sunspot" type). However, when assumptions are introduced to avoid indeterminacy, then it has been shown that asset risk premia can be decomposed in two parts, a standard (S)EU risk premium component that tends to be proportional to the covariance between asset returns and (appropriate functions of) the rate of growth of fundamentals or the market portfolio, and an ambiguity premium. This dual structure greatly helps in developing a unified and elegant solution to the equity premium and risk-free rate puzzles."

consequences. First, indifference to the timing of resolution of uncertainty, as shown in Strzalecki [46], a MaxMin investor is indifferent between two prospects that differs only in the time at which uncertainty is resolved. Secondly, absolute and relative ambiguity aversion are both *constant,* i.e. the utility of continuation plans is not affected by the scale and position of the payoffs. Similarly, if we assume robust preferences *a lá* Hansen and Sargent [27], we commit, at the same time, to a preference for early resolution of uncertainty (see Strzalecki [46]), to a "probabilistically sophisticated investor" and to a constant absolute ambiguity aversion.

The present work, proposes a *unified* approach to asset pricing under the ambiguity aversion, which abstracts from the specific functional form of the investor's utility. In addition, we allow the possibility that the investor changes her attitude toward ambiguity across time in a dynamically consistent fashion. For example, we allow for an investor to be a MaxMin EU maximizer today and a expected utility maximizer tomorrow.

We show that ambiguity aversion may cause market non-participation and price indeterminacy, regardless of the specific functional form we adopt. Moreover, we prove that the main determinant of assets prices is a set of "relevant" beliefs that can be identified in all models used in applications (see Ghirardato and Siniscalchi [20]). If relevant beliefs vary continuously with information's arrival, an equilibrium exists (Theorem 4). This is an important result since we let the investor to change her attitude toward ambiguity across time.

We use a recursive utility that includes many standard recursive models of decision under ambiguity (as Klibanoff et al. [32], Epstein and Schneider [12], Maccheroni et al. [35], Siniscalchi [44]) and it is given by

$$V_t(c) = u(c_t) + \beta I_t(V_{t+1}(c))$$

where $I_t$ is an ambiguity averse certainty equivalent operator.

The results follows from the general Euler's (in)equalities we obtain, they are of the form

$$\min_{p\in C_t} \beta E_p \left[ \frac{u'(c_{t+1})}{u'(c_t)}(q_{t+1} + d_{t+1}) \right] \leq q_t \leq \max_{p\in C_t} \beta E_p \left[ \frac{u'(c_{t+1})}{u'(c_t)}(q_{t+1} + d_{t+1}) \right]$$

Despite their similarity with Epstein and Wang [16], the inequalities above are not generated by a MaxMin intertemporal utility, but they hold for a much larger class of ambiguity averse preferences. The main element, the set $C_t$ of "time-t relevant priors", contains those beliefs which are relevant to *identifies candidate solutions* conditional on the observed realizations. Since we study non-participation equilibria, $C_t$ controls the amplitude of the non-participation interval.

A first consequence is indeterminacy, namely, a multiplicity of prices that supports (non-participation) equilibrium (see Lemma 3). This extends Epstein and Wang [16] no-trade result to a much wider class of preferences.

Secondly, prices may be discontinuous in equilibrium. It follows from the multiplicity of priors and their evolution over time. More precisely, not all the priors in $C_t$ generate prices, but only the ones solving the Euler's equation, therefore, continuity of the whole set $C_t$ with respect to information's arrival is not sufficient to guarantee continuity of prices. Indeed, we need the stronger assumption, namely, *the set of priors solving Euler's inequality* has to vary (almost) continuously with information. In particular, "explosion" of such beliefs is not allowed for a small change in information (see Theorem 5).

Another point to highlight concerns changing tastes. The results show that, as long as dynamic consistency is granted, the exact form of the certainty equivalent operator $I_t$ is irrelevant to the existence of equilibria.

Lastly, Euler's inequalities generates a family of ambiguity stochastic discount factors (SDF), $M^a_{t+1}$, that enrich the standard pricing formula to

$$1 = E_p\left[ M^a_{t+1} M_{t+1} R_{t+1} \middle| \mathscr{F}_t \right] \tag{SDF}$$

The structure of $M^a_{t+1}$ is completely determined by the optimizing behavior of an agent. Since, different models imply a different nature of the probabilities contained in $C_t$, each model is attached to a particular stochastic discount factors $M^a_{t+1}$.

The augmented pricing formula (SDF) allows to study the relation between prices and the ambiguity stochastic discount factor. In particular, excess returns and the Hansen-Jagannathan bound. Concerning excess returns, we provide a formula that clearly separates the role of risk and ambiguity. The expected excess return of a portfolio depends on the covariance of the return with the standard SDF plus, the covariance of the ambiguous discount factor with the risk neutral evaluation of the portfolio. Finally, the ratio between the standard deviation and the expected value of the ambiguity SDF is greater than the risk neutral Sharpe ratio attained by any portfolio.

The first part of the paper is devoted to study the existence and the properties of equilibrium prices. In the second part we calculate equilibrium prices for specific recursive models, hence we explicit the form of the ambiguity discount factor $M^a_{t+1}$.

The paper is a generalization of Epstein and Wang [16][4], where it is assumed that the agent is MaxMin expected utility maximizer. The literature on asset pricing under ambiguity is extensive (see Guidolin and Rinaldi [23] for a survey). A recent paper is Collard et al. [8], where they use recursive smooth ambiguity to address the historical equity premium puzzle assuming the economy evolves according to a hidden-state variable process. In a static framework, Bossaerts et al. [2] employ the $\alpha$-MaxMin model. The effect of ambiguity information arrival when investors are ambiguity averse in the form of MaxMin has been studied in Epstein and Schneider [15], Ozsoylev and Werner [40] and Illeditsch [29], where it is proved that ambiguity aversion amplifies price volatility when the agents receive unexpected (bad) news and it may lead to portfolio inertia. Asset pricing with robust preferences is studied in Hansen and Sargent [27], Hansen et al. [28] and recently by Maenhout [36]. The literature on market non-participation under ambiguity started with Dow and Werlang [10] in the static framework and Epstein and Wang [16] in a dynamic Lucas' economy, with MaxMin preferences in both cases. Easley and O'Hara [11] studied the role of market regulation to enhance market participation when investors are ambiguity averse. Recently, Ju and Miao [30] developed an asset pricing model under ambiguity using a recursive utility who allows a three-way separation of ambiguity aversion, risk aversion and inter-temporal substitution, through a non-linear aggregator of current and continuation utility. The result is a three-way decomposition of the stochastic discount factor that takes into account, ambiguity, risk and inter-temporal substitution.

## 2 Intertemporal utility: existence and uniqueness

In this section we set the framework in which inter-temporal consumption-investment problem are defined and we give conditions for the existence and uniqueness of the inter-

[4]The same authors extended these results to more general state space and consumption processes in Epstein and Wang [17]. A continuous time version is studied in Chen and Epstein [6].

temporal utility we use to price assets.

Let the set of states $\Omega$ be a compact metric space endowed with its Borel $\sigma$-algebra $\mathscr{B}(\Omega)$. Denote $\mathscr{M}_1^+(\Omega)$ the space of probability measures on $\Omega$, it is a compact Polish space[5]. Time is discrete and the decision maker observes at time $t \in \mathbb{N}$ a realization $\omega_t \in \Omega$. The measurable space given by $(\Omega^\infty, \mathscr{B}(\Omega^\infty))$, where $\mathscr{B}(\Omega^\infty)$ is the product Borel $\sigma$-algebra, represents possible paths (or histories). Let denote $\omega^t \triangleq (\omega_1, \ldots, \omega_t)$ a path up to time $t$ and, for all $t$, $\Omega^t$ the set of all such paths. It is canonically embedded in $\Omega^\infty$, so it becomes a measurable space when endowed with the (cylinders) product $\sigma$-algebra $\mathscr{B}(\Omega^t)$. A consumption process $c = (c_t)$ is *adapted* if $c_t : \Omega^\infty \to \mathbb{R}^n$ is $\mathscr{B}(\Omega^t)$-measurable for all $t$. Moreover, we say it is *continuous* if $c_t$ is continuous for all $t$ and *real-valued* if $c_t : \Omega^\infty \to \mathbb{R}$. The space of adapted continuous real-valued processes is denoted by $\mathrm{AC}(\Omega^\infty, \mathscr{B}(\Omega^\infty))$, it becomes a Banach space when endowed with the weighted sup-norm

$$\|c\| \triangleq \sup_t \sup_{\omega^t} \frac{\left|c_t(\omega^t)\right|}{b^t} \quad \text{for some } b \geq 1$$

The space $\mathscr{D}_{\text{cont}}$ of consumption processes is given by

$$\mathscr{D}_{\text{cont}} \triangleq \left\{c \in \mathrm{AC}(\Omega^\infty, \mathscr{B}(\Omega^\infty)) : \|c\| < \infty \;\text{ and }\; c_t(\omega^t) \geq 0, \text{ for all } t \geq 1 \text{ and } \omega^t \in \Omega^t\right\}$$

Another important space is the space of adapted real-valued processes, denoted by $\mathrm{AD}(\Omega^\infty, \mathscr{B}(\Omega^\infty))$. It becomes a Banach space when endowed with the essential weighted sup-norm[6]

$$\|c\|_\infty \triangleq \operatorname*{ess\,sup}_{t,\omega^t} \frac{\left|c_t(\omega^t)\right|}{b^t} \quad \text{for some } b \geq 1$$

Let $\mathscr{D}$ be

$$\mathscr{D} \triangleq \left\{c \in \mathrm{AD}(\Omega^\infty, \mathscr{B}(\Omega^\infty)) : \|c\|_\infty < \infty \;\text{ and }\; c_t(\omega^t) \geq 0, \text{ for all } t \geq 1 \text{ and } \omega^t \in \Omega^t\right\}$$

We assume the decision maker evaluates infinite consumption streams according to the recursive utility

$$V_t(c, \omega^t) = u\left(c_t(\omega^t)\right) + \beta I\left(V_{t+1}(c, \omega^t); \omega^t\right) \tag{1}$$

[5]In the weak* topology $\sigma(\mathscr{M}_1^+(\Omega), C(\Omega))$.

[6]See Marinacci and Montrucchio [37] for details. The definition of $\mathrm{AD}(\Omega^\infty, \mathscr{B}(\Omega^\infty))$ implicitly assumes the existence of a measure $\mu$ that fixes the null events.

where $I$ is a locally Lipschitz, quasiconcave[7] and normalized[8] and monotone (w.r.t. pointwise order) certainty equivalent operator. Quasiconcavity represent ambiguity aversion of an investor or preference for *hedging* (see Cerreia-Vioglio et al. [4] for a discussion and a representation). The dependence of $I(\cdot,\omega^t)$ from $\omega^t \in \Omega^t$ allows the investor to change the certainty equivalent operator across time. Although we use a time-dependent certainty equivalent, this is not a model of changing tastes (as in Caplin and Leahy [3]). Indeed, we will assume time-consistency, hence, the possibly different certainty equivalents agree on conditional preferences. A weak assumption we impose is a form of continuity of the certainty equivalent operator with respect to histories of observed states. Namely, continuity of the map

$$\omega^t \longmapsto I(\cdot;\omega^t) \tag{Cont.}$$

Condition (Cont.) is a form of continuity of tastes with respect to accumulated evidence. If two histories up to time $t$ are close enough, meaning they are similar, then the utilities of the continuation value are close as well. It is always satisfied when $\Omega$ is a finite space. As a technical assumption, it is necessary to prove the existence of a continuous recursive utility. In the case of a functional form that does not vary with time i.e. $I(\cdot;\omega^t) = I(\cdot)$, continuity is automatically satisfied (for example in Strzalecki [46]). Allowing for a time-varying utility we are not assuming IID ambiguity (in the sense of Epstein and Schneider [13]), that is, the investor may learn about the underlying ambiguity.

Turning to the special cases in the literature, in Epstein and Wang [16] the intertemporal utility is given by

$$V_t(c,\omega^t) = u\left(c_t(\omega^t)\right) + \beta \min_{p\in\mathscr{P}(\omega^t)} \int V_{t+1}(c,\omega^t)\,dp$$

a dynamic extension of MaxMin expected utility of Gilboa and Schmeidler [21]. Recently, Maccheroni et al. [35] proposed an axiomatization of recursive variational preferences, which generalizes the recursive multiple priors of Epstein and Schneider [12]. In the recursive case, the intertemporal utility is given by

$$V_t(c,\omega^t) = u\left(c_t(\omega^t)\right) + \beta \min_{p\in\mathcal{M}_1^+(\Omega)} \left(\int V_{t+1}(c,\omega^t) + \eta_t(\omega^t,p)\right)$$

[7]A functional $I : C(\Omega) \to \mathbb{R}$ is said quasiconcave if upper contour sets, $\{f \in C(\Omega) : I(f) \geq a\}$, are convex, for all $a \in \mathbb{R}$.

[8]We say that $I$ is normalized if $I(1_\Omega k) = k$ for all $k \in \mathbb{R}$.

where $\eta_t(\omega^t, p)$ is a penalization function. When $\eta_t(\omega^t, p) = D(p\|q_t)$ is the Kullback-Leibler divergence, we recover the robust preferences of Hansen and Sargent [27]. Similarly, a recursive version of the smooth ambiguity model of Klibanoff et al. [31] has been recently axiomatized by Klibanoff et al. [32]. The intertemporal utility is of the form

$$V_t(c,\omega^t) = u\left(c_t(\omega^t)\right) + \beta\phi^{-1}\left[\int_\Theta \phi\left(\int V_{t+1}(c,\omega^t)\,dp_\theta(\omega_{t+1}|\omega^t)\right)d\mu(\theta|\omega^t)\right]$$

where $\phi(\cdot)$ characterizes ambiguity attitude and $\Theta$ is a (finite) set of parameters that determines the transition probability $p_\theta(\omega_{t+1}|\omega^t)$. In general, Klibanoff et al. [31] preferences are not locally Lipschitz[9], however, a behavioral condition to impose loc. Lipschitz can be found in Ghirardato and Siniscalchi [20, Online Appendix]. An additional model is the recursive version axiomatized by Siniscalchi [44], of the Vector expected utility of Siniscalchi [43] . The form of intertemporal utility is

$$V_t(c,\omega^t) = u(c_t(\omega^t)) + \beta E_{p_t}\left[V_{t+1}(c,\omega^t)\right] + A_{t,\omega^t}\left(\beta E_p\left[\xi^{t,\omega^t}\cdot V_{t+1}(c,\omega^t)\right]\right)$$

where $p_t$ is a baseline probability, $\xi^{t,\omega^t}$ are the adjustment factors and $A_{t,\omega^t}$ is an adjustment function.

The present approach can be generalized to include other interesting cases of recursive utilities that separates risk attitude and intertemporal substitution. Since we focus on the role of ambiguity only, asset pricing with general intertemporal utility with nonlinear aggregators as in Marinacci and Montrucchio [37], may represent an insightful extension of this paper. In that case, current and the continuation utilities are aggregated through a non-linear function, i.e.

$$V_t(c,\omega^t) = W\left(c_t(\omega^t), I_t(V_{t+1}(c,\omega^t);\omega^t)\right)$$

in the spirit of Epstein and Zin [18]. Such utility allows a separation of ambiguity attitude, risk aversion and intertemporal substitution. As mentioned above, Ju and Miao [30] model asset prices using a special case of the non-linear recursive utility given by

$$V_t(c,\omega^t) = \left[(1-\beta)c_t^{1-\rho} + \beta\left[I_t(V_{t+1}(c,\omega^t);\omega^t)\right]^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

[9] In section 6.3 we will give a sufficient conditions.

and

$$I_t(c_t;\omega^t) = \left(E_{\mu_t}\left[E_{\pi_t}\left[V_{t+1}^{1-\gamma}(c,\omega^t)\right]\right]^{\frac{1-\eta}{1-\gamma}}\right)^{\frac{1}{1-\eta}}$$

hence with a *three way separation* of risk aversion, ambiguity aversion and intertemporal substitution.

## 2.1 Time and preferences

Dynamic models of decision under ambiguity are characterized by two distinct features: dynamic consistency and attitude toward the timing of resolution of uncertainty. In this work we assume that recursive utility given in Eq. (1) is dynamically consistent. Clearly, the recursive intertemporal utility we adopt is intuitively dynamically consistent, however, we need a precise definition, since we allow for a time-dependent certainty equivalent operator.

We say that the recursive utility of Eq. (1) is dynamically consistent if, for all $t \in \mathbb{N}$ and $\omega^t \in \Omega^t$, $c', c \in \mathscr{D}$ and $T \geq t$, $V_t(c',\omega^t) \geq V_t(c,\omega^t)$ if:

DC1. $c'_\tau = c_\tau$ for $\tau = t,\dots T-1$

DC2. $V_T(c',\omega^T) \geq V_T(c,\omega^T)$ for all $\omega^T \in \Omega^T$

and $V_t(c',\omega^t) > V_t(c,\omega^t)$ if DC2 is strict in a non-null event.

Dynamic consistency is a normatively appealing, but restrictive requirement. For example, in a finite state-space setting, Gumen and Savochkin [24] proved that many classes of preference are dynamically unstable, meaning, that after conditioning they collapse to expected utility.

Dynamic consistency is always satisfied under recursive expected utility, however, for non-expected utility models, additional assumptions on the beliefs' updating procedure are necessary. In the case of recursive multiple priors this reduces to a property of the set of priors called *rectangularity*[10] as provided in Epstein and Schneider [12]. General conditions that guarantee dynamic consistency *along the tree* are proposed in Maccheroni et al. [35] or in Siniscalchi [44]. In general these implies state separability of the penalization or adjustment function.[11]

[10] Equivalent properties have been proposed in the literature on dynamic risk measures see Delbaen [9] and Riedel [42].

[11] Ambiguity sensitive preferences display a trade-off between dynamic consistency and consequentialism. Beyond the Epstein and Schneider [12] approach, other methods of updating ambiguous beliefs are proposed,

As mentioned in the in introduction, the second feature of dynamic preferences is the preference for earlier or later resolution of uncertainty as studied in Strzalecki [46]. He shows that preferences of the form

$$V_t = u(c_t) + \beta I(V_{t+1})$$

with a concave $I$ always display a preference for early resolution of uncertainty (and aversion to long run risk), suitably defined in the ambiguity setting, with MaxMin being the "knife edge" model of indifference. The consequence for asset pricing is a preference for assets whose uncertainty is resolved earlier. The importance of this issue is highlighted by Bansal and Yaron [1], where the attitude toward long-run risk offers a possible explanation for the equity premium puzzle. In this work we assume a quasiconcave certainty equivalent operator, hence, we are agnostic about the preference for the timing of resolution of uncertainty.

## 2.2 Existence and uniqueness

This section is devoted to establish sufficient conditions for the existence and uniqueness of a utility function that solves Eq. (1). Existence of a recursive utility satisfying (1) is guaranteed by monotonicity (w.r.t. pointwise order) and an application of Tarski's fixed point theorem[12], once the set of consumption processes $\mathscr{D}_{\text{cont}}$ is endowed with the pointwise order. Then, the recursive utility in Eq. (1) *always* exists.

Considering uniqueness, in the case of a globally Lipschitz (for example in Epstein and Wang [16], Maccheroni et al. [35] and Siniscalchi [44]) certainty equivalent $I_t$, uniqueness follows from standard contraction techniques. Our assumption of a locally Lipschitz certainty equivalent precludes the contraction argument, nonetheless, next theorem shows that under an appropriate relation between the discount rate $\beta$ and the rate of growth of consumption processes, a unique recursive utility exists.

**Theorem 1.** *Suppose $I$ is locally Lipschitz continuous, if $\beta b < 1$ then, for each $c \in \mathscr{D}_{\text{cont}}$ there exists a unique $V(c) \in \mathscr{D}_{\text{cont}}$ satisfying Eq.* (1).

namely the dynamic consistent, but not consequentialist approach of Hanany and Klibanoff [25] and the sophisticated approach of Siniscalchi [45].

[12] Tarski's fixed point theorem actually implies the existence of a complete lattice of solutions to (1) (see Marinacci and Montrucchio [37].)

The intuition behind the proof is that, standard contraction arguments prove at the same time existence and uniqueness. In this case existence is given by monotonicity and Tarski's fixed point theorem, therefore, we only need uniqueness. But uniqueness is implied by the boundedness of the consumption processes plus the additional requirement on the growth rate and intertemporal discount.

The last result of this section relates the recursive utility of Eq. (1) to the properties of the underlying process. In particular, we say that a process $c \in \mathscr{D}_{\text{cont}}$ has a time-homogeneous Markov structure, if there is a function $c^*$ such that

$$c_t(\omega^t) = c^*(\omega_t), \quad t \geq 1, \;\; \omega^t \in \Omega^t$$

Then we have the following result:

**Proposition 2.** *If $c$ is a time-homogeneous Markov process, so is $V(c)$ that solves Eq.* (1).

Therefore, intertemporal utility evolves according to the underlying consumption process. For easy of notation, if a process $c$ is time-homogeneous and Markov, the related utility is written as $V_t(c) \triangleq V_t(c, \omega^t)$ for all $t$.

# 3 The economy

We consider the same extension of Lucas' pure exchange economy of Epstein and Wang [16]. Assume $u' > 0, u'(0) = \infty$. There is a single good with total supply available at any time and state modeled by an endowment process $e = (e_t) \in \mathscr{D}_{\text{cont}}$. We assume that the endowment process has a time-homogeneous Markov structure, so there is a function $e^*$ such that

$$e_t(\omega^t) = e^*(\omega_t), \quad t \geq 1, \;\; \omega^t \in \Omega^t$$

with $e^*(\omega) > 0$ for all $\omega \in \Omega$. In each period, $N$ securities are traded at prices $q_i = \left(q_{i,t}\right) \in \mathscr{D}$, $i \in 1, \ldots, N$. Each security pays a dividend $d_1 = \left(d_{i,t}\right) \in \mathscr{D}_{\text{cont}}$ and it is available in zero net supply at all times and states of the world. The consumer optimizes her intertemporal utility choosing consumption and portfolio allocations at the current and future periods. A pair $(c, \theta)$, with $c \in \mathscr{D}_{\text{cont}}$ and $\theta = (\theta_t)$ is a continuous process, represents a plan of consumption and portfolio allocations at each period $t$. We say that $(c, \theta)$ is $(t, \omega^t)$*-feasible* if for all $\tau \geq t$,

$$q_\tau \theta_\tau + c_\tau = \theta_{\tau-1}[q_\tau + d_\tau] + e_\tau, \quad \theta_{t-1}(\omega^{t-1}) \triangleq 0$$

$$\inf_{i,\tau,\omega^\tau} \theta_{i,\tau}(\omega^\tau) > -\infty$$

The first condition is an intertemporal budget constraint, whereas the second is a weak restriction on short sales. Let ${}^t c \triangleq (c_t, c_{t+1}, \ldots)$ denote the continuation consumption process and denote ${}^t c|\omega^t \triangleq (c_\tau)_{\tau=t}^\infty$ the continuation of $c$ conditional to history $\omega^t$. A $(t,\omega^t)$-feasible plan $(c,\theta)$ is $(t,\omega^t)$*-optimal*, if

$$V_t({}^t c|\omega^t, \omega^t) \geq V_t({}^t c'|\omega^t, \omega^t), \quad \text{for all } (t,\omega^t)\text{-feasible plan } (c',\theta')$$

An *equilibrium* is a price process $(q_t) \in \mathscr{D}^n_{\text{cont}}$ such that $(e,0)$ is $(t,\omega^t)$-optimal for all $t \geq 1$ and $\omega^t \in \Omega^t$. Moreover we call a price process $(q_t)$ a *weak equilibrium* if it belongs to $\mathscr{D}^n$.

## 3.1 Ambiguous CCAPM

This section is devoted to derive the Euler's equations. As in Epstein and Wang [16], we use a two-periods perturbation of an equilibrium process. The intuition is that the investor is always at her optimum along the equilibrium path, therefore, any perturbation must be sub-optimal. Next result is a differential characterization of optimality with respect to a today-tomorrow perturbation. To avoid technical complications, we assume that $I(\cdot,\omega_t)$ is nice[13] at $e^*$ for all $t$ and $\omega_t$.

**Lemma 3.** *Let $e \in \mathscr{D}_{\text{cont}}$ be a positive, Markovian and time-homogeneous consumption process with $e_t(\omega_t) = e^*(\omega_t)$. Let $h = (h_t)_1^\infty$, with $h_1 \in \mathbb{R}$, $h_2 \in C(\Omega)$ and $h_t = 0$ for all $t \neq 1,2$. Then, a necessary condition for optimality of inertia i.e.*

$$0 \in \operatorname*{argmax}_\xi V_t(e^* + \xi(h_1, h_2, 0, \ldots), \omega^t)$$

*is*

$$0 = u'(e^*(\omega_t))h_1 + \beta \int u'(e^*) h_2 \, dp \quad \textit{for some } p \in \partial I(V_{t+1}(e^*); \omega_t) \tag{2}$$

[13] A functional $I(\cdot,\omega_t)$ is nice at $c$ if $0 \notin \partial I(c,\omega_t)$. Niceness excludes that an investor considers possible the degenerate measure that assigns zero measure to *all* events.

Where $\partial I(V_{t+1}(e^*);\omega_t)$ is the Clarke subdifferential (see appendix A for the definition) of $I(\cdot;\omega_t)$ at $V_{t+1}(e^*)$.

Similarly to Epstein and Wang [16] and Maccheroni et al. [35],we can identify an element of $\partial V_t(e^*)$ as an element of $\mathbb{R}\times\mathcal{M}_1^+(\Omega)$ given by

$$\partial V_t(e^*) = \left\{\left(u'(e^*(\omega_t)), u'(e^*)dp\right) : \exists p \in \partial I\left(V_{t+1}(e^*);\omega_t\right)\right\}$$

Where $u'(e^*)dp$ is a measure with density $u'(e^*)$ with respect to $p$.

Assume $q$ is a equilibrium. Given $(t,\omega_t)$, let a perturbation $(c,\theta)$ defined as $c_\tau = e_\tau$ and $\theta_\tau = 0$ for all $\tau \neq t, t+1$, for $\Delta \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, $c_t = e_t - \xi(\Delta\cdot q_t)$, $\theta_t = \xi\Delta$, $\theta_{t+1} = 0$ and $c_{t+1} = e_{t+1} + \xi\Delta\cdot(q_{t+1}+d_{t+1})$. $\xi$ and $\Delta$ represent, respectively, the size and direction of the perturbation. Optimality of the initial policy means that any perturbation must leave the decision maker worst off. Let $h_t \triangleq -\Delta\cdot q_t$ and $h_{t+1} \triangleq \Delta\cdot(q_{t+1}+d_{t+1})$. By Lemma 3, Eq. (2) can be rewritten as

$$0 = -u'(e^*(\omega_t))\Delta\cdot q_t + \beta\int u'(e^*)[\Delta\cdot(q_{t+1}+d_{t+1})]dp \quad \text{for some } p \in \partial I(V_{t+1}(e^*);\omega_t)) \tag{3}$$

If we normalize the subdifferential, we can define a correspondence $C(e^*;\omega_t):\Omega \rightrightarrows \mathcal{M}_1^+(\Omega)$, as

$$C(e^*;\omega_t) \triangleq \left\{\frac{p}{p(\Omega)} : p \in \partial I(V_{t+1}(e^*);\omega_t)\right\} \tag{4}$$

by the assumption of local niceness and the properties of Clarke subdifferential (see Proposition 8), $C(e^*;\omega_t) \subseteq \mathcal{M}_1^+(\Omega)$ is a weak* compact and convex correspondence. The set $C(e^*;\omega_t)$ contains the *time-t relevant beliefs*. They are definable regardless of the functional form of intertemporal utility and they are the main information needed to price assets. In the next section we will impose a condition on this set to prove existence of equilibria.

Eq. (3) allows to define an interval of prices given by

$$\min_{p\in C(e^*;\omega_t)} \beta\int u'(e^*)\Delta\cdot(q_{t+1}+d_{t+1})dp \leq u'(e^*(\omega_t))\Delta\cdot q_t \leq \max_{p\in C(e^*;\omega_t)} \beta\int u'(e^*)\Delta\cdot(q_{t+1}+d_{t+1})dp$$

Or equivalently

$$\min_{p\in C(e^*;\omega_t)} \left\{\beta E_p\left[\frac{u'(e_{t+1})}{u'(e^*(\omega_t)}\Delta\cdot(q_{t+1}+d_{t+1})\right] - \Delta\cdot q_t\right\} \leq 0, \quad \forall\Delta\in\mathbb{R}^N \tag{5}$$

The previous equation is more general than the corresponding equation in Epstein and Wang [16] since we do not assume any specific functional form for $I$, beyond (locally Lipschitz) continuity and quasiconcavity. However, the interpretation is similar, *optimality of market non-participation is consistent with a non-degenerated interval of prices.* We can conclude that market non-participation depends on ambiguity aversion itself and it is not a consequence of the extremity of the MaxMin decision rule.

Proceeding as in Epstein and Wang [16], we can rewrite Eq. (5) in a more tractable form as

$$\sup_{\Delta} \min_{p \in C(e^*;\omega_t)} F(p,\Delta) \leq 0$$

by linearity of $F(p,\cdot)$ we have

$$\sup_{\Delta \in \Gamma} \min_{p \in C(e^*;\omega_t)} F(p,\Delta) \leq 0$$

where $\Gamma \triangleq \mathrm{co}\,\{\pm i\text{th unit coordinate vector } i = 1,\dots,n\}$. By Fan MinMax (Theorem 14)

$$\min_{p \in C(e^*;\omega_t)} \max_{\Delta \in \Gamma} F(p,\Delta) \leq 0$$

Berge's Maximum (Theorem 12) and the same argument of Epstein and Wang [16] leads to the equality

$$\min_{p \in C(e^*;\omega_t)} \max_i \left\{ \left| \beta E_p \left[ \frac{u'(e_{t+1})}{u'(e^*(\omega_t))} (q_{i,t+1} + d_{i,t+1}) \right] - q_{i,t} \right| \right\} = 0 \tag{6}$$

Equation (6) is more transparent in the case $N = 1$, i.e. a unique asset is traded, then it becomes

$$\min_{p \in C(e^*;\omega_t)} \beta E_p \left[ \frac{u'(e_{t+1})}{u'(e^*(\omega_t))} (q_{t+1} + d_{t+1}) \right] \leq q_t \leq \max_{p \in C(e^*;\omega_t)} \beta E_p \left[ \frac{u'(e_{t+1})}{u'(e^*(\omega_t))} (q_{t+1} + d_{t+1}) \right] \tag{7}$$

Inequalities (7) are the Euler's inequalities that an equilibrium price has to satisfy.

# 4 Equilibria

As mentioned in the introduction, existence and characterization of equilibria require additional assumptions on the behavior of time-t relevant beliefs with respect to information arrival. In particular, the set $C(e^*,\omega_t)$ has to vary smoothly with information's arrival. The assumption of time-homogeneity and Markov allows us to impose a continuity condition only with respect to the information observed at that node. We call this assumption *weak Feller* since it is weaker than the corresponding assumption in Epstein and Wang [16].

**Assumption 1** (Weak Feller)**.** *The map* $\omega_t \longmapsto C(e^*;\omega_t)$ *is continuous.*

Weak Feller is a weak and intuitive requirement. It is trivially satisfied with a finite state-space or under recursive expected utility or, more importantly, when the certainty equivalent operator $I(\cdot,\omega^t)$ does not depend on $\omega^t$, namely IID ambiguity. Although being a continuity requirement, Weak Feller is only sufficient to prove that a *weak* equilibrium exists.

To sketch the proof, Weak Feller assumption implies, by Berge's Maximum Theorem, that the correspondence

$$\omega \longmapsto \operatorname*{argmin}_{p\in C(e^*;\omega)} \int f\,dp$$

is upper hemicontinuous for all $f$, hence it admits a measurable selection (by Th. 13). In turn, a selection induces an equilibrium price. We are ready to state the main theorem of this section:

**Theorem 4** (Characterization and Existence of Equilibria)**.**

*(i)* *$q$ is an weak equilibrium, if and only if, it satisfies Eq.* (6).

*(ii)* *Under Weak Feller (Assumption 1), there exists a weak equilibrium.*

Theorem 4 fully characterizes weak equilibria as the prices satisfying Eq. (6). Indeterminacy of prices is a consequence of the possible different selections that solves the sequence of Euler's inequalities. Equilibrium prices may be discontinuous indeed, under ambiguity, a small change in information may induce overreaction of the investor even though *beliefs vary smoothly with information.* This follows from the distinction between the time-t relevant beliefs $C(e^*;\omega_t)$ and the ones used to select optimal continuation plans, namely the "argmin" that solves Euler's inequalities and generates prices. In the next section, we show that imposing a stronger continuity requirement on this smaller set of priors guarantees continuity of the equilibrium price.

It is worth noticing that, in the particular case of IID ambiguity and a time-homogeneous and Markov return of the asset, any fluctuation of prices can only be caused by indeterminacy, hence there is room for volatility of the sunspot type. Again, this is not a phenomenon due to the MaxMin decision rule, but it holds under general ambiguity aversion.

By Theorem 4, a weak equilibrium price is given by the recursive relation

$$q_t^* = \beta E_{p_t}\left[\frac{u'(e_{t+1})}{u'(e^*(\omega_t))}(q_{t+1}^* + d_{t+1})\right]$$

The existence of the ambiguity stochastic discount factor $M_{t+1}^a$ we announced in the introduction, is hidden in the probability $p_t$. In Section 6, explicit furmulae for the ambiguity stochastic discount factor are provided.

## 4.1 Continuous equilibrium prices

In this section we strengthen the Weak Feller assumption to impose continuous equilibrium prices. The interpretation of the new assumption is that, the subset of time-t relevant beliefs *that are used to define equilibrium prices* has to vary (weakly) continuously to information's arrival. In particular, we do not allow a sudden "explosion" of beliefs. Given this, we can prove the existence of a continuous equilibrium price. This is not sufficient to eliminate indeterminacy unless preferences are differentiable. We can now state the stability assumption:

**Assumption 2** (Stability)**.** *The map*

$$\omega \longmapsto \operatorname*{argmin}_{p \in C(e^*;\omega)} \int f\, dp$$

*is* lower hemicontinuous *for all* $f \in C(\Omega)$.

Intuitively, beliefs solving the intertemporal optimization problem do not overreact to small changes in information, indeed, lower hemicontinuity is not compatible with a sudden enlargement of the set of beliefs the investor uses to generate prices.

The following is the main theorem of this section.

**Theorem 5.** *Under and Assumption 2 (Stability), there exists an equilibrium.*

Theorem 5 only asserts the existence of a continuous equilibrium, when there is indeterminacy, alternative equilibrium prices may be discontinuous.

To sum up, by strengthening the assumption on the "smoothness" of beliefs with respect to information's arrival, we can control the smoothness of prices. Notably, the conditions we impose to prove the existence of equilibrium and weak equlibria are independent of the specific functional form we can adopt. We can conclude that the existence and continuity of equilibrium prices primarily *depend on the reaction of beliefs to new information*, regardless of the variability of investor's tastes across time.

# 5 Hansen-Jagannathan bound under ambiguity

We claimed that ambiguity augments the standard pricing formula to (SDF) however, the previous sections provide no intuitions on the derivation of $M^a_{t+1}$. Next example hopefully fills the gap.

**Example 1** (Smooth ambiguity)**.** *Let*

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$

*and $u(c) = \log c$ for $\gamma = 1$. Assume a parametric smooth ambiguity certainty equivalent*[14]*:*

$$V_t(c,\omega^t) = \frac{c_t(\omega^t)^{1-\gamma}}{1-\gamma} + \beta\phi^{-1}\left[\int_\Theta \phi\left(\int V_{t+1}(c,\omega^t)\,dp_{t,\theta}\right)d\mu(\theta|\omega^t)\right]$$

*where $\Theta$ is an (unobservable) space of parameters $\theta \in \Theta$. $p_{t,\theta} \triangleq p_\theta(\omega_{t+1}|\omega^t)$ is the probability under $p_\theta$ that the next observed state of the world is $\omega_{t+1}$ given $\omega^t$ and $\mu$ represents the investor's prior on $\Theta$. Assuming an everywhere differentiable $\phi$, Equation (7) becomes*

$$q_t = \beta E_{\mu(\theta|\omega^t)}\left[\xi_t(\theta)E_{p_{t,\theta}}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right]$$

*where $\xi_t(\theta)$ is a distortion affecting $\mu(\theta|\omega^t)$ given by:*

$$\xi_t(\theta) \triangleq \frac{\phi'\left(E_{p_{t,\theta}}\left[V_{t+1}(e^*)\right]\right)}{\phi'\left(\phi^{-1}\left(E_{\mu(\theta|\omega^t)}\left[\phi\left(E_{p_{t,\theta}}\left[V_{t+1}(e^*)\right]\right)\right]\right)\right)}$$

We will prove that ambiguity affects excess return and the Hansen-Jagannathan bound (Hansen and Jagannathan [26]) through the additional stochastic discount factor $M^a_{t+1}$.

[14] In section 6.3 we prove that if $\phi$ and $\phi^{-1}$ are both locally Lipschitz, then the whole representation is locally Lipschitz. In the example we implicitly assume that the condition is satisfied.

Let's start with an expected result: the ratio between standard deviation and expected value of the joint discount factor $M_t^a M_t$, exceeds the Sharpe ratio of any portfolio. Let define the risk and ambiguity free rate

$$r_{t+1}^f = \frac{1}{E_p\left[M_{t+1}^a M_{t+1}\right]}$$

notice that the risk and ambiguity free rate is not unique as it depends on $M_{t+1}^a$. Then, given an arbitrary portfolio $h$, $E_p\left[M_{t+1}^a M_{t+1}\left(R_{t+1}^h - r_{t+1}^f\right)\right] = 0$. At this point we can break the previous equality as

$$\mathrm{Cov}_p\left(M_{t+1}^a,\, M_{t+1}\left(R_{t+1}^h - r_{t+1}^f\right)\right) = -E_p\left[M_{t+1}^a\right] E_p\left[M_{t+1}\left(R_{t+1}^h - r_{t+1}^f\right)\right] \tag{8}$$

or

$$\mathrm{Cov}_p\left(M_{t+1}^a M_{t+1},\, R_{t+1}^h - r_{t+1}^f\right) = -E_p\left[M_{t+1}^a M_{t+1}\right] E_p\left[R_{t+1}^h - r_{t+1}^f\right] \tag{9}$$

By standard argument, Equation (9) implies the usual Hansen-Jagannathan bound for the joint risk-ambiguity SDF

$$\frac{\sigma_p(M_{t+1}^a M_{t+1})}{E_p\left[M_{t+1}^a M_{t+1}\right]} \geq \sup_h \left| \frac{E_p\left[\left(R_{t+1}^h - r_{t+1}^f\right)\right]}{\sigma_p\left[R_{t+1}^h\right]} \right|$$

and since it is true for all $M_{t+1}^a$, it is also true if we take the infimum, then

$$\inf_{M_{t+1}^a} \frac{\sigma_p(M_{t+1}^a M_{t+1})}{E_p\left[M_{t+1}^a M_{t+1}\right]} \geq \sup_h \left| \frac{E_p\left[\left(R_{t+1}^h - r_{t+1}^f\right) h\right]}{\sigma_p\left(R_{t+1}^h\right)} \right| \tag{10}$$

Then, the standard deviation of any admissible joint stochastic discount factor exceeds the Sharpe ratio of any portfolio, meaning that ambiguity always increases the volatility of the SDF.

Differently, Equation (8) allows to study the Hansen-Jagannathan bound for the ambiguity discount factor only. First, we provide a new formula for the $p$-expected excess return of an asset, that clearly separates the role of risk and ambiguity. $p$-Excess return is the sum of two parts: the covariance between the return and the risk discount factor plus the covariance of the ambiguity discount factor with the risk neutral evaluation of the return:

$$E_p\left[R_{t+1}^h - r_{t+1}^f\right] = -\frac{\mathrm{Cov}_p\left(M_{t+1}, R_{t+1}^h\right)}{E_p\left[M_{t+1}\right]} - \frac{\mathrm{Cov}_p\left(M_{t+1}^a, M_{t+1}\left(R_{t+1}^h - r_{t+1}^f\right)\right)}{E_p\left[M_{t+1}^a\right] E_p\left[M_{t+1}\right]} \tag{11}$$

Finally, we can derive a bound for the ratio of the standard deviation and the expected value of the ambiguity SDF, in the spirit of Hansen and Jagannathan [26]:

$$\inf_{M^a_{t+1}} \frac{\sigma_p(M^a_{t+1})}{E_p\left[M^a_{t+1}\right]} \geq \sup_h \left| \frac{E_p\left[M_{t+1}\left(R^h_{t+1} - r^f_{t+1}\right)\right]}{\sigma_p\left(M_{t+1}R^h_{t+1}\right)} \right|$$

The ratio between the standard deviation and expected utility of the ambiguity discount factor exceeds the Sharpe ratio of any portfolio:

# 6 Calculation for specific models

In this section we calculate the equilibrium price of assets, assuming specific certainty equivalent in Equation (1) and we explicit the ambiguity stochastic discount factor $M^a_t$. We assume that the Bernoulli utility has a power form

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$

and for $\gamma = 1$, we have $u(c) = \log c$. For simplicity, we consider an economy with a single asset.

As anticipated in the introduction, to each model we can specify an ambiguity stochastic discount factor $M^a_{t+1}$ that satisfies the formula

$$1 = E_p\left[M^a_{t+1} M_{t+1} R_{t+1} \middle| \mathscr{F}_t\right]$$

$M^a_{t+1}$ is fully characterized by the priors contained in $C_t(e^*, \omega_t)$, since each model corresponds to a different formulation for the time-t relevant priors.

## 6.1 Uncertainty averse preferences

We first focus on the most general class of Uncertainty Averse preference (UA) of Cerreia-Vioglio et al. [4]. They impose a minimal restriction to the ambiguity attitude of a DM, such as the uncertainty aversion axiom of Gilboa and Schmeidler. It means that if a continuation plan $V_t(c_t, \omega^t)$ is weakly preferred to another $V_t(c'_t, \omega^t)$, then any mixture of them is weakly preferred to the worst. Many models of decision under ambiguity are special cases[15] of UA.

[15] Variational preferences, MaxMin, Confidence preferences, uncertainty averse Smooth Ambiguity.

Each act is evaluated by

$$I(f) = \inf_{p \in \mathcal{M}_1^+(\Omega)} G\left(\int f\,dp, p\right)$$

where $u$ is a Bernoulli utility, $G : \mathbb{R} \times \mathcal{M}_1^+(\Omega) \to (-\infty, \infty]$ is a quasi-convex, monotone in the first argument, normalized and lower-semicontinuous function. UAP preferences are not locally Lipschitz continuous in general, however, we can add to the axioms of Cerreia-Vioglio et al. [4] the locally Lipschitz continuity axiom of Ghirardato and Siniscalchi [20, Online Appendix]. Let define

$$\operatorname{dom} G \triangleq \bigcup \{p \in \mathcal{M}_1^+(\Omega) : G(x,p) < \infty\}$$

and let $\mathscr{C}^* = \operatorname{cl}(\operatorname{dom} G)$. Then, a dynamic formulation is given by

$$V_t(c,\omega^t) = \frac{c_t(\omega^t)^{1-\gamma}}{1-\gamma} + \beta \min_{p \in \mathscr{C}_t^*} G_t\left(\int V_{t+1}(c,\omega^t)\,dp, p\right) \tag{12}$$

By Ghirardato and Siniscalchi [19], the Clarke subdifferential of a certainty equivalent representing uncertainty averse preferences, can be written as:[16]

$$\partial I(V(e^*)) = \operatorname*{argmin}_{p \in \mathcal{M}_1^+(\Omega)} G\left(\int V(e^*)\,dp, p\right)$$

Therefore Eq. (7) can be rewritten as

$$\min_{p_t \in C_t(e^*,\omega_t)} \beta E_{p_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right] \le q_t \le \max_{p_t \in C_t(e^*,\omega_t)} \beta E_{p_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]$$

and

$$C(e^*,\omega_t) = \left\{\frac{p}{p(\Omega)} : p \in \Pi(V(e^*))\right\}$$

where

$$\Pi(V(e^*)) = \operatorname*{argmin}_{p \in \mathscr{C}^*} G\left(\int V(e^*)\,dp, p\right)$$

With UA preferences, the relevant priors are the ones minimizing the cost function $G$, this generalizes the result of Epstein and Wang [16] An equilibrium price is

$$q_t = \beta E_{p_t^*}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]$$

[16] Assuming niceness of $I$ at $V(e^*)$.

The difference with a price coming out from rational expectation is in the prior $p_t^*$. With an uncertainty averse certainty equivalent, the priors used in equilibrium are those who locally rationalize the preferences, indeed, $\Pi(V(e^*))$ can be rewritten (see Cerreia-Vioglio et al. [4])

$$\Pi(V(e^*)) = \left\{p \in \mathcal{M}_1^+(\Omega) : E_p\left[V(e^*)\right] \geq E_p\left[V(c)\right] \text{ implies } I(V(e^*) \geq I(V(c))\right\}$$

We can exploit the additional structure of our setting to give a new representation of the set $C_t(e^*, \omega_t)$.

In particular, assuming that $V(e^*) \mapsto G\left(\int V(e^*)dp, p\right)$ is regular and locally Lipshitz for all $p \in \Pi(V(e^*))$, a combination of Equation (12) and Theorem 10 gives the following characterization:

**Lemma 6.** *Let $V(e^*) \in C(\Omega)$, and $I(V(e^*);\omega) = \min_{p\in\mathcal{C}^*} G\left(\int V(e^*)dp, p\right)$.*

$$C(e^*, \omega) \subseteq \left\{\frac{\int_{\mathcal{C}^*} \alpha_p p\, \mu(dp)}{\int_{\mathcal{C}^*} \alpha_p\, \mu(dp)} : \text{ for some } \mu \in \mathcal{M}_1^+\left(\Pi(V(e^*))\right)\right\} \tag{13}$$

*where $p \mapsto \alpha_p$ is a measurable selection from $p \mapsto \partial_{[\mathcal{C}^*]} G\left(\int V(e^*)dp, p\right)$.*

Where $\partial_{[\mathcal{C}^*]} = \overline{\text{co}}^*\left(q \in \text{rca}(\Sigma) : q_i \in \partial G(V_n, p_n), V_n \to V(e^*), p_n \to p, p_n \in \mathcal{C}^*\right)$ and $q$ is a weak* cluster point of $q_i$.

The main novelty of the result is the existence of a second-order prior that averages the first-order priors used in the intertemporal optimization. Then, Eq. (7) becomes

$$\begin{cases} q_t \geq \min_{\mu_t \in \mathcal{M}_1^+(\Pi(V_{t+1}(e^*)))} \beta E_{\mu_t}\left[\xi_{p_t,\mu_t} E_{p_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right] \\ q_t \leq \max_{\mu_t \in \mathcal{M}_1^+(\Pi(V_{t+1}(e^*)))} \beta E_{\mu_t}\left[\xi_{p_t,\mu_t} E_{p_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right] \end{cases}$$

where $\xi_{p_t,\mu_t}$ is a Radon-Nikodym derivative affecting $\mu_t$ and given by

$$\xi_{p_t,\mu_t} = \frac{\alpha_p}{\int_{\mathcal{C}_t^*} \alpha_p \mu_t(dp)}$$

Since $G(\cdot, p)$ is quasi-convex $\alpha_p$ is high when the continuation value is low so that $\xi_{p_t,\mu_t}$ gives more weight to the probabilities of observing a bad outcome, among the locally rationalizing probabilities, reflecting pessimism. The previous expression is similar to the one related to (Almost) Smooth certainty equivalent, although here the second-order prior averages a particular family of first-order priors, namely the ones rationalizing local preferences. Whereas in the smooth case, the first-order priors are those in the support of the second-order prior and they represent the models an investor considers possible. This different nature clearly leads to a different ambiguity stochastic discount factor.

## 6.2 Special cases: homothetic and variational

A further specialization of uncertainty averse preferences, shed new lights on the nature of the ambiguous stochastic discount factor $M^a_{t+1}$. If we assume that the agent has homothetic recursive preference in the sense of Chateauneuf and Faro [5], then

$$V_t(c,\omega^t) = \frac{c_t(\omega_t)^{1-\gamma}}{1-\gamma} + \beta \min_{p\in\mathscr{C}^*_t} \frac{\int V_{t+1}(c,\omega^t)dp}{\eta_t(\omega^t,p)}$$

for some concave and upper hemicontinuos function $\eta_t : \mathscr{C}^*_t \to [0,1]$. Then, Equation (13) becomes

$$C_t(e^*,\omega_t) = \left\{ \frac{\int_{\mathscr{C}^*_t} (\eta(\omega,p))^{-1} p\ \mu_t(dp)}{\int_{\mathscr{C}^*_t} (\eta(\omega,p))^{-1} \mu_t(dp)} : \text{ for some } \ \mu_t \in \mathscr{M}^+_1(\Pi(V_{t+1}(e^*))) \right\}$$

in this case $\Pi(V(e^*)) = \text{argmin}_{p\in\mathscr{C}^*} \frac{\int V(e^*)dp}{\eta(\omega,p)}$. Therefore, Eq. (7) becomes

$$\begin{cases} q_t \geq \min\limits_{\mu_t\in\mathscr{M}^+_1(\Pi(V_{t+1}(e^*)))} \beta E_{\mu_t}\left[\xi_{p,\mu_t} E_{p_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right] \\ q_t \leq \max\limits_{\mu_t\in\mathscr{M}^+_1(\Pi(V_{t+1}(e^*)))} \beta E_{\mu_t}\left[\xi_{p,\mu_t} E_{p_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right] \end{cases}$$

where $\xi_{p,\mu_t}$ is a Radon-Nikodym derivative affecting $\mu_t$ given by

$$\xi_{p,\mu_t} = \frac{(\eta_t(\omega_t,p))^{-1}}{\int_{\mathscr{C}^*_t}(\eta_t(\omega_t,p)^{-1}\mu_t(dp)}$$

so that an equilibrium price is given by

$$q_t = \beta E_{\mu_t}\left[\xi_{p,\mu_t} E_{p_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right)$$

With homothetic preferences, the distortion $\xi_{p,\mu_t}$ affecting the second order beliefs, depends on the investor's "confidence" about the likelihood of the belief. If her confidence in $p$ is "low", $\eta_t(\omega_t,p)$ is close to zero and $\xi_{p,\mu_t}$ is high, increasing the probability of observe a "low" confidence prior, again reflecting pessimism.

In the case of recursive Variational preferences,

$$V_t(c,\omega^t) = \frac{c_t(\omega_t)^{1-\gamma}}{1-\gamma} + \beta \min_{p\in\mathcal{M}_1^+(\Omega)} \left( \int V_{t+1}(c,\omega^t) + \eta_t(\omega^t,p) \right)$$

we have

$$C_t(e^*,\omega_t) = \left\{ \int_{\mathcal{C}_t^*} p\mu_t(dp) : \text{ for some } \quad \mu_t \in \mathcal{M}_1^+(\Pi(V_{t+1}(e^*))) \right\}$$

in this case $\Pi(V(e^*)) = \text{argmin}_{p\in\mathcal{C}^*} \int V(e^*)dp + \eta(\omega,p)$. Then, a similar computation as above implies

$$\begin{cases} q_t \geq \min\limits_{\mu_t\in\mathcal{M}_1^+(\Pi(V_{t+1}(e^*)))} \beta E_{\mu_t}\left[E_{p_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right] \\ q_t \leq \max\limits_{\mu_t\in\mathcal{M}_1^+(\Pi(V_{t+1}(e^*)))} \beta E_{\mu_t}\left[E_{p_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right] \end{cases}$$

and an equilibrium price is

$$q_t = \beta E_{\mu_t}\left[E_{p_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right]$$

An interesting subcase is represented by recursive multiplier preferences, where the cost function is given by $\eta_t(\omega_t,p) \triangleq \kappa D(p\|\hat{p}_t)$, for some $\kappa \in [0,\infty]$ where

$$D(p\|\hat{p}_t) = \int p \log \frac{p}{\hat{p}_t}, \text{ if } p \ll \hat{p}_t$$

and $D(p\|\hat{p}_t) = \infty$ otherwise, is the Kullback-Leibler distance of $p$ with respect to $\hat{p}_t$ and $\hat{p}_t$ is a reference measure. It is well known that

$$V_t(c,\omega^t) = \frac{c_t(\omega_t)^{1-\gamma}}{1-\gamma} + \beta \min_{p\in\mathcal{M}_1^+(\Omega)} \left( \int V_{t+1}(c,\omega^t)dp + \kappa D(p\|\hat{p}_t) \right)$$

is observationally equivalent to

$$V_t(c,\omega^t) = \frac{c_t(\omega_t)^{1-\gamma}}{1-\gamma} - \kappa\beta \log\left( \int \exp\left( -\frac{V_{t+1}(c,\omega^t)}{\kappa} \right) d\hat{p}_t \right)$$

Since $D(\cdot\|\hat{p}_t) : \mathcal{M}_1^+(\Omega) \to [0,\infty]$ is strictly convex, $C(e^*,\omega)$ is a singleton and there is a unique equilibrium price $q$. It is determined by (see Maccheroni et al. [35])

$$q_t = \beta \frac{E_{\hat{p}_t}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\exp\left(-\frac{V_{t+1}(e^*)}{\kappa}\right)\right]}{E_{\hat{p}_t}\left[\exp\left(-\frac{V_{t+1}(e^*)}{\kappa}\right)\right]}$$

the usual ambiguity stochastic discount factor related to robust control preferences.

## 6.3 (Almost) Smooth Ambiguity Model

In Example 1, we derived of ambiguity SDF for the case of Smooth Ambiguity model of Klibanoff et al. [31]. The form of intertemporal utility is given by

$$V_t(c,\omega^t) = \frac{c_t(\omega^t)^{1-\gamma}}{1-\gamma} + \beta\phi^{-1}\left[\int_\Theta \phi\left(\int V_{t+1}(c,\omega^t)dp_{t,\theta}\right)d\mu(\theta|\omega^t)\right] \tag{14}$$

again, $\Theta$ is an (unobservable) space of parameters $\theta \in \Theta$. $p_{t,\theta} \triangleq p_\theta(\omega_{t+1}|\omega^t)$ is the probability under $p_\theta$ that the next observed state of the world is $\omega_{t+1}$ given $\omega^t$ and $\mu$ represents the DM prior on $\Theta$. It is worth noticing the extreme case in which $\phi$ is the identity, since it clearly explains the strength of the rational expectation hypothesis. When the utility is given by

$$V_t(c,\omega^t) = \frac{c_t(\omega^t)^{1-\gamma}}{1-\gamma} + \beta\left[\int_\Theta \left(\int V_{t+1}(c,\omega^t)dp_{t,\theta}\right)d\mu(\theta|\omega^t)\right]$$

it is observationally equivalent to recursive expected utility. However, the price of the asset conveys more information than the expected utility case, indeed,

$$q_t = \beta E_{\mu(\theta|\omega^t)}\left[E_{p_{t,\theta}}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right]$$

or equivalently

$$q_t = \beta E_{\nu_t(\omega^t,\cdot)}\left[\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]$$

where

$$\nu_t(\omega^t, A) = \int_\Theta p_{t,\theta}(A)d\mu(\theta|\omega^t)$$

Moreover, if $\nu_t(\omega^t,\cdot)$ is not continuous in $\omega^t$, the unique equilibrium price can be discontinuous.

Now, we generalize the result of Example 1, assuming a non-smooth second-order function $\phi$. In the following we assume that $\phi$ is not continuously differentiable but only Clarke regular[17] and both $\phi$ and $\phi^{-1}$ are locally Lipschitz . The last condition guarantees that the whole representation is locally Lipschitz(see Lemma 16 in appendix B).

We first calculate the generalized gradient of the functional

$$I(c_t;\omega^t) = \phi^{-1}\left[\int_\Theta \phi\left(\int V_{t+1}(c,\omega^t)dp_{t,\theta}\right)d\mu(\theta|\omega^t)\right]$$

Assume that $\phi$ and $\phi^{-1}$ are regular and let $h_t(V_{t+1}(c)) = \int_\Theta \phi\left(\int V_{t+1}(c,\omega^t)dp_{t,\theta}\right)d\mu(\theta|\omega^t)$ and $g(x) = \phi^{-1}(x)$, then by Clarke [7, Theorem 2.3.9]

$$\partial I(c_t;\omega^t) = \overline{\text{co}}\left(\hat{\alpha}(p_{t,\theta})\gamma_t : \hat{\alpha}(p_{t,\theta}) \in \partial h_t(V_{t+1}(c)), \gamma_t \in \partial g(h_t(c))\right)$$

From Ghirardato and Siniscalchi [19], each element of $\partial h_t(V_{t+1}(c))$ can be represented as

$$\hat{\alpha}(p_{t,\theta})(f) = \int_\Theta \alpha(p_{t,\theta})\left(E_{p_{t,\theta}}[f]\right)d\mu(\theta|\omega^t)$$

where $p_{t,\theta} \mapsto \alpha(p_{t,\theta})$ is a measurable selection from

$$p_{t,\theta} \longmapsto \partial\phi\left(E_{p_{t,\theta}}[V_{t+1}(c)]\right)$$

and $\gamma_t \in \partial(\phi^{-1}(h_t(c))$. Equation (7) becomes

$$\begin{cases} q_t \geq \min\limits_{\tilde{\xi}_t(\theta)\in\Lambda(V_{t+1}(e^*))} \beta E_{\mu(\theta|\omega^t)}\left[\tilde{\xi}_t(\theta)E_{p_{t,\theta}}\left[\left(\dfrac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right] \\ q_t \leq \max\limits_{\tilde{\xi}_t(\theta)\in\Lambda(V_{t+1}(e^*))} \beta E_{\mu(\theta|\omega^t)}\left[\tilde{\xi}_t(\theta)E_{p_{t,\theta}}\left[\left(\dfrac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]\right] \end{cases}$$

Where

$$\Lambda(V_{t+1}(e^*)) \triangleq \left\{\gamma_t\alpha(p_{t,\theta}) : \gamma_t \in \partial\phi^{-1}(h_t(e^*)), \alpha(p_{t,\theta}) \in \partial\phi\left(E_{p_{t,\theta}}\left[V_{t+1}(e^*)\right]\right)\right\}$$

is the set of distortions affecting a specific model $\theta$ and $k \mapsto \gamma_t(k)$ is a measurable selection from $\partial\phi^{-1}(k)$. Therefore, we generalize the conditions of Collard et al. [8]. If $\phi$ is concave hence the investor is ambiguity averse, $\alpha(p_{t,\theta}) \in \partial\phi\left(E_{p_{t,\theta}}[V_{t+1}(e^*)]\right)$ is high when the expected value of the continuation is low, so that the ambiguity stochastic discount factor increases the probability of observing a bad outcome, reflecting pessimism.

[17] See Appendix A for a definition.

The above calculations show that also with a regular but non-smooth certainty equivalent, indeterminacy of prices may result.

### 6.4 Vector expected utility

Recursive vector expected utility has been axiomatized in Siniscalchi [44], where it is provided an application to a Lucas' economy similar to the present work. In the case of recursive vector expected utility, the form of intertemporal utility is given by

$$V_t(c,\omega^t) = \frac{c_t(\omega_t)^{1-\gamma}}{1-\gamma} + \beta E_{p_t(\omega_t,\cdot)}\left[V_{t+1}(c,\omega^t)\right] + A_{t,\omega^t}\left(\beta E_{p_t(\omega_t,\cdot)}\left[\xi^{t,\omega^t}\cdot V_{t+1}(c,\omega^t)\right]\right)$$

By Ghirardato and Siniscalchi [19] and Siniscalchi [44], Equation (7) becomes

$$q_t = \beta E_{p_t(\omega_t,\cdot)}\left[\left\{1 + \sum_{0\le i\le M} \frac{\partial A_{t,\omega_t}\left(E_{p_t(\omega_t,\cdot)}[\beta V_{t+1}(e^*)\cdot \xi^{t,\omega_t}]\right)}{\partial \phi_i}\xi_i^{t,\omega_t}\right\}\left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}(q_{t+1}+d_{t+1})\right]$$

where $\phi_i \in \mathbb{R}^n$ and $\partial A/\phi_i$ is a partial derivative. The functional form of vector expected is particularly suitable for a direct comparison with the EU case. Here ambiguity attaches an additional piece to the standard stochastic discount factor who exacerbates the effects of high or low dividend. The analysis follows that of Siniscalchi [44]. To gain intuition, assume a Clarke regular $A$ and let $n = 1$, so that only an adjustment factor exists. Under ambiguity aversion, $A$ is concave, together with the fact that $A(0) = 0$ is a maximum, $A'_{t,\omega_t}\left(E_{p_t(\omega_t,\cdot)}[\beta V_{t+1}(e^*)\cdot \xi^{t,\omega_t}]\right) \le 0$. To see the effect on discounting, if the investor expects a high endowment tomorrow, $u'$ is low, and this effect is reinforced by $-A'(\cdot)$. Therefore, ambiguity aversion increases the discount of good outcomes, with respect to the benchmark EU case.

# Appendix A. Non-smooth optimization and set-valued analysis

For completeness we report the main results that are used in the proofs.

**Definition 7.** *Let $U \subset X$ an open subset of a Banach space $X$. Let $f$ a locally Lipschitz functional $f : U \to \mathbb{R}$. For every $x \in U$ and $y \in X$, the Clarke (upper) derivative of $f$ in $x$ in the direction $y$ is*

$$f^\circ(x;y) = \limsup_{\substack{z\to x\\ t\downarrow 0}} \frac{f(z+ty)-f(y)}{t}$$

*The Clarke (sub)differential of $f$ at $x$ is the set*

$$\partial f(x) = \left\{x^* \in X^* : \langle x^*, x\rangle \le f^\circ(x; y), \forall y \in X\right\}$$

Next proposition collects some useful properties of Clarke differential (see Clarke [7]).

**Proposition 8.** *Let $f : U \to \mathbb{R}$ be locally Lipschitz. Then:*

1. *$\partial f(x)$ is a nonempty, convex, weak* compact subset of $X^*$.*
2. *For every $v \in X$, $f^\circ(x; v) = \max_{x^* \in \partial f(x)} \langle x^*, v\rangle$*
3. *If $f$ is convex, $\partial f(x)$ is the usual subdifferential of convex analysis.*
4. *(Lebourg mean value theorem) For any $x, y \in X$, there exists $\gamma \in (0,1)$, such that*

$$f(x) - f(y) \in \langle \partial f(\gamma x + (1-\gamma)y), x - y\rangle$$

A function $f : X \to \mathbb{R}$ on a Banach space $X$ is Clarke regular at $x$ if for all $v \in X$,

$$f^\circ(x; v) = f'(x; v)$$

where $f'(x; v) = \lim_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t}$ is the usual directional derivative.

**Theorem 9** (Clarke [7], Chain Rule)**.** *Let $F$ be a map from a Banach space $X$ to another Banach space $Y$ and $g$ a real-valued function on $Y$. Suppose $F$ is strictly differentiable at $x$ and $g$ is locally Lipschitz at $F(x)$. Then $f = g \circ F$ is Lipschitz at $x$ and*

$$\partial f(x) \subseteq \partial g(F(x)) \circ D_s F(x)$$

*Equality holds if $g$ or $-g$ is regular at $F(x)$ or, if $F$ maps every neighborhood of $x$ to a set which is dense in a neighborhood of $F(x)$.*

The interpretation (see Clarke [7, Remark 2.3.11], is that every element $z^* \in \partial f(x)$ can be represented as a composition of a map $y^* \in \partial g(F(x))$ and $D_s F(x)$ such that $\langle z^*, v\rangle = \langle y^*, D_s F(x)(v)\rangle$.

Let $f_t$ be a family of functions on a topological vector space $X$, parametrized by $t \in T$, with $T$ a topological space. Assume that each $f_t$ is locally Lipschitz. The following definitions and results are from Clarke [7]. Denote $\partial_{[T]} f_t(x)$ the set

$$\partial_{[T]} f_t(x) \triangleq \overline{\text{co}}^* \left(x^* \in X^* : x_i^* \in \partial f_{t_i}(x_i),\ x_i \to x,\ t_i \to t,\ t_i \in T,\ x^* \text{ is a weak* cluster point of } x_i^*\right)$$

and $\overline{\text{co}}^*$ denotes the weak*-closed convex hull operator. The correspondence $(\tau, y) \longmapsto \partial f_\tau(y)$ is said to be (weak*) closed at $(t, x)$, provided $\partial_{[T]} f_t(x) = \partial f_t(x)$.

Consider the following assumption to hold in the following.

(i) $T$ is sequentially compact.

(ii) For some neighborhood $U$ of $x$, the map $t \mapsto f_t(y)$ is upper semicontinuous for each $y \in U$.

(iii) Each $f_t$, $t \in T$ is Lipschitz of a given rank $K$ on $U$ and $\left\{f_t(x) : t \in T\right\}$ is bounded.

Define the function $f : X \to \mathbb{R}$ as

$$f(x) \triangleq \max_{t\in T} f_t(x)$$

then $f$ is Lipschitz on $U$, it is finite and the maximum is attained. Denote $M(x) \triangleq \left\{t \in T : f_t(x) = f(x)\right\}$, $M(x)$ is closed and non-empty for each $x \in X$. For a given subset $S \subseteq T$, write $\mathscr{M}_1^+(S)$ the collection of Radon probability measures supported on $S$.

**Theorem 10** (Clarke [7], Th. 2.8.2)**.** *In addition to (i)-(iii) assume that either* *(iv)* $X$ *is separable, or*

*(iv)'* $T$ *is metrizable*

*Then one has*

$$\partial f(x) \subset \left\{ \int_T \partial_{[T]} f_t(x) \mu(dt) : \; \mu \in \mathcal{M}_1^+(M(x)) \right\} \tag{15}$$

*If* $(\tau, y) \mapsto \partial_\tau f(y)$ *is closed at* $(t,x)$ *for each* $t \in M(x)$*, and if* $f_t$ *is regular at* $x$ *for each* $t \in M(x)$*, then* $f$ *is regular at* $x$ *and equality holds in expression* (15) *(with* $\partial_{[T]} f_t(x) = \partial f_t(x)$*).*

For the interpretation of the right-hand side of Eq. (15), an element $x^*$ of that set is an element of $X^*$ to which corresponds $t \mapsto x_t^* \in \partial_{[T]} f_t(x)$ from $T$ to $X^*$ and an element $\mu \in \mathcal{M}_1^+(M(x))$ such that $t \mapsto \langle x_t^*, v\rangle$ is $\mu$-integrable for all $v \in X$ and

$$\langle x^*, v\rangle = \int_T \langle x_t^*, v\rangle \mu(dt)$$

The next two theorems are reported for completeness.
A set-valued map $\Gamma : X \rightrightarrows Y$ between topological spaces is said to be lower hemicontinuous if the set $\{x \in X : \Gamma(x) \cap V \neq \emptyset\}$ is open for all open $V \subset Y$.

**Theorem 11** (Michael's Theorem)**.** *Suppose* $X$ *is a paracompact space,* $Y$ *a topological linear space and* $Z$ *a convex closed subset of* $Y$ *containing* $0$ *that has a base* $\{B_n\}$ *for the neighborhoods of* $0$*, consisting of symmetric and convex sets such that* $B_{n+1} \subset \frac{1}{2} B_n$ *Suppose that* $\Gamma : X \to Y$ *is a lower hemicontinuous set-valued map with closed convex values with* $\Gamma(X) + B_n \subset Z$*. Suppose that for each* $y \in \Gamma(X)$*,* $y + B_n$ *is open in* $Z$*. Then the map* $\bar{\Gamma}$ *defined by* $\bar{\Gamma}(x) = \overline{\Gamma(x)}$ *admit a continuous selection.*

As noted in Epstein and Wang [16], the proof of this stronger version of Michael's selection theorem, follows from Michael [39].

**Theorem 12** (Berge's Maximum Th.)**.** *Let* $\Gamma : X \rightrightarrows Y$ *be a continuous set-valued map between topological spaces with nonempty compact values, and suppose that* $f : \mathrm{Gr}\,\Gamma \to$ *is continuous. Let* $m : X \to \mathbb{R}$ *be*

$$m(x) = \max_{y \in \Gamma(x)} f(x,y)$$

*and let the correspondence* $\mu : X \rightrightarrows Y$ *of maximizers be*

$$\mu(x) = \left\{ y \in \Gamma(x) : f(x,y) = m(x) \right\}$$

*then*

*i.* *m is a continuous function*

*ii.* $\mu$ *has nonempty compact values*

*iii.* *if* $Y$ *is Hausdorff, then* $\mu$ *is upper hemicontinuous.*

Next theorem gives a sufficient condition for the existence of a measurable selection. A set-valued map $\Gamma$ from a measurable space $(X, \Sigma)$ to a topological space $Y$ is said to be weakly measurable if $\{x \in X : \Gamma(x) \cap V \neq \emptyset\} \in \Sigma$, for each closed $V \subseteq Y$

**Theorem 13** (Kuratowski-Ryll-Nardzewki Selection Th.)**.** *A weakly measurable set-valued map with nonempty closed values from a measurable space into a Polish space admits a measurable selection.*

In particular, an upper hemicontinous correspondences admits a measurable selection. Last theorem is a well-known result of Fan.

**Theorem 14** (Fan's MinMax)**.** *Let* $X$ *and* $Y$ *be metrizable convex and compact subsets of some linear topological space and* $f$ *a continuous real-valued function on* $X \times Y$ *that satisfies*

1. *$f(\cdot,y)$ is concave on $X$ for each $y$*
2. *$f(x,\cdot)$ is convex on $Y$ for each $x$*

*then:*

$$\max_x \min_y f(x,y) = \min_y \max_x f(x,y)$$

# Appendix B. Proofs.

We need the following preliminary result:

**Lemma 15.** *Fix $c \in \mathscr{D}_{\text{cont}}$ and let $V_t$ given as in Eq. (1). Define for each $T \in \mathbb{N}$, $\{V_t^T(c)\}_{t=1}^{\infty} \in \mathscr{D}_{\text{cont}}$, $V_t^T \triangleq 0$ if $t > T$ and*

$$V_t^T(c,\omega_t) \triangleq u(c_t(\omega_t)) + \beta I\left(V_{t+1}^T(c,\omega_t)\right)$$

*for $0 \le t \le T$. If $\beta b < 1$, then $\lim_{T\to\infty} V_t^T(c,\omega_t) = V_t(c,\omega_t)$ for all $t$ and $\omega_t$.*

*Proof.* Of Lemma 15. For any $t \le T$ and $\omega_t$, notice that $V_t^T(c,\omega_t) \le V_t(c,\omega_t) \le V_t^T(c,\omega_t) + \|V(c)\|\,\beta^{T-t+1} b^{T+1}$. □

*Proof.* Of Theorem 1. Take $V(c)$, $V'(c)$ that solve Eq. (1), (by Tarksi's fixed point theorem at least one exists). Fix an arbitrary $t$ then,

$$\begin{aligned}
V_t(c,\omega^t) - \bar{V}_t^T(c,\omega^t) &= \beta E_{p_t}\left[V_{t+1}(c,\omega^t) - \bar{V}_{t+1}^T(c,\omega^t)\right] \quad \text{by Lebourg MVT.}\\
&= \beta E_{p_t}\left[\beta I(V_{t+1}(c,\omega^t)) - \beta I(\bar{V}_{t+1}^T(c,\omega^t))\right] \quad \text{by definition}\\
&= \beta E_{p_t}\left[\beta E_{p_{t+1}}\left[V_{t+1}(c,\omega^t) - \bar{V}_{t+1}^T(c,\omega^t)\right]\right] \quad \text{by Lebourg MVT.}\\
&\le \beta E_{p_t}\left[\cdots\left[\beta E_{p_{t+T+1}}\left[V_{t+T+1}(c,\omega^{t+T})\right]\right]\right]\\
&\le (\beta b)^{T-t+1} b^t \|V(c)\| \xrightarrow{T\to\infty} 0 \quad \text{by Lemma 15}
\end{aligned}$$

Applying the same argument to $\bar{V}_t(c,\omega^t) - V_t^T(c,\omega^t)$ concludes the proof. □

*Proof.* Of Proposition 2. By definition any $c \in \mathscr{D}_{\text{cont}}$ is Borel measurable, by assumption (4.1), $I(\cdot;\omega^t)$ is continuous, therefore the composite map $I(c;\omega^t)$ is continuous, hence Borel. Let $\mathscr{D}_{\text{cont}}^M$ be the set of continuous time-homogeneous Markov processes endowed with the pointwise order. If $c \in \mathscr{D}_{\text{cont}}^M$, then, the map

$$(Tc)_t(\omega^t) \triangleq u(c_t(\omega_t)) + I\left(c_{t+1}(\cdot,\omega_t);\omega^t\right)$$

maps $\mathscr{D}_{\text{cont}}^M$ into itself, since $I(c,\omega^t)$ is Borel. Monotonicity w.r.t. pointwise order of $I$ implies that of $T$ therefore, Tarski's fixed point theorem implies the existence of a fixed point. □

*Proof.* Of Lemma 3. By definition

$$V_t(e^* + \xi(h_1,h_2,0,\ldots),\omega^t) = u(e^*(\omega_t) + \xi h_1) + \beta I\left(V_{t+1}(e^* + \xi h_2,\omega^t);\omega_t\right)$$

which is equal to

$$V_t(e^* + \xi(h_1,h_2,0,\ldots),\omega^t) = u(e^*(\omega_t) + \xi h_1) + \beta I\left(u(e^* + \xi h_2 + \phi(\omega_{t+1});\omega_t\right)$$

where $\phi(\omega_{t+1}) = I\left(V_{t+2}(e^*,\omega_t);\omega^{t+1}\right)$. Optimality of $\xi = 0$ implies by Theorem 9 and the consequent discussion that

$$0 = u'(e^*(\omega_t))h_1 + \beta \int u'(e^*)h_2\,dp, \quad \text{for some } p \in \partial I(u(e^*) + \phi(\omega_{t+1});\omega_t)$$

The proof is concluded noticing that $u(e^*)+\phi(\omega_{t+1})=V_{t+1}(e^*)$. □

We first introduce a concept of equilibrium that is weaker then global equilibrium. First, given $(t,\omega^t)$, let $(c',\theta')$ a plan with $c'=\{c'_t\}_1^\infty$, $c'_\tau=0$ for all $\tau\neq t,t+1$, $c'_t\in\mathbb{R}$, $c'_{t+1}\in C(\Omega)$, $\theta'_\tau=0$ for all $\tau\neq t$, $\theta'_t=\Delta$ for $\Delta\in\mathbb{R}^n$. We define a *two-periods perturbation* of a $(t,\omega^t)$-feasible plan $(c,\theta)$ as $(c+\xi c',\theta+\xi\theta')$ for sufficiently small $\xi\in\mathbb{R}$ such that $(c+\xi c',\theta+\xi\theta')$ is $(t,\omega^t)$-feasible. We say a $(t,\omega^t)$-feasible plan is *myopically optimal* if

$$V_t({}^tc|\omega^t,\omega^t)\geq V_t({}^tc'|\omega^t,\omega^t),\quad \text{for all two-periods perturbation } (c',\theta')$$

Then we call $\{q_t\}\in\mathscr{D}_n$ a *weak myopic equilibrium* if $(e,0)$ is myopically optimal for all $t\geq 1$, $\omega^t\in\Omega^t$.

*Proof.* Of Theorem 4.
(i). We need the following fact.

*Fact* 1. If $q$ satisfies Eq. (6) then it is a weak myopic equilibrium.

*Proof.* Of Fact 1. First note that, if $q$ is a weak myopic equilibrium, then $V_t(e)\geq V_t(c,\omega_t)$ for all two-periods perturbations $c$ and it is equivalent to $e$ being a local maximum for $\xi\mapsto V(e+\xi c,\omega_t)$. By Clarke [7, Proposition 2.3.2], we have $0\in\partial V(e+\xi c,\omega_t)$ at $\xi=0$. To prove Fact 1, assume that $q$ is a weak myopic equilibrium, and Eq. (6) is not satisfied. Let

$$\Gamma^*(e^*,\omega_t)\triangleq\operatorname*{argmin}_{p\in C(e^*,\omega_t)}\int\frac{u'(e_{t+1})}{u'(e_t)}(q_{t+1}+d_{t+1})dp$$

Then, for some $t\in\mathbb{N}$ and $\omega^t\in\Omega^t$ with $\theta_t=\Delta$,

$$\beta E_{\pi_t(\omega^t;\cdot)}\{u'(e_{t+1})\theta_t\cdot(q_{t+1}+d_{t+1})\}>\theta_t\cdot q_t u'(e_t)\tag{16}$$

where $\pi_t:\Omega_t\to\mathscr{M}^+(\Omega_t)$, with $\pi_t(\omega^t;\cdot)\in\Gamma^*(e^*,\omega_t)$ for each $t$ and $\omega_t$. Now let $c_\tau=e_\tau$ for all $\tau\neq t,t+1$ $c_t\triangleq e_t-\theta_t\cdot q_t$, $c_{t+1}=e_{t+1}+\theta_t\cdot(q_{t+1}+d_{t+1})$ and $\theta_\tau=0$ for all $\tau\neq t$ and , it is $(t,\omega_t)$-feasible, but inequality Eq. 16 is equivalent to

$$\frac{d}{d\xi}V_t(e+\xi c,\omega_t)\Big|_{\xi=0}>0$$

for a two-periods perturbation $c$, contradicting the assumption that $q$ is a weak myopic equilibrium. □

To conclude the proof of (i), we will show that weak myopic equilibria are weak global. Assume by contradiction that $q$ is a weak myopic equilibrium that is not weak global. Then, there is a $(t,\omega^t)$-feasible path (for all $t,\omega^t$) $c\in\mathscr{D}$ (assume for simplicity that $c$ is time-homogeneous and Markov) such that, for some $t$ and some $\omega_t$,

$$V_t(c,\omega_t)\geq V_t(e^*)+2\epsilon$$

for some $\epsilon>0$. Choose an integer $N$, such that, if a consumption plan $c'_\tau=c_\tau$ for $\tau=t,\ldots,t+N$, then $V_t(c',\omega_t)\geq V(c,\omega_t)+\epsilon$. It follows that for such paths

$$V_t(c',\omega_t)\geq V_t(e^*)+\epsilon$$

Define a family of $N$ consumption processes $\hat{c}^{t+i}$, $i=0,\ldots,N-1$, such that

$$\hat{c}^{t+i}\triangleq\begin{cases}c_{t+j+1} & \text{for all } j\in[0,i]\\ e_t & \text{otherwise}\end{cases}$$

Then at $t$,

$$V_t(\hat{c}^{t+N-1}, \omega_t) > V_t(e^*) \tag{17}$$

Moreover notice that, $\hat{c}^{t+N-2}$ coincides with $e$ from $t-N-1$ onward, whereas $\hat{c}^{t+N-1}$ differs from $e$ only at $t+N-1$. Since $q$ is a weak myopic equilibrium, it is robust for a two-periods perturbation, hence, a posteriori for a one-period perturbation, then

$$V_{t+N-1}(\hat{c}^{t+N-2}) = V_{t+N-1}(e^*) \geq V_{t+N-1}(\hat{c}^{t+N-1})$$

Dynamic consistency DC1-DC2 implies

$$V_t(\hat{c}^{t+N-2}) \geq V_t(\hat{c}^{t+N-1}) \tag{18}$$

Together Eq. (17) and Eq. (18) imply

$$V_t(\hat{c}^{t+N-2}, \omega_t) > V_t(e^*)$$

Repeating the argument until time $t$, we get $V_t(\hat{c}^t, \omega_t) > V_t(e^*)$ but $\hat{c}^t$ is a one-period perturbation of $e$, and this contradict the fact that $q$ is a weak myopic equilibrium.
(ii). Given Assumption 1, Berge Maximum theorem implies upper hemicontinuity of $\omega_t \mapsto \Gamma^*(e^*, \omega_t)$ hence, for all $t$, by Theorem 13, there is a measurable selection, , $\hat{\pi}_t(\omega^t; \cdot) \in \Gamma^*(e^*, \omega_t)$ and $\omega_t \in \Omega^t$, such that

$$q_{i,t} = \beta E_{\hat{\pi}(\omega^t;\cdot)}\left[\frac{u'(e_{t+1})}{u'(e^*(\omega_t))}(q_{t+1} + d_{t+1})\right]$$

Then define an operator $T : \mathscr{D} \to \mathscr{D}$ as

$$(T_i f)_t(\omega_t) = \beta E_{\hat{\pi}(\omega^t;\cdot)}\left[f_{t+1} + u'(e_{t+1}) d_{i,t+1}\right]$$

It clearly has a unique fixed point $T_i \hat{f}_i = \hat{f}_i$, now let $\hat{f}_{i,t} = u'(e^*(\omega_t)) q_{i,t}$. Then, by part (i) it is a weak global equilibrium. □

*Proof.* Of Theorem 5. The argument follows the one of Epstein and Wang [16, Part (b) of Th. 2]. By Assumption 2 and Theorem 11 there exists a continuous selection $\pi_t(\omega^t, \cdot)$ from $\Gamma^*(e^*, \omega_t)$ then, by Theorem 4 part (i), it defines an equilibrium $q$, by continuity of $\pi_t(\omega^t, \cdot)$, $q$ is also continuous.

□

*Proof.* Of Lemma 6. First note that,

$$I(V(e^*); \omega) = \min_{p \in \mathscr{C}^*} G\left(\int V(e^*) dp, p\right) = -\max_{p \in \mathscr{C}^*} -G\left(\int f dp, p\right)$$

Moreover, by Clarke [7, Proposition 2.3.1], $\partial(-I)(V(e^*); \omega) = -\partial I(V(e^*); \omega)$. Let first check the conditions before Theorem 10 with $T = C^*$ and for $\hat{I}(V(e^*); \omega) \triangleq \max_{p \in C^*} -G\left(\int V(e^*) dp, p\right)$. By definition $\mathscr{C}^* = \mathrm{cl}(\mathrm{dom}\, G)$. We need to show that $\mathscr{C}^*$ is sequentially (weak*) compact, but this follows from separability of $C(\Omega)$, which implies that weak* compact subsets of $C(\Omega)^* = \mathrm{rca}(\mathscr{B})$ are weak* sequentially compact. Clearly, $p \mapsto -G(y, p)$ is upper hemicontinuous for all $y$ in a neighborhood of $x$, and by the locally Lipschitzianity assumption, each $-G(\cdot, p)$ is locally Lipschitz. Moreover $\{-G(x, p) : p \in \mathscr{C}^*\}$ is bounded by definition. $\mathscr{C}^*$ is a closed subset of a polish space so it is separable then, we can apply Theorem 10 noticing that

$$\begin{aligned}\left\{\int_{\mathscr{C}^*} \partial_{[\mathscr{C}^*]} - G\left(\int V(e^*) dp, p\right) \mu(dp) : \mu \in \mathscr{M}_1^+\left(\Pi(V(e^*); \omega)\right)\right\} &\supset \partial \hat{I}(V(e^*); \omega) \\ &= \partial(-I)(V(e^*); \omega) = -\partial I(V(e^*); \omega)\end{aligned}$$

By assumption each $G(\int V(e^*), p)$ is regular at $\Pi(V(e^*))$ then, by Clarke [7, Theorem 2.3.9], using the notation $a \mapsto G(\int a dp, p)$ as $a \mapsto (G \circ p)(a)$, we have

$$\partial \left(G \circ p\right)(V(e^*)) = \left\{\alpha_p p : \alpha \in \partial_{[C_t^*]} G\left(\int V(e^*) dp, p\right)\right\}$$

where

$$\partial_{[C_t^*]} G\left(\int a dp, p\right) = \overline{\text{co}}^* \left(p \in \text{rca}(\Omega) : p_n \in \partial G(a_n, q_n),\ a_n \to a,\ q_n \to q,\ q_n \in C^*,\ p_n \overset{w*-cp}{\longrightarrow} p\right)$$

where $p_n \overset{w^*-cp}{\longrightarrow} p$ means that $p$ is a weak* cluster point of $p_n$. Hence, the result follows after normalization. □

Next lemma gives a sufficient condition for smooth ambiguity preferences to have a locally Lipschitz representation:

**Lemma 16.** *If $\phi$ and $\phi^{-1}$ are locally Lipschitz, then*

$$I(f) = \phi^{-1}\left(\int_\Theta \phi\left(\int_\Omega f dp_\theta\right) d\mu(\theta)\right)$$

*is locally Lipschitz for all $f \in C(\Omega)$.*

*Proof.* Of Lemma 16. Let $f, g \in C(\Omega)$, then

$$\begin{aligned}
\|I(f) - I(g)\| = &\left\|\phi^{-1}\int_\Theta \phi(\langle f, p_\theta\rangle) d\mu(\theta) - \phi^{-1}\int_\Theta \phi(\langle g, p_\theta\rangle) d\mu(\theta)\right\| \\
\leq &k\left\|\int_\Theta \phi(\langle f, p_\theta\rangle) d\mu(\theta) - \int_\Theta \phi(\langle g, p_\theta\rangle) d\mu(\theta)\right\| \\
\leq &k\int_\Theta \|\phi(\langle f, p_\theta\rangle) d\mu(\theta) - \phi(\langle g, p_\theta\rangle)\| \, d\mu(\theta) \\
\leq &kk' \|\langle f, p_\theta\rangle - \langle g, p_\theta\rangle\| \leq kk' \|f - g\|
\end{aligned}$$

where the first and third inequalities follows from the assumption.

□

# References


[1] R. Bansal and A. Yaron. Risks for the long run: a potential resolution of asset pricing puzzles. *The Journal of Finance*, 59(4):1481–1509, 2004.

[2] P. Bossaerts, P. Ghirardato, S. Guarnaschelli, and W. R. Zame. Ambiguity in asset markets: Theory and experiment. *Review of Financial Studies*, 23(4):1325–1359, April 2010.

[3] A. Caplin and J. Leahy. The recursive approach to time inconsistency. *Journal of Economic Theory*, 131(1):134–156, November 2006.

[4] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio. Uncertainty averse preferences. *Journal of Economic Theory*, 146(4):1275–1330, July 2011.

[5] A. Chateauneuf and J. H. Faro. Ambiguity through confidence functions. *Journal of Mathematical Economics*, 45(9-10):535–558, September 2009.

[6] Z. Chen and L. G. Epstein. Ambiguity, risk, and asset returns in continuous time. *Econometrica*, 70(4):1403–1443, July 2002.

[7] F. H. Clarke. *Optimization and Nonsmooth Analysis*. J. Wiley, New York, 1983.

[8] F. Collard, S. Mukerji, K. Sheppard, and J. Tallon. Ambiguity and the historical equity premium. Documents de travail du Centre d'Economie de la Sorbonne 11032, Université Panthéon-Sorbonne (Paris 1), Centre d'Economie de la Sorbonne, May 2011.

[9] F. Delbaen. The structure of m-Stable sets and in particular of the set of risk neutral measures. In Michel Émery and Marc Yor, editors, *In Memoriam Paul-André Meyer*, volume 1874 of *Lecture Notes in Mathematics*, pages 215–258. Springer Berlin Heidelberg, 2006. ISBN 978-3-540-30994-9.

[10] J. Dow and S. R. da Costa Werlang. Uncertainty aversion, risk aversion, and the optimal choice of portfolio. *Econometrica*, 60(1):197–204, January 1992.

[11] D. Easley and M. O'Hara. Ambiguity and nonparticipation: The role of regulation. *Review of Financial Studies*, 22(5):1817–1843, 2009.

[12] L. Epstein and M. Schneider. Recursive multiple-priors. *Journal of Economic Theory*, 113(1):1–31, November 2003.

[13] L. G. Epstein and M. Schneider. IID: independently and indistinguishably distributed. *Journal of Economic Theory*, 113(1):32–50, November 2003.

[14] L. G. Epstein and M. Schneider. Learning under ambiguity. *Review of Economic Studies*, 74(4):1275–1303, 2007.

[15] L. G. Epstein and M. Schneider. Ambiguity, information quality, and asset pricing. *Journal of Finance*, 63(1):197–228, 02 2008.

[16] L. G. Epstein and T. Wang. Intertemporal asset pricing under knightian uncertainty. *Econometrica*, 62(2):283–322, March 1994.

[17] L. G. Epstein and T. Wang. Uncertainty, risk-neutral measures and security price booms and crashes. *Journal of Economic Theory*, 67(1):40–82, October 1995.

[18] L. G. Epstein and S. Zin. Substitution, risk aversion and the temporal behavior of consumption asset returns: a theoretical framework. *Econometrica*, 57(4):937–969, July 1989.

[19] P. Ghirardato and M. Siniscalchi. A more robust definition of multiple prior. Carlo Alberto Notebook 144, Collegio Carlo Alberto, 2010.

[20] P. Ghirardato and M. Siniscalchi. Ambiguity in the small and in the large. *Econometrica*, 80:2827–2847, November 2012.

[21] I. Gilboa and D. Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, April 1989.

[22] S. J. Grossman and R. J. Shiller. The determinants of the variability of stock market prices. *American Economic Review*, 71(2):222–27, May 1981.

[23] M. Guidolin and F. Rinaldi. Ambiguity in asset pricing and portfolio choice: a review of the literature. *Theory and Decision*, 74(2):183–217, February 2013.

[24] A. Gumen and A. Savochkin. Dynamically stable preferences. *Journal of Economic Theory*, 148(4):1487–1508, 2013.

[25] E. Hanany and P. Klibanoff. Updating preferences with multiple priors. *Theoretical Economics*, 2(3):261–298, 2007.

[26] L. P. Hansen and R. Jagannathan. Implications of security market data for models of dynamic economies. *Journal of Political Economy*, 99(2):pp. 225–262, 1991.

[27] L. P. Hansen and T. J. Sargent. Robust control and model uncertainty. *The American Economic Review*, 91(2):pp. 60–66, 2001.

[28] L. P. Hansen, T. J. Sargent, and T. D. Jr. Tallarini. Robust permanent income and pricing. *Review of Economic Studies*, 66(4):873–907, October 1999.

[29] P. K. Illeditsch. Ambiguous information, portfolio inertia, and excess volatility. *The Journal of Finance*, 66(6):2213–2247, 2011.

[30] N. Ju and J. Miao. Ambiguity, learning, and asset returns. *Econometrica*, 80(2):559–591, 03 2012.

[31] P. Klibanoff, M. Marinacci, and S. Mukerji. A smooth model of decision making under ambiguity. *Econometrica*, 73(6):1849–1892, November 2005.

[32] P. Klibanoff, M. Marinacci, and S. Mukerji. Recursive smooth ambiguity preferences. *Journal of Economic Theory*, 144(3):930–976, May 2009.

[33] M. Leippold, F. Trojani, and P. Vanini. Learning and asset prices under ambiguous

information. *Review of Financial Studies*, 21(6):2565–2597, November 2008.

[34] R. E. Jr Lucas. Asset prices in an exchange economy. *Econometrica*, 46(6):1429–45, November 1978.

[35] F. Maccheroni, M. Marinacci, and A. Rustichini. Dynamic variational preferences. *Journal of Economic Theory*, 128(1):4–44, May 2006.

[36] P. Maenhout. Robust portfolio rules and asset pricing. *Review of Financial Studies*, 17 (4):951–983, 2004.

[37] M. Marinacci and L. Montrucchio. Unique solutions for stochastic recursive utilities. *Journal of Economic Theory*, 145(5):1776–1804, September 2010.

[38] R. Mehra and E. C. Prescott. The equity premium: A puzzle. *Journal of Monetary Economics*, 15(2):145–161, March 1985.

[39] E. Michael. Continuous selections. I. *Annals of Mathematics*, 63(2):361–382, March 1956.

[40] H. Ozsoylev and J. Werner. Liquidity and asset prices in rational expectations equilibrium with ambiguous information. *Economic Theory*, 48(2):469–491, October 2011.

[41] L. Pastor and R. F. Stambaugh. Are stocks really less volatile in the long run? *The Journal of Finance*, 67(2):431–478, 2012.

[42] F. Riedel. Optimal stopping with multiple priors. *Econometrica*, 77(3):857–908, 2009.

[43] M. Siniscalchi. Vector expected utility and attitudes toward variation. *Econometrica*, 77(3):801–855, 05 2009.

[44] M. Siniscalchi. Recursive vector expected utility. Mimeo, 2011.

[45] M. Siniscalchi. Dynamic choice under ambiguity. *Theoretical Economics*, 6(3), September 2011.

[46] T. Strzalecki. Temporal resolution of uncertainty and recursive models of ambiguity aversion. *Econometrica*, 81(3):1039–1074, 2013.

[47] P. Weil. The equity premium puzzle and the risk-free rate puzzle. *Journal of Monetary Economics*, 24(3):401–421, November 1989.