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Abstract

We study unique and globally attracting solutions of a general nonlinear equation that has as special cases some recursive equations widely used in Economics.

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1 Introduction

Consider a standard intertemporal stochastic setting, where uncertainty is modelled by a probability space (Ω, Σ, P) and information by an increasing filtration $\{\Sigma_t\}_t$ defined on Ω.

In such settings, most economic applications are based on a ranking of some relevant stochastic payoff streams, like for example a consumption stream $c = \{c_t\}_t$, adapted to the given filtration. The ranking is made through a stream of preference functionals $V = \{V_t(c)\}_t$, also adapted to the filtration, and their basic form is

$$V_t(c) = \mathbb{E}_t\left(\sum_{\tau \geq t} \beta^{\tau - t} u(c_t)\right) \tag{1}$$

where $\mathbb{E}_t$ is the expectation conditional on Σ_t, $\beta \in (0,1)$ is a discount factor, and u is an instantaneous utility function, constant over time.

This basic representation is analytically very tractable and has the following main behavioral features: it is dynamically consistent and independent of both unrealized alternatives and past consumption levels. It has, however, an important drawback: it is unable to disentangle risk attitudes from the intertemporal elasticity of substitution.

In order to overcome this difficulty, starting from Kreps and Porteus [20] and Epstein and Zin [10], the more general preference functional

$$V_t(c) = W(c_t, \mathcal{M}_t(V_{t+1}(c))) \tag{2}$$

has been widely used, where $W : \mathbb{R}^2 \to \mathbb{R}$ is a (temporal) aggregator and $\mathcal{M}_t$ is a time t certainty equivalent of future utility V_{t+1}. Clearly, (2) generalizes the additive case (1):

$$V_t(c) = u(c_t) + \beta \mathbb{E}_t[V_{t+1}(c)]. \tag{3}$$

Standard specifications are, for example, the CES aggregator $W(x,y) = (x^\rho + \beta y^\rho)^{\frac{1}{\rho}}$ and the quasi-arithmetic certainty equivalent

$$\mathcal{M}_t(V_{t+1}) = \phi^{-1}(\mathbb{E}_t(\phi \circ V_{t+1})), \tag{4}$$

where ϕ is a strictly increasing function. The preference functional (3), with $u(c_t) = c_t$, is the special case in which $\rho = 1$ and $\phi(x) = x$.

The recursive preference functional (2) allows a separation between risk attitudes and the degree of intertemporal substitution. It is still analytically tractable and retains the main behavioral features of the basic preference functional (1), that is, dynamic consistency and independence of both unrealized alternatives and past consumption levels.

Because of its tractability and theoretical soundness, the recursive preference functional (2) has become very popular in finance and macroeconomics applications. These works typically specify an aggregator W and a certainty equivalent $\mathcal{M}$, and the stochastic payoff streams $c = \{c_t\}_t$ are then evaluated by using the stochastic process V that solves the recursive equation (2).

For this reason a key feature of the equation (2) is the existence of a solution V for a given specification of W, $\mathcal{M}$, and c. Even more important, to be economically meaningful such solution has to be unique and globally attractive.

In fact, uniqueness is crucial both to properly interpret the solution as an evaluation of the stream c and to carry out comparative statics exercises, which describe how the solution/evaluation varies as

$\mathcal{M}$, W and c vary. On the other hand, global attractivity is needed to find the solution iteratively by starting from any possible initial point.

Together, uniqueness and global attractivity make the solution economically meaningful and computable. For this reason in this paper we investigate in depth the uniqueness and global attractivity of the solution of the recursive equation (2). In particular, our aim is to find conditions directly on W and $\mathcal{M}$, so that a direct check of the properties of W and $\mathcal{M}$ is enough to determine whether there is a unique and globally attracting solution. This direct verifiability is important in applications. For example, for the specification (4) this means to have conditions directly on the function ϕ.

Despite the importance of (2) in applications, surprisingly little seems to be known about the existence of unique and globally attracting solutions of these recursive equations. Our results therefore fill a significant gap in the literature that uses them.

In particular, in our analysis we consider two types of aggregators W, which we call Thompson and Blackwell aggregators. Thompson aggregators are based on condition (W-iii) below, that is,

$$W(x, \alpha y) \geq \alpha W(x, y) + (1 - \alpha) W(x, 0), \qquad \forall \alpha \in [0, 1], \forall x, y \in \mathbb{R}_+,$$

a simple concavity condition on W, while Blackwell aggregators use condition (W-v) below, that is,

$$|W(x, y) - W(x, y')| \leq \beta |y - y'|, \qquad \forall x, y, y' \in \mathbb{R}_+.$$

a standard Lipschitzian condition often imposed on aggregators.[1] As we show in Section 3.1, for some common specifications of the aggregators the conditions that make them either Thompson or Blackwell nicely complement each other, and together cover a wide range of values of the parameters of these specifications.

For example, in the classic CES case $W(x, y) = (x^\rho + \beta y^\rho)^{\frac{1}{\rho}}$ we have

Thompson	if $\rho \leq 1$ and $\beta < 1$
Blackwell	if $\beta < 1 < \rho$

and so the Thompson and Blackwell conditions are fully complementary: they cover the cases $\rho = 1$ and $\rho > 1$, respectively. Together, they allow us to consider all values of the parameter ρ.

For the standard specification (4), the recursive equation (2) takes the form

$$V_t = W\left(c_t, \phi^{-1}\left(\mathbb{E}_t\left(\phi \circ V_{t+1}\right)\right)\right) \tag{5}$$

and our Theorems 3 and 4 establish the existence of a unique and globally attracting solution to this version of equation (2) provided either W is Thompson and ϕ exhibits increasing relative risk aversion (IRRA) or W is Blackwell and ϕ exhibits increasing absolute risk aversion (IARA). These well known conditions on ϕ include some of the most popular specifications of ϕ, such as CRRA, CARA, and the quadratic.

In particular, for the CES aggregator equation (5) becomes

$$V_t = \left(c_t^\rho + \beta\left(\phi^{-1}\left(\mathbb{E}_t\left(\phi \circ V_{t+1}\right)\right)\right)^\rho\right)^{\frac{1}{\rho}} \tag{6}$$

and Corollary 2 shows that there is a unique and globally attracting solution to equation (6) provided either $\beta < 1 < \rho$ and ϕ is IARA or $\rho \leq 1$, $\beta^\rho < 1$, and ϕ is IRRA.[2] The latter set of conditions include

[1] See, e.g., Stokey and Lucas [26, condition (W3) p. 115].

[2] To be precise, this is true when the stream c is bounded, but Corollary 2 shows that a similar result holds for suitably unbounded c

the classic CRRA specification of ϕ, which gives the most popular version of (6), that is,

$$V_t = \left(c_t^{\rho} + \beta \left(\mathbb{E}_t \left(V_{t+1}^{1-\gamma} \right) \right)^{\frac{\rho}{1-\gamma}} \right)^{\frac{1}{\rho}} . \tag{7}$$

These results are special cases of much more general results, Theorems 1 and 2, the main results of the paper. They solve the general nonlinear equation (10) below, which includes (2) as a very special case. Besides its generality, the advantage of studying the general equation (10) is that it makes it possible to focus on the essential features that lead to uniqueness and global attractivity, without being distracted by the peculiar features that some more special cases might have.

On a technical level, our main novel contribution is the introduction of Thompson aggregators. For this "concave" aggregators the nonlinear equation (2) cannot be solved via standard contraction arguments à la Blackwell. Instead, we need to use different contraction techniques based on Thompson [28] (this also motivates the terminology). The use of these techniques to study the recursive equation (2) is a secondary contribution of our paper.

We close by discussing some related works. In their seminal paper, Epstein and Zin [10, Theorem 3.1] proved the existence of a solution for equation (2) when W is a CES aggregator. They established the uniqueness of such solution only when, in addition, $\mathcal{M}_t$ has the quasi-arithmetic form (4) with a power specification of ϕ ([10, p. 963]). Our results therefore substantially generalize their original findings.

More recently, Ozaki and Streufert [24] studied related issues, although with different goals and modelling. Their paper provides a comprehensive study of the existence of optima through dynamic programming techniques. In place of (3), they start from the additive intertemporal relation

$$V_t\left(z^t\right) = u\left(c\left(z^t\right)\right) + \beta \int V_{t+1}\left(z^t, z_{t+1}\right) Q\left(dz_{t+1} \mid z_t\right), \tag{8}$$

where $z^t = (z_0, z_1, ..., z_t)$ denotes time-t history of the exogenous shocks and $Q\left(dz_{t+1} \mid z_t\right)$ is a time homogeneous Markov stochastic kernel. The recursive version of (8) they study is $V_t\left(z^t\right) = W\left(c\left(z^t\right), \mathcal{M}_{z_t}\left(V_{t+1}\left(z^t, \cdot\right)\right)\right)$. This setting is particularly suited for Dynamic Programming, where the standard Markov operator is replaced by the nonadditive operator $f\left(z'\right) \to \mathcal{M}_z\left[f\left(z'\right)\right]$.

Our approach is more traditional, related to the classic contraction techniques in function spaces. This allows us to give conditions that are rather simple to check, a main feature of our analysis, while the conditions that [24] consider for (8) are sometimes difficult to check (see, for instance, some of their conditions N.1-N.12).

The paper is organized as follows. After some preliminaries in Section 2, we present the general nonlinear equation (10) in Section 3, and we solve it in the bounded case in Section 4 and in the unbounded case in Section 5. The special case (2) is then studied in Section 6, while Section 7 considers another application (based on [15]). The Appendix contains all proofs, as well as some related material.

2 Preliminaries

2.1 Uncertainty and Information

We model uncertainty and information in a standard way.[3] Specifically, uncertainty is modelled by a probability space (Ω, Σ, P), where Σ is an event σ-algebra of a state space Ω and $P : \Sigma \to [0, 1]$ is a

[3]See, e.g., Stokey and Lucas [26] for canonical interpretations of this setting in terms of shocks/observations.

countably additive probability measure. Given a discrete time horizon $\mathbb{N} = \{1, ..., t, ...\}$, information is modelled through an increasing filtration $\{\Sigma_t\}_{t\geq 1}$ of σ-algebras contained in Σ.

A real-valued adapted stochastic process $X = \{X_t\}_{t\geq 1}$ is a collection of Σ_t-measurable functions $X_t : \Omega \to \mathbb{R}$. We denote by L the space of all adapted processes. Henceforth any process will be adapted to the filtration $\{\Sigma_t\}_{t\geq 1}$, even if not explicitly mentioned.

A process $X \in L$ can be regarded as a (suitably measurable)[4] function $X : \Omega \times \mathbb{N} \to \mathbb{R}$. As such, we can consider on L the pointwise order $\leq$, i.e., given any $X', X'' \in L$, we write $X' \leq X''$ if $X'_t \leq X''_t$ P-a.e. for all $t \geq 1$. In particular, we write $[X', X''] = \{X \in L : X' \leq X \leq X''\}$. We also set $L_+ = \{X \in L : X \geq 0\}$. A process will be said to be integrable if $\mathbb{E}(X_t) < \infty$ for all $t \geq 1$.

We denote by L^∞ the set of all essentially bounded processes, that is, $X \in L^\infty$ if

$$\|X\|_\infty \equiv \operatorname{ess} \sup_{(\omega,t)\in\Omega\times\mathbb{N}} |X_t(\omega)| < +\infty.$$

In particular, D denotes the subset of L^∞ consisting of all (almost) deterministic processes; that is, $X \in D$ if there exists $d : \mathbb{N} \to \mathbb{R}$ such that $X_t(\omega) = d(t)$ for all $t \geq 1$ and P-a.e. $\omega \in \Omega$.

Endowed with the essential supnorm $\|\cdot\|_\infty$, L^∞ becomes a Banach space (under the usual identification of its P-a.e. equal elements).

Notation. (i) With a slight abuse of notation, k denotes the constant process X_t such that $X_t(\omega) = k \in \mathbb{R}$ for all $t \geq 1$ and P-a.e. $\omega \in \Omega$. (ii) $\mathbb{E}_t(X_{t+1})$ denotes the conditional expectation $\mathbb{E}^P(X_{t+1} \mid \Sigma_t)$, provided X is an integrable process; moreover, equalities and inequalities between Σ-measurable functions are understood to hold P-a.e. even where not stated explicitly. (iii) Given $X \in L_+$, we set $[X]_\infty = \operatorname{ess}\inf_{(\omega,t)\in\Omega\times\mathbb{N}} X_t(\omega)$.

2.2 Weighted Norms

In order to deal with unbounded processes, it is useful to consider weighted supnorms, a standard generalization of the supnorm (see, e.g., Wessels [29] and Boyd [5]). A weight function is a deterministic process $w \in L_+$ with $w \geq 1$, and it induces an (essential) weighted supnorm $\|\cdot\|_w : L \to [0, \infty]$ by

$$\|X\|_w \equiv \operatorname{ess} \sup_{(\omega,t)\in\Omega\times\mathbb{N}} \frac{|X_t(\omega)|}{w_t}.$$

Set $L^w = \{X \in L : \|X\|_w < +\infty\}$. The pair $(L^w, \|\cdot\|_w)$ is easily seen to be a Banach space. Note that for any element $X \in L^w$ we have that X_t is essentially bounded for every t. Consequently, all $X \in L^w$ are integrable.

Since $\|X\|_w = \|w^{-1}X\|_\infty$, the norms $\|\cdot\|_w$ and $\|\cdot\|_\infty$ are equivalent, and so $L^w = L^\infty$, when w is essentially bounded. Weighted supnorms therefore become relevant when w is unbounded.

Example 1 Given $a > 1$, let $w_t = a^t$ be the exponential weight, for all $t \geq 1$. In this case,

$$\|X\|_w = \operatorname{ess} \sup_{(\omega,t)\in\Omega\times\mathbb{N}} \frac{|X_t(\omega)|}{a^t}.$$

▲

A deterministic function $w : \mathbb{N} \to \mathbb{R}$ is said to be an *admissible weight function* if there is a positive constant $a_w > 1$ such that $w(t+1) \leq a_w w(t)$ for all $t \geq 1$.

Exponential weights a^t are clearly admissible, with $a = a_w$. On the other hand, $w(t) = t$ is an important example of an admissible weight function, with $a_w = 2$, that is not exponential.

[4] More details are given in Appendix A.

Example 2 Given $0 < d < u$, consider a binomial process $Z \in L_+$ such that $Z_t(\omega) \in \{d, u\}$ for all $(\omega, t) \in \Omega \times \mathbb{N}$. Define $X \in L_+$ as

$$X_t = \sum_{\tau=1}^{t} Z_\tau, \qquad P\text{-a.e.},$$

for each $t \geq 1$, so that X_t is an additive process ([14, p. 884]). For example, X_t is a random walk when the Z_t are i.i.d..

Since $dt \leq X_t(\omega) \leq ut$, for each $n \geq 1$ the weight function $w_n(t) = t \vee n$ is admissible with $a_{w_n} = 1 + n^{-1}$, and

$$\left[w_n^{-1} X\right]_\infty = \frac{d}{n} \quad \text{and} \quad \|X\|_{w_n} = u. \tag{9}$$

Hence, $X \in L_+^{w_n}$. This additive process will be studied in Section 5.3. ▲

2.3 Risk Attitudes

In Decision Theory there are two classic indices associated to a given continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}$, twice differentiable on $(0, \infty)$:

$$A_\phi(t) = -\frac{\phi''(t)}{\phi'(t)} \qquad \text{and} \qquad R_\phi(t) = -\frac{t\phi''(t)}{\phi'(t)}$$

for all $t > 0$. The index A_ϕ is the coefficient of *absolute risk aversion*, while R_ϕ is the coefficient of *relative risk aversion*. The interpretation of these classic indices is well known and we refer the reader to the original articles of de Finetti [9], Arrow [1], and Pratt [25], as well as to Kreps [19].

We can classify the functions ϕ using these indices:

- a function $\phi : \mathbb{R}_+ \to \mathbb{R}$ is *decreasing absolute risk averse* (DARA) if its index $A_\phi : (0, \infty) \to \mathbb{R}$ is non-increasing, it is *increasing absolute risk averse* (IARA) if its index $A_\phi : (0, \infty) \to \mathbb{R}$ is non-decreasing, and it is *constant absolute risk averse* (CARA) if it is both DARA and IARA;
- a function $\phi : \mathbb{R}_+ \to \mathbb{R}$ is *decreasing relative risk averse* (DRRA) if its index $R_\phi : (0, \infty) \to \mathbb{R}$ is non-increasing, it is *increasing relative risk averse* (IRRA) if its index $R_\phi : (0, \infty) \to \mathbb{R}$ is non-decreasing, and it is *constant relative risk averse* (CRRA) if it is both DRRA and IRRA.

In applications the most important classes of functions are the DARA and IRRA (see, e.g., [1, p. 96]), which in turn include the CARA and CRRA.

We shall introduce in Section 8.2 another interesting class: the strongly decreasing absolute risk averse (SDARA).

Lemma 1 *Consider a strictly increasing $\phi : \mathbb{R}_+ \to \mathbb{R}$, twice differentiable on $(0, \infty)$. If ϕ is concave and IARA, then it is IRRA, while if ϕ is convex and IRRA, then it is IARA.*

Examples of functions belonging to these classes are well known, and for brevity we refer the reader to [1], [25], and [19].

3 A General Nonlinear Operator Equation

The nonlinear equations used in Economic Dynamics discussed in the Introduction are special cases of the following general nonlinear operator equation defined on the space of positive adapted processes L_+:

$$X = W(Y, \mathcal{M}(X)), \tag{10}$$

that is, for all $t \geq 1$,

$$X_t(\omega) = W(Y_t(\omega), \mathcal{M}_t(X)(\omega)), \qquad P\text{-a.e.},$$

where $Y \in L_+$ is a given stochastic process, $\mathcal{M} : L_+ \to L_+$ is an operator, and $W : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a function.[5]

For example, equation (2) is a special case of (10) where $Y_t = c_t$, $X_t = V_t$, and $\mathcal{M}_t(X)$ only depends on X_{t+1}, i.e.,

$$X_{t+1} = X'_{t+1} \Longrightarrow \mathcal{M}_t(X) = \mathcal{M}_t(X'), \qquad \forall X, X' \in L_+. \tag{11}$$

Equation (10) can be written in the form:

$$X = T(X),$$

where $T : L_+ \to L_+$ is the operator given by[6]

$$T(X) = W(Y, \mathcal{M}(X)), \tag{12}$$

that is, for all $t \geq 1$,

$$T_t(X)(\omega) = W(Y_t(\omega), \mathcal{M}_t(X)(\omega)), \qquad P\text{-a.e.}$$

A solution of equation (10) is thus a fixed point of the operator T, and so the resolution of this equation can be reduced to finding the fixed points of the operator T.

Existence of a solution is, of course, the first issue to deal with when considering (10). But, as we discussed in the Introduction, in economic applications the mere existence of a solution is not enough and two further properties are required: uniqueness and global attractivity. In terms of the fixed points of T, this amounts to requiring the existence of a unique fixed point X^* for the operator T, with $T^n(X) \to X^*$ in some suitable metric for every initial condition X.[7]

3.1 Aggregator Functions

We will solve equation (10) for suitable $\mathcal{M}$ and W. Specifically, in the spirit of Lucas and Stokey [21], we say that $W : \mathbb{R}^2_+ \to \mathbb{R}$ is an *aggregator function* if it satisfies the following two properties:

(W-i) W is nonnegative and monotone, i.e., $0 \leq W(x, y) \leq W(x', y')$ if $x \leq x'$ and $y \leq y'$,

(W-ii) there is a sequence $\{x_n, y_n\} \subseteq \mathbb{R}^2_+$, with $x_n \uparrow \infty$, such that $W(x_n, y_n) \leq y_n$ for each n.

These are standard assumptions. In particular, (W-ii) is for example satisfied when, for each x, the function $W(x, \cdot) : \mathbb{R}_+ \to \mathbb{R}$ has a fixed point.

In some results we will need some of the following further properties:

(W-iii) W is concave in the second variable at 0, i.e.,

$$W(x, \alpha y) \geq \alpha W(x, y) + (1 - \alpha) W(x, 0)$$

for each $\alpha \in [0, 1]$ and each $x, y \in \mathbb{R}_+$,

[5] We are using the notation $\mathcal{M}_t(X)(\omega) \equiv (\mathcal{M}X)(\omega, t)$. Further, with a slight abuse of notation, we write $W(Y, \mathcal{M}(X))$ in place of $W \circ (Y, \mathcal{M}(X))$, where we regard each element of the pair $(Y, \mathcal{M}(X))$ as a function from $\Omega \times \mathcal{T}$ into $\mathbb{R}^2$.

[6] We implicitly assume that W is Borel measurable, something that will be guaranteed by the monotonicity assumption (W-i). Moreover, to ease notation we write T in place of T_Y.

[7] Attractivity of the fixed point implies many important consequences for the preferences. See Propositions 3 and 4.

(W-iv) $W(x,0) > 0$ for each $x > 0$,

(W-v) W is a contraction in y, i.e., there is some $\beta \in (0,1)$ such that

$$|W(x,y) - W(x,y')| \leq \beta |y - y'|, \qquad \forall x, y, y' \in \mathbb{R}_+. \tag{13}$$

By standard contraction arguments, any $W : \mathbb{R}_+^2 \to \mathbb{R}$ that satisfies the Lipschitzian condition (W-v) has a unique fixed points in y for each $x \geq 0$. Hence, (W-v) implies (W-ii).

In our main results the aggregators are required to satisfy either both (W-iii) and (W-iv) or (W-v). In the former case, our results rely on a fixed point for suitably concave functions based on Thompson [28] , while in the latter case we use a standard contraction argument à la Blackwell [4]. This motivates the following terminology.

Definition 1 *An aggregator that satisfies both properties (W-iii) and (W-iv) is called a Thompson aggregator, while an aggregator that satisfies (W-v) is called a Blackwell aggregator.*

The last property we consider is a generalization of subhomogeneity:

(W-vi) W is γ-subhomogeneous, i.e., there is some $\gamma \in [0,1]$ such that:

$$W(\alpha^\gamma x, \alpha y) \geq \alpha W(x,y),$$

for each $\alpha \in (0,1]$ and each $x, y \in \mathbb{R}_+$.

Standard subhomogeneity corresponds to $\gamma = 1$. For example, the "asymmetric" CES function $W(x,y) = (x + \beta y^\rho)^{\frac{1}{\rho}}$, with $\beta, \rho \in (0,1)$, is ρ-subhomogeneous.

In Appendix D we give some simple technical properties of aggregators. Here we illustrate with few examples the conditions on W that we have just introduced.

Example 3 The family of functions

$$W(x,y) = (x^\eta + \beta y^\sigma)^{\frac{1}{\rho}},$$

with $\eta, \sigma, \rho, \beta > 0$, includes many interesting cases. Conditions (W-i) and (W-iv) are always satisfied. Condition (W-ii) holds if either $\sigma < \rho$ or $\sigma = \rho$ and $\beta < 1$.

The concavity condition (W-iii) holds iff $\sigma \leq 1$ and $\sigma \leq \rho$. The contraction property (W-v) is satisfied iff $\sigma = \rho \geq 1$ and $\beta < 1$.[8]

Summing up:

Thompson	if $\sigma \leq 1$ and either $\sigma < \rho$ or $\sigma = \rho$ and $\beta < 1$
Blackwell	if $\beta < 1 \leq \sigma = \rho$

Hence, Thompson and Blackwell's conditions share only the very special case $\sigma = \rho = 1 > \beta$. They are otherwise different, and they nicely complement each other.

Finally, as to (W-vi), W is clearly γ-subhomogeneous, with $\gamma = \sigma/\eta$, as long as $\sigma \leq \eta$. ▲

[8] In fact,

$$\frac{\partial}{\partial y} W(x,y) = \beta \left(\frac{y^\rho}{x^\eta + \beta y^\rho} \right)^{\frac{1-\rho}{\rho}} = \beta^{\frac{1}{\rho}} \left(\frac{\beta y^\rho}{x^\eta + \beta y^\rho} \right)^{\frac{1-\rho}{\rho}} \leq \beta^{\frac{1}{\rho}} < 1,$$

provided $\beta < 1$.

Example 4 Some classic aggregators are special cases of Example 3. For instance:

(i) The CES function $W(x,y) = (x^\rho + \beta y^\rho)^{\frac{1}{\rho}}$ is the special case $\rho = \sigma = \eta$. Here:

Thompson	if $\rho \leq 1$ and $\beta < 1$
Blackwell	if $\beta < 1 < \rho$

In the CES special case there is no overlap between Thompson and Blackwell's conditions, which therefore fully complement each other. In particular, Thompson covers the case $\rho \leq 1$, while Blackwell covers the case $\rho > 1$.

(ii) The "asymmetric" CES function $W(x,y) = (x + \beta y^\rho)^{\frac{1}{\rho}}$ enjoys the same property of the CES, with the only difference that (W-vi) is true iff $\rho \in (0,1)$.

(iii) The function $W(x,y) = (x + \beta y)^{\frac{1}{\rho}}$ is the special case $\eta = \sigma = 1$. They are aggregator functions provided either $\rho > 1$ or $\rho = 1$ and $\beta \in (0,1)$. They always satisfy (W-iii), (W-iv), and (W-vi). The contraction condition (W-v) fails unless $\rho = 1$. In sum:

Thompson	if either $\rho > 1$ or $\rho = 1 > \beta$
Blackwell	if $\rho = 1 > \beta$

Thompson's conditions are here much more general than Blackwell's ones. ▲

Example 5 Koopmans, Diamond, and Williamson [17, p. 97] study the aggregator

$$W(x,y) = \frac{1}{\theta} \log\left(1 + \eta x^\delta + \beta y\right)$$

with $\theta, \beta, \delta, \eta > 0$. Conditions (W-i) and (W-ii) are satisfied, and so W is an aggregator. Moreover, W always satisfies (W-iii), (W-iv), and (W-vi), while it satisfies (W-v) iff $\beta < \theta$. Hence:

Thompson	always satisfied
Blackwell	if $\beta < \theta$

▲

We close with an example of aggregators that are neither Blackwell nor Thompson.

Example 6 Consider the following variation of Example 3:

$$W(x,y) = \begin{cases} (x^\eta + \beta y^\sigma)^{\frac{1}{\rho}} & \text{if } x, y \in \mathbb{R}_{++} \\ 0 & \text{else} \end{cases} \tag{14}$$

with $\eta, \sigma, \rho < 0$ and $\beta > 0$. Condition (W-i) always holds, while (W-ii) is true if $\sigma > \rho$ or $\sigma = \rho$ and $\beta < 1$. Condition (W-iii) is valid if $\rho \leq \sigma$, while (W-v) never holds.

Condition (W-vi) holds if $\rho \leq \sigma$ and $\eta \leq \sigma$. A serious drawback of the functions (14) is that condition (W-iv), which plays a key role in our uniqueness results, fails.

Similar properties hold for the Cobb-Douglas function $W(x,y) = x^\eta y^\sigma$, which can be regarded as the $\rho = 0$ special case of (14). ▲

3.2 Certainty Equivalent Operators

As to the operator $\mathcal{M}$, we say that $\mathcal{M} : L_+ \to L_+$ is a *certainty equivalent operator* if it is adapted (i.e., it maps adapted processes into adapted processes) and:

(M-i) $\mathcal{M}(k) = k$ for all $k \geq 0$,

(M-ii) $\mathcal{M}$ is monotone, i.e., $\mathcal{M}(X') \leq \mathcal{M}(X'')$ if $X' \leq X''$.

Besides the basic properties (M-i) and (M-ii), other properties of the operator $\mathcal{M}$ will play a key role. In particular, $\mathcal{M} : L_+ \to L_+$ is:

(M-iii) *constant subadditive* if $\mathcal{M}(X + k) \leq \mathcal{M}(X) + k$ for all $k \geq 0$ and all $X \in L_+$,

(M-iv) *subhomogeneous* if $\mathcal{M}(\alpha X) \geq \alpha \mathcal{M}(X)$ for all $\alpha \in [0, 1]$ and all $X \in L_+$,

(M-v) a *shift operator* if $\mathcal{M}(d) = S(d)$ for all $d \in D_+$, where $S : D \to D$ is the shift operator,[9]

(M-vi) *shift subadditive* if $\mathcal{M}(X + d) \leq \mathcal{M}(X) + S(d)$ for all $d \in D_+$ and all $X \in L_+$.

Since $S(k) = k$ for all $k \geq 0$, (M-v) implies (M-i) and (M-vi) implies (M-iii).

Example 7 Consider the operator $\mathcal{M} : L_+^\infty \to L_+^\infty$ given by:

$$\mathcal{M}_t(X) = \mathbb{E}_t(X_{t+1}), \qquad P\text{-a.e.},$$

for each $t \geq 1$. This operator clearly satisfies all properties (M-i)-(M-vi). ▲

Example 8 Given a strictly increasing function $\phi : [0, \infty) \to \mathbb{R}$, consider the operator $\mathcal{M} : L_+^\infty \to L_+^\infty$ given by:

$$\mathcal{M}_t(X) = \phi^{-1}(\mathbb{E}_t(\phi \circ X_{t+1})), \qquad P\text{-a.e.},$$

for each $t \geq 1$. When $\phi(t) = t$ we get back to the previous example. This operator satisfies (M-i), (M-ii) and (M-v) and so it is a certainty equivalent operator.

Using the results we prove in Appendix C we can say when the operator $\mathcal{M}$ also satisfies properties (M-iv) and (M-vi) (and so (M-iii)). In fact, suppose $\phi' > 0$. By Theorem 8 the operator $\mathcal{M}$ satisfies (M-vi) whenever ϕ is IARA, and $\mathcal{M}$ satisfies (M-iv) whenever ϕ is IRRA. ▲

Example 9 We can generalize the previous example by taking a countable set C of probability distributions $Q : \Sigma \to [0, 1]$, all absolutely continuous w.r.t. P, and by considering the operator $\mathcal{M} : L_+^\infty \to L_+^\infty$ given by:

$$\mathcal{M}_t(X) = \inf_{Q \in C} \phi^{-1}\left(\mathbb{E}_t^Q(\phi \circ X_{t+1})\right), \qquad P\text{-a.e.},$$

for each $t \geq 1$. When C is a singleton we get back to Example 8.[10] The properties of this "multiple priors" operator are similar to those of the operator of Example 8. In particular, it is a certainty equivalent operator because it satisfies (M-i), (M-ii) and (M-v), and, when $\phi' > 0$, it satisfies (M-vi) whenever ϕ is IARA, while it satisfies (M-iv) whenever ϕ is IRRA. ▲

[9] That is, $S(d)(t) = d(t+1)$ for all $t \geq 1$ and all deterministic processes $d \in D$.

[10] We restricted C to be countable to avoid measurability problems. When each Σ_t is finite these problems do not arise and C can have any cardinality.

4 The Bounded Case

In this section we study in depth the bounded case L^∞. In the next section we will extend the results to the unbounded case.

We begin with a simple existence result, which shows that fixed points exist under very general conditions.

Proposition 1 *Consider the operator T given by (12), where W is an aggregator function and $\mathcal{M}$ is a certainty equivalent operator. If $Y \in L_+^\infty$, then T has a fixed point in L_+^∞.*

Turn now to uniqueness, the main property which we are interested in. It is easy to check (see Lemma 3 of Appendix D) that there exist unique $y_*, y^* \in \mathbb{R}_+$ such that:

$$W\left([Y]_\infty, y_*\right) = y_* \qquad \text{and} \qquad W\left(\|Y\|_\infty, y^*\right) = y^*, \tag{15}$$

provided W is either Blackwell or Thompson.

Example 10 Consider the CES function $W(x, y) = (x^\rho + \beta y^\rho)^{\frac{1}{\rho}}$ with $\beta, \rho > 0$. Suppose $Y \in L_+^\infty$, with $[Y]_\infty > 0$. Simple algebra shows that

$$W(x, y) = y \Longleftrightarrow y = (1-\beta)^{-\frac{1}{\rho}} x, \qquad \forall x \in \mathbb{R}_+,$$

so that

$$y^* = (1-\beta)^{-\frac{1}{\rho}} \|Y\|_\infty \quad \text{and} \quad y_* = (1-\beta)^{-\frac{1}{\rho}} [Y]_\infty . \tag{16}$$

For the root aggregator $W(x, y) = \sqrt{x + \beta y}$ we have:

$$y^* = \beta + \sqrt{4\|Y\|_\infty + \beta^2} \quad \text{and} \quad y_* = \beta + \sqrt{4[Y]_\infty + \beta^2}.$$

▲

We can now state the uniqueness theorem for the bounded case.

Theorem 1 *Consider the operator T given by (12), where W is an aggregator function and $\mathcal{M}$ is a certainty equivalent operator. Given any $Y \in L_+^\infty$, then T has a unique fixed point $\widehat{X}$ in L_+^∞ provided at least one of the following conditions holds:*

(i) $\mathcal{M}$ is constant subadditive and W is Blackwell;

(ii) $\mathcal{M}$ is subhomogeneous, W is Thompson, and $[Y]_\infty > 0$.

Moreover, $\widehat{X}$ belongs to $[y_*, y^*]$ and is globally attracting on L_+^∞, that is,

$$\left\| T^n(X) - \widehat{X} \right\|_\infty \to 0, \qquad \forall X \in L_+^\infty. \tag{17}$$

Example 11 Given $Y \in L_+^\infty$, consider the CES operator

$$T_t(X) = \left(Y_t^\rho + \beta \left[\mathcal{M}_t(X)\right]^\rho\right)^{\frac{1}{\rho}}, \qquad P\text{-a.e.},$$

where $\beta, \rho \in (0, 1)$ and $\mathcal{M}$ is a subhomogeneous certainty equivalent operator. If $[Y]_\infty > 0$, then by Theorem 1 there exists a unique and globally attracting fixed point $\widehat{X}$ of T, with

$$\widehat{X} \in \left[(1-\beta)^{-\frac{1}{\rho}} [Y]_\infty, (1-\beta)^{-\frac{1}{\rho}} \|Y\|_\infty\right]$$

▲

Example 12 Given $Y \in L_+^\infty$, consider the root operator

$$T_t(X) = \sqrt{Y_t + \beta \mathcal{M}_t(X)}, \qquad P\text{-a.e.},$$

where $\beta \in (0,1)$ and $\mathcal{M}$ is any subhomogeneous certainty equivalent operator. If $[Y]_\infty > 0$, then by Theorem 1 there exists a unique and globally attracting fixed point $\widehat{X}$ of T, with

$$\widehat{X} \in \frac{1}{2}\left[\beta + \sqrt{4[Y]_\infty + \beta^2}, \beta + \sqrt{4\|Y\|_\infty + \beta^2}\right]$$

▲

5 The Unbounded Case

5.1 Existence

This section studies the unbounded case. Also here we begin with a general existence result. Recall that $\mathcal{M}$ is a shift operator if it satisfies (M-v); moreover, the limit in (18) exists by Lemma 4 of Appendix D.

Proposition 2 *Consider the operator T given by (12), where W is a γ-subhomogeneous aggregator function and $\mathcal{M}$ is a shift certainty equivalent operator. If $Y \in L_+^w$ for some admissible weight function w, then T has a fixed point in L_+^w provided*

$$a_w \left(\lim_{y\to+\infty} \frac{W(\|Y\|_w, y)}{y}\right)^\gamma < 1. \tag{18}$$

Remark. When the aggregator W satisfies (W-v) for some $\beta \in (0,1)$, then it is easy to see that (18) holds whenever $a_w \beta^\gamma < 1$.

This existence result is based on condition (18), which will play a key role in this section. For example, for the positively homogeneous CES aggregator $W(x,y) = (x^\rho + \beta y^\rho)^{\frac{1}{\rho}}$ with $\rho \neq 0$, we have

$$\lim_{y\to+\infty} \frac{W(\|Y\|_w, y)}{y} = \lim_{y\to+\infty} \frac{(\|Y\|_w^\rho + \beta y^\rho)^{\frac{1}{\rho}}}{y} = \beta^{\frac{1}{\rho}}. \tag{19}$$

and so in this case condition (18) is satisfied whenever $a_w \beta^{\frac{1}{\rho}} < 1$. This existence condition for the CES aggregator can be found in Epstein and Zin [10, Theorem 3.1], which is therefore a special case of Proposition 2.

Next we illustrate Proposition 2 with other examples.

Example 13 Here we generalize the CES example we just discussed. Consider the family of functions $W(x,y) = (x^\eta + \beta y^\sigma)^{\frac{1}{\rho}}$ with $\eta, \sigma, \rho, \beta > 0$, discussed in Example 3. They are aggregators if either $\sigma < \rho$ or $\sigma = \rho$ and $\beta < 1$, and they satisfy (W-vi) if $\sigma \leq \eta$. Observe that, if $\sigma < \rho$, $\lim_{y\to+\infty} W(x,y)/y = 0$ for each $x \in \mathbb{R}_+$, while, if $\sigma = \rho$,

$$\lim_{y\to+\infty} \frac{W(x,y)}{y} = \beta^{\frac{1}{\rho}}, \qquad \forall x \in \mathbb{R}_+.$$

As a result, in the case $\sigma < \rho$ and $\sigma \leq \eta$, Proposition 2 implies that the operator T has a fixed point for a given $Y \in L_+$ provided there is an admissible w such that $\|Y\|_w < \infty$ and provided $\mathcal{M}$ is a shift operator.

In the case $\sigma = \rho \leq \eta$ and $\beta < 1$, the same is true under the additional condition

$$a_w^{\frac{\eta}{\sigma}} \beta^{\frac{1}{\rho}} < 1, \tag{20}$$

which generalizes the above condition $a_w \beta^{\frac{1}{\rho}} < 1$. ▲

Example 14 Consider the aggregator

$$W(x, y) = \frac{1}{\vartheta} \log \left(1 + \eta x^{\delta} + \beta y\right)$$

with $\vartheta, \beta, \delta, \eta > 0$, studied in Example 5. We have $\lim_{y \to +\infty} W(x, y)/y = 0$ for each $x \in \mathbb{R}_+$. By Proposition 2, the operator T has a fixed point for a given $Y \in L_+$ provided there is an admissible w such that $\|Y\|_w < \infty$ and provided $\mathcal{M}$ is a shift operator. ▲

5.2 Uniqueness

We now study uniqueness, our main object of interest. In order to do so, we first provide a generalization of the quantity y^* defined in (15).

Suppose $Y \in L_+^w$ for some admissible weight function w. It is easy to check (see Corollary 5 of Appendix D) that there is a unique y^w be such that

$$W\left(\|Y\|_w, y^w\right) = a_w^{-\frac{1}{\gamma}} y^w \tag{21}$$

provided W is either Thompson or Blackwell with $\beta < a_w^{-\frac{1}{\gamma}}$.

Moreover, $y^w = y^*$ when $a_w = 1$ (and when y^* exists unique). Hence, y^w generalizes y^* to the case $a_w > 1$.

Example 15 Consider the CES function $W(x, y) = (x^{\rho} + \beta y^{\rho})^{\frac{1}{\rho}}$ with $\beta, \rho \in (0, 1)$. Suppose $Y \in L_+^w$ for some admissible weight function w. Simple algebra shows that

$$\frac{W(x, y)}{y} = \frac{1}{a_w} \Longleftrightarrow y = a_w x \left(1 - \beta a_w^{\rho}\right)^{-\frac{1}{\rho}}, \qquad \forall x \in \mathbb{R}_+$$

so that

$$y^w = a_w \left(1 - \beta a_w^{\rho}\right)^{-\frac{1}{\rho}} \|Y\|_w . \tag{22}$$

When $a_w = 1$, we get $y^w = y^*$, where y^* is given in (16). ▲

Theorem 2 *Consider the operator T given by (12), where W is a γ-subhomogeneous aggregator function and $\mathcal{M}$ is a shift certainty equivalent operator. If $Y \in L_+^w$ for some admissible weight functions w, then T has a unique fixed point $\widehat{X}$ in L_+^w provided Y satisfies (18) and at least one of the following conditions holds:*

(i) $\mathcal{M}$ is shift subadditive and W is Blackwell for some $\beta \in (0, 1)$ such that $a_w \beta < 1$.

(ii) $\mathcal{M}$ is subhomogeneous, W is Thompson, and

$$\left[W(Y, 0)\, w^{-\frac{1}{\gamma}}\right]_{\infty} > 0. \tag{23}$$

Moreover, $\widehat{X}$ belongs to $\left[y_*, y^w w^{\frac{1}{\gamma}}\right]$ and is globally attracting, that is,

$$\left\|T^n(X) - \widehat{X}\right\|_w \to 0, \qquad \forall X \in L_+^w. \tag{24}$$

Remark. As we remarked after Proposition 2, when the aggregator W satisfies (W-v) for some $\beta \in (0,1)$, then (18) holds if $a_w \beta^\gamma < 1$. Since $a_w \beta \leq a_w \beta^\gamma$, we conclude that under (W-v) the condition $a_w \beta^\gamma < 1$ implies both (18) and $a_w \beta < 1$.

We now illustrate Theorem 2 with a couple of examples.

Example 16 Given $Y \in L_+^w$ for some admissible w, consider the CES aggregator

$$T_t(X) = \left(Y_t^\rho + \beta \mathcal{M}_t(X)^\rho\right)^{\frac{1}{\rho}},$$

where $\beta \in (0,1)$, $\rho \in (0,1]$, and $\mathcal{M}$ is any subhomogeneous shift certainty equivalent operator.

This aggregator is positively homogeneous and Thompson. By (19), condition (18) is satisfied if $a_w \beta^{\frac{1}{\rho}} < 1$. Hence, if $\left[w^{-1}Y\right]_\infty > 0$ and $a_w \beta^{\frac{1}{\rho}} < 1$, then by Theorem 2 there exists a unique and globally attracting fixed point $\widehat{X} \in L_+^w$, and, by (16) and (22),

$$\widehat{X} \in \left[(1-\beta)^{-\frac{1}{\rho}} [Y]_\infty, a_w \left(1 - \beta a_w^\rho\right)^{-\frac{1}{\rho}} \|Y\|_w\right]. \tag{25}$$

▲

Example 17 Given $Y \in L_+^w$ for some admissible w, consider the root aggregator

$$T_t(X) = \sqrt{Y_t + \beta \mathcal{M}_t(X)},$$

where $\beta \in (0,1)$ and $\mathcal{M}$ is any subhomogeneous shift certainty equivalent operator.

This aggregator is subhomogeneous and Thompson. If $\left[w\sqrt{Y}\right]_\infty > 0$, then by Theorem 2 there exists a unique and globally attracting fixed point $\widehat{X}$ of T, with

$$\widehat{X} \in \frac{1}{2}\left[\beta + \sqrt{4[Y]_\infty + \beta^2}, a_w^2 \beta + \sqrt{4a_w^2 \|Y\|_w + a_w^4 \beta^2}\right] \tag{26}$$

▲

5.3 An Additive Process

In the previous examples we considered Theorem 2 under different specifications of the aggregator. To be more concrete, in this last subsection we will show what form Theorem 2 takes when Y is the additive process of Example 2.

We need some notation. Given the weight function $w_n(t) = t \vee n$ for some n, set $L_+^w = \bigcup_n L_+^{w_n}$. Moreover, as in (15), let $y_d, y^u \in \mathbb{R}_+$ be such that $W(u, y_d) = y_d$ and $W(u, y_u) = y_u$.

Corollary 1 *Consider the operator T given by (12), where W is a γ-subhomogeneous aggregator function and $\mathcal{M}$ is a shift certainty equivalent operator. If Y is the additive process of Example 2, then T has a unique fixed point $\widehat{X}$ in L_+^w provided*

$$\lim_{y \to +\infty} \frac{W(u,y)}{y} < 1 \tag{27}$$

and at least one of the following conditions holds:

(i) $\mathcal{M}$ is shift subadditive and W is Blackwell for some $\beta \in (0,1)$ such that $\beta < 1$.

(ii) $\mathcal{M}$ *is subhomogeneous,* W *is Thompson, and*

$$\inf_t \frac{W(dt,0)}{t^{\frac{1}{\gamma}}} > 0. \tag{28}$$

Moreover, $\widehat{X}$ *belongs to* $[y_d, y^u]$ *and is globally attracting over* L^w_+ *(i.e., (24) holds).*

Thanks to the specific properties of the additive process, Corollary 1 is sharper than Theorem 2. In particular, conditions (18) and (23) take the sharper forms (27) and (28). Interestingly, $\widehat{X}$ belongs to $[y_d, y^u]$ by Corollary 1, and so we get here the same range $[y_d, y^u]$ we would had if Y were a bounded processes (and so Theorem 1 could have been invoked).

Turn back to Examples 16 and 17, and assume now that Y is the additive process. By Corollary 1, in the CES Example 16 there exists a unique and globally attracting fixed point $\widehat{X}$ provided $\beta^{\frac{1}{\rho}} < 1$, and (14) becomes

$$\widehat{X} \in \left[(1-\beta)^{-\frac{1}{\rho}} d, (1-\beta)^{-\frac{1}{\rho}} u\right].$$

As to the root Example 17, by Corollary 1 there always exists a unique and globally attracting fixed point $\widehat{X}$, and (26) becomes

$$\widehat{X} \in \frac{1}{2}\left[\beta + \sqrt{4d+\beta^2}, \beta + \sqrt{4u+\beta^2}\right]$$

6 A Recursive Equation

We now apply the results of the previous section to recursive equations of the form (2), discussed in the Introduction.

Given a strictly monotone function $\phi : [0,\infty) \to \mathbb{R}$, define $\mathcal{M} : L^w_+ \to L^w_+$ by:

$$\mathcal{M}_t(X) = \phi^{-1}\left(\mathbb{E}_t(\phi \circ X_{t+1})\right), \qquad P\text{-a.e.}, \tag{29}$$

for all $t \geq 1$. This is the certainty equivalent operator we introduced in Example 8.

Under this specification of $\mathcal{M}$, the nonlinear operator equation (10) takes the recursive form:

$$X_t = W\left(Y_t, \phi^{-1}\left(\mathbb{E}_t(\phi \circ X_{t+1})\right)\right), \qquad P\text{-a.e.}, \tag{30}$$

for all $t \geq 1$, and the associated operator $T : L^w_+ \to L^w_+$ is given, for all $t \geq 1$, by:

$$T_t(X) = W\left(Y_t, \phi^{-1}\left(\mathbb{E}_t(\phi \circ X_{t+1})\right)\right), \qquad P\text{-a.e.} \tag{31}$$

The nonlinear equation (30) is the quasi-arithmetic specification (4) of the classic recursive equation (4) discussed in the Introduction, which is widely used in macroeconomics and finance.

By Proposition 1, a solution for (30) exists under very general conditions. As to uniqueness, the issue here is to derive conditions directly on the function ϕ that make it possible to apply Theorem 1. In this way we can tell whether a unique solution for equation (30) exists by a direct check of the properties of the function ϕ.

The next result provides such condition for the bounded case. It is based on Theorem 1 and on results on quasi-arithmetic means derived in Appendix C (already mentioned in Example 8). A piece of notation: $\mathcal{C}^2_m(\mathbb{R}_+)$ denotes the set of all functions $\phi : \mathbb{R}_+ \to \mathbb{R}$ that are twice differentiable on $(0,\infty)$, with either $\phi'(t) < 0$ for all $t > 0$ or $\phi'(t) > 0$ for all $t > 0$.

Theorem 3 *Consider the operator* T *given by (31), where* $Y \in L^\infty_+$, W *is an aggregator, and* $\phi \in \mathcal{C}^2_m(\mathbb{R}_+)$. *Then,* T *has a unique fixed point* $\widehat{X}$ *provided at least one of the following conditions holds:*

(i) ϕ is IARA and W is Blackwell;

(ii) ϕ is IRRA, W is Thompson, and $[Y]_\infty > 0$.

Moreover, $\widehat{X}$ belongs to $[y_*, y^*]$ and is globally attracting over L_+^∞ (i.e., (17) holds).

For the unbounded case we have the following result, based on Theorem 2 and on the mentioned results on quasi-arithmetic means.

Theorem 4 *Consider the operator T given by (31), where W is a γ-subhomogeneous aggregator function and $\phi \in \mathcal{C}_m^2(\mathbb{R}_+)$. Suppose that $Y \in L_+^w$ for some admissible weight function. Then, T has a unique fixed point $\widehat{X}$ in L_+^w provided at least one of the following conditions holds:*

(i) ϕ is IARA and W is Blackwell for some $\beta \in (0,1)$ such that $a_w^{\frac{1}{\gamma}} \beta < 1$.

(ii) ϕ is IRRA, W is Thompson, and Y satisfies (18) and (23).

Moreover, $\widehat{X}$ belongs to $\left[y_*, y^w w^{\frac{1}{\gamma}}\right]$ and is globally attracting over L_+^w (i.e., (24) holds).

Eq. (7) in the Introduction, that is,

$$X_t = \left(Y_t^\rho + \beta \left(\mathbb{E}_t \left(X_{t+1}^{1-\gamma}\right)\right)^{\frac{\rho}{1-\gamma}}\right)^{\frac{1}{\rho}}, \tag{32}$$

features a CES aggregator and a CRRA certainty equivalent, and is a special case of the recursive equation (30) widely used in applications (see, e.g., [7] and [8]).

By condition (ii) of Theorem 4, equation (32) has a unique and globally attracting solution if

$$\rho \in (0,1], \, a_w \beta^{\frac{1}{\rho}} < 1, \text{ and } [wY]_\infty > 0. \tag{33}$$

In fact, $\rho \in (0,1]$ ensures that W is Thompson, while $a_w \beta^{\frac{1}{\rho}} < 1$ and $[wY]_\infty > 0$ are the forms that conditions (18) and (23) take in this case, respectively. Moreover, by (16) and (22), this unique solution belongs to

$$\left[(1-\beta)^{-\frac{1}{\rho}} [Y]_\infty, a_w (1-\beta a_w^\rho)^{-\frac{1}{\rho}} \|Y\|_w\right]. \tag{34}$$

For instance, when Y is the additive process of Example 2, by Corollary 1 we can refine the conditions in (33) by saying that equation (32) has a unique and globally attracting solution if

$$\rho \in (0,1] \text{ and } \beta^{\frac{1}{\rho}} < 1.$$

The next corollary generalizes what we just observed about equation (32) by giving the version of Theorem 4 for the CES aggregator.

Corollary 2 *Given $Y \in L_+^w$ for some admissible w and given $\phi \in \mathcal{C}_m^2(\mathbb{R}_+)$, the recursive equation*

$$X_{t+1} = \left(Y_t^\rho + \beta \left(\phi^{-1}\left(\mathbb{E}_t(\phi \circ X_{t+1})\right)\right)^\rho\right)^{\frac{1}{\rho}},$$

where $\beta \in (0,1)$, $\rho > 0$ and $\gamma \neq 1$, has a unique and globally attracting solution $\widehat{X}$ provided at least one of the following conditions holds:

(i) ϕ is IARA and $a_w \beta < 1$.

(ii) ϕ is IRRA, $\rho \in (0,1]$, $a_w \beta^{\frac{1}{\rho}} < 1$, and $[wY]_\infty > 0$.

Moreover, $\widehat{X}$ belongs to (34).

We close with an example that further illustrates Theorem 4.

Example 18 Given $Y \in L_{+}^{w}$ for some admissible w, consider the operator $T(X) = W(Y, \mathcal{M}(X))$, where $W(x,y) = \sqrt{x+\beta y}$ and $\mathcal{M}$ is the HARA certainty equivalent operator given by the operator of Example 8 with $\phi(t) = \alpha^{-1}(t+k)^{\alpha}$, where $\alpha, \beta \in (0,1)$ and $k \geq 0$.

If $[Y]_{\infty} > 0$ and $a_w \beta^{\frac{1}{\rho}} < 1$, by Theorem 4 the operator T has a unique and globally attracting fixed point $\widehat{X}$, with

$$\widehat{X} \in \frac{1}{2}\left[\beta + \sqrt{4d+\beta^2}, a_w^2 \beta + \sqrt{4a_w^2 \|Y\|_w + a_w^4 \beta^2}\right].$$

▲

Two final remarks: (i) For the operator given by (31) the most relevant case is when ϕ is concave. Though in this case IARA implies IRRA (see Lemma 1), condition (ii) requires $[Y]_{\infty} > 0$ and it delivers a weaker (though very useful) attracting property. For this reason, condition (i) is interesting even when ϕ is concave. (ii) Theorems 3 and 4 are easily seen to hold also for the certainty equivalent operator of Example 9.

7 Some Properties of the Solutions

7.1 Correspondence Functions

In this section we briefly study the dependence of the solution of our equation (10) on the given process Y, an important issue in view of both optimization and comparative statics exercises.

In order to do this, following Koopmans [16, p. 297] we introduce correspondence functions. Specifically, given an aggregator W and a certainty equivalent operator $\mathcal{M}$, consider the nonlinear equation (10), that is,

$$X = W(Y, \mathcal{M}(X)), \tag{35}$$

and the operators $\{T_Y\}_{Y \in L_+}$ given by $T_Y(X) = W(Y, \mathcal{M}(X))$ for all $X \in L_+$. The *solution domain* of equation (35), denoted $sol(T)$, is the collection of all $Y \in L_+$ such that T_Y has a unique and globally attracting fixed point $\widehat{X}_Y$. The set $sol(T)$ or, at least parts of it, can be determined for example via Theorems 1 and 2.

The function $\varphi : sol(T) \to L_+$ such that $\varphi(Y) = \widehat{X}_Y$ for all $Y \in sol(T)$ is the *correspondence function* induced by T.

Example 19 In the preference equation $V_t(c) = W(c_t, \mathcal{M}_t(V_{t+1}(c)))$ we have $Y_t = c_t$ and $\varphi_t(Y) = V_t(c)$. That is, $\varphi_t(Y)$ is the evaluation $V_t(c)$ at time t of the consumption stream c.[11]

Suppose W is a CES aggregator, so that $V_t(c) = (c_t^{\rho} + \beta[\mathcal{M}_t(V_{t+1}(c))]^{\rho})^{1/\rho}$. If $\rho \in (0,1]$, and $\mathcal{M}_t$ is a shift and subhomogeneous certainty equivalent operator, then by Theorem 2 and Example 16 we have:

$$\bigcup_{\left\{w : a_w \beta^{\frac{1}{\rho}} < 1\right\}} \left\{Y \in L_+ : \|Y\|_w < \infty \text{and } \left[w^{-1}Y\right]_{\infty} > 0\right\} \subseteq sol(T) \tag{36}$$

For example, if there exist $m, M > 0$ and $a \geq 1$ such that $ma^t \leq Y_t \leq Ma^t$ for all $t \geq 1$, then $Y \in sol(T)$. ▲

[11] With a slight abuse of notation, in this section we write $\mathcal{M}_t(V_{t+1}(c))$ to indicate that $\mathcal{M}_t$ depends only on $V_{t+1}(c)$ (see (11)).

The dependence of $\widehat{X}_Y$ on Y can be studied through the properties of the correspondence function. The next result is an instance of this.

Proposition 3 *The correspondence function is monotone, that is,*

$$Y' \leq Y'' \Longrightarrow \varphi(Y') \leq \varphi(Y''), \qquad \forall Y', Y'' \in sol(T).$$

If, in addition, W and $\mathcal{M}$ are concave on their domains, then φ is concave.

We say that an operator $\mathcal{M} : L_+ \to L_+$ is *history independent* if, given any $X', X'' \in L_+$,

$$X'_\tau = X''_\tau \quad \forall \tau \geq t+1 \Longrightarrow \mathcal{M}_t(X') = \mathcal{M}_t(X'')$$

for each $t \geq 1$.

In words, $\mathcal{M}$ is history independent when it is forward looking: $\mathcal{M}_t(X)$ depends only on the values that the process X takes from $t+1$ on. History independence is an important property in applications and, for instance, the operators in Examples 7, 8, and 9 all satisfy it.

The next simple result, whose simple proof is omitted, shows that under history independence only the future values of Y matter.

Proposition 4 *If $\mathcal{M}$ is history independent, then, given any $Y', Y'' \in sol(T)$,*

$$Y'_\tau = Y''_\tau \quad \forall \tau \geq t \Longrightarrow \varphi_t(Y') = \varphi_t(Y'')$$

for each $t \geq 1$.

In the equation $V_t(c) = W(c_t, \mathcal{M}_t(V_{t+1}(c)))$ the operator $\mathcal{M}_t$ is history independent as it only depends on V_{t+1}. Proposition 4 then implies that $V_t(c) = V_t({}_tc)$, where ${}_tc = (c_t, c_{t+1}, ...)$.

We now study what happens to the solutions of the equation (10) in a stationary environment. For this reason, unlike the rest of the paper, here we consider a doubly infinite setting, with a filtration $\{\Sigma_t\}_{t\in\mathbb{Z}}$. Let $\sigma : \Omega \to \Omega$ be an automorphism (the time shift) such that $\sigma(\Sigma_t) = \Sigma_{t-1}$ for all $t \in \mathbb{Z}$. The probability P is *stationary* if, for any $A \in \Sigma$, we have $P(A) = P\left(\sigma^{-1}(A)\right)$.

This implies $P(A) = P(\sigma^t(A))$ for all $t \in \mathbb{Z}$. Given any Σ-measurable function $\xi : \Omega \to \mathbb{R}$, define the shift $\sigma\xi$ of ξ as $\sigma\xi = \xi \circ \sigma^{-1}$.[12] Analogously, set $\sigma^t\xi = \xi \circ \sigma^{-t}$ for all $t \in \mathbb{Z}$.

From $P(A) = P\left(\sigma^{-1}(A)\right)$ it follows that all the random variables $\sigma^t\xi$ have the same distribution. Moreover, if ξ is Σ_t-measurable, then $\sigma\xi$ is Σ_{t-1}-measurable and $\sigma^t\xi$ is Σ_0-measurable (this is true even for t negative).

We say that an operator $\mathcal{M} : L_+ \to L_+$ is *commutative* if $\sigma \circ \mathcal{M} = \mathcal{M} \circ \sigma$. For instance, the operators of Examples 7, 8, and 9 are all commutative. In fact, consider for instance $\mathcal{M}_t(X) = \phi^{-1}(\mathbb{E}_t(\phi \circ X_{t+1}))$ of Example 8. By [3, p. 109], $\phi \circ (\sigma X) = \sigma(\phi \circ X)$, and so

$$\begin{aligned}
\mathcal{M}_t(\sigma X) &= \phi^{-1} \circ \mathbb{E}_t(\phi \circ (\sigma X_{t+2})) = \phi^{-1} \circ \mathbb{E}_t[\sigma(\phi \circ X_{t+2})] \\
&= \phi^{-1} \circ \sigma \mathbb{E}_{t+1}[\phi \circ X_{t+2}] = \sigma\left(\phi^{-1} \circ \mathbb{E}_{t+1}[\phi \circ X_{t+2}]\right) = \sigma \mathcal{M}_t(X).
\end{aligned}$$

Next we show that under stationarity the value of φ_t is uniquely determined by the initial value φ_0 once we take suitable shifts of Y.

[12] Observe that, if $\xi = 1_A$, then $\sigma\xi = 1_A \circ \sigma^{-1} = 1_{\sigma A}$.

Proposition 5 *If $\mathcal{M}$ is commutative, then*

$$\varphi_t(Y) = \varphi_0\left(\sigma^t Y\right) \circ \sigma^t, \tag{37}$$

for each $t \in \mathbb{N}$ and each $Y \in sol(T)$.

For the equation $V_t(c) = W(c_t, \mathcal{M}_t(V_{t+1}(c)))$, Eq. (37) becomes

$$V_t(c)(\omega) = V_0\left(\sigma^t c\right)\left(\sigma^t \omega\right) = V_0({}_t c)({}_t\omega),$$

and the value of $V_t(c)$ is thus determined by the initial value V_0, computed at t shifted values of both c and ω.

7.2 Optimization

Suppose each Σ_t is finite and $P(A) > 0$ for all $A \in \Sigma_t$. Given a finite sequence $\left\{Y_1^*, ..., Y_{t-1}^*\right\}$, let $\Gamma_t(Y_1, ..., Y_{t-1}) : \Omega \to 2^{{}_t sol(T)}$ be a suitably measurable multivalued function from Ω to ${}_t sol(T)$.[13]

Consider the following general class of dynamic optimization problems:

$$\sup_{{}_tY(\omega) \in \Gamma_t(Y_1,...,Y_{t-1})(\omega)} \varphi_t({}_tY\,; Y_1, ..., Y_{t-1})(\omega) \qquad \forall t \geq 1 \tag{DP1}$$

At each node (ω, t), the correspondence function $\varphi_t({}_tY; Y_1, ..., Y_{t-1})(\omega)$ is maximized over all processes ${}_tY$ that satisfy the constraint ${}_tY(\omega) \in \Gamma_t(Y_1, ..., Y_{t-1})(\omega)$.

At each node an optimal continuation process ${}_tY \in_t sol(T)$ is selected, provided it exists. We call ${}_tY$ a *solution plan* and its first component Y_t a *solution choice*. In fact, out of the whole plan ${}_tY$, the first component Y_t is, in general, the choice actually carried out before getting to the next choice node.

An adapted process $Y^* = \{Y_t^*\}_t$ is a *(solution) choice process* if for each t there is ${}_{t+1}\widetilde{Y} \in \Gamma_t\left(Y_1^*, ..., Y_{t-1}^*\right)$ such that:

$$\varphi_t\left({}_{t+1}\widetilde{Y}\,, Y_t^*; Y_1^*, ..., Y_{t-1}^*\right)(\omega) \geq \varphi_t\left({}_tY\,; Y_1^*, ..., Y_{t-1}^*\right)(\omega), \qquad \forall_t Y \in \Gamma_t\left(Y_1^*, ..., Y_{t-1}^*\right).$$

In other words, given the earlier choices $\left\{Y_1^*, ..., Y_{t-1}^*\right\}$, at each choice node there is a solution plan $\left({}_{t+1}\widetilde{Y}\,, Y_t^*\right)$ whose first component is Y_t^*. The plan $\left({}_{t+1}\widetilde{Y}\,, Y_t^*\right)$ "justifies" the choice of Y_t^*.

A choice process $Y^* = \{Y_t^*\}_t$ is *consistent* if

$$\varphi_t\left({}_tY^*\,; Y_1^*, ..., Y_{t-1}^*\right)(\omega) \geq \varphi_t\left({}_tY\,; Y_1^*, ..., Y_{t-1}^*\right)(\omega),$$

for each t and each ${}_tY \in \Gamma_t\left(Y_1^*, ..., Y_{t-1}^*\right)$. That is, a choice process Y^* is consistent if and only if Y^* is a solution plan of the time 1 problem such that its continuations ${}_tY^*$ keep being solution plans of all subsequent time t problems. Therefore, in this case the choice behavior of the decision maker is fully consistent with his intertemporal plans.

Example 20 For the equation $V_t(c) = W(c_t, \mathcal{M}_t(V_{t+1}(c)))$ the optimization problem DP1 becomes:

$$\sup_{{}_tc(\omega) \in B_t(c_1,...,c_{t-1})(\omega)} V_t({}_tc\,; c_1, ..., c_{t-1})(\omega) \qquad \forall t \geq 1, \tag{DC1}$$

where $B_t(c_1, ..., c_{t-1}) : \Omega \to 2^{sol(T)}$ is an intertemporal budget constraint that depends on the earlier consumption $(c_1, ..., c_{t-1})$. The value function $V_t({}_tc\,; c_1, ..., c_{t-1})$ can also depend on the earlier consumption (because, for example, of consumption habits developed in the past).

[13] Notation: given $Y \in L_+$, set ${}_tY = \{Y_t, Y_{t+1}, ...\}$. Moreover, ${}_tsol(T) = \{{}_tY : Y \in sol(T)\}$.

At each node the consumer selects a consumption plan ${}_t c = \{c_\tau\}_{\tau \geq t}$. In particular, a choice consumption process $c^* = \{c_t^*\}_t$ is consistent if

$$V_t\left({}_t c^*\, ; c_1^*, ..., c_{t-1}^*\right)(\omega) \geq V_t\left({}_t c\, ; c_1^*, ..., c_{t-1}^*\right)(\omega),$$

for each t and all ${}_t c \in \Gamma_t\left(c_1^*, ..., c_{t-1}^*\right)$. ▲

Problem DP1 becomes significantly simpler when $\mathcal{M}$ is history independent. In fact, by Proposition 4 in this case we have $\varphi_t\left({}_t Y; Y_1, ..., Y_{t-1}\right) = \varphi_t\left({}_t Y\right)$ and problem DP1 takes the simpler form

$$\sup_{{}_t Y(\omega) \in \Gamma_t(Y_1,...,Y_{t-1})(\omega)} \varphi_t\left({}_t Y\right)(\omega) \qquad \forall t \geq 1 \tag{DP2}$$

For example, the equation $V_t(c) = W\left(c_t, \mathcal{M}_t\left(V_{t+1}(c)\right)\right)$ history independence holds, and so problem DC1 can be written as:

$$\sup_{{}_t c(\omega) \in B_t(c_1,...,c_{t-1})(\omega)} V_t\left({}_t c\right)(\omega) \qquad \forall t \geq 1 \tag{DC2}$$

For the simplified problem DP2 we now give simple conditions under which a consistent choice process exists. Say that $\Gamma = \{\Gamma_t\left(Y_1, ..., Y_{t-1}\right)\}_t$ is *recursive* if, for each $t \geq 1$,

$${}_t Y \in \Gamma_t\left(Y_1, ..., Y_{t-1}\right) \Longrightarrow \left(Y_{t-1}, {}_t Y\right) \in \Gamma_{t-1}\left(Y_1, ..., Y_{t-2}\right), \qquad \forall Y \in sol\,(T).$$

Moreover, say that φ is *dynamically consistent* if, for each $t \geq 1$,[14]

$$Y_t = Y_t' \text{ and } \varphi_{t+1}(Y) \geq (>)\, \varphi_{t+1}(Y') \Longrightarrow \varphi_t(Y) \geq (>)\, \varphi_t(Y'), \qquad \forall Y, Y' \in sol\,(T).$$

Definition 2 *Problem DP2 is* time consistent *if* Γ *is recursive and* φ *is dynamically consistent.*

Time consistent problems admit consistent choice processes. In other words, they satisfy Bellman's Principle of Optimality.

Proposition 6 *Given a time consistent problem DP2, any solution of the time* 1 *problem is a consistent choice process.*

Example 21 If $\mathcal{M}$ satisfies (11) and is strictly monotone,[15] then the correspondence function is dynamically consistent. For instance, the process $V_t(c)$ such that $V_t(c) = W\left(c_t, \mathcal{M}_t\left(V_{t+1}(c)\right)\right)$ is dynamically consistent provided M is strictly monotone. ▲

The Principle of Optimality we just established allows to solve time consistent dynamic problems using dynamic programming techniques. This is a topic that is beyond the scope of the present paper. Sometimes, however, these problems can be, at least partly, solved without using dynamic programming techniques, but more classic analytic arguments. We close this section by studying a classic example where this can be done, first studied by Epstein and Zin [10] and [11] from a dynamic programming standpoint.

Consider a consumption-saving problem of an infinitely lived representative agent with recursive utility specified by

$$V_t(c) = \left(c_t^\rho + \beta\left[\mathcal{M}_t\left(V_{t+1}(c)\right)\right]^\rho\right)^{1/\rho} \tag{38}$$

where $\rho \in (0, 1]$ and $\mathcal{M}_t$ is a shift and history independent certainty equivalent operator. There is no exogenous income, and the dynamics of wealth is given by

$$A_{t+1} = \left(1 + R_{t+1}\right)\left(A_t - c_t\right), \tag{39}$$

[14] Given any two Σ-measurable functions $X, X' : \Omega \to \mathbb{R}$, we write $X > X'$ if $X \geq X'$ and there exists $A \in \Sigma$ with $P(A) > 0$ such that $X(\omega) > X'(\omega)$ for all $\omega \in A$.

[15] That is, given any $X, X' \in L_+$, $X_t > X_t'$ implies $\mathcal{M}_t\left(X_{t+1}\right) > \mathcal{M}_t\left(X_{t+1}'\right)$ for each $t \geq 1$.

where $R = \{R_t\}_t \in L_+$ is the process of the rates of return of money, and $A_1 \geq 0$ is given. The intertemporal budget constraint $B_t(c_1, ..., c_{t-1}) \equiv B(A_t, c_1, ..., c_{t-1}, {}_tR)$ is given by:

$$\left\{ {}_tc = (c_t, ...) : \exists \{A_\tau\}_{\tau > t} \in L_+ \text{ s.t. } A_{\tau+1} = (1 + R_{\tau+1})(A_\tau - c_\tau) \text{ and } 0 \leq c_\tau \leq A_\tau \ \forall \tau \geq t \right\}$$

and the consumer problem DC2 becomes

$$\sup_{{}_tc \in B_t(c_1, ..., c_{t-1})} V_t({}_tc) \tag{40}$$

Since this specification of V_t is dynamically consistent, it is actually enough to consider the problem

$$\sup_{c \in B_1} V_1(c) \tag{41}$$

where $B_1 \equiv B(A_1, R)$ is given by

$$\left\{ c : \exists \{A_t\}_{t > 1} \in L_+ \text{ s.t. } A_{t+1} = (1 + R_{t+1})(A_t - c_t) \text{ and } 0 \leq c_t \leq A_t \ \forall t \geq 1 \right\}.$$

In fact, given a solution c^* of problem (41), by dynamic consistency its continuation ${}_tc^*$ is a solution of (40)

Suppose $a \equiv \sup_{t \geq 1}(1 + R_t) < \infty$. By (39), $c_t \leq A_1 a^t$ for all $t \geq 1$, and so, if $a\beta^{\frac{1}{\rho}} < 1$, the part of *sol* (T) determined by Theorem 2 and relevant for problem (41) is given by

$$\bigcup_{\{w : a_w \leq a\}} \left\{ Y \in L_+ : \|Y\|_w < \infty \text{ and } \left[w^{-1}Y\right]_\infty > 0 \right\}.$$

Theorem 5 *Suppose $\mathcal{M}_t$ is measurably homogeneous, that is,*

$$\mathcal{M}_t[\psi_t V_{t+1}] = \psi_t \mathcal{M}_t[V_{t+1}], \qquad \forall t \geq 1, \forall \psi \in L_+. \tag{42}$$

Then, an adapted process $c \in$ sol (T), with $0 < c_t < A_t$, is a solution of problem (41) only if it satisfies the Euler equation

$$1 = \mathcal{M}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{\rho - 1} (1 + R_{t+1}) \right]^{1/\rho}. \tag{43}$$

The classic specification $\mathcal{M}_t[V_{t+1}] = \left(\mathbb{E}_t\left(V_{t+1}^\alpha\right)\right)^{1/\alpha}$ is measurably homogeneous. In this case (43) becomes

$$1 = \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{\frac{\rho-1}{\alpha}} (1 + R_{t+1})^{\frac{1}{\alpha}} \right]^{\alpha/\rho},$$

which is a classic Euler equation in recursive optimization, due to [10] and [11], and widely used in applications (see, e.g., [7] and [8]). The multiple priors operator

$$\mathcal{M}_t(V_{t+1}) = \inf_{Q \in C} \left(\mathbb{E}_t^Q \left(V_{t+1}^\alpha \right) \right)^{1/\alpha}$$

is a more general example of a measurably homogeneous operator. In this case, (43) becomes

$$1 = \inf_{Q \in C} \mathbb{E}_t^Q \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{\frac{\rho-1}{\alpha}} (1 + R_{t+1})^{\frac{1}{\alpha}} \right]^{\alpha/\rho}.$$

8 Further Analysis

8.1 A Variation on Theorem 1 and a Pathological Example

Condition (W-iv), that is, the requirement that $W(x,0) > 0$ for each $x > 0$, may not be satisfied in some important cases. For example, the Cobb-Douglas aggregator $W(x,y) = x^{\alpha}y^{\beta}$, with $\alpha, \beta > 0$, does not satisfy (W-iv). It also fails for the aggregators in (14).

To see how the failure of (W-iv) can generate multiple fixed points, suppose that $W(x,0) = 0$ for each $x > 0$ (and so, by (W-i), $W(0,0) = 0$). In this case $T(0) = 0$, and so there always exists a trivial fixed point.

Fortunately, the next result, a variation of Theorem 2, shows that even without (W-iv) there exist unique and globally attracting fixed points provided we consider processes that are uniformly bounded away from 0 (that is, that belong to the interior of L_+^{∞}).

Theorem 6 *Consider the operator T given by (12), where W is an aggregator function and $\mathcal{M}$ is a certainty equivalent operator. Given any $Y \in L_+^{\infty}$, then T has a unique fixed point $\widehat{X}$ in $\operatorname{int} L_+^{\infty}$ provided:*

(i) there exists $k > 0$ such that, given any $\varepsilon > 0$ small enough,

$$W(k\varepsilon, \varepsilon) > \varepsilon \tag{44}$$

and, for all $x > 0$, there is $y \geq \varepsilon$ for which $W(k\varepsilon + x, y) \leq y$,

(ii) $W(x, \cdot)$ is concave,

(iii) $\mathcal{M}$ is subhomogeneous and $[Y]_{\infty} > 0$.
The unique fixed point $\widehat{X}$ in $\operatorname{int} L_+^{\infty}$ is globally attracting over $\operatorname{int} L_+^{\infty}$, that is,

$$\left\| T^n(X) - \widehat{X} \right\|_{\infty} \to 0, \qquad \forall X \in \operatorname{int} L_+^{\infty}. \tag{45}$$

Theorem 6 guarantees uniqueness only over $\operatorname{int} L_+^{\infty}$ and there might well exist multiple fixed points among the X that do not belong to $\operatorname{int} L_+^{\infty}$. The next example illustrates what happens in such a case, and it also shows the pathological nature that fixed points can have when they are not globally attracting.

Example 22 The Cobb-Douglas aggregator $W = x^{\alpha}y^{\beta}$ satisfies conditions (i) and (ii) of Theorem 6 whenever $\alpha + \beta \leq 1$. Let us look at fixed points in the deterministic setting; i.e., $\Omega = \{\omega\}$ and $L^{\infty}(\Omega \times \mathbb{N}) = l^{\infty}(\mathbb{N}) \equiv l^{\infty}$. If $\mathcal{M}$ is a shift operator, then (10) takes the form[16]

$$X_t = Y_t^{1/2} X_{t+1}^{1/2}$$

This equation has the trivial solution $X_t \equiv 0$. Moreover, there is also the solution

$$X_t = Y_t^{1/2} Y_{t+1}^{1/4} Y_{t+2}^{1/8} \cdots = \prod_{\tau \geq 0} Y_{t+\tau}^{2^{-\tau}}. \tag{46}$$

provided $Y \in \operatorname{int} l_+^{\infty}$. This solution belongs to $\operatorname{int} l_+^{\infty}$ and, by Theorem 6, it is the unique fixed point in $\operatorname{int} l_+^{\infty}$. It does not attract all the points of l_+^{∞}. For instance, if the initial condition is $X = (X_0, X_1, \ldots)$, with $X_{\tau} = 0$ for some τ, then $T^n(X) \to 0$.

[16] For convenience, we set here $\mathbb{N} = \{0, 1, \ldots\}$ instead of $\mathbb{N} = \{1, 2, \ldots\}$.

There are plenty of fixed points. In fact, by inverting the equation $X_t = Y_t^{1/2} X_{t+1}^{1/2}$, we get $X_{t+1} = Y_t^{-1} X_t^2$, which can be solved iteratively. In particular, simple algebra shows the existence of a continuum of solutions given by

$$X_t = \frac{X_0^{2^t}}{Y_{t-1} Y_{t-1}^2 Y_{t-2}^4 \cdots Y_0^{2^{t-1}}}, \tag{47}$$

for all $t \geq 0$ and $Y \in \operatorname{int} l_+^\infty$, where X_0 is a fixed function. This process is bounded if $X_0 \leq Y_0^{1/2} Y_1^{1/4} Y_2^{1/8} \cdot \cdots$. As it might also be checked directly, Theorem 6 implies that all these solutions do not belong to $\operatorname{int} l_+^\infty$, unless they agree with (46). Observe that the solutions (47) are generally neither stationary nor forward looking. ▲

8.2 Linear Aggregators

Here we give a uniqueness result that is peculiar to equation (30) and it does not follow from earlier results. However, we will see that it requires W to be linear. In order to state the result, define for a function $\phi : \mathbb{R}_+ \to \mathbb{R}$ the index:

$$S_\phi(t) = -\frac{\phi''(t)}{\left[\phi'(t)\right]^2}, \qquad \forall t \geq 1.$$

Say that ϕ is *strongly decreasing absolute risk averse* (SDARA) if its index $S_\phi : (0, \infty) \to \mathbb{R}$ is non-increasing. Since $A_\phi(t) = S_\phi(t)\,\phi'(t)$, SDARA implies DARA in the relevant case $\phi' > 0$. Clearly, the converse is false: consider for instance $\phi(t) = -e^{-t}$.

Theorem 7 *Consider the operator T given by (31), where $W(x, y) = x + \beta y$ for some $\beta \in (0, 1)$ and $Y \in L_+^\infty$. Suppose $\phi : \mathbb{R}_+ \to \mathbb{R}$ is concave and twice differentiable on $(0, \infty)$, with $\phi'(t) > 0$ for all $t > 0$. Then, T has a unique fixed point $\widehat{X}$ provided ϕ is SDARA and at least one of the following two conditions hold:*

(i) $\sup_{t \geq 0} R_\phi(t) < 1.$*;*

(ii) $[Y]_\infty > 0$ *and ϕ is IRRA*

Under (i), $\widehat{X}$ is globally attracting, while under (ii) we have:

$$\left\| T^n(X) - \widehat{X} \right\|_\infty \to 0, \qquad \forall X \in \left[0, (1-\beta)^{-1} \|Y\|_\infty\right].$$

Remark. When ϕ is IRRA, condition (ii) can be replaced with $\lim_{t \to +\infty} R_\phi(t) < 1$.

We illustrate Theorem 7 with few examples.

Example 23 Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be the concave and strictly increasing function given by $\phi(t) = \alpha^{-1} t^\alpha$, with $\alpha \in (0, 1)$. We have,

$$R_\phi(t) = 1 - \alpha \qquad \text{and} \qquad S_\phi(t) = (1 - \alpha)\, t^{-\alpha}$$

and so ϕ is SDARA and satisfies condition (ii). By Theorem 7, equation (30) has a unique and globally attracting solution when it features this function ϕ, provided $W(x, y) = x + \beta y$ for some $\beta \in (0, 1)$.▲

Example 24 Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be the concave and strictly increasing function given by $\phi(t) = (a + t^\alpha)^{1/\alpha}$, with $a > 0$ and $\alpha \in (0,1)$. We have

$$\begin{aligned} S_\phi(t) &= a(1-\alpha)t^{-\alpha}(a+t^\alpha)^{-1/\alpha} \\ R_\phi(t) &= a(1-\alpha)(a+t^\alpha)^{-1}. \end{aligned}$$

Both these indices are decreasing, and $\sup_{t\geq 0} R_\phi(t) \leq 1-a$. The function ϕ is SDARA and satisfies condition (ii). By Theorem 7, equation (30) has a unique and globally attracting solution when it features this function ϕ, provided $W(x,y) = x + \beta y$ for some $\beta \in (0,1)$. A similar example is given by $\phi(t) = at + bt^\alpha$. ▲

Example 25 Consider the concave and strictly increasing HARA function $\phi : \mathbb{R}_+ \to \mathbb{R}$ given by $\phi(t) = \alpha^{-1}(t+k)^\alpha$ with $\alpha \in (0,1)$. We have

$$R_\phi(t) = (1-\alpha)\frac{t}{t+k} \qquad \text{and} \qquad S_\phi(t) = \frac{1-\alpha}{(t+k)^\alpha},$$

and so ϕ is SDARA and IRRA, with $\lim_{t\to\infty} R_\phi(t) = 1-\alpha < 1$. By Theorem 7, equation (30) has a unique and globally attracting solution when it features this function ϕ, provided $W(x,y) = x + \beta y$ for some $\beta \in (0,1)$. ▲

8.3 A Second Application

Let $\{P_\theta\}_{\theta\in\Theta}$ be a family of probability distributions on Σ, where Θ is a finite parameter space. Set $P = |\Theta|^{-1}\sum_{\theta\in\Theta} P_\theta$. At each node (ω,t), define a distribution $\mu_{(\omega,t)}$ on 2^Θ, i.e., $\mu_{(\omega,t)} : 2^\Theta \to [0,1]$. All such distributions can be related via conditioning.

We now consider a second recursive equation in this more general setting. Given again a strictly increasing function $\phi : [0,\infty) \to \mathbb{R}$, define $\mathcal{M} : L_+ \to L_+$ by:

$$\mathcal{M}_t(X)(\omega) = \mathbb{E}^{\mu_{(\omega,t)}}\left(\phi^{-1}\left(\mathbb{E}_t^{P_\theta}(\phi \circ X_{t+1})\right)\right), \qquad P\text{-a.e,}$$

for each $t \geq 1$.

Under this second specification of $\mathcal{M}$, equation (10) takes the recursive form:

$$X_t(\omega) = W\left(Y_t(\omega), \mathbb{E}^{\mu_{(\omega,t)}}\left(\phi^{-1}\left(\mathbb{E}_t^{P_\theta}(\phi \circ X_{t+1})\right)\right)\right), \qquad P\text{-a.e,} \tag{48}$$

for each $t \geq 1$, which has been recently studied by Klibanoff, Marinacci, and Mukerji [15].

The associated operator $T : L_+^\infty \to L_+^\infty$ is given by:

$$T_t(X)(\omega) = W\left(Y_t(\omega), \mathbb{E}^{\mu_{(\omega,t)}}\left(\phi^{-1}\left(\mathbb{E}_t^{P_\theta}(\phi \circ X_{t+1})\right)\right)\right), \qquad P\text{-a.e,}$$

for each $t \geq 1$.

Although for brevity we omit the details, it can be shown that all results established in the previous section for the recursive equation (30) hold *verbatim* for equation (48).

A The Space of Adapted Processes

We can look at the space L of all adapted processes, as the space of measurable functions with respect a suitable σ-algebra. Such a construction is rather standard in probability theory and we just outline it. Denote by λ the counting measure on $(\mathbb{N}, 2^\mathbb{N})$. Consider the measure space $(\Omega \times T, \Sigma \otimes 2^T, P \otimes \lambda)$. Define the σ-algebra $\mathcal{F} \subset \Sigma \otimes 2^T$ as

$$A \in \mathcal{F} \Longleftrightarrow A_t \in \Sigma_t \text{ for all } t \geq 1$$

where A_t denotes the t-section of the set $A \subset \Omega \times T$. It is easy to check that the process $X(\omega, t)$ is adapted iff X is $\mathcal{F}$-measurable. Therefore, we may identify L with the space $L(\Omega \times T, \mathcal{F}, P \otimes \lambda)$, where two elements $X, X' \in L(\Omega \times T, \mathcal{F}, P \otimes \lambda)$ are identified iff $X_t = X'_t$ P-a.e. for all t.

The spaces L^∞ and L^w can then be regarded as subspaces of $L(\Omega \times T, \mathcal{F}, P \otimes \lambda)$. An important property is that L^w turns out to be a Banach lattice. Moreover, the positive cone L^w_+ is normal (see [18] and [28]). Actually, if $X_1 \leq X_2$ in L^w_+, it follows $\|X_1\|_w \leq \|X_2\|_w$.

B A Contraction Theorem

In this appendix we present a contraction theorem based on the Thompson metric, which adapts to the setting of this paper some results due to Montrucchio [22]. The results below hold in any ordered normed space with a complete and normal positive cone. For concreteness, however, we will use the space L^w, which suffices for our purposes.

Following [28] (see also [23]), two adapted processes X and Y of L^w_+ are said to be *comparable* if there exist scalars $\alpha > 0$ and $\beta > 0$ such that $\alpha X \leq Y \leq \beta X$. This is an equivalence relation on L^w_+ and $C_X = \{Y : Y \text{ is comparable to } X\}$ denotes the component containing X.

Given two comparable elements X and Y, set

$$M(Y \mid X) = \inf \{\alpha > 0;\ Y \leq \alpha X\}. \tag{49}$$

Observe that the inf in (49) is a minimum, so that $M(Y \mid X) > 0$.

If X and Y are two comparable elements in L^w_+, their Thompson distance d_τ is defined as

$$d_\tau(X, Y) = \max \{\ln M(X \mid Y), \ln M(Y \mid X)\}.$$

If C is a component of L^w_+, one can easily prove that d_τ gives a metric on C. Moreover, Thompson [28] proves the following result.

Theorem 8 *Let C be a component of L^w_+. Then C is a complete metric space with respect to the metric d_τ.*

This theorem relies basically on the fact that the cone L^w_+ is complete and normal.

Next we show when an operator is a contraction with respect to the Thompson metric.

Theorem 9 *Suppose an operator $T : [0, X_1] \to [0, X_1]$, with $[0, X_1] \subseteq L^w_+$, has the following properties:*

(i) T is monotone, i.e., $X \leq Y$ implies $T(X) \leq T(Y)$ for all $X, Y \in [0, X_1]$;

(ii) $T(0) = X_0$ is comparable to X_1, i.e., $X_0 \in C_{X_1}$;

(iii) $T(\alpha X) \geq \alpha T(X) + (1 - \alpha) T(0)$ for all $\alpha \in [0, 1]$ and $X \in [X_0, X_1]$.

Then, T is a contraction over $[X_0, X_1]$ with respect to the Thompson distance d_τ; that is,

$$d_\tau(T(X), T(Y)) \leq \gamma d_\tau(X, Y), \qquad \forall X, Y \in [X_0, X_1],$$

where $\gamma = 1 - \mu^{-1} < 1$ and $\mu = M(X_1 \mid X_0)$.

Proof. We begin with the following easy claim, a variant of Bernoulli's inequality.

Claim It holds $t^{\gamma} \leq 1 - \gamma + \gamma t$ for all $t \geq 0$ and all $0 \leq \gamma \leq 1$.

Proof of the Claim: The function t^{γ} is concave over $t \geq 0$. From the superdifferentiability property at $t = 1$, it follows the desired inequality. □

The result is trivial if $X_0 = X_1$. Suppose that $X_0 \neq X_1$. Set $\mu = M(X_1 \mid X_0)$. Clearly, $\mu > 1$. Otherwise, $X_1 \leq \mu X_0 \leq X_0$. Hence, $X_0 \geq X_1$. As $X_0 \leq X_1$, it would be $X_0 = X_1$, a contradiction.

Consider two non-identical elements X, $Y \in [X_0, X_1]$. They are clearly comparable. Set $\beta_1 = M(X \mid Y)$, $\beta_2 = M(Y \mid X)$ and $\beta = \max\{\beta_1, \beta_2\}$. As X and Y are distinct, $\beta > 1$. Suppose that $\beta = \beta_1 = M(X \mid Y)$. This means $\beta Y \geq X$, i.e., $Y \geq \beta^{-1} X$ with $\beta^{-1} < 1$. By (ii) and (iii),

$$T(Y) \geq T\left(\beta^{-1} X\right) \geq \beta^{-1} T(X) + \left(1 - \beta^{-1}\right) X_0. \tag{50}$$

From $\mu = M(X_1 \mid X_0)$ it follows $X_0 \geq \mu^{-1} X_1$. Plugging this into (50),

$$\begin{aligned} T(Y) &\geq \beta^{-1} T(X) + \left(1 - \beta^{-1}\right) \mu^{-1} X_1 \\ &\geq \beta^{-1} T(X) + \left(1 - \beta^{-1}\right) \mu^{-1} T(X) \\ &= \left[\beta^{-1} + \mu^{-1}\left(1 - \beta^{-1}\right)\right] T(X), \end{aligned}$$

where in the second inequality we are using the fact that $T(X) \leq X_1$.

By setting $t = \beta^{-1}$ and $\gamma = 1 - \mu^{-1}$ in the Claim, we obtain

$$\beta^{-1} + \mu^{-1}\left(1 - \beta^{-1}\right) = \mu^{-1} + \beta^{-1}\left(1 - \mu^{-1}\right) \geq \beta^{-\gamma}.$$

Therefore, we have $\beta^{\gamma} T(Y) \geq T(X)$. In view of (49), $M(T(X) \mid T(Y)) \leq \beta^{\gamma}$ and thus

$$\ln M(T(X) \mid T(Y)) \leq \gamma \ln M(X \mid Y) = \gamma d_{\tau}(X, Y).$$

If also $\beta_2 > 1$, the same argument leads to

$$\ln M(T(Y) \mid T(X)) \leq \gamma \ln M(Y \mid X) \leq \gamma d_{\tau}(X, Y).$$

This implies $d(TX, TY) \leq \gamma d(X, Y)$, which ends the proof. Suppose, on the contrary, that $\beta_2 \leq 1$. We then have $X \geq Y$. Thus $TX \geq TY$, and so $M(TY \mid TX) \leq 1$. Its logarithm is nonpositive and therefore the same result holds. ■

Theorem 9 implies the following contraction result.

Theorem 10 *Suppose the conditions of Theorem 9. Then, T has a unique fixed point $\widehat{X} \in [0, X_1]$. Moreover, $\widehat{X}$ is globally attracting on $[0, X_1]$; i.e.,*

$$\left\| T^n(X) - \widehat{X} \right\|_w \to 0, \qquad \forall X \in [0, X_1]. \tag{51}$$

Proof. We begin with a claim, which is essentially Lemma 1 of [28].

Claim If X and Y are two comparable elements of L_+^w and $\|X\|_w, \|Y\|_w \leq a$, then

$$\|X - Y\|_w \leq a\left(\exp d_{\tau}(X, Y) - 1\right).$$

Proof of the Claim. Set $\lambda = M(X \mid Y)$, $\nu = M(Y \mid X)$ and $\mu = \max\{\lambda, \nu\}$. If $X \neq Y$, then $\mu > 1$. It follows $\mu Y \geq X$ and $\mu X \geq Y$. Therefore, $|X - Y| \leq (\mu - 1)(X \vee Y)$, and so

$$\frac{|X-Y|}{w} \leq (\mu - 1)\frac{X \vee Y}{w} \leq (\mu - 1)\left(\|X\|_w \vee \|Y\|_w\right) \leq (\mu - 1)\, a.$$

In turn, this implies $\|X - Y\|_w \leq (\mu - 1)\, a = a\left(\exp d_\tau (X, Y) - 1\right)$, because $\mu = \exp d_\tau (f, g)$. □

As $T(0) \geq 0$ and T is monotone, as $X \in [0, X_1]$, we have $T(X) \in [X_0, X_1]$. Consequently, the fixed points lie in $[X_0, X_1]$. On the other hand, $T : [X_0, X_1] \to [X_0, X_1]$ is a contraction mapping w.r.t. the Thompson distance. By hypothesis, $[X_0, X_1]$ is contained in a component C of L_+^w. The interval $[X_0, X_1]$ is closed and bounded with respect to the weighted supnorm $\|\cdot\|_w$. By the Claim, $[X_0, X_1]$ is then closed in C w.r.t. the Thompson metric, and so $([X_0, X_1], d_\tau)$ is a complete metric space. By the Banach Contraction Mapping Theorem, there exists a unique $\widehat{X}$ such that $T\left(\widehat{X}\right) = \widehat{X}$. The Claim guarantees that $\left\|T^n(X) - \widehat{X}\right\|_w \to 0$ for any $X \in [0, X_1]$. ■

Theorem 11 *Suppose the monotone operator $T : L_+^w \to L_+^w$ has the following properties:*

(i) $[T(0)/w]_\infty > 0$*;*

(ii) there is some $H \in L_+^w$ such that $T(H) \leq H$;

(iii) $T(\alpha X) \geq \alpha T(X) + (1 - \alpha) T(0)$ *for all $\alpha \in [0, 1]$ and all $X \in L_+^w$.*

Then, T has a unique fixed point $\widehat{X}$ in L_+^w which belongs to $[0, H]$ and it is such that

$$\left\|T^n(X) - \widehat{X}\right\|_w \to 0, \qquad \forall X \in [0, H].$$

Proof. Clearly, $X \in [0, H]$ implies $T(X) \in [0, H]$. Let $[T(0)/w]_\infty = \alpha > 0$. It follows $T(0) \geq \alpha w$. As $H \in L_+^w$, $H \leq \beta w$ for some $\beta > 0$. Consequently, $T(0) \geq \alpha\beta^{-1} H$ and $T(0)$ and H are comparable. By Theorem 10 there exists a unique fixed point $\widehat{X} \in [0, H]$ and, by (51),

$$\left\|T^n(0) - \widehat{X}\right\|_w \to 0. \tag{52}$$

Suppose there exists another fixed point $\widetilde{X} \in L_+^w$. The same argument previously used (replacing H by $\widetilde{X}$) we get that $\widetilde{X}$ is the unique fixed point in $\left[0, \widetilde{X}\right]$. As $\left\|T^n(0) - \widetilde{X}\right\|_w \to 0$, we deduce that $\widehat{X} = \widetilde{X}$. ■

C Quasi-Arithmetic Means

Let $\Delta_{n+1} = \left\{q \in \mathbb{R}_+^n : \sum_{i=1}^n q_i = 1\right\}$ be the unit simplex of $\mathbb{R}^n$. Given a strictly monotone function $\phi : \mathbb{R}_+ \to \mathbb{R}$ and a vector $q \in \Delta_{n+1}$, define $M_\phi(\cdot; q) : \mathbb{R}_+^n \to \mathbb{R}$ by

$$M_\phi(x; q) = \phi^{-1}\left(\sum_{i=1}^n \phi(x_i)\, q_i\right), \qquad \forall x \in \mathbb{R}_+^n.$$

The function M_ϕ is often called a quasi-arithmetic mean (see, e.g., [12, Ch. III] and [6, Ch. IV]).

In the paper two properties of these means play a key role:

- $M_\phi(\cdot; q) : \mathbb{R}_+^n \to \mathbb{R}$ is *constant subadditive* if

  $$M_\phi(x + k; q) \leq M_\phi(x; q) + k, \qquad \forall x, k \in \mathbb{R}_+^n,$$

  for any $q \in \Delta_n$ and any $n \geq 1$, where k denotes both a scalar and the corresponding constant vector $(k, ..., k)$.

- $M_\phi(\cdot;q):\mathbb{R}^n_+\to\mathbb{R}$ is *subhomogeneous* if

  $$\alpha M_\phi(x;q)\le M_\phi(\alpha x;q),\qquad \forall x\in\mathbb{R}^n_+,\forall\alpha\in[0,1],$$

  for any $q\in\Delta_n$ and any $n\ge 1$.

These two properties are closely connected, and next we provide a duality between them (we omit the simple proof). Here e^x for $x\in\mathbb{R}^n$ stands for the vector $(e^{x_1},...,e^{x_n})$, with the convention $e^{-\infty}=0$.

Proposition 7 *The quasi-arithmetic mean M_ϕ is subhomogeneous iff $M_{\widehat{\phi}}(x;q)=\log M_\phi(e^x;q)$ is constant subadditive, where*

$$\widehat{\phi}(t)=\phi\left(e^t\right)\qquad \forall t\ge 0.$$

If, in addition, ϕ is twice differentiable on $(0,\infty)$, then

$$A_{\widehat{\phi}}(t)=R_\phi\left(e^t\right)-1.$$

The duality $\phi\mapsto\widehat{\phi}$ is a one-to-one correspondence between IARA and IRRA functions, as well as between DARA and DRRA functions, and CARA and CRRA functions. In other words, this duality preserves the classification of functions according to absolute and relative risk aversion mentioned in Section 2. Observe that this duality also preserves monotonicity, that is, ϕ is increasing iff $\widehat{\phi}$ does.

We begin by characterizing constant subadditive quasi-arithmetic means. Using the duality of Proposition 7, we will then derive a characterization of subhomogeneous quasi-arithmetic means.

Theorem 12 *Suppose $\phi:\mathbb{R}_+\to\mathbb{R}$ is twice differentiable on $(0,\infty)$, with either $\phi'(t)<0$ for all $t>0$ or $\phi'(t)>0$ for all $t>0$. Then, $M_\phi(\cdot;q):\mathbb{R}^n_+\to\mathbb{R}$ is constant subadditive iff ϕ is IARA.*

Proof. W.l.o.g., assume $\phi(0)=0$, so that $\phi(0)=\phi^{-1}(0)=0$. Following Beck (1970) (see [6, p. 251]), define $f:\phi(\mathbb{R}_+)\to\mathbb{R}$ by

$$f(s)=\phi\left(\phi^{-1}(s)+k\right),\qquad \forall s\in\phi(\mathbb{R}_+).$$

Claim. Suppose ϕ is strictly increasing. Then, $M_\phi(\cdot;q)$ is constant subadditive iff f is concave on $\phi(\mathbb{R}_{++})$.

Proof of the Claim. Let f be concave on $\phi(\mathbb{R}_{++})$. On the other hand, we have $f(s)\ge f(0)$ because ϕ is strictly increasing. Hence, f is concave on the entire $\phi(\mathbb{R}_+)$.

Let $\{s_i\}_{i=1}^n\subseteq\phi(\mathbb{R}_+)$ and let $x_i=\phi^{-1}(s_i)$. Then,

$$\begin{aligned}\sum_{i=1}^n\phi(x_i+k)\,q_i &= \sum_{i=1}^n\phi\left(\phi^{-1}(s_i)+k\right)q_i=\sum_{i=1}^n f(s_i)\,q_i\\ &\le f\left(\sum_{i=1}^n s_iq_i\right)=\phi\left(\phi^{-1}\left(\sum_{i=1}^n s_iq_i\right)+k\right)\\ &= \phi\left(\phi^{-1}\left(\sum_{i=1}^n\phi(x_i)\,q_i\right)+k\right),\end{aligned}$$

and so, being ϕ^{-1} increasing,

$$\phi^{-1}\left(\sum_{i=1}^n\phi(x_i+k)\,q_i\right)\le\phi^{-1}\left(\sum_{i=1}^n\phi(x_i)\,q_i\right)+k.$$

This shows that M_ϕ is constant subadditive.

As to the converse, assume all M_ϕ, given any $q \in \Delta_n$ and any $n \geq 1$, are constant subadditive. Let $s_1, s_2, \in \phi(\mathbb{R}_{++})$ with $x_i = \phi^{-1}(s_i)$, and let $\alpha \in [0,1]$. Then,

$$\begin{aligned} f(ts_1 + (1-t)s_2) &= \phi\left(\phi^{-1}(ts_1 + (1-t)s_2) + k\right) \\ &= \phi\left(\phi^{-1}(t\phi(x_1) + (1-t)\phi(x_2)) + k\right) \\ &= \phi(M_\phi(x;(t,1-t)) + k) \\ &\geq \phi(M_\phi(x+k);(t,1-t)) \\ &= t\phi(x_1+k) + (1-t)\phi(x_2+k) \\ &= tf(s_1) + (1-t)f(s_2), \end{aligned}$$

and so f is concave on $\mathbb{R}^n_{++}$. □

In view of the Claim, we need to show that f is concave on $\phi(\mathbb{R}_{++})$. We first assume that $\phi' > 0$ on $\mathbb{R}_{++}$. Set $\psi = \phi^{-1}$, so that

$$f(s) = \phi(\psi(s) + k) \tag{53}$$

for all $s \in \phi(\mathbb{R}_{++})$. For all $x > 0$, we have

$$\phi'(x) = \frac{1}{\psi'(\phi(x))} \quad \text{and} \quad \phi''(x) = -\frac{\psi''(\phi(x))}{\left(\psi'(\phi(x))\right)^3}, \tag{54}$$

so that

$$\frac{(\psi')^2}{\psi''} = -\frac{\phi'}{\phi''} \circ \psi \tag{55}$$

The function f is concave iff $f'' \leq 0$. We have:

$$f''(s) = \phi''(\psi(s)+k)\left(\psi'(s)\right)^2 + \phi'(\psi(s)+k)\psi''(s),$$

and so $f'' \leq 0$ iff

$$\phi''(\psi(s)+k)\left(\psi'(s)\right)^2 \leq -\phi'(\psi(s)+k)\psi''(s). \tag{56}$$

Hence, inequality (56) holds iff

$$\frac{\phi''(\psi(s)+k)}{\phi'(\psi(s)+k)} \leq -\frac{\psi''(s)}{\left(\psi'(s)\right)^2}$$

that is,

$$-\frac{\phi''(\psi(s)+k)}{\phi'(\psi(s)+k)} \geq -\frac{\phi''(\psi(s))}{\phi'(\psi(s))}$$

This holds iff ϕ is IARA, as $\mathbb{R}_+$ is the domain of ϕ. We conclude that $f'' \leq 0$ iff ϕ is IARA, as desired.

Finally, if $\phi' < 0$, then consider $\varphi = -\phi$. It holds $A_\varphi = A_\phi$ and $M_\varphi(x;q) = M_\phi(x;q)$. As $\varphi' > 0$, by what we just proved we have:

$$M_\varphi(x+k;q) = M_\phi(x+k;q) \leq M_\phi(x;q) + k = M_\varphi(x;q) + k,$$

as desired. ■

Corollary 3 *Under the hypotheses of Proposition 12, $M_\phi(\cdot;q) : \mathbb{R}^n_+ \to \mathbb{R}$ is Lipschitz of degree one for any $q \in \Delta_n$ and any $n \geq 1$, i.e.,*

$$|M_\phi(x;q) - M_\phi(y;q)| \leq \|x-y\|_\infty, \qquad \forall x,y \in \mathbb{R}^n_+, \tag{57}$$

for any $q \in \Delta_n$ and any $n \geq 1$.

Proof. By Proposition 12,

$$M_\phi(x+k;q) \le M_\phi(x;q)+k, \qquad \forall x,k\in\mathbb{R}^n_+.$$

Since ϕ is strictly monotonic, M_ϕ is non-decreasing on $\mathbb{R}^n_+$. Then, given any $x,y\in\mathbb{R}^n_+$, the inequality $x\le y+\|x-y\|_\infty$ implies

$$M_\phi(x;q)\le M_\phi\left(y+\|x-y\|_\infty;q\right)\le M_\phi(y;q)+\|x-y\|_\infty,$$

which in turn implies (57). ■

Theorem 12 and a simple application of the duality established in Proposition 7 give the next characterization of subhomogeneous quasi-arithmetic means.

Corollary 4 *Assume $\phi:\mathbb{R}_+\to\mathbb{R}$ is twice differentiable on $(0,\infty)$, with either $\phi'(t)<0$ for all $t>0$ or $\phi'(t)>0$ for all $t>0$. Then, $M_\phi(\cdot;q):\mathbb{R}^n_+\to\mathbb{R}$ is subhomogeneous iff ϕ is IRRA.*

We close by showing what form Theorem 12 and Corollary 4 take in our more general probability setting (we omit the proof, which is similar to that of Theorem 12 using the Jensen inequalities for conditional expectations).

Proposition 8 *Let $X\in L_+$ and let Σ' be a σ-algebra contained in Σ. Suppose $\phi:\mathbb{R}_+\to\mathbb{R}$ is twice differentiable on $(0,\infty)$, with either $\phi'(t)<0$ for all $t>0$ or $\phi'(t)>0$ for all $t>0$. Then, ϕ is IARA iff*

$$\phi^{-1}\circ\mathbb{E}\left(\phi\circ(X+k)\mid\Sigma'\right)\le\phi^{-1}\circ\mathbb{E}\left(\phi\circ X\mid\Sigma'\right)+k, \qquad P\text{-a.e.},$$

for all $k\ge 0$, while ϕ is IRRA iff

$$\alpha\phi^{-1}\circ\mathbb{E}\left(\phi\circ X\mid\Sigma'\right)\le\phi^{-1}\circ\mathbb{E}\left(\phi\circ(\alpha X)\mid\Sigma'\right), \qquad P\text{-a.e.},$$

for all $\alpha\in[0,1]$.

D Some Basic Properties of Aggregators

Because of the Lipschitzian property (W-v), Blackwell aggregators $W(x,\cdot):\mathbb{R}_+\to\mathbb{R}$ are obviously continuous for each $x\ge 0$. Also Thompson aggregators are easily seen to be continuous.

Lemma 2 *Aggregators $W(x,\cdot):\mathbb{R}_+\to\mathbb{R}$ that satisfy (W-iii) are continuous for each $x\ge 0$.*

Proof. Given any $x\ge 0$, set $\varphi(y)=W(x,y)$ for all $y\ge 0$. Suppose $y_n\uparrow y$. There exists $\{\alpha_n\}_n\subseteq[0,1]$, with $\alpha_n\uparrow 1$, such that $y_n=\alpha_n y$ for each n. By (W-i) and (W-iii),

$$\varphi(y_n)=\varphi(\alpha_n y)\ge\alpha_n\varphi(y)+(1-\alpha_n)\varphi(0)\longrightarrow\varphi(y)\ge\varphi(y_n)$$

and so $\varphi(y)=\lim_n\varphi(y_n)$. Suppose next $y_n\downarrow y$. Then, there exists $\{\alpha_n\}_n\subseteq[0,1]$, with $\alpha_n\uparrow 1$, such that $y=\alpha_n y_n$ for each n. By (W-i) and (W-iii),

$$\varphi(y_n)\ge\varphi(y)=\varphi(\alpha_n y_n)\ge\alpha_n\varphi(y_n)+(1-\alpha_n)\varphi(0)$$

and so $\liminf_n\varphi(y_n)\ge\varphi(y)\ge\limsup_n\varphi(y_n)$. ■

In Section 3.1 we observed that the Blackwell condition (W-v) implies the existence of unique and globally attracting fixed points in y for each $x\ge 0$. For Thompson aggregators we have the following result.

Lemma 3 *The aggregator $W(x,\cdot):\mathbb{R}_+\to\mathbb{R}$ has a fixed point $\widehat{y}$ for each $x\geq 0$. If, in addition, W is Thompson, then such fixed point is unique for each $x>0$, and $\lim_n W^n(x,0)=\widehat{y}$.*

Proof. By (W-ii), given any $x\geq 0$ there are two positive real numbers x_n and y_n such that $x\leq x_n$ and $W(x_n,y_n)\leq y_n$. By (W-i), for each $y\in[0,y_n]$ we then have:

$$0\leq W(x,y)\leq W(x_n,y_n)\leq y_n,$$

and so we can write $W(x,\cdot):[0,y_n]\to[0,y_n]$. Since $W(x,\cdot)$ is monotone, by the Tarski Fixed Point Theorem (see [27, Thm 1]) there is $y^*\in[0,y_n]$ such that $W(x,y^*)=y^*$.

As to uniqueness, given $x>0$ set $\varphi(y)=W(x,y)$ for each $y\geq 0$. By (W-iv), $\varphi(0)>0$. Suppose there exist $y',y''\in\mathbb{R}_+$ such that $\varphi(y')=y'$ and $\varphi(y'')=y''$, with $0<y'<y''$ (observe that, by (W-iv), 0 cannot be a fixed point). Then, there is $t\in(0,1)$ such that $y'=ty''$, so that (W-iii) implies:

$$\begin{aligned} y' &= \varphi(y')=\varphi((1-t)0+ty'')\geq(1-t)\varphi(0)+t\varphi(y'') \\ &= (1-t)\varphi(0)+ty''=(1-t)\varphi(0)+y'. \end{aligned}$$

This is a contradiction because $\varphi(0)>0$.

It remains to prove that $\lim_n\varphi^n(0)=\widehat{y}$. By (W-i), $\varphi(0)>0$ implies $\varphi(\varphi(0))>\varphi(0)$. Hence, $\varphi^{n+1}(0)=\varphi(\varphi^n(0))>\varphi^n(0)$ for all n. Moreover, $\widehat{y}\geq 0$ implies $\widehat{y}=\varphi(\widehat{y})\geq\varphi(0)$. In turn, this easily implies $\varphi^n(0)\leq\widehat{y}$ for all n. We conclude that $\{\varphi^n(0)\}_n$ is a bounded monotone sequence. Set $\alpha=\lim_n\varphi^n(0)$. By Lemma 2, φ is continuous, and so:

$$\varphi(\alpha)=\varphi\left(\lim_n\varphi^n(0)\right)=\lim_n\varphi(\varphi^n(0))=\lim_n\varphi^{n+1}(0)=\alpha.$$

Hence, $\widehat{y}=\alpha$, as desired. ■

Corollary 5 *Given any $\theta>0$ and $x>0$, there exists a unique y solving equation $W(x,y)=\theta y$ provided either W is Thompson or it is Blackwell for some $\beta\in(0,1)$ such that $\beta<\theta$.*

Proof. It follows from Lemma 3 once we consider the aggregator $\theta^{-1}W$. ■

The next lemma gives another simple but important property of Thompson aggregators.

Lemma 4 *Given $x>0$, suppose the aggregator $W(x,\cdot):\mathbb{R}_+\to\mathbb{R}$ satisfies either (W-iii) or (W-vi). Then, the function $W(x,y)/y$ is decreasing in y. If, in addition, $W(x,\cdot)$ satisfies (W-iv), then $W(x,y)/y$ is strictly decreasing in y, that is,*

$$y<y'\Longrightarrow\frac{W(x,y)}{y}>\frac{W(x,y')}{y'},\qquad\forall y,y'\in\mathbb{R}_{++}\tag{58}$$

Proof. Observe that both (W-iii) and (W-vi) imply that W is subhomogeneous in y.

Let $y'>y>0$. Then, there exists $t\in(0,1)$ such that $y=ty'$, and so

$$\frac{W(x,y)}{y}=\frac{W(x,ty')}{ty'}\geq\frac{tW(x,y')}{ty'}=\frac{W(x,y')}{y'}.$$

If (W-iv) holds, then $W(x;0)>0$ and so $W(x,y)/y>W(x,y')/y'$, as desired. ■

Two consequences of Lemma 4 are noteworthy. First, the limit

$$\lim_{y\to+\infty}\frac{W(x,y)}{y}$$

exists and is finite because $W(x,y)/y$ is decreasing in y. Second, let $\widehat{y}$ be the unique fixed point of $W(x,\cdot)$ provided by Lemma 3. Since $W(x,\widehat{y})/\widehat{y}=1$, (58) implies

$$(W(x,y)-y)(y-\widehat{y})<0,\qquad\forall y\neq\widehat{y}.\tag{59}$$

In other words, we have $W(x,y)<y$ iff $y>\widehat{y}$, and $W(x,y)>y$ iff $y<\widehat{y}$.

E Proofs in the Main Text

Proof of Proposition 1. By Lemma 3, $W(x,\cdot):\mathbb{R}_+\to\mathbb{R}_+$ has a fixed point for each $x\geq 0$. Let y^* be any fixed point of $W(\|Y\|_\infty,\cdot)$. Next we show that $T(X)\in[0,y^*]$ whenever $X\in[0,y^*]$. In fact, given $X\in[0,y^*]$, (W-i) and (M-ii) imply that

$$T(X)=W(Y,\mathcal{M}(X))\leq W(\|Y\|_\infty,y^*)=y^*.$$

We can therefore write $T:[0,y^*]\to[0,y^*]$. Observe that $[0,y^*]$ is a complete lattice with respect to the pointwise order $\leq$. Since the operator $T:[0,y^*]\to[0,y^*]$ is monotone, by the Tarski Fixed Point Theorem we conclude that T has a fixed point. ■

Proof of Theorem 1. Suppose that (i) holds. A routine argument based on (W-v) shows that $\|T(X_1)-T(X_2)\|_\infty\leq\beta\|X_1-X_2\|_\infty$ for all $X_1,X_2\in L_+^\infty$. By the Banach Contraction Mapping Theorem, there is a unique and globally attractive fixed point $\widehat{X}\in L_+^\infty$. Since

$$[Y]_\infty\leq Y\leq\|Y\|_\infty, \tag{60}$$

from Proposition 3 it follows that $\widehat{X}\in[y_*,y^*]$ because $\varphi([Y]_\infty)=y_*$ and $\varphi(\|Y\|_\infty)=y^*$.

Suppose that (ii) holds. We have $W(Y_t,\mathcal{M}_t(X))\leq W(\|Y\|_\infty,\mathcal{M}_t(y^*))=y^*$, and so we can write $T:[0,y^*]\to[0,y^*]$. Let us show that the conditions of Theorem 11 are satisfied (by setting $w\equiv 1$). Condition (i) of Theorem 11 is satisfied as $[Y]_\infty>0$, while condition (ii) is satisfied by $H=y^*$. Moreover, for all $\alpha\in[0,1]$ and all $X\in L_+^\infty$, we have:

$$\begin{aligned}T(\alpha X)&=W(Y,\mathcal{M}(\alpha X))\geq W(Y,\alpha\mathcal{M}(X))=W(Y,\alpha\mathcal{M}(X)+(1-\alpha)\mathcal{M}(0))\\&\geq\alpha W(Y,\mathcal{M}(X))+(1-\alpha)W(Y,\mathcal{M}(0))=\alpha T(X)+(1-\alpha)T(0),\end{aligned}$$

and so also condition (iii) of Theorem 11 is satisfied. Hence, by Theorem 11 there exists a unique fixed point $\widehat{X}\in[0,y^*]$, with

$$\left\|T^n(X)-\widehat{X}\right\|_\infty\to 0,\qquad\forall X\in[0,y^*].$$

We complete the proof by proving (17), which also implies that $\widehat{X}$ is the unique fixed point in L_+^∞. Let $\widetilde{X}\in L_+^\infty$ and set $\widetilde{y}=\left\|\widetilde{X}\right\|_\infty\vee y^*$. We have $\widetilde{X}\in[0,\widetilde{y}]$ and we can write $T:[0,\widetilde{y}]\to[0,\widetilde{y}]$. In fact, suppose $X\in[0,\widetilde{y}]$. Then, by (59), $\widetilde{y}\geq y^*$ implies:

$$T(X)\leq T(\widetilde{y})=W(Y,\widetilde{y})\leq W(\|Y\|_\infty,\widetilde{y})\leq\widetilde{y}$$

and so $T(X)\in[0,\widetilde{y}]$. Putting $H=\widetilde{y}$, by proceeding as before we can show that

$$\left\|T^n(X)-\widehat{X}\right\|_\infty\to 0,\qquad\forall X\in[0,\widetilde{y}],$$

and so $\left\|T^n\left(\widetilde{X}\right)-\widehat{X}\right\|_\infty\to 0$, as desired.

Finally, also here (60) and Proposition 3 imply $\widehat{X}\in[y_*,y^*]$. ■

Proof of Theorem 6. As $[Y]_\infty>0$, we can find an $\bar{\varepsilon}>0$ such that $[Y]_\infty>k\bar{\varepsilon}$. Define the new aggregator

$$W_1(x,y)=W(x+k\bar{\varepsilon},y+\bar{\varepsilon})-\bar{\varepsilon}.$$

W_1 is clearly monotone and, condition (i) implies that W_1 satisfies (W-ii). Hence W_1 is an aggregator that, in addition, satisfies (W-iv). Actually, $W_1(x,0) = W(x + k\overline{\varepsilon}, \overline{\varepsilon}) - \overline{\varepsilon} \geq W(k\overline{\varepsilon}, \overline{\varepsilon}) - \overline{\varepsilon} > 0$. To conclude, the concavity condition (ii) implies that W_1 is concave at 0. Consequently, W_1 is Thompson.

Let us study the fixed points of the operator

$$T_1(X) = W_1(Y - k\overline{\varepsilon}, \mathcal{M}(X)) = W(Y, \mathcal{M}(X + \overline{\varepsilon})) - \overline{\varepsilon}$$

By construction $[Y - k\overline{\varepsilon}]_\infty > 0$. Consequently, by Theorem 1 there is a unique positive solution, globally attracting on L^∞_+. Let $\widehat{X}_1$ be such a solution. It satisfies

$$\begin{aligned} \widehat{X}_1 &= W\left(Y, \mathcal{M}\left(\widehat{X}_1 + \overline{\varepsilon}\right)\right) - \overline{\varepsilon} \\ \widehat{X}_1 + \overline{\varepsilon} &= W\left(Y, \mathcal{M}\left(\widehat{X}_1 + \overline{\varepsilon}\right)\right). \end{aligned}$$

Hence $\widehat{X} = \widehat{X}_1 + \overline{\varepsilon}$ is a fixed point of our initial problem and it satisfies $\widehat{X} \geq \overline{\varepsilon}$. Clearly, it is unique for the function greater or equal to $\overline{\varepsilon}$ and it is globally convergent therein. Actually, let $X_0 \geq \overline{\varepsilon}$, and consider $\widetilde{X}_0 = X_0 - \overline{\varepsilon}$. Iterating T_1, we get $\widehat{X}_1$ and therefore the iterates of T approach $\widehat{X}$.

As $\overline{\varepsilon}$ can be chosen arbitrarily small, the desired result follows. ■

Proof of Proposition 3. Let $Y_1 \leq Y_2$. We prove by induction that $T^n_{Y_1}(0) \leq T_{Y_2}(0)$ for all $n \geq 1$. Clearly,

$$T_{Y_1}(0) = W(Y_1, 0) \leq W(Y_2, 0) = T_{Y_2}(0)$$

and the claim is then true when $n = 1$. Suppose that it is true for n; then,

$$\begin{aligned} T^{n+1}_{Y_1}(0) &= T_{Y_1}\left(T^n_{Y_1}(0)\right) = W\left(Y_1, \mathcal{M}\left(T^n_{Y_1}(0)\right)\right) \\ &\leq W\left(Y_2, \mathcal{M}\left(T^n_{Y_2}(0)\right)\right) = T^{n+1}_{Y_2}(0), \end{aligned}$$

as desired. Since $\widehat{X}_{Y_1}$ and $\widehat{X}_{Y_2}$ are globally attracting, we have $T^n_{Y_1}(0) \uparrow \widehat{X}_{Y_1}$ and $T^n_{Y_2}(0) \uparrow \widehat{X}_{Y_2}$, and so $\widehat{X}_{Y_1} \leq \widehat{X}_{Y_2}$.

Suppose W and $\mathcal{M}$ are concave on their domains. Set $\overline{\alpha} = 1 - \alpha$. We have

$$\begin{aligned} T_{\alpha Y_1 + \overline{\alpha} Y_2}(0) &= W(\alpha Y_1 + \overline{\alpha} Y_2, 0) \geq \alpha W(Y_1, 0) + \overline{\alpha} W(Y_2, 0) \\ &= \alpha T_{Y_1}(0) + \overline{\alpha} T_{Y_2}(0). \end{aligned}$$

By induction,

$$\begin{aligned} T^{n+1}_{\alpha Y_1 + \overline{\alpha} Y_2}(0) &= T_{\alpha Y_1 + \overline{\alpha} Y_2}\left(T^n_{\alpha Y_1 + \overline{\alpha} Y_2}(0)\right) \geq T_{\alpha Y_1 + \overline{\alpha} Y_2}\left(\alpha T^n_{Y_1}(0) + \overline{\alpha} T^n_{Y_2}(0)\right) \\ &= W\left(\alpha Y_1 + \overline{\alpha} Y_2, \mathcal{M}\left[\alpha T^n_{Y_1}(0) + \overline{\alpha} T^n_{Y_2}(0)\right]\right) \\ &\geq W\left(\alpha Y_1 + \overline{\alpha} Y_2, \alpha \mathcal{M} T^n_{Y_1}(0) + \overline{\alpha} \mathcal{M} T^n_{Y_2}(0)\right) \\ &\geq \alpha W\left(Y_1, \mathcal{M} T^n_{Y_1}(0)\right) + \overline{\alpha} W\left(Y_2, \mathcal{M} T^n_{Y_2}(0)\right) \\ &= \alpha T^{n+1}_{Y_1}(0) + \overline{\alpha} T^{n+1}_{Y_2}(0). \end{aligned}$$

Taking limits, we get the desired result. ■

Proof of Proposition 5. By the definition of ϕ we have

$$\phi_t(Y)(\omega) = W(Y_t(\omega), \mathcal{M}_t(\phi(Y))(\omega)),$$

and so, by replacing ω by $\sigma^{-1}\omega$ and t by $t+1$, we get

$$\begin{aligned} \phi_{t+1}(Y)\left(\sigma^{-1}\omega\right) &= W\left(Y_{t+1}\left(\sigma^{-1}\omega\right), \mathcal{M}_{t+1}(\phi(Y))\left(\sigma^{-1}\omega\right)\right) \\ \sigma\phi(Y) &= W(\sigma Y, \sigma \mathcal{M}(\phi(Y))) \end{aligned}$$

As the solution is unique, we have

$$\begin{aligned} \sigma\phi(Y) &= \phi(\sigma Y) \\ \phi(Y) &= \sigma^{-1}\phi(\sigma Y) \end{aligned} \tag{61}$$

and $\phi_1(Y) = \phi_0(\sigma Y) \circ \sigma$. We can generalize (61) as

$$\phi(Y) = \sigma^{-t}\phi\left(\sigma^t Y\right), \qquad \forall t \in \mathbb{Z}.$$

In turn this implies the desired result. ■

Proof of Proposition 2. By (18) and by Lemma 4, there is $\widetilde{y}$ such that

$$\frac{W\left(\|Y\|_w, y\right)}{y} \leq a_w^{-\frac{1}{\gamma}}, \qquad \forall y \geq \widetilde{y}. \tag{62}$$

Let $X \in \left[0, \widetilde{y} w^{\frac{1}{\gamma}}\right]$. Properties (W-i), (W-vi), (M-ii), and (M-v) imply that

$$\begin{aligned} \frac{T_t(X)(\omega)}{w_t^{\frac{1}{\gamma}}} &= \frac{1}{w_t^{\frac{1}{\gamma}}} W\left(Y_t(\omega), \mathcal{M}_t(X)(\omega)\right) \leq W\left(w_t^{-1} Y_t(\omega), w_t^{-\frac{1}{\gamma}} \mathcal{M}_t(X)(\omega)\right) \\ &\leq W\left(w_t^{-1} Y_t(\omega), w_t^{-\frac{1}{\gamma}} \mathcal{M}_t\left(\widetilde{y} w^{\frac{1}{\gamma}}\right)(\omega)\right) \\ &= W\left(w_t^{-1} Y_t(\omega), w_t^{-\frac{1}{\gamma}} S\left(\widetilde{y} w^{\frac{1}{\gamma}}\right)(\omega)\right) \\ &= W\left(w_t^{-1} Y_t(\omega), w_t^{-\frac{1}{\gamma}} \widetilde{y} w(t+1)^{\frac{1}{\gamma}}\right) \\ &\leq W\left(\|Y\|_w, a_w^{\frac{1}{\gamma}} \widetilde{y}\right) \leq a_w^{-\frac{1}{\gamma}}\left(\widetilde{y} a_w^{\frac{1}{\gamma}}\right) = \widetilde{y}, \end{aligned}$$

where the last inequality follows from (62) because $\widetilde{y} a_w^{\frac{1}{\gamma}} \geq \widetilde{y}$.

Hence, $T(X) \in \left[0, \widetilde{y} w^{\frac{1}{\gamma}}\right]$ and we can therefore write $T : \left[0, \widetilde{y} w^{\frac{1}{\gamma}}\right] \to \left[0, \widetilde{y} w^{\frac{1}{\gamma}}\right]$. By the Tarski Fixed Point Theorem, T has a fixed point because T is monotone and $\left[0, \widetilde{y} w^{\frac{1}{\gamma}}\right]$ is a complete lattice. ■

Proof of Theorem 2. Suppose that (i) holds. Let $X_1, X_2 \in L_+^w$. As $X_1 \leq X_2 + \|X_1 - X_2\|_w w$ and $X_2 \leq X_1 + \|X_1 - X_2\|_w w$, we have

$$\begin{aligned} \mathcal{M}(X_1) &\leq \mathcal{M}\left(X_2 + \|X_1 - X_2\|_w w\right) \leq \mathcal{M}(X_2) + \|X_1 - X_2\|_w S(w) \\ &\leq \mathcal{M}(X_2) + a_w \|X_1 - X_2\|_w w, \\ \mathcal{M}(X_2) &\leq \mathcal{M}\left(X_1 + \|X_1 - X_2\|_w w\right) \leq \mathcal{M}(X_1) + \|X_1 - X_2\|_w S(w) \\ & \quad\ \mathcal{M}(X_1) + a_w \|X_1 - X_2\|_w w, \end{aligned}$$

and so $|\mathcal{M}(X_1) - \mathcal{M}(X_2)| \leq a_w \|X_1 - X_2\|_w w$. Hence, for all $t \geq 1$ and P-a.e. $\omega \in \Omega$,

$$\begin{aligned} |T_t(X_1)(\omega) - T_t(X_2)(\omega)| &= |W\left(Y_t(\omega), \mathcal{M}_t(X_1)(\omega)\right) - W\left(Y_t(\omega), \mathcal{M}_t(X_2)(\omega)\right)| \\ &\leq \beta |\mathcal{M}_t(X_1)(\omega) - \mathcal{M}_t(X_2)(\omega)| \leq a_w \beta \|X_1 - X_2\|_w w, \end{aligned}$$

so that $\|T(X_1) - T(X_2)\|_w \leq a_w \beta \|X_1 - X_2\|_w$. This shows that T is a contraction w.r.t. $\|\cdot\|_w$ on L_+^w, and so the uniqueness and global attractivity of $\widehat{X}$ follows from the Banach Contraction Mapping Theorem.

Next suppose that (ii) holds. Using (18), we can proceed as in the proof of Proposition 2 to show that we can write $T:\left[0,\widetilde{y}w^{\frac{1}{\gamma}}\right]\to\left[0,\widetilde{y}w^{\frac{1}{\gamma}}\right]$. Let $\widetilde{y}$ be defined as in the proof of Proposition 2, and let $X\in\left[0,\widetilde{y}w^{\frac{1}{\gamma}}\right]$. Again by proceeding as in the proof of Theorem 2 we can show that $T(X)\in\left[0,\widetilde{y}w^{\frac{1}{\gamma}}\right]$, so that we can write $T:\left[0,\widetilde{y}w^{\frac{1}{\gamma}}\right]\to\left[0,\widetilde{y}w^{\frac{1}{\gamma}}\right]$.

It is now enough to show that the conditions of Theorem 11 are satisfied. Condition (i) of Theorem 11 is satisfied because of (23), while condition (ii) is satisfied by $H=\widetilde{y}w^{\frac{1}{\gamma}}$. In fact,

$$
\begin{aligned}
\frac{T_t\left(\widetilde{y}w^{\frac{1}{\gamma}}\right)(\omega)}{w_t^{\frac{1}{\gamma}}} &= \frac{1}{w_t^{\frac{1}{\gamma}}}W\left(Y_t(\omega),\mathcal{M}\left(\widetilde{y}w^{\frac{1}{\gamma}}\right)(\omega)\right)\\
&\le W\left(w_t^{-\frac{1}{\gamma}}Y_t(\omega),w_t^{-\frac{1}{\gamma}}\mathcal{M}\left(\widetilde{y}w^{\frac{1}{\gamma}}\right)(\omega)\right)\\
&= W\left(w_t^{-\frac{1}{\gamma}}Y_t(\omega),w_t^{-\frac{1}{\gamma}}S\left(\widetilde{y}w^{\frac{1}{\gamma}}\right)(\omega)\right)=W\left(w_t^{-\frac{1}{\gamma}}Y_t(\omega),w_t^{-\frac{1}{\gamma}}\widetilde{y}w(t+1)^{\frac{1}{\gamma}}\right)\\
&\le W\left(\|Y\|_w,\widetilde{y}a_w^{\frac{1}{\gamma}}\right)\le a_w^{-\frac{1}{\gamma}}\left(\widetilde{y}a_w^{\frac{1}{\gamma}}\right)=\widetilde{y},
\end{aligned}
$$

where the last inequality follows from (62) because $\widetilde{y}a_w^{\frac{1}{\gamma}}\ge\widetilde{y}$.

Finally, condition (iii) is satisfied thanks to (M-iv). We conclude that all conditions of Theorem 11 are satisfied, so that there exists a unique fixed point $\widehat{X}$ in $\left[0,\widetilde{y}w^{\frac{1}{\gamma}}\right]$. Moreover,

$$
\left\|T^n(X)-\widehat{X}\right\|_w\to 0,\qquad\forall X\in\left[0,\widetilde{y}w^{\frac{1}{\gamma}}\right],
$$

and by proceeding as in the last part of the proof of Theorem 1 we can show that (24) holds and that $\widehat{X}$ is the unique fixed point in L_+^w.

Finally, it remains to show that $\widehat{X}\in\left[y_*,y^ww^{\frac{1}{\gamma}}\right]$. First observe that $\widehat{X}\in\left[0,\widetilde{y}w^{\frac{1}{\gamma}}\right]$ for any $\widetilde{y}>y^w$. This easily implies $\widehat{X}\in\left[0,y^ww^{\frac{1}{\gamma}}\right]$, as desired. Moreover, if $X\ge y_*$, then

$$
T(X)=W(Y,\mathcal{M}(X))\ge W([Y]_\infty,y_*)=y_*
$$

and so we can write $T:\left[y_*,\widetilde{y}w^{\frac{1}{\gamma}}\right]\to\left[y_*,\widetilde{y}w^{\frac{1}{\gamma}}\right]$. Under (i), by the Banach Contraction Mapping Theorem we conclude that $\widehat{X}\in\left[y_*,y^ww^{\frac{1}{\gamma}}\right]$ because $\left[y_*,y^ww^{\frac{1}{\gamma}}\right]$ is a closed subset of L_+^w. Under (ii), observe that

$$
\widehat{X}=T_Y^n\left(\widehat{X}\right)\ge T_{[Y]_\infty}^n(0)\quad\text{and}\quad T_{[Y]_\infty}^n(0)\uparrow y_*,
$$

and so $\widehat{X}\ge y_*$, as desired. ■

Proof of Corollary 1. Suppose (ii) holds. As observed in Example 2, $a_{w_n}=1+n^{-1}$. By (27), there is $\overline{n}$ large enough so that $\left(1+n^{-1}\right)\left(\lim_{y\to+\infty}y^{-1}W(u,y)\right)^\gamma<1$. By (28), $\inf_t(t\vee n)^{-\frac{1}{\gamma}}W(dt,0)>0$ for all $n\ge\overline{n}$. Hence, by Theorem 2, for all $n\ge\overline{n}$ there exists a unique and globally attracting fixed point $\widehat{X}_n\in L_+^{w_n}$. Since $\|\cdot\|_{w_{n+1}}\le\|\cdot\|_{w_n}$, we have $L^{w_n}\subseteq L^{w_{n+1}}$. Hence, $\widehat{X}_n=\widehat{X}_{n+1}$ for all $n\ge\overline{n}$. In fact, by (24), $\left\|\widehat{X}_{n+1}-\widehat{X}_n\right\|_{w_{n+1}}=\left\|\widehat{X}_{n+1}-T^m\left(\widehat{X}_n\right)\right\|_{w_{n+1}}\overset{m}{\to}0$.

In view of all this, we can set $\widehat{X}=\widehat{X}_n$ for some $n\ge\overline{n}$. We have $\widehat{X}\in L_+^w$ and $\widehat{X}=T\left(\widehat{X}\right)$. Moreover, $\widehat{X}\in\left[y_d,y^n\left(1+n^{-1}\right)^\gamma\right]$ for all $n\ge\overline{n}$, where $W(u,y^n)=y^n\left(1+n^{-1}\right)^\gamma$. Set $W_n=\left(1+n^{-1}\right)^\gamma W$, so that $W_n(u,y^n)=y^n$. For all $n\ge\overline{n}$, we have $W_n\le W_{n+1}\le W$ and $\lim_m W_n^m(u,0)=y^n$ (see Lemma 3). Hence, $y^n\le y^{n+1}\le y^u$ for all $n\ge\overline{n}$. Let $y^n\uparrow y^*$. For all $y\in[y_1,y^u]$, we have

$$
0\le W(u,y)-W_n(u,y)\le W(u,y)\left[1-\left(1+n^{-1}\right)^\gamma\right]\le W(u,y^u)\left[1-\left(1+n^{-1}\right)^\gamma\right].
$$

Moreover,

$$\begin{aligned} W(u,y^*) - W_n(u,y^n) &= W(u,y^*) - W(u,y^n) + W(u,y^n) - W_n(u,y^n) \\ &\leq W(u,y^*) - W(u,y^n) + W(u,y^u)\left[1-\left(1+n^{-1}\right)^{\gamma}\right] \xrightarrow{n} 0 \end{aligned}$$

because W is continuous by Lemma 3. Hence, $W(u,y^*) = \lim_n W_n(u,y^n) = \lim_n y^n = y^*$, and we conclude that $y^* = y^u$.

We conclude that $\widehat{X} \leq y^n\left(1+n^{-1}\right)^{\gamma}$for all $n \geq \overline{n}$ implies $\widehat{X} \leq y^u$, and so $\widehat{X} \in [y_d, y^u]$.

As to (i), we can proceed as before by taking $\overline{n}$ large enough so that $\left(1+n^{-1}\right)\beta < 1$ and $\left(1+n^{-1}\right)\left(\lim_{y\to+\infty} y^{-1}W(u,y)\right)^{\gamma} < 1$. ■

Proof of Theorem 3. In view of Theorem 1, it is enough to observe that by Theorem 12 the operator $\mathcal{M}$ given by (29) is constant subadditive if ϕ is IARA, while by Corollary 4 $\mathcal{M}$ is subhomogeneous if ϕ is IRRA. ■

Proof of Theorem 4. In view of Theorem 2, it is enough to observe that $\mathcal{M}$ given by (29) satisfies (M-v), and by Theorem 12 it is constant subadditive if ϕ is IARA, while by Corollary 4 it is subhomogeneous if ϕ is IRRA. ■

Proof of Proposition 6. Suppose $Y^* = \{Y_t^*\}_t$ is a solution plan at $t=1$. We want to show that ${}_tY^*$ are solution plans of all time t problems.

By induction, suppose $\{{}_\tau Y^*\}_{\tau=1}^{t-1}$ is optimal for the time $\tau \leq t-1$ problem. Suppose ${}_t\widetilde{Y} \in \Gamma_t\left(Y_1^*,...,Y_{t-1}^*\right)$ is optimal for the time t problem; that is,

$$\varphi_t\left({}_t\widetilde{Y}\right) \geq \varphi_t\left({}_tY\right), \qquad \forall {}_tY \in \Gamma_t\left(Y_1^*,...,Y_{t-1}^*\right). \tag{63}$$

Since Γ is recursive, $\left(Y_{t-1}^*, {}_t\widetilde{Y}\right) \in \Gamma_{t-1}\left(Y_1^*,...,Y_{t-2}^*\right)$, and so $\varphi_{t-1}\left({}_{t-1}Y^*\right) \geq \varphi_{t-1}\left(Y_{t-1}^*, {}_t\widetilde{Y}\right)$. We then have $\varphi_t\left({}_tY^*\right) = \varphi_t\left({}_t\widetilde{Y}\right)$. In fact, by (63), $\varphi_t\left({}_tY^*\right) \neq \varphi_t\left({}_t\widetilde{Y}\right)$ implies $\varphi_t\left({}_tY^*\right) < \varphi_t\left({}_t\widetilde{Y}\right)$, and so, by dynamic consistency, $\varphi_{t-1}\left({}_{t-1}Y^*\right) < \varphi_{t-1}\left(Y_{t-1}^*, {}_t\widetilde{Y}\right)$, a contradiction. We conclude that $\varphi_t\left({}_tY^*\right) = \varphi_t\left({}_t\widetilde{Y}\right)$, and so ${}_tY^*$ is optimal for the time t problem. ■

Proof of Theorem 5. Let (c_t^*, A_t^*) be an optimal adapted solution, given an initial condition $A_1 \geq 0$. Set $J_t = V_t(c^*)$ and $c_t^* = \chi_t A_t^*$. In view of (39), $\chi_t \in [0,1]$. As observed by [10, p. 955], there is an adapted process $\{\xi_t\}_t$ such that $J_t = \xi_t A_t^*$. To ease notation, in the sequel we just write c and A in place of c^* and A^*. Clearly,

$$J_t = \left(c_t^\rho + \beta\left[\mathcal{M}_t\left(J_{t+1}\right)\right]^\rho\right)^{1/\rho}. \tag{64}$$

From (64), we have

$$\begin{aligned} J_t^\rho &= c_t^\rho + \beta\left[\mathcal{M}_t\left(J_{t+1}\right)\right]^\rho \\ \xi_t^\rho A_t^\rho &= c_t^\rho + \beta\left[\mathcal{M}_t\left(\xi_{t+1}A_{t+1}\right)\right]^\rho \\ \xi_t^\rho A_t^\rho &= c_t^\rho + \beta\left(A_t - c_t\right)^\rho\left[\mathcal{M}_t\left(\xi_{t+1}\left(1+R_{t+1}\right)\right)\right]^\rho \end{aligned}$$

Setting $\mu_t = \mathcal{M}_t\left(\xi_{t+1}\left(1+R_{t+1}\right)\right)$, we then have

$$\xi_t^\rho A_t^\rho = c_t^\rho + \beta\left(A_t - c_t\right)^\rho \mu_t^\rho. \tag{65}$$

By optimality, c_t must maximize the concave[17] scalar function $c \to c^\rho + \beta (A_t - c)^\rho \mu_t^\rho$ on $(0, A_t)$. The first order conditions give

$$\begin{aligned} c_t^{\rho-1} &= \beta (A_t - c_t)^{\rho-1} \mu_t^\rho \\ \chi_t^{\rho-1} &= \beta (1 - \chi_t)^{\rho-1} \mu_t^\rho . \end{aligned} \tag{66}$$

Note that (64) can be written as

$$\xi_t^\rho = \chi_t^\rho + \beta (1 - \chi_t)^\rho \mu_t^\rho . \tag{67}$$

Eliminating μ_t^ρ from (66) and (67), we get $\xi_t^\rho \chi_t^{1-\rho} = 1$. Namely,

$$\xi_t = \chi_t^{1-1/\rho} \tag{68}$$

Plugging (68) into μ_t of (66), we have

$$\begin{aligned} \chi_t^{\rho-1} &= \beta (1 - \chi_t)^{\rho-1} \left[\mathcal{M}_t \left(\chi_{t+1}^{1-1/\rho} (1 + R_{t+1}) \right) \right]^\rho \\ \chi_t^{1-1/\rho} &= \beta^{1/\rho} (1 - \chi_t)^{1-1/\rho} \mathcal{M}_t \left(\chi_{t+1}^{1-1/\rho} (1 + R_{t+1}) \right) . \end{aligned} \tag{69}$$

Putting $\chi_t = c_t / A_t$ and $\chi_{t+1} = c_{t+1} / A_{t+1}$ into (69), some tedious algebra finally gives

$$1 = \mathcal{M}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{\rho-1} (1 + R_{t+1}) \right]^{1/\rho} .$$

■

F Proof of Theorem 7

Theorem 7 follows from the results we prove in this section. We divide the argument depending on whether condition (i) or (ii) holds.

F.1 Condition (i)

Define the map

$$\Phi : L_+^\infty \to L_+^\infty$$

as $X \to \phi \circ X$ where ϕ is continuous, strictly increasing and $\phi(0) = 0$. The map $\Phi : L_+^\infty \to L_+^\infty$ is an homeomorphism. Actually, fix $X \in L_+^\infty$ and $\varepsilon > 0$. We have

$$\begin{aligned} -\|X - Y\| &\leq Y - X \leq \|X - Y\| \\ X - \|X - Y\| &\leq Y \leq X + \|X - Y\| \\ \phi(X - \|X - Y\|) &\leq \phi(Y) \leq \phi(X + \|X - Y\|) \\ \phi(X - \|X - Y\|) - \phi(X) &\leq \phi(Y) - \phi(X) \leq \phi(X + \|X - Y\|) - \phi(X) . \end{aligned}$$

As the range of values of X is bounded, by the uniform continuity of ϕ on bounded intervals, there exists a number $\delta > 0$ such that $\|X - Y\| < \delta$ implies $\phi(X + \|X - Y\|) - \phi(X) < \varepsilon$ and $\phi(X - \|X - Y\|) - \phi(X) > -\varepsilon$. Consequently,

$$|\phi(Y) - \phi(X)| < \varepsilon$$

[17]Here $\rho \leq 1$ is crucial, otherwise this function would be convex and the first order condition would characterize minima.

which means $\|\phi \circ Y - \phi \circ X\| < \varepsilon$. Note further that the inverse map of Φ is $X \to \phi^{-1} \circ X$.

Endow L_+^∞ with the new distance $d_\phi(X, Y) = \|\Phi f - \Phi g\|$. Clearly, this distance is equivalent to the norm-distance (they generate the same topology). However, the homeomorphism $\Phi : L_+^\infty \to L_+^\infty$ is not necessarily bi-Lipschitz. Therefore the contraction properties are rather different in the two metrics.

Given $\mathcal{M} : L_+^\infty \to L_+^\infty$, define the "conjugate" operator $\widetilde{\mathcal{M}} = \Phi \circ \mathcal{M} \circ \Phi^{-1}$. We are a position to formulate an abstract contraction theorem for the operator T.

Theorem 13 *Let $T(X) = Y + \beta\mathcal{M}(X)$. Then, the operator T has a unique and globally attracting fixed point provided:*

(i) $\widetilde{\mathcal{M}}$ *is non-expansive, i.e.,* $\left\|\widetilde{\mathcal{M}}(X_1) - \widetilde{\mathcal{M}}(X_2)\right\| \leq \|X_1 - X_2\|$,

(ii) the homeomorphism $X \to \beta f$ is a contraction for the metric d_ϕ, i.e., $d_\phi(\beta f_1, \beta f_2) \leq \mu d_\phi(X_1, X_2)$, with $\mu \in (0, 1)$

(iii) ϕ concave.

Proof. *Step 1.* From (i),

$$\begin{aligned}
\left\|\widetilde{\mathcal{M}}(X_1) - \widetilde{\mathcal{M}}(X_2)\right\| &\leq \|X_1 - X_2\| \\
\left\|\Phi\mathcal{M}\Phi^{-1}X_1 - \Phi\mathcal{M}\Phi^{-1}X_2\right\| &\leq \|X_1 - X_2\| \\
\|\Phi\mathcal{M}(X_1') - \Phi\mathcal{M}(X_2')\| &\leq \|\Phi f_1' - \Phi f_2'\| \\
d_\phi(\mathcal{M}(X_1'), \mathcal{M}(X_2')) &\leq d_\phi(X_1', X_2').
\end{aligned}$$

Consequently, the operator $\mathcal{M}$ is non-expansive for the metric d_ϕ as well.

Step 2. Let us prove that the translation $X \to X + Y$, with $Y \geq 0$, is non-expansive for the metric d_ϕ. Actually,

$$\begin{aligned}
\phi(X(\omega) + Y(\omega)) - \phi(X_1(\omega) + Y(\omega)) &\leq \phi(X(\omega)) - \phi(X_1(\omega)) \\
&= |\phi(X(\omega)) - \phi(X_1(\omega))|
\end{aligned}$$

if $X(\omega) \geq X_1(\omega)$, because the concave function ϕ has decreasing increments. Likewise, if $X(\omega) \leq X_1(\omega)$ we have

$$\phi(X(\omega) + Y(\omega)) - \phi(X_1(\omega) + Y(\omega)) \geq -|\phi(X(\omega)) - \phi(X_1(\omega))|.$$

Therefore,

$$|\phi(X + Y) - \phi(X_1 + Y)| \leq |\phi(X) - \phi(X_1)|$$

which implies $d_\phi(X + Y, X_1 + Y) \leq d_\phi(X, X_1)$ for all $X, X_1, Y \in L_+^\infty$.

Step 3. We prove that the operator T is contractive for the metric d_ϕ. Let $X_1, X_2 \in L_+^\infty$,

$$\begin{aligned}
d_\phi(T(X_1), T(X_2)) &= d_\phi(Y + \beta\mathcal{M}(X_1), Y + \beta\mathcal{M}(X_2)) \leq d_\phi(\beta\mathcal{M}(X_1), \beta\mathcal{M}(X_2)) \\
&\leq \mu d_\phi(\mathcal{M}(X_1), \mathcal{M}(X_2)) \leq \mu d_\phi(X_1, X_2).
\end{aligned}$$

We have hence the existence of a unique fixed point $\overline{X}$. Further, $d_\phi\left(T^n(X_0), \overline{X}\right) \to 0$, that is, $\left\|\Phi T^n(X_0) - \Phi\overline{X}\right\| \to 0$. In view of the fact that Φ is an homeomorphism, it follows $\left\|T^n(X_0) - \overline{X}\right\| \to 0$. ■

It remains to investigate the relation between conditions (i) and (ii) of this theorem, and the properties of the function ϕ.

Proposition 9 *A sufficient condition for (i) of Theorem 13 to hold is that ϕ be SDARA, i.e.,*

$$t \to -\frac{\phi''(t)}{\left[\phi'(t)\right]^2} \tag{70}$$

be non-increasing.

Proof. By definition of $\widetilde{\mathcal{M}}$, we have

$$\left(\widetilde{\mathcal{M}}_t X\right)(\omega) = \mathbb{E}_t\left(\phi \circ X_{t+1}\right).$$

Therefore, a sufficient condition for $\widetilde{\mathcal{M}}$ be non-expansive is that the generalized mean functions

$$M(x) = \phi\left[\sum_i p_i \phi^{-1}(x_i)\right]$$

be constant subadditive. By the Claim in the proof of Theorem 12, this is equivalent to the condition that the functions

$$g(t) = \phi^{-1}\left[x + \phi(t)\right]$$

are concave for every $x \geq 0$. From $x + \phi(t) = \phi(g(t))$, differentiating twice we get easily

$$g"(t) = \frac{\phi''(t)\left[\phi'(g(t))\right]^2 - \phi''(g(t))\left[\phi'(t)\right]^2}{\left[\phi'(g(t))\right]^3}.$$

Hence the condition is

$$-\frac{\phi''(t)}{\left[\phi'(t)\right]^2} \geq -\frac{\phi''(g(t))}{\left[\phi'(g(t))\right]^2}. \tag{71}$$

Observe that $g(t) \geq t$ for all $x \geq 0$. Consequently, (70) implies (71) and the claim is proven. ■

Proposition 10 *A sufficient condition to have condition (ii) of Theorem 13 is that*

$$\sup_{t \geq 0} \frac{\phi'(\beta t)}{\phi'(t)} < \beta^{-1}. \tag{72}$$

Proof. Let

$$h(t) = \phi\left(\beta\phi^{-1}(t)\right) \tag{73}$$

be over $[0, +\infty]$. It is increasing and its derivative is

$$h^{'}(t) = \frac{\beta\phi'(\beta u)}{\phi'(u)} \quad \text{with } u = \phi^{-1}(t).$$

From (72) it follows that h is Lipschitz with constant less than 1. Namely,

$$\left|\phi\left(\beta\phi^{-1}(t_1)\right) - \phi\left(\beta\phi^{-1}(t_2)\right)\right| \leq \mu\left|t_1 - t_2\right|.$$

Hence,

$$\begin{aligned}
\left|\phi\left(\beta\phi^{-1}(X_1)\right) - \phi\left(\beta\phi^{-1}(X_2)\right)\right| &\leq \mu\left|X_1 - X_2\right| \\
\left|\phi(\beta X_1') - \phi(\beta X_2')\right| &\leq \mu\left|\phi(X_1') - \phi(X_2')\right| \\
\left\|\phi(\beta X_1') - \phi(\beta X_2')\right\| &\leq \mu\left\|\phi(X_1') - \phi(X_2')\right\| \\
d_\phi(\beta X_1', \beta X_2') &\leq \mu d_\phi(X_1', X_2').
\end{aligned}$$

as desired. ■

We have thus concluded that:

Corollary 6 *Under the conditions: ϕ concave, SDARA, and*

$$\sup_{t\geq 0}\frac{\phi'(\beta t)}{\phi'(t)}<\beta^{-1}, \tag{74}$$

the operator T has a unique globally attracting fixed point.

The condition that ϕ be SDARA is compatible with both IRRA and DRRA (see examples below). In these cases the condition (72) it is easier to check in that the function $h(t)$ in (73) is concave or convex. Therefore (72) is equivalent to

$$\lim_{t\to+\infty}\frac{\phi'(\beta t)}{\phi'(t)}<\beta^{-1}$$

provided ϕ exhibits an increasing relative risk aversion. Similarly, the condition is

$$\lim_{t\to 0}\frac{\phi'(\beta t)}{\phi'(t)}<\beta^{-1}$$

if ϕ exhibits a decreasing relative risk aversion.

Condition (74) is somewhat related to the theory of regularly varying functions (see Huberman and Ross [13]) and it can be related as well to the relative risk aversion index, as shown for example by [13].

Proposition 11 *If ϕ of class C^2, a sufficient condition to have condition (74) is that*

$$\sup_{t\geq 0}R_\phi(t)<1.$$

If ϕ is IRRA, the condition

$$\lim_{t\to+\infty}R_\phi(t)<1$$

is a necessary and sufficient in order (74) to hold.

Proof. We have:

$$\phi'(t)=\phi'(1)\exp\left\{-\int_1^t\frac{R_\phi(\tau)}{\tau}d\tau\right\},$$

and so

$$\frac{\phi'(\beta t)}{\phi'(t)}=\exp\left\{\int_{\beta t}^t\frac{R_\phi(\tau)}{\tau}d\tau\right\}.$$

Set $\sup_{t\geq 0}R_\phi(t)=\lambda<1$. By the integral mean value theorem,

$$\frac{\phi'(\beta t)}{\phi'(t)}\leq\exp\left\{\lambda\int_{\beta t}^t\frac{d\tau}{\tau}\right\}=\exp\left\{\lambda\left(\log t-\log\beta t\right)\right\}.$$

Hence,

$$\frac{\phi'(\beta t)}{\phi'(t)}\leq\beta^{-\lambda}<\beta^{-1},$$

and the first claim is proved. Clearly, if $R_\phi(t)$ increases $\sup_{t\geq 0}R_\phi(t)=\lim_{t\to\infty}R_\phi(t)$ and the sufficiency is proved as well. Concerning the necessity, assume by contradiction that $\lim_{t\to\infty}R_\phi(t)=$

$\rho \geq 1$. Then for any arbitrarily small $\varepsilon > 0$, we have $R_\phi(t) \geq 1 - \varepsilon$ for $t \geq t_0$. Fixed a $\beta \in (0,1)$ for t large enough, $[\beta t, t] \subset [t_0, \infty]$. Therefore,

$$\begin{aligned}\int_{\beta t}^{t} \frac{r(\tau)\, d\tau}{\tau} &\geq -(1-\varepsilon)\log\beta \\ \frac{\phi'(\beta t)}{\phi'(t)} &\geq \exp\{-(1-\varepsilon)\log\beta\} \\ \frac{\phi'(\beta t)}{\phi'(t)} &\geq \beta^{-(1-\varepsilon)}.\end{aligned}$$

This last contradicts (74). Actually, from (74) it follows

$$\sup_{t\geq 0} \frac{\phi'(\beta t)}{\phi'(t)} < \lambda_1 < \beta^{-1}$$

for some λ_1. By setting

$$\bar{\varepsilon} = 1 + \frac{\log \lambda_1}{\log \beta} > 0,$$

we get

$$\sup_{t\geq 0} \frac{\phi'(\beta t)}{\phi'(t)} < \beta^{-(1-\bar{\varepsilon})},$$

a contradiction. ∎

F.2 Condition (ii)

Consider the conjugate operator $\widetilde{T} = \Phi \circ T \circ \Phi^{-1}$, namely

$$\begin{aligned}\widetilde{T}X &= \phi\left[Y + \beta\mathcal{M}\phi^{-1}(X)\right] \\ &= \phi\left[Y + \beta\phi^{-1}\widetilde{\mathcal{M}}X\right].\end{aligned}$$

Since ϕ is IRRA, $\widetilde{\mathcal{M}}$ is subhomogeneous. Since ϕ is SDARA, the operator $X \to \phi\left[Y + \beta\phi^{-1}(X)\right]$ is concave. Hence,

$$\begin{aligned}\widetilde{T}(\alpha X) &= \phi\left[Y + \beta\phi^{-1}\widetilde{\mathcal{M}}(\alpha X)\right] \geq \phi\left[Y + \beta\phi^{-1}\left(\alpha\widetilde{\mathcal{M}}(X)\right)\right] \\ &= \phi\left[Y + \beta\phi^{-1}\left(\alpha\widetilde{\mathcal{M}}(X) + (1-\alpha)\,0\right)\right] \\ &\geq \alpha\phi\left[Y + \beta\phi^{-1}\left(\widetilde{\mathcal{M}}(X)\right)\right] + (1-\alpha)\,\phi(Y) \\ &= \alpha\widetilde{T}(X) + (1-\alpha)\,\widetilde{T}(0).\end{aligned}$$

Moreover, as $T : \left[0, (1-\beta)^{-1}\|Y\|_\infty\right] \to \left[0, (1-\beta)^{-1}\|Y\|_\infty\right]$, we have

$$\begin{aligned}\widetilde{T}\left(\phi\left((1-\beta)^{-1}\|Y\|_\infty\right)\right) &= \phi\left(Y + \beta\phi^{-1}\left(\phi\left((1-\beta)^{-1}\|Y\|_\infty\right)\right)\right) \\ &= \phi\left(T\left((1-\beta)^{-1}\|Y\|_\infty\right)\right) \leq \phi\left((1-\beta)^{-1}\|Y\|_\infty\right).\end{aligned}$$

On the other hand,

$$\widetilde{T}(0) = \phi(Y) \geq \phi([Y]_\infty) > \phi(0) \geq 0,$$

so that all conditions of Theorem 11 holds. Then $\widetilde{T}$ has a unique fixed point $\widehat{X}$ with

$$\left\|\widetilde{T}^n(X) - \widehat{X}\right\|_\infty \to 0, \qquad \forall X \in \left[0, \phi\left((1-\beta)^{-1}\|Y\|_\infty\right)\right].$$

Hence, $\Phi^{-1} \circ \widehat{X}$ is the unique fixed point of T. As

$$X \in \left[0, (1-\beta)^{-1} \|Y\|_\infty\right] \Longleftrightarrow \phi(X) \in \left[0, \phi\left((1-\beta)^{-1} \|Y\|_\infty\right)\right],$$

we have:

$$\left\|\Phi \circ T^n(X) - \Phi \circ \Phi^{-1}\left(\widehat{X}\right)\right\|_\infty \to 0, \qquad \forall X \in \left[0, \frac{\|Y\|_\infty}{1-\beta}\right],$$

and so

$$\left\|T^n(X) - \Phi^{-1}\left(\widehat{X}\right)\right\|_\infty \to 0, \qquad \forall X \in \left[0, (1-\beta)^{-1} \|Y\|_\infty\right],$$

as desired. ■

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