THE CARLO ALBERTO NOTEBOOKS

Collegio Carlo Alberto

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Working Paper No. 96
December 2008
www.carloalberto.org

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Abstract This paper explores the dynamic dependence properties of a Lévy process, the Variance Gamma, which has non Gaussian marginal features and non Gaussian dependence. In a static context, such a non Gaussian dependence should be represented via copulas. Copulas, however, are not able to capture the dynamics of dependence. By computing the distance between the Gaussian copula and the actual one, we show that even a non Gaussian process, such as the Variance Gamma, can "converge" to linear dependence over time. Empirical versions of different dependence measures confirm the result.
Journal of Economic Literature Classification: C16, G12.

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 We thank Claudio Patrucco for having provided the marginal calibrations and Ksenia Rulik for computational assistance. Financial support from MIUR is gratefully acknowledged.

Introduction

In the financial literature, different univariate Lévy processes have been applied in order to model stock returns (as a reference for Lévy processes see for example Sato [7]). Their multivariate extensions are still under investigation and represent an open field of research. One of the most popular Lévy process in Finance is the Variance Gamma introduced by Madan and Seneta [5]. A multivariate extension has been introduced by Madan and Seneta themselves. It has been discussed, among others, by Cont and Tankov [1] and calibrated on data by Luciano and Schoutens [3]. A generalization of this multivariate process named α-VG has been introduced by Semeraro [8]. The generalization is able to capture independence and to span a wide range of dependence. For fixed margins it also allows to model various levels of dependence. This was impossible under the previous VG model.

The purpose of the present work is to provide an empirical analysis of the dependence structure of the α-VG process and its evolution over time. The α-VG process depends on three parameters for each margin $(\mu_j, \sigma_j, \alpha_j)$ and an additional common parameter a. The linear correlation coefficient is known in closed formula and its expression is independent of time. It can be proved (see Luciano and Semeraro [4]) that the process has also non linear dependence.

Powerful tools to study non linear dependence between random variables are copulas. In a seminal paper, Embrechts, Mc Neil and Straumann [2] invoked their use to represent both linear and non linear dependence. Copulas, which had been introduced in the late fifties in Statistics and had been used mostly by Actuaries, do answer static dependence representation needs. However, they cannot cover all the joint representation issues in Finance. One can always impose a specific copula, at a specific time, on given marginal distributions: however, for given infinitely divisible marginal distributions the condition that a copula has to satisfy in order to provide an infinitely divisible joint distribution are not known (see Cont and Tankov [1]).

At the opposite, if one starts from a multivariate stochastic process as a primitive entity, he seldom has the corresponding copula in closed form at every point in time. Indeed, copula knowledge at a single point in time does not help in representing dependence at later maturities. Apart from specific cases, such as the traditional Black Scholes or lognormal process, the copula of the process is time dependent. And reconstructing it from the evolution equation of the underlying process is not an easy task.

In order to describe the evolution of dependence over time we need a family of copulas $\{C_t, t \geq 0\}$. Most of the times, as in the VG case, it is neither possible to derive C_t from the expression of C_1 not to get C_1 in closed form. Anyway, via Sklar's Theorem [9], a numerical approximation of the copula at any time t can be obtained. The latter argument, together with the fact that for the α- VG process the linear correlation is constant in time, leads us to compare the α-VG empirical copula for different tenures t with the Gaussian one. We study the evolution over time of the distance between the empirical and the Gaussian copula as a measure of the corresponding evolution of non-linear dependence. The paper is organized as follows: section one reviews the VG model and its dependence; it illustrates how we reconstruct the empirical copula. Section two compares the approximating and actual copula, while section three concludes.

1 VG models

The VG univariate model for financial returns $X(t)$ has been introduced by Madan and Seneta [5]. It is a natural candidate for exploring multivariate extensions of Lévy processes and copula identification problems outside the Black Scholes case for a number of reasons:

- it can be written as a time changed Wiener process: before any time change then its copula is known in closed form;
- it is one of the simplest Lévy processes which present non Gaussian features at the marginal level, such as asymmetry and kurtosis;
- there is a well developed tradition of risk management implementations for it, at least in the univariate case.

Formally, let us recall that the VG is a three parameter Lévy process (μ,σ,α) with characteristic function

$$\psi_{X_{VG(t)}}(u) = [\psi_{X_{VG(1)}}(u)]^t = (1 - iu\mu\alpha + \frac{1}{2}\sigma^2\alpha u^2)^{-\frac{t}{\alpha}}. \tag{1}$$

The VG process has been generalized to the multivariate setting by Madan and Seneta themselves [5] and calibrated on data by Luciano and Schoutens [4]. This multivariate generalization has some drawbacks: it cannot generate independence and it has a dependence structure determined by the marginal parameters, one of which (α) must be common to each marginal process.

To overcome the problem, the multivariate VG process has been generalized to the α-VG process (Semeraro [8]). The latter can be obtained by time changing a multivariate Brownian motion with independent components by a multivariate subordinator with gamma margins.

Let Y_i, $i = 1,...,n$ and Z be independent real gamma processes with parameter respectively $(\frac{1}{\alpha_i} - a, \frac{1}{\alpha_i}), i = 1,...,n$ and $(a,1)$. The multivariate subordinator $\{\mathbf{G}(t), t \geq 0\}$ is defined by the following

$$\mathbf{G}(t) = (G_1(t),...,G_n(t))^T = (Y_1(t) + \alpha_1 Z(t),...,Y_n(t) + \alpha_n Z(t))^T, \tag{2}$$

where $\alpha_j > 0$ $j = 1,...,n$ are real parameters. Let W_i be independent Brownian motions with drift μ_i and variance σ_i. The $\mathbf{R}^n$ valued process $\mathbf{Y} = \{\mathbf{Y}(t), t > 0\}$ defined as:

$$\mathbf{X}(t) = (W_1(G_1(t)),...,W_n(G_n(t)))^T \tag{3}$$

where $\mathbf{G}$ is independent from $\mathbf{W}$ is an α-VG process.

It depends on three marginal parameters $(\mu_j, \sigma_j, \alpha_j)$ and an additional common parameter a. Its characteristic function is the following

$$\psi_{\mathbf{X}(t)}(\mathbf{u}) = \prod_{j=1}^{n}(1-\alpha_j(i\mu_j u_j - \tfrac{1}{2}\sigma_j^2 u_j^2))^{-t(\frac{1}{\alpha_j}-a)}(1-\sum_{j=1}^{n}\alpha_j(i\mu_j u_j - \tfrac{1}{2}\sigma_j^2 u_j^2))^{-ta}. \quad (4)$$

The usual multivariate VG obtains for $\alpha_j = \alpha$, $j = 1,...,n$ and $a = \frac{1}{\alpha}$.

For the sake of simplicity from now on we consider the bivariate case.

Since the marginal processes are VG, the corresponding distributions F_t^1 and F_t^2 can be obtained in a standard way, i.e. conditioning with respect to the marginal time change:

$$F_t^i(x_i) = \int_0^{+\infty}\Phi\left(\frac{x_i-\mu_i(w_i+\alpha_i z)}{\sigma_i\sqrt{w_i+\alpha_i z}}\right)f_{G(t)}(z)dz, \quad (5)$$

where $f_{G(t)}^i$ is the density of a gamma distribution with parameters $(\frac{t}{\alpha_i}, \frac{t}{\alpha_i})$. The expression for the joint distribution at time t, $F_t = F_{\mathbf{X}(t)}$, is:

$$F_t(x_1,x_2) = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\Phi\left(\frac{x_1-\mu_1(w_1+\alpha z)}{\sigma_1\sqrt{w_1+\alpha_1 z}}\right)\Phi\left(\frac{x_2-\mu_2(w_2+\beta z)}{\sigma_2\sqrt{w_2+\alpha_2 z}}\right) \quad (6)$$

$$\cdot f_{Y_1(t)}(w_1)f_{Y_2(t)}(w_2)f_{Z(t)}(z)dw_1dw_2dz, \quad (7)$$

where Φ is a standard Normal distribution function and $f_{Y_1(t)}$, $f_{Y_2(t)}$, $f_{Z(t)}$ are gamma distributions with parameters respectively: $(t(\frac{1}{\alpha_1}-a), \frac{1}{\alpha_1})$, $(t(\frac{1}{\alpha_2}-a), \frac{1}{\alpha_2})$ and $(ta, 1)$ (see Semeraro [8]).

1.1 Dependence Structure

In this section, we investigate the dependence or association structure of the α-VG process.

We know from Sklar's Theorem that there exists a copula such that any joint distribution can be written in terms of the unconditional marginal ones:

$$F_t(x_1,x_2) = C_t(F_t^1(x_1), F_t^2(x_2)). \quad (8)$$

The copula C_t can be obtained as:

$$C_t(u_1,u_2) = F_t((F_t^1)^{-1}(u_1), (F_t^2)^{-1}(u_2)), \quad (9)$$

where $(F_t^i)^{-1}$ is the generalized inverse of F_t^i, $i = 1,2$.

Since the joint and marginal distributions in 5 and 6 cannot be written in closed form, the corresponding copula and non linear dependence measures, such as Spearman's rho and Kendall's tau, cannot be obtained analytically.

The only measure of dependence one can find in closed form is the linear correlation coefficient, whose expression follows:

$$\rho^{\mathbf{X}(t)} = \frac{\mu_1\mu_2\alpha_1\alpha_2 a}{\sqrt{(\sigma_1^2+\mu_1^2\alpha_1)(\sigma_2^2+\mu_2^2\alpha_j)}}. \tag{10}$$

such a coefficient is independent of time, but depends on both the marginal and the common parameter a. For given marginal parameters the correlation is increasing in the parameter a. Since a has to satisfy the following bounds: $0 \le a \le min(\frac{1}{\alpha_1}, \frac{1}{\alpha_2})$ the maximal correlation allowed by the model corresponds to $a = min(\frac{1}{\alpha_1}, \frac{1}{\alpha_2})$.

However, it can be proved that linear dependence in not exhaustive, since even when $\rho = 0$ the components of the process can be dependent (see Luciano and Semeraro [4]). In order to study the whole dependence we should evaluate empirical versions of the copula obtained from 9 using the integral expression of the marginal and joint distributions in 5 and 6. A possibility which is open to the researcher, in order to find an approximating copula for the process at time t, is then the following:

- Fix a grid (u_i, v_i), $i = 1, ...N$ on the square $[0,1]^2$;
- for each $i = 1,...,N$ compute $(F_t^1)^{-1}(u_i)$ and $(F_t^2)^{-1}(v_i)$ by numerical approximation of the integral expression: let $(\tilde{F}_t^1)^{-1}(u_i)$ and $(\tilde{F}_t^2)^{-1}(v_i)$ be the numerical results;
- find a numerical approximation for the integral expression 6, let it be $\hat{F}_t(x_i, y_i)$;
- Find the approximated value of $C_t(u_i, v_i)$:

$$\hat{C}_t(u_i, v_i) = \hat{F}((\tilde{F}_t^1)^{-1}(u_i), (\tilde{F}_t^1)^{-1}(u_i)),\ i = 1, ...N.$$

We name the copula $\hat{C}_t$ numerical copula of the α-VG distribution at time t.

In order to discuss the behavior of non linear dependence we compare the empirical copula and the Gaussian one with the appropriate linear correlation coefficient, for different tenors t. We use the classical L^1 distance:

$$d_t(C_t, C_t') = \int_0^1 |C_t(u,v) - C_t'(u,v)| dudv, \tag{11}$$

It is easy to demonstrate that the distance d is coherent with concordance order, i.e. $C_t \prec C_t' \prec C_t''$ implies $d(C_t, C_t') \le d(C_t, C_t'')$ (see Nelsen [6]). It follows that the nearer the copulas are in terms of concordance, the nearer they are in terms of d_t. In fact the stance d_t has been used in order to define measure of dependence such as Schweizer and Wolff's sigma, see Nelsen [6]. Observe that the maximal distance among two copulas is $\frac{1}{6}$, i.e. the distance between the upper and lower Fréchet bounds.

Therefore for each t we:

- fix the marginal parameters and a linear correlation coefficient;
- find the numerical approximation $\hat{C}_t$ of the copula of the process over the prespecified grid;
- compute the distance between the numerical and Gaussian copula[2].

[2] Since we have the empirical copula only on a grid we use the discrete version of the previous distance.

2 Empirical investigation

2.1 Data

The procedure outlined above has been applied to a sample of seven major stock indices: S&P, Nasdaq, CAC 40, FTCE, Nikkei, Dax, Hang Seng. For each index we estimated the marginal VG parameters under the risk neutral measure, using Bloomberg quotes of the corresponding options with three months to expiry. For each index, six strikes (the closest to the initial price) were selected, and the corresponding option prices were monitored over a one hundred days window, from 7/14/06 to 11/30/06.

2.2 Selection of the α-VG parameters

We estimated the marginal parameters as follows: using the quotes of the first day only, we obtained the parameter values which minimized the mean square error between theoretical and observed prices, the theoretical ones being obtained by FRFT. We used the results as guess values for the second day, the second day results as guess values for the third day, and so on. The marginal parameters used here are the averags of the estimates over the etnire period. The previous procedure is intended to provide marginal parameters which are actually not dependent on an initial arbitrary guess and are representative of the corresponding stock index price, under the assumption that the latter is stationary over the whole time window. The marginal values for the VG processes are reported in the following table:

Asset	μ	σ	α
S&P	-0.65	0.22	0.10
Nasdaq	-0.67	0.11	0.13
CAC 40	-0.46	0.10	0.11
FTCE	-0.59	0.045	0.031
Nikkei	-0.34	0.16	0.10
DAX	-0.27	0.13	0.14
Hang Seng	-1.68	0.8	0.03

We perform our analysis using the marginal parameter reported above and the maximal correlation allowed by the model.

For each pair of assets the following table gives the values of ρ (upper entry) and $a = min\{\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\}$ (lower one) obtained using 10 corresponding to the maximal correlation. The correlation coefficients are:

2.3 Approximation results

We computed the empirical copula $\hat{C}_t$ for the following tenors: $t = 0.1, 1, 10, 100$. We report in table 2.3 below the distances d_t corresponding to each pair of stocks and each time t. In order to give a qualitative idea of the distances obtained we

Pair	0.1	1	10	100
S&P/Nasdaq	0.015	0.0098	0.0098	0.0097
S&P/CAC 40	0.022	0.0098	0.0097	0.0097
S&P/FTCE	0.0101	0.0085	0.0085	0.0085
S&P/Nikkei	0.037	0.0094	0.0091	0.0089
S&P/DAX	0.034	0.0092	0.0088	0.0087
S&P/Hang Seng	0.011	0.0083	0.0084	0.0084
Nasdaq/CAC 40	0.020	0.0095	0.0095	0.0094
Nasdaq/FTCE	0.010	0.0079	0.0079	0.0079
Nasdaq/Nikkei	0.0263	0.0088	0.0085	0.0083
Nasdaq/DAX	0.035	0.0092	0.0088	0.0087
Nasdaq/Hang Seng	0.010	0.0078	0.0079	0.0079
CAC 40/FTCE	0.010	0.0079	0.0079	0.0079
CAC 40/Nikkei	0.0261	0.0088	0.0085	0.0085
CAC 40/DAX	0.0273	0.0088	0.0085	0.0083
CAC 40/Hang Seng	0.010	0.0078	0.0079	0.0079
FTCE/Nikkei	0.0170	0.0078	0.0074	0.0072
FTCE/DAX	0.0165	0.0077	0.0072	0.0071
FTCE/Hang Seng	0.0097	0.0098	0.0098	0.0098
Nikkei/DAX	0.0201	0.0078	0.0074	0.0073
Nikkei/Hang Seng	0.012	0.0071	0.0071	0.0071
DAX/Hang Seng	0.0115	0.0069	0.0069	0.0069

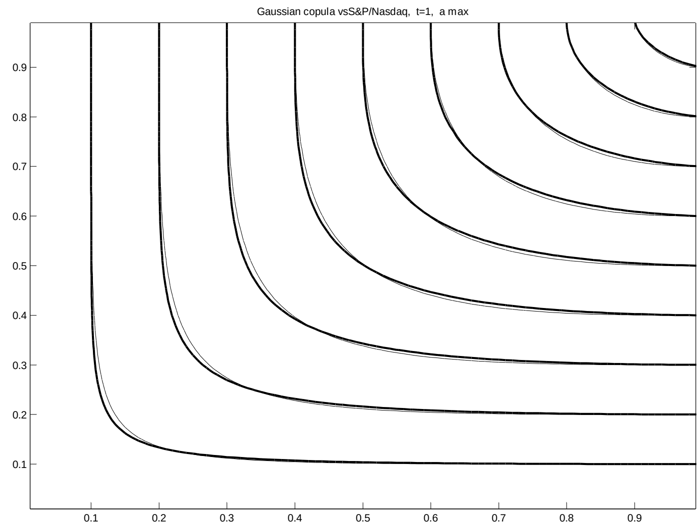
provide a graphical representation of the copula level curves for the pair Nasdaq and *S&P* at time $t = 1$.

Generally speaking, we observe that the distance in table 2.3 is very low and decreasing in time. Therefore the Gaussian copula seems to be a good approximation of the true copula, at least for long horizons.

2.4 Measures of dependence

In order to confirm our results we also compare two non linear dependence measures obtained simulating the copula with the corresponding ones of the Gaussian copula. For $t = 0.1, 1, 10, 100$ we computed the simulated values of Spearman's rho, $\tilde{\rho}_S(t)$, and Kendal's tau, $\tilde{\tau}(t)$. We found the analytic values of the Gaussian copula corresponding to each pair, by means of the relationships:

$$\rho_S = \frac{6}{\pi} \arcsin \frac{\rho}{2}; \quad \tau = \frac{2}{\pi} \arcsin \rho. \tag{12}$$



The results obtained are coherent with respect to our results, we report below the results concerning the first pair, the other behave in a similar way.

Pair		$\hat{C}_{0.1}$	$\hat{C}_1$	$\hat{C}_{10}$	$\hat{C}_{100}$	Gauss
$S\&P$/Nasdaq	$\tilde{\rho}_s$	0.74	0.78	0.79	0.79	0.79
	$\tilde{\tau}$	0.54	0.59	0.58	0.59	0.59

We observe that for t increasing both measures of dependence tend towards corresponding values for the Gaussian copula. This is coherent with the numerical result of the previous section.

3 Conclusions and further research

This paper measures the non linear dependence of the VG process, calibrated to a set of stock market data, by means of a distance between its empirical copula at time t and the corresponding Gaussian one, which is characterized by the (constant) correlation coefficient of the process.

The marginal parameters used in our analysis are the estimates obtained from seven major stock indices. The common parameter a is selected so as to consider maximal dependence.

Our empirical analysis suggests that non linear dependence is decreasing in time, since the approximation given by the Gaussian copula improves in time. Empirical non linear dependence coefficients confirm the result. The tentative conclusion is that, similarly to marginal non gaussianity, which is usually stronger on high frequencies than on weakly or monthly returns, joint non linear dependence and non gaussian -as captured by time changed processes like the VG- fades over time.

Appendix

α-VG process realizations (time 1).

- Simulate N realizations from the independent laws $\mathscr{L}(Y_1)$, $\mathscr{L}(Y_2)$, $\mathscr{L}(Z)$; let them be respectively x_1^n, x_2^n, z^n for $n = 1,...,N$;
- obtain N realizations (w_1^n, w_2^n) of $\mathbf{W}$ through the relations $W_1 = Y_1 + Z$ and $W_2 = Y_2 + Z$;
- generate N independent random draws from each of the independent random variables M_1 and M_2 with laws $N(0, W_1)$ and $N(0, W_2)$. The draws for M_1 in turn are obtained from N normal distributions with zero mean and variance w_1^n, namely
$$M_1(n) = N(0, w_1^n)$$
The draws for M_2 are from normal distributions with zero mean and variance w_2^n, namely
$$M_2(n) = N(0, w_2^n)$$
- obtain N realizations (y_1^n, y_2^n) of $\mathbf{Y}(1)$ by means of the relations
$$\begin{aligned} x_1^n &= \mu_1 w_1^n + \sigma_1 M_1(n) \\ x_2^n &= \mu_2 w_2^n + \sigma_2 M_2(n) \end{aligned}$$
where the parameters μ_j and σ_j, $j = 1,2$ depend on the specific model under consideration.

  (x_1^n, x_2^n) are the realizations of the pair
$$\begin{aligned} x_1(1) &= \mu_1 W_1 + \sigma_1 M_1 \\ x_2(1) &= \mu_2 W_2 + \sigma_2 M_2 \end{aligned}$$

Simulated measure of dependence

The simulated version of Spearman's rho, $\tilde{\rho}_S$, and Kendal's tau, $\tilde{\tau}_C(t)$, at time t, can be obtained from N realizations of the processes at time t $(X_1^i(t), X_2^i(t))$, $i = 1,...,N$:

$$\tilde{\rho}_S = 1 - 6\frac{\sum_{i=1}^{n}(R_i - S_i)^2}{n(n^2-1)}, \tag{13}$$

where $R_i = Rank(X_1^i(t))$ and $S_i = Rank(X_2^i(t))$.

$$\tilde{\tau}_C(t) = \frac{c-d}{\binom{N}{2}}, \tag{14}$$

where c is the number of concordance pairs of the sample and d the number of discordant ones. A pair $(X_1^i(t), X_2^i(t))$ is said to be discordant [concordant] if $X_1^i(t)X_2^i(t) \leq 0$ $[X_1^i(t)X_2^i(t) \geq 0]$. The N realizations of the process are obtained by the method described in the following section.

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