Doubts or Variability?

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Costs of Fluctuations

Question: what are the benefits of *further* reductions in aggregate fluctuations?

$$\sum_{t=0}^{\infty} \sum_{\epsilon^t} \beta^t \frac{C_t(\epsilon^t)^{1-\gamma}}{1-\gamma} \pi(\epsilon^t)$$

$$\log C_{t+1} = \mu + \log C_t + \sigma \epsilon_{t+1}$$

$$\epsilon_{t+1} \sim \mathcal{N}(0,1)$$

γ is coefficient of relative risk aversion.

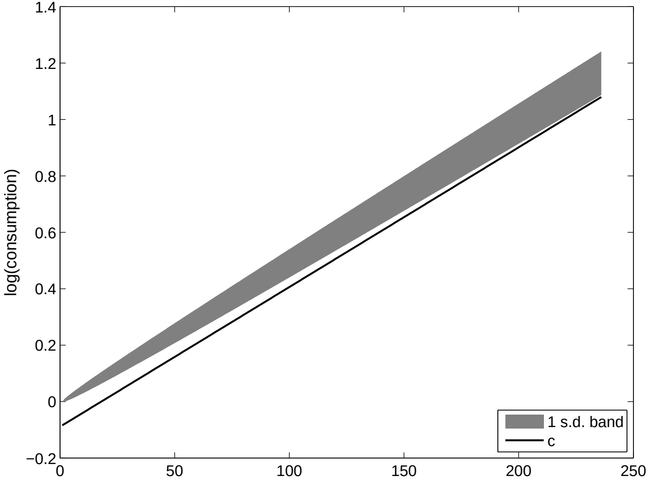

Figure: An elimination-of-risk experiment for the random walk model.

Key parameter γ

- You can use risk premia on risky assets to deduce γ.
- But Lucas (1987, 2003) doesn't like using asset prices to restrict γ because . . .

Motivation

"No one has found risk aversion parameters of 50 or 100 in the diversification of individual portfolios, in the level of insurance deductibles, in the wage premiums associated with occupations with high earnings risk, or in the revenues raised by state-operated lotteries. It would be good to have the equity premium resolved, but I think we need to look beyond high estimates of risk aversion to do it."

Robert E. Lucas, Jr., January 10, 2003

Outline

- Tallarini's graphs with time-separable CRRA and Kreps-Porteus preferences.
- Martingale representation of model uncertainty and its price.
- Reinterpreting Tallarini's γ: they are not prices of risk.
- Disciplining γ with detection error probabilities.
- High model uncertainty prices are silent about Lucas's experiment that eliminates *risk*.
- High model uncertainty prices are informative about a distinct experiment that eliminates *model specification uncertainty*.

The equity premium and risk-free rate puzzles

Formula for pricing one-period asset:

$$p_t = E_t \left(m_{t+1,t} x_{t+1} \right).$$

For CRRA preferences, $m_{t+1,t}$ is the marginal rate of substitution:

$$m_{t+1,t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

where γ is the coefficient of relative risk aversion. The risk free rate is

$$\frac{1}{r_t^f} = E_t \left[m_{t+1,t} \right] = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right].$$

Hansen and Jagannathan (1991) bound:

$$\frac{\left| E\left[\xi\right] \right|}{\sigma\left(\xi\right)} \leq \frac{\sigma\left(m\right)}{E\left[m\right]} \equiv MPR.$$

CRRA preferences and risk-free rate puzzle

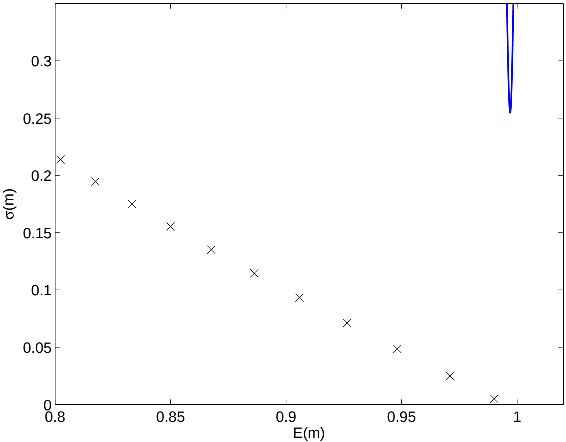

Figure: Solid line: Hansen-Jagannathan volatility bounds for quarterly returns on the value-weighted NYSE and Treasury Bill, 1948-2005. Crosses: Mean and standard deviation for intertemporal marginal rate of substitution for CRRA time separable preferences. The coefficient of relative risk aversion, γ takes on the values 1, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 and the discount factor β=0.995.

Shocks and consumption plans

- $c_t = \log C_t$
- $\varepsilon_{t+1}, t \geq 0$ a sequence of random shocks with conditional densities $\pi(\varepsilon_{t+1}|\varepsilon^t, x_0)$ and an implied joint density $\pi(\varepsilon^\infty|x_0)$.
- $\mathcal{C}$ set of consumption plans with time t component c_t a measurable function of (ε^t, x_0).
- Recursive representation of consumption plans:

$$\begin{aligned} x_{t+1} &= Ax_t + B\varepsilon_{t+1} \\ c_t &= Hx_t \end{aligned}$$

implies

$$c_t = H(B\varepsilon_t + AB\varepsilon_{t-1} + \cdots + A^{t-1}B\varepsilon_1) + HA^t x_0$$

Two good-fitting consumption processes

1. geometric random walk:

$$c_t = c_0 + t\mu + \sigma_\varepsilon(\varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_1), t \geq 1$$

$$\varepsilon_t \sim \pi(\varepsilon_t) \sim \mathcal{N}(0,1)$$

2. geometric trend stationary:

$$c_t = \rho^t c_0 + t\mu + \sigma_\varepsilon(\varepsilon_t + \rho\varepsilon_{t-1} + \cdots + \rho^{t-1}\varepsilon_1)$$

$$\varepsilon_t \sim \pi(\varepsilon_t) \sim \mathcal{N}(0,1)$$

Parameter estimates for geometric RW and TS

Table: Estimates from quarterly U.S. data 1948:2-2005:4

Parameter	Random Walk	Trend Stationary
μ	0.004974	0.004741
σ_ε	0.005111	0.005069
ρ	-	0.980008

Four types of agents I, II, III, and IV

1. Kreps-Porteus-Epstein-Zin.
2. Ambiguity averse Hansen-Sargent multiplier preferences.
3. Ambiguity averse Hansen-Sargent constraint preferences.
4. Pessimistic *ex post* Bayesian.

Objects defining preferences - Agent I

- a discount factor $\beta \in (0, 1)$.
- an intertemporal elasticity of substitution IES equal to unity.
- a risk aversion parameter $\gamma \geq 1$.
- a conditional density $\pi(\varepsilon_{t+1})$ for ε_{t+1} and an implied joint distribution $\pi(\varepsilon^\infty)$.

Objects defining preferences – Agent II

- a discount factor $\beta \in (0, 1)$.
- an intertemporal elasticity of substitution IES equal to unity.
- a risk aversion parameter $\gamma = 1$
- a conditional density $\pi(\varepsilon_{t+1})$ for ε_{t+1} and an implied joint distribution $\pi(\varepsilon^\infty)$.
- a parameter θ that penalizes the entropy associated with a minimizing player's perturbation of $\pi(\varepsilon^\infty)$.

Objects defining preferences – Agent III

- a discount factor $\beta \in (0, 1)$.
- an intertemporal elasticity of substitution IES equal to unity.
- a risk aversion parameter 1.
- a conditional density $\pi(\varepsilon_{t+1})$ for ε_{t+1} and an implied joint distribution $\pi(\varepsilon^\infty)$.
- a parameter η that measures the discounted relative entropy of perturbations to $\pi(\varepsilon^\infty)$ allowable to a minimizing player.

Objects defining preferences – Agent IV

- a discount factor $\beta \in (0, 1)$.
- an intertemporal elasticity of substitution IES equal to unity.
- a risk aversion parameter $\gamma = 1$.
- a unique pessimistic joint probability density $\hat{\pi}(\varepsilon^\infty; A, B, H, \theta)$ indexed by A, B, H, θ.

Observational equivalences

- Agents I and II are observationally equivalent in the strong sense that they have identical preferences over $\mathcal{C}$.
- Agent III and IV are observationally equivalent with I and II in the more restricted, but for us still very useful, sense that their valuations of risky assets coincide at the endowment process.

Behavior of agents

- Agent I: high precautionary savings motivated by high *risk aversion*.
- Agents II and III: high precautionary savings motivated by *fear of model misspecification*.
- Agent IV: high savings due to *pessimism* about rate of growth of consumption.
- In our pure endowment economy, these alternative saving motives get reflected in asset prices.

Agent I: Kreps-Porteus-Epstein-Zin

$$\log V_t = (1-\beta)c_t + \beta \log\left(E_t V_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}$$

IES $= 1$, atemporal risk-aversion $= \gamma \geq 1$. Define

$$U_t \equiv \log V_t / (1-\beta)$$

and

$$\theta = \frac{-1}{(1-\beta)(1-\gamma)} \quad \text{or} \quad \gamma = 1 + \frac{1}{(1-\beta)\theta}$$

Then

$$\boxed{U_t = c_t - \beta\theta \log E_t \left[\exp\left(\frac{-U_{t+1}}{\theta}\right)\right]}$$

$\gamma = 1$ (or $\theta = +\infty$) implies

$$U_t = c_t + \beta E_t U_{t+1}$$

Agent I cont'd

Bellman equation:

$$U(x) = c - \beta\theta \log E \exp\left(\frac{-U(Ax + B\varepsilon)}{\theta}\right)$$

Agent I cont'd

Solutions of Bellman equation:

Random walk:

$$U_t = \frac{\beta}{(1-\beta)^2}\left[\mu - \frac{\sigma_\varepsilon^2}{2\theta(1-\beta)}\right] + \frac{1}{1-\beta}c_t$$

Trend stationary:

$$U_t = \frac{\beta\mu}{(1-\beta)^2} - \frac{\sigma_\varepsilon^2\beta}{2\theta(1-\beta)(1-\beta\rho)^2} + \frac{\gamma\beta(1-\rho)}{(1-\beta\rho)(1-\beta)}t + \frac{1}{1-\beta\rho}c_t.$$

Arrow securities and SDF for Agent I

Price of Arrow security:

$$\left(\beta \frac{C_t}{C_{t+1}(\varepsilon_{t+1})}\right)\left(\frac{\exp\left(-U_{t+1}(\varepsilon_{t+1})/\theta\right)}{E_t\left[\exp\left(-U_{t+1}(\varepsilon_{t+1})/\theta\right)\right]}\right)\pi(\varepsilon_{t+1})$$

SDF:

$$m_{t+1,t} = \left(\beta \frac{C_t}{C_{t+1}}\right)\left(\frac{\exp\left(-U_{t+1}/\theta\right)}{E_t\left[\exp\left(-U_{t+1}/\theta\right)\right]}\right).$$

Formulas for risk-free rate and MPR

Random Walk Model:

$$r^f = \frac{1}{\beta} \exp \left[\mu - \frac{\sigma_\varepsilon^2}{2} (2\gamma - 1) \right]$$

$$\frac{\sigma(m)}{E[m]} = \left\{ \exp\left[\sigma_\varepsilon^2 \gamma^2\right] - 1 \right\}^{\frac{1}{2}}$$

A graph of Tallarini

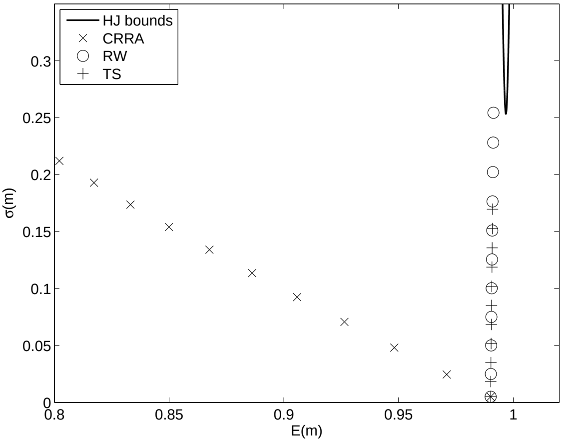

Figure: *Solid line:* Hansen-Jagannathan volatility bounds for quarterly returns on the value-weighted NYSE and Treasury Bill, 1948-2005. *Circles:* Mean and standard deviation for intertemporal marginal rate of substitution generated by Epstein-Zin preferences with random walk consumption. *Pluses:* Mean and standard deviation for stochastic discount factor generated by Epstein-Zin preferences and trend stationary consumption. *Crosses:* Mean and standard deviation for intertemporal marginal rate of substitution for CRRA time separable preferences. The coefficient of relative risk aversion, γ takes on the values 1, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 and the discount factor β=0.995.

Lucas's objection

"No one has found risk aversion parameters of 50 or 100 in the diversification of individual portfolios, in the level of insurance deductibles, in the wage premiums associated with occupations with high earnings risk, or in the revenues raised by state-operated lotteries. It would be good to have the equity premium resolved, but I think we need to look beyond high estimates of risk aversion to do it."

Our response: *distrust* . . ., meaning that we ask:

Can a small amount of distrust of the specification of a stochastic model substitute for a large amount of risk-aversion on the part of the representative consumer?

Martingale representation of model misspecifications

- Likelihood ratio $\{G_t : t \geq 0\}$ a nonnegative martingale where $G_0 = 1$.
- Form $\hat{\pi}(\varepsilon^t | x_0) \equiv G_t(\varepsilon^t)\pi(\varepsilon^t)$.
- Form the martingale increments

$$g_{t+1} = \begin{cases} \frac{G_{t+1}}{G_t} & \text{if} \quad G_t > 0 \\ 1 & \text{if} \quad G_t = 0. \end{cases}$$

- Note that $g_{t+1} \geq 0$, $E_t g_{t+1} = 1$, and $G_{t+1} = g_{t+1} G_t$ and

$$G_t = G_0 \prod_{j=1}^{t} g_j.$$

Martingales (contd)

$$\int W(\varepsilon^t, x_0)\hat{\pi}(\varepsilon^t, x_0)d\varepsilon^t = \int G(\varepsilon^t)W(\varepsilon^t, x_0)\pi(\varepsilon^t, x_0)d\varepsilon^t$$

or

$$\hat{E}W = EGW$$

g distorts conditional density, so

$$\hat{E}(W_{t+1}|\varepsilon^t, x_0) = E(g_{t+1}W_{t+1}|\varepsilon^t, x_0)$$

Entropy

Conditional entropy:

$$E[g_{t+1} \log g_{t+1} | \varepsilon^t, x_0]$$

Discounted entropy ball:

$$\beta E \left[\sum_{t=0}^{\infty} \beta^t G_t E(g_{t+1} \log g_{t+1} | \varepsilon^t, x_0) \middle| x_0 \right] \leq \eta$$

Agent II: ambiguity averse *multiplier* preferences

$$W(x_0) = \min_{\{g_{t+1}\}} \sum_{t=0}^{\infty} E\Big\{\beta^t G_t \Big[c_t + \beta\theta E(g_{t+1}\log g_{t+1}|\varepsilon^t, x_0)\Big]\Big|x_0\Big\}$$

$$\begin{aligned} x_{t+1} &= Ax_t + B\varepsilon_{t+1} \\ c_t &= Hx_t, \quad x_0 \ \text{given} \\ G_{t+1} &= g_{t+1}G_t, \ E[g_{t+1}|\varepsilon^t, x_0] = 1, g_{t+1} \geq 0, G_0 = 1 \end{aligned}$$

Bellman equation:

$$GW(x) = \min_{g(\varepsilon)\geq 0} G\Big(c + \beta \int \Big(g(\varepsilon)W(Ax + B\varepsilon) + \theta g(\varepsilon)\log g(\varepsilon)\Big)\pi(\varepsilon)d\varepsilon\Big)$$

Agent II: cont'd

Bellman equation:

$$W(x) = c + \min_{g(\varepsilon)\geq 0} \left(\beta \int \Big(g(\varepsilon) W(Ax + B\varepsilon) + \theta g(\varepsilon) \log g(\varepsilon)\Big) \pi(\varepsilon) d\varepsilon\right)$$

or

$$\boxed{W(x) = c - \beta\theta \log E \exp\left(\frac{-W(Ax + B\varepsilon)}{\theta}\right)}$$

Therefore,

$$W(x) \equiv U(x)$$

Agents I and II have identical preferences over elements of $\mathcal{C}$.

Agent III: ambiguity averse *constraint* preferences

$$\widetilde{W}(x_0, \eta) = \min_{\{g_{t+1}\}} \sum_{t=0}^{\infty} E\left[\beta^t G_t c_t \middle| x_0\right]$$

subject to

$$\beta E\left[\sum_{t=0}^{\infty} \beta^t G_t E(g_{t+1} \log g_{t+1} | \varepsilon^t, x_0) \middle| x_0\right] \leq \eta$$

$$\begin{aligned}
x_{t+1} &= Ax_t + B\varepsilon_{t+1} \\
c_t &= Hx_t, \quad x_0 \text{ given} \\
G_{t+1} &= g_{t+1} G_t, \; E[g_{t+1}|\varepsilon^t, x_0] = 1, g_{t+1} \geq 0, G_0 = 1.
\end{aligned}$$

Level curves

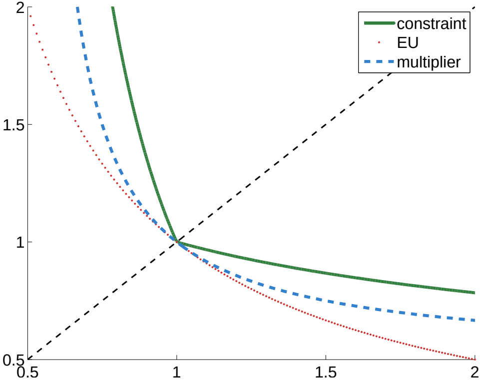

Figure: Indifference curves for expected utility, multiplier (smooth), and constraint (kinked at 45 degree line) preferences. The worst case probability $\hat{\pi}_1 < .5$ when $c_1 > c_2$ and $\hat{\pi}_1 > .5$ when $c_1 < c_2$.

Level curves

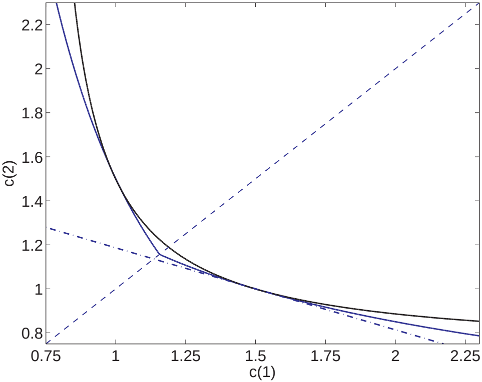

Figure: Indifference curves for multiplier (smooth) and constraint (kinked at 45 degree line) preferences. The worst case probability $\hat{\pi}_1 < .5$ when $c_1 > c_2$ and $\hat{\pi}_1 > .5$ when $c_1 < c_2$.

Minimizing distortion

Exponential tilting:

$$\hat{g}_{t+1} = \left(\frac{\exp\left(-W(Ax_t + B\varepsilon_{t+1})/\theta\right)}{E_t\left[\exp\left(-W(Ax_t + B\varepsilon_{t+1})/\theta\right)\right]} \right)$$

This is a stochastic version of Murphy's Law: events occur with probabilities in inverse proportion to their desirability.

For the geometric random walk model:

$$\hat{g}_{t+1} \propto \exp\left(\frac{-\sigma_\varepsilon \varepsilon_{t+1}}{(1-\beta)\,\theta}\right)$$

Worst case conditional density

The distorted conditional density is

$$\hat{\pi}\left(\varepsilon_{t+1}\right) \propto \exp\left(\frac{-\varepsilon_{t+1}^2}{2}\right)\exp\left(\frac{-\sigma_\varepsilon \varepsilon_{t+1}}{(1-\beta)\,\theta}\right)$$

Completing the square we get

$$\hat{\pi}(\varepsilon_{t+1}) \sim \mathcal{N}\left(\frac{-\sigma_\varepsilon}{(1-\beta)\,\theta}, 1\right)$$

Worst case models

Worst case conditional densities are $\hat{\pi}(\varepsilon_t) \sim \mathcal{N}(w(\theta), 1)$.
For the random walk model

$$w(\theta) = -\frac{\sigma_\varepsilon}{(1-\beta)\,\theta}$$

For the trend stationary model

$$w(\theta) = -\frac{\sigma_\varepsilon}{(1-\rho\beta)\,\theta}$$

Transmission across frequencies

Concerns about misspecification transform shocks that are *transitory* under the approximating models into shocks that are *permanent* under the worst-case models.

Agent IV: An *ex post* Bayesian

$$J(x_0) = E_0 \sum_{t=0}^{\infty} \beta^t c_t$$

subject to

$$\begin{aligned} x_{t+1} &= Ax_t + B\varepsilon_{t+1} \\ c_t &= Hx_t \end{aligned}$$

where

$$\varepsilon_{t+1} \sim \mathcal{N}(w(\theta), 1)$$

Agents III and IV have different preferences than Agents I and II, but . . .

For a *fixed* $\hat{\pi}(\varepsilon^\infty | A, B, H, \theta)$, the type IV pessimistic agent makes different choices over $\mathcal{C}$ (i.e., plans $(\tilde{A}, \tilde{B}, \tilde{H}) \neq (A, B, H)$) than does the type I or type II agent.

For the *particular* A, B, H plan and θ used to derive $\hat{\pi}(\varepsilon^\infty)$, the type IV agent evaluates risky claims exactly as does a type I or II agent.

Arrow securities prices for type IV agent

Price of Arrow security:

$$\left(\beta \frac{C_t}{C_{t+1}(\varepsilon_{t+1})}\right)\hat{\pi}(\varepsilon_{t+1}) = \left(\beta \frac{C_t}{C_{t+1}(\varepsilon_{t+1})}\right)\hat{g}(\varepsilon_{t+1})\pi(\varepsilon_{t+1})$$

Market price of model uncertainty (MPU)

SDF under distorted model:

$$m_{t+1,t} = \beta \frac{C_t}{C_{t+1}}.$$

SDF under approximating model:

$$m_{t+1,t} = \beta \frac{C_t}{C_{t+1}} \hat{g}(\varepsilon_{t+1}).$$

$$\mathrm{MPU} \equiv \mathrm{std}_t(\hat{g}) = [\exp(w(\theta)' w(\theta) - 1)]^{\frac{1}{2}} \approx |w(\theta)|$$

Risk free rate (random walk)

- Agents I, II, and III:

$$r^f = \frac{1}{\beta} \underbrace{\exp\left(\mu + \frac{1}{2}\sigma_\varepsilon^2\right)}_{\text{growth}} \underbrace{\exp\left(-\gamma\sigma_\varepsilon^2\right)}_{\text{extra precautionary saving}}$$

- Agent IV:

$$r^f = \frac{1}{\beta} \overbrace{\exp\left(\mu + \frac{1}{2}\sigma_\varepsilon^2 - \gamma\sigma_\varepsilon^2\right)} \overbrace{1}$$

Conditional and discounted entropy

$$\pi \sim \mathcal{N}(0,1), \quad \hat{\pi} \sim \mathcal{N}(-w(\theta),1)$$

$$\int \left(\log \hat{\pi}(\varepsilon) - \log \pi(\varepsilon)\right) \hat{\pi}(\varepsilon) d\varepsilon = \frac{1}{2} w'(\theta) w(\theta)$$

Discounted entropy:

$$\eta = \beta E\left[\sum_{t=0}^{\infty} \beta^t G_t E(g_{t+1} \log g_{t+1} | \varepsilon^t, x_0) \Big| x_0\right] = \frac{\beta}{2(1-\beta)} w'(\theta) w(\theta)$$

$$MPU = \sqrt{\frac{2\eta(1-\beta)}{\beta}}$$

Entropy constraint equalizing θ's

The following values of θ imply identical values of discounted entropy η for the TS and RW models:

$$\theta_{\text{TS}} = \frac{1-\beta}{1-\rho\beta}\theta_{\text{RW}}$$

Calibrating θ

"Beware of theorists bearing free parameters."
Robert E. Lucas, Jr.

We use a statistical theory of model selection to define a mapping from the entropy penalty parameter θ to the probability of a detection error in a statistical test for discriminating between the approximating model and the worst case model associated with that θ.

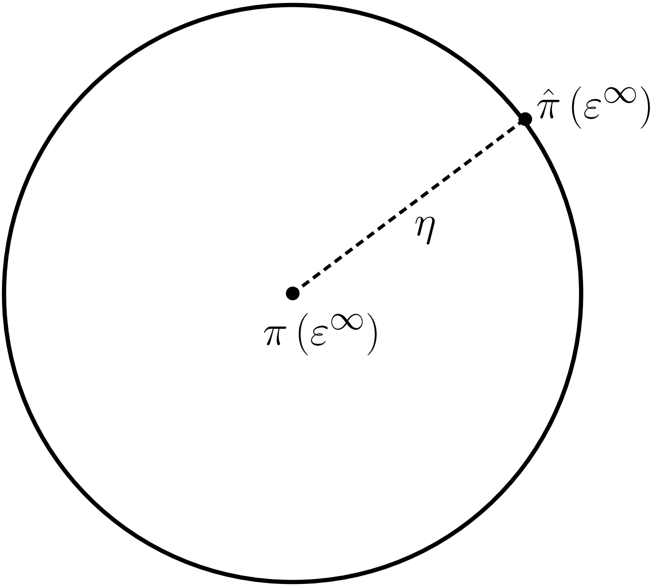

Figure: Detection error probabililites

Detection error probabilities

Form the log likelihood ratio

$$\log \frac{L_A}{L_B}$$

Likelihood ratio selects model A when $\log \frac{L_A}{L_B} > 0$ and model B when $\log \frac{L_A}{L_B} < 0$.

$$p_A = \Pr\left(\log \frac{L_A}{L_B} < 0\right) |A$$

$$p_B = \Pr\left(\log \frac{L_A}{L_B} > 0\right) |B$$

Probability of a detection error is

$$p(\theta) = \frac{1}{2}(p_A + p_B).$$

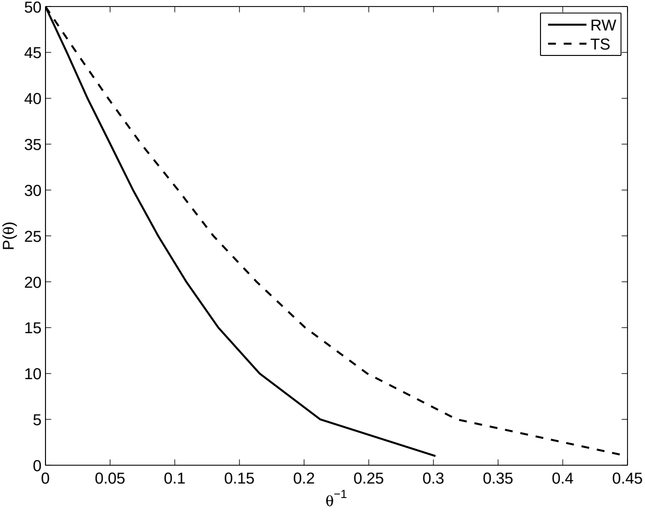

Figure: Detection probabilities versus θ^{-1} for the random walk and trend stationary models.

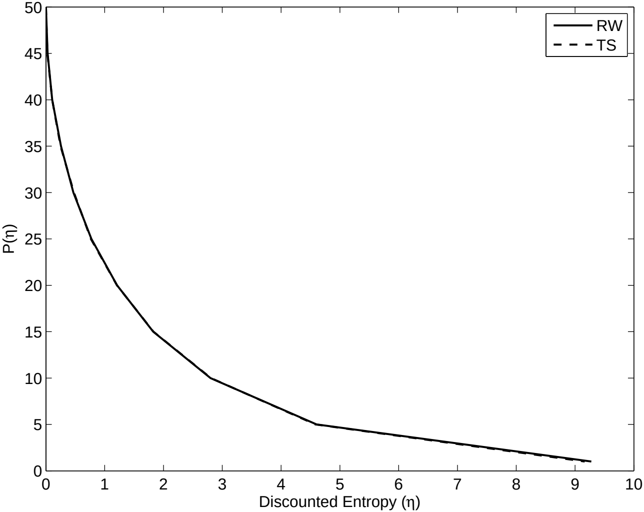

Figure: Detection probabilities versus discounted entropy η for the random walk and trend stationary models (the two curves coincide).

Tallarini's graph with detection error probabilities

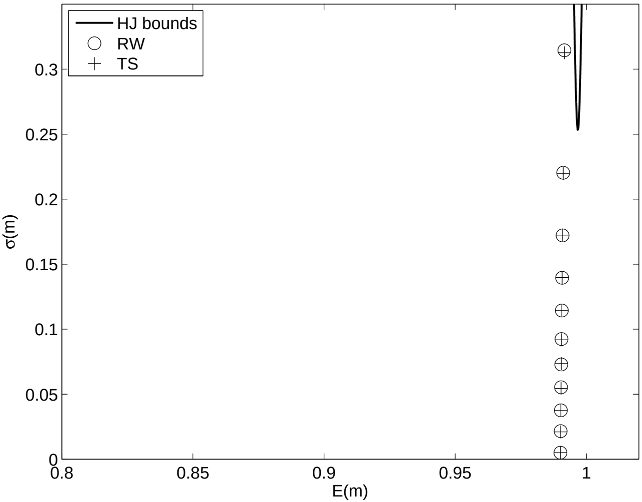

Figure: Reciprocal of risk free rate, market price of risk pairs for the random walk ($\circ$) and trend stationary ($+$) models for values of $p(\gamma)$ of 50, 45, 40, 35, 30, 25, 20, 15, 10, 5 and 1 percent.

An elimination-of-model uncertainty experiment

The MPU has information about the type II representative consumer's attitude about eliminating model uncertainty, not risk.

The type II representative consumer evaluates consumption plans *as if* he is a type IV agent who completely trusts a type II agent's consumption-plan-specific worst-case model.

For the particular consumption plan used to derive $\hat{\pi}$, the type II agent evaluates risky assets just like the type IV agent.

An elimination-of-model uncertainty experiment

Therefore, in so far as pricing risky assets *at the* (A, B, H) *consumption plan* is concerned, the type II representative consumer acts as if he is indifferent between:

1. The approximating model together with its model uncertainty (a big entropy η ball).
2. The worst case model without model uncertainty (a null $\eta = 0$ entropy ball).

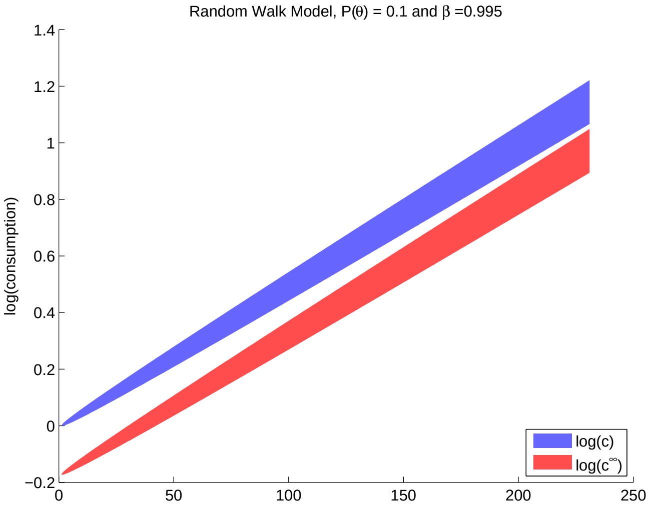

Figure: An elimination-of-model-uncertainty experiment for the random walk model. The type II agent is indifferent between the higher consumption distribution with model uncertainty and the lower consumption distribution without model uncertainty.

Figure: An elimination-of-model-uncertainty experiment for the random walk model. The type II agent is indifferent between the low growth consumption distribution implied by his worst case model and the high growth consumption distribution implied by his approximating model without model uncertainty.

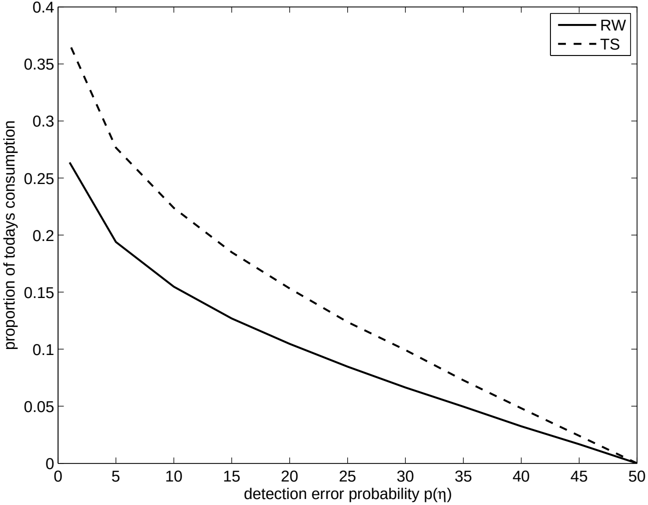

Figure: Proportion of initial consumption that a representative type II consumer would be willing to give up to live in a world without model uncertainty

Beyond high estimates of risk aversion

In response to Lucas's call that we "go beyond high estimates of risk aversion" to explain the equity premium, we have reinterpreted Tallarini's quantitative findings in terms of aversion to model specification uncertainty rather than to well understood risks.

Doing that destroys the Hansen-Sargent-Tallarini-Alvarez-Jermann link between the equity premium and Lucas-like measures of the benefits of eliminating business cycle fluctuations.

It creates a link between the equity premium and an experiment about eliminating model uncertainty.

Dogmatism and learning

Our consumer investigates the robustness of his valuations to alternative dogmatic priors on mean consumption growth.

Why not non-dogmatic priors and learning? Effects of robustness then?
How should you learn when you don't trust your model?
These are taken up in Hansen and Sargent 'Fragile Beliefs and the Market Price of Model Uncertainty'.