Exercises for Week 3: Suggested Solutions

Exercise 3.1:

We will show this by contradiction. Suppose, therefore, $\mathbf{y} \succ \mathbf{x}$ and recall that $A^+_{\mathbf{y}}$ and $A^-_{\mathbf{x}}$ denote the upper contour set of $\mathbf{y}$ and the lower contour set of $\mathbf{x}$, respectively.

Since both of these sets are closed (by assumption), their respective complements $X \setminus A^+_{\mathbf{y}} = \{\mathbf{z} \in X : \mathbf{y} \succ \mathbf{z}\}$ and $X \setminus A^-_{\mathbf{x}} = \{\mathbf{z} \in X : \mathbf{z} \succ \mathbf{x}\}$ are open. Notice, moreover, that $\mathbf{x} \in X \setminus A^+_{\mathbf{y}}$ and $\mathbf{y} \in X \setminus A^-_{\mathbf{x}}$. Now, if a point z lies within an open set, then any sequence that converges to z has a subsequence that lies entirely in the open set. Thus, for some $n_1, n_2 \in \mathbb{N}$, we have $\{\mathbf{x}^n\}_{n=n_1}^{\infty} \in X \setminus A^+_{\mathbf{y}}$ and $\{\mathbf{y}^n\}_{n=n_2}^{\infty} \in X \setminus A^-_{\mathbf{x}}$.

Let us now focus on the sequence $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ and on the upper contour set $A^+_{\mathbf{y}}$. Since a sequence has (countably) infinitely-many members, we will have either infinitely-many or finitely-many elements of $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ in the set $A^+_{\mathbf{y}}$. Of course, in the latter case, we will have finitely-many elements of $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ in the complement set $X \setminus A^+_{\mathbf{y}}$. More precisely, we have one of the following two cases:

(i) There exists $n_3 \in \mathbb{N}$ such that $\{\mathbf{y}^n\}_{n=n_3}^{\infty} \in A^+_{\mathbf{y}}$. But the elements of this subsequence give us a contradiction: for $n \geq \max\{n_1, n_3\}$, we have $\mathbf{y}^n \succsim \mathbf{y}$ and $\mathbf{y} \succ \mathbf{x}^n$; that is, $\mathbf{y}^n \succ \mathbf{x}^n$.

(ii) There exists a subsequence, say $\left\{\mathbf{y}^{k(n)}\right\}_{n\in\mathbb{N}}$, such that $\mathbf{y} \succ \mathbf{y}^{k(n)}$ for all $n \in \mathbb{N}$. But the set $X \setminus A^-_{\mathbf{y}^{k(n)}}$ is open and contains $\mathbf{y}$. Therefore, there exists some $n_4 \in \mathbb{N}$ such that the subsequence $\{\mathbf{y}^n\}_{n=n_4}^{\infty}$ gives $\mathbf{y}^n \succ \mathbf{y}^{k(n)}$ for all $n \geq n_4$. In other words, the subsequence $\{\mathbf{y}^n\}_{n=n_4}^{\infty}$ lies in the upper contour set of $\mathbf{y}^{k(n)}$ which is closed and should, therefore, contain also its limit; i.e. $\mathbf{y} \succsim \mathbf{y}^{k(n)}$ for all $n \in \mathbb{N}$. But this is a contradiction for any $k(n) \geq n_4$.

NOTE: A much easier way of proving the claim of this exercise can be constructed if we assume that the preference relation $\succsim$ is monotone. Recall that a rational, monotone, and continuous preference relation admits a continuous utility function. If you read carefully through the proof of this result, you will realize that, with respect to the continuity of preferences, we actually made use only of the closeness of the upper and lower contour sets. Hence, given that the upper and lower contour sets are closed and that the preference relation is rational and monotone, we have a continuous utility function.

It remains, therefore, to show that, if a preference relation admits a continuous utility function, it must be continuous (which is actually Exercise MWG 3.C.2). To see this, let $u : X \to \mathbb{R}$ represent $\succsim$ on X and suppose that $u(\cdot)$ is continuous on X. Take a sequence of pairs $\{\mathbf{x}^n, \mathbf{y}^n\}_{n\in\mathbb{N}}$ such that $\mathbf{x}^n \to \mathbf{x}$, $\mathbf{y}^n \to \mathbf{y}$, and $\mathbf{x}^n \succsim \mathbf{y}^n$ for every n. Since $u(\mathbf{x}^n) \geq u(\mathbf{y}^n)$ for all n, the continuity of $u(\cdot)$ implies that $u(\mathbf{x}) \geq u(\mathbf{y})$; equivalently, $\mathbf{x} \succsim \mathbf{y}$ as required.[1]

Exercise 3.2:

Part (i) is trivial since $\mathbf{x} > \mathbf{y}$ is a stronger vector relation than $\mathbf{x} \geq \mathbf{y}$. For part (ii), take any $\epsilon > 0$ and consider the open ball $B_{\mathbf{y}}(\epsilon)$. Observe that $\mathbf{x} \in \mathbb{R}^L_+$ with $x_l = y_l + \frac{\epsilon}{2}$ for $l \in \{1, ..., L\}$ lies in $B_{\mathbf{y}}(\epsilon)$. Moreover, $\mathbf{x} > \mathbf{y}$ and, since the preference relation is monotone, $\mathbf{x} \succ \mathbf{y}$.

Exercise 3.3:

Take $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ and suppose that $u : \mathbb{R}^L_+ \to \mathbb{R}$ represents preferences while $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing. If $\mathbf{x} \geq \mathbf{y}$, by $u(\cdot)$ being increasing, we get $u(\mathbf{x}) \geq u(\mathbf{y})$. Thus, by $f(\cdot)$ being strictly increasing, $f(u(\mathbf{x})) \geq f(u(\mathbf{y}))$. This establishes that $f(u(\cdot))$ is also increasing on $\mathbb{R}^L_+$.

If $u : \mathbb{R}^L_+ \to \mathbb{R}$ is quasi-concave, we have $u(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq \min\{u(\mathbf{x}), u(\mathbf{y})\}$, for all $\alpha \in [0, 1]$. Without loss of generality, let $u(\mathbf{x}) = \min\{u(\mathbf{x}), u(\mathbf{y})\}$. Since $f(\cdot)$ is strictly increasing, $u(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq u(\mathbf{x})$ implies $f(u(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})) \geq f(u(\mathbf{x}))$ and $f(u(\mathbf{x})) = \min\{f(u(\mathbf{x})), f(u(\mathbf{y}))\}$. That is, $f(u(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})) \geq \min\{f(u(\mathbf{x})), f(u(\mathbf{y}))\}$ which establishes that $f(u(\cdot))$ is also quasi-concave on $\mathbb{R}^L_+$.

[1] Here, we are making use of the following result: if the sequence of real numbers $\{u_n\}_{n\in\mathbb{N}}$ is such that $u_n \geq 0$ for every $n \in \mathbb{N}$, then $\lim_{n\to\infty} u_n \geq 0$. This is trivial to show by contradiction. Consider the real sequence $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \geq 0$ for all $n \in \mathbb{N}$ and let $u_n \to u_0$. If $u_0 < 0$, then there must exist $M < 0$ such that $u_0 < M$. Moreover, $|u_n - u_0| = u_n - u_0 > u_n - M \geq -M$ which contradicts the fact that the sequence converges to u_0.

Exercise 3.4:

(a). Suppose first that $u(\cdot)$ is is homogenous of degree one (HoD(1)). Let $\alpha > 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ with $\mathbf{x} \sim \mathbf{y}$. Since $u(\mathbf{x}) = u(\mathbf{y})$, we get $\alpha u(\mathbf{x}) = \alpha u(\mathbf{y})$. By $u(\cdot)$ being HoD(1), the last equality is equivalent to $u(\alpha\mathbf{x}) = u(\alpha\mathbf{y})$. Thus, $\alpha\mathbf{x} \sim \alpha\mathbf{y}$.

Suppose now that the preference relation $\succsim$ is homothetic. Consider the utility function constructed in MWG Proposition 3.C.1. We have $u(\mathbf{x})\mathbf{e} \sim \mathbf{x}$ and $u(\alpha\mathbf{x})\mathbf{e} \sim \alpha\mathbf{x}$. By the homotheticity and transitivity of $\succsim$, moreover, $\alpha\mathbf{x} \sim \alpha u(\mathbf{x})\mathbf{e}$. Thus, by transitivity, $\alpha u(\mathbf{x})\mathbf{e} \sim u(\alpha\mathbf{x})\mathbf{e}$. And by construction of the function $u(\cdot)$, $u(\alpha\mathbf{x}) = \alpha u(\mathbf{x})$.

(b). Suppose first that $\succsim$ is represented by a utility function of the form $u(\mathbf{x}) = x_1 + \phi(\mathbf{x}_{-1})$.[2] Let $\alpha \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ with $\mathbf{x} \sim \mathbf{y}$. Then $u(\mathbf{x}) = u(\mathbf{y})$ and $\alpha + u(\mathbf{x}) = \alpha + u(\mathbf{y})$. Moreover,

$$\begin{aligned} u(\mathbf{x}) + \alpha &= x_1 + \alpha + \phi(\mathbf{x}_{-1}) = u(\mathbf{x} + \alpha\mathbf{e}_1) \\ u(\mathbf{y}) + \alpha &= y_1 + \alpha + \phi(\mathbf{y}_{-1}) = u(\mathbf{y} + \alpha\mathbf{e}_1) \end{aligned}$$

Since $u(\mathbf{x} + \alpha\mathbf{e}_1) = u(\mathbf{y} + \alpha\mathbf{e}_1)$, we get $\mathbf{x} + \alpha\mathbf{e}_1 \sim \mathbf{y} + \alpha\mathbf{e}_1$.

Suppose now that $\succsim$ is quasilinear with respect to the first commodity. We will proceed along the same line of argument as in the proof of Proposition MWG 3.C.1. Specifically, we will construct the utility function of the desired form by reducing the general problem of comparing different commodity bundles in $\mathbb{R}^L_+$ to the one of comparing commodity bundles on a line. We will do this by finding the bundles of indifference on this line and assigning utility levels along it.

The proof comprises of two steps. First, we will establish that comparison of bundles can be reduced to a line parallel to the vector $\mathbf{e}_1$. This requires Claims 1-3 below (more precisely, the argument is really based on Claim 3 but its proof uses Claim 2 whose proof, in turn, makes use of Claim 1).

Let $v(\cdot)$ be a utility function for the preferences $\succsim$ (its existence is guaranteed by Proposition MWG 3.C.1; yet, it need not be of the quasilinear form).

[2] In the text, I use the following notation $\mathbf{z}_{-l} = (x_1, ..., x_{l-1}, x_{l+1}, ..., x_L)$ for a L-dimensional vector $\mathbf{x}$. Moreover, $\mathbf{e}_l$ is the vector that has 1 at the lth entry and zeros everywhere else.

Consider varying the quantity of the first commodity for a given bundle of the remaining L-1 commodities. More precisely, given $\mathbf{x}_{-1}^* \in \mathbb{R}_+^{L-1}$, define $\mathrm{I}\left(\mathbf{x}_{-1}^*\right) = \left\{v\left(x_1, \mathbf{x}_{-1}^*\right) \in \mathbb{R} : x_1 \in \mathbb{R}_+\right\}$. Since $\succsim$ is continuous on $\mathbb{R}_+^{L-1}$ and strongly monotone along $\mathbf{e}_1$, $\mathrm{I}\left(\mathbf{x}_{-1}^*\right)$ is an non-empty interval in $\mathbb{R}$.

Claim 1 *For any* $\mathbf{x}_{-1}^*, \mathbf{x}_{-1}' \in \mathbb{R}_+^{L-1}$,

$$I\left(\mathbf{x}_{-1}^*\right) \neq I\left(\mathbf{x}_{-1}'\right) \quad \Rightarrow \quad I\left(\mathbf{x}_{-1}^*\right) \cap I\left(\mathbf{x}_{-1}'\right) = \varnothing$$

Proof. Let $\mathrm{I}\left(\mathbf{x}_{-1}^*\right) \neq \mathrm{I}\left(\mathbf{x}_{-1}'\right)$. Since none of $\mathrm{I}\left(\mathbf{x}_{-1}^*\right), \mathrm{I}\left(\mathbf{x}_{-1}'\right)$ is empty, there must exist $v_0 \in \mathbb{R}$ such that $v_0 \in \mathrm{I}\left(\mathbf{x}_{-1}^*\right)$ but $v_0 \notin \mathrm{I}\left(\mathbf{x}_{-1}'\right)$. Therefore, either $v_0 \geq \sup \mathrm{I}\left(\mathbf{x}_{-1}'\right)$ or $v_0 \leq \inf \mathrm{I}\left(\mathbf{x}_{-1}'\right)$ (since $\mathrm{I}\left(\mathbf{x}_{-1}'\right)$ is an open interval, $\sup \mathrm{I}\left(\mathbf{x}_{-1}'\right), \inf \mathrm{I}\left(\mathbf{x}_{-1}'\right) \notin \mathrm{I}\left(\mathbf{x}_{-1}'\right)$).

Suppose that $v_0 \geq \sup \mathrm{I}\left(\mathbf{x}_{-1}'\right)$ (the argument for the case $v_0 \leq \inf \mathrm{I}\left(\mathbf{x}_{-1}'\right)$ is trivially similar). Take x_1^0 such that $v_0 = v\left(x_1^0, \mathbf{x}_{-1}^*\right)$. Since $v_0 \geq \sup \mathrm{I}\left(\mathbf{x}_{-1}'\right)$, we have $v\left(x_1^0, \mathbf{x}_{-1}^*\right) > v\left(x_1, \mathbf{x}_{-1}'\right)$ for every $x_1 \in \mathbb{R}_+$. Equivalently, $\left(x_1^0, \mathbf{x}_{-1}^*\right) \succ \left(x_1, \mathbf{x}_{-1}'\right)$ for every $x_1 \in \mathbb{R}_+$. Take now any $y_1, y_1' \in \mathbb{R}$. We have $\left(x_1^0, \mathbf{x}_{-1}^*\right) \succ \left(y_1 + x_1^0 - y_1', \mathbf{x}_{-1}'\right)$ and, by the quasilinearity of $\succsim$, $\left(x_1^0 - \left(x_1^0 - y_1'\right), \mathbf{x}_{-1}^*\right) \succ \left(y_1, \mathbf{x}_{-1}'\right)$. That is, $\left(y_1', \mathbf{x}_{-1}^*\right) \succ \left(y_1, \mathbf{x}_{-1}'\right)$ for any $y_1', y_1 \in \mathbb{R}_+$. Equivalently, $v\left(y_1', \mathbf{x}_{-1}^*\right) > v\left(y_1, \mathbf{x}_{-1}'\right)$ for any $y_1', y_1 \in \mathbb{R}_+$. Since every member of the interval $\mathrm{I}\left(\mathbf{x}_{-1}^*\right)$ exceeds every one of $\mathrm{I}\left(\mathbf{x}_{-1}'\right)$, $\mathrm{I}\left(\mathbf{x}_{-1}^*\right) \cap \mathrm{I}\left(\mathbf{x}_{-1}'\right) = \varnothing$. ∎

For each $\mathbf{x}_{-1} \in \mathbb{R}_+^{L-1}$ define $\mathrm{E}\left(\mathbf{x}_{-1}\right) = \left\{\mathbf{x}_{-1}^* \in \mathbb{R}_+^{L-1} : \mathrm{I}\left(\mathbf{x}_{-1}\right) = \mathrm{I}\left(\mathbf{x}_{-1}^*\right)\right\}$.

Claim 2 *For every* $\mathbf{x}_{-1} \in \mathbb{R}_+^{L-1}$, *the set* $E\left(\mathbf{x}_{-1}\right)$ *is open in* $\mathbb{R}_+^{L-1}$.

Proof. Take any $\mathbf{x}_{-1}^* \in \mathrm{E}\left(\mathbf{x}_{-1}\right)$. For $x_1 \in \mathbb{R}_+$, let $v = v\left(x_1, \mathbf{x}_{-1}^*\right) \in \mathrm{I}\left(\mathbf{x}_{-1}^*\right) = \mathrm{I}\left(\mathbf{x}_{-1}\right)$ and take $\epsilon > 0$ such that $(v - \epsilon, v + \epsilon) \subseteq \mathrm{I}\left(\mathbf{x}_{-1}^*\right)$. Since $v(\cdot)$ is continuous, there exists $\delta > 0$ such that, if $\mathbf{y}_{-1} \in \mathbb{R}_+^{L-1}$ and $||\mathbf{x}_{-1}^* - \mathbf{y}_{-1}|| < \delta$, then $|v\left(x_1, \mathbf{x}_{-1}^*\right) - v\left(x_1, \mathbf{y}_{-1}\right)| < \epsilon$. Thus, $(v - \epsilon, v + \epsilon) \cap \mathrm{I}\left(\mathbf{y}_{-1}\right) \neq \varnothing$ and, consequently, $\mathrm{I}\left(\mathbf{x}_{-1}^*\right) \cap \mathrm{I}\left(\mathbf{y}_{-1}\right) \neq \varnothing$. By the contrapositive statement of Claim 1, this gives $\mathrm{I}\left(\mathbf{x}_{-1}^*\right) = \mathrm{I}\left(\mathbf{y}_{-1}\right)$. Therefore, $\mathbf{y}_{-1} \in \mathrm{E}\left(\mathbf{x}_{-1}\right)$ for every $\mathbf{y}_{-1} \in \mathbb{R}_+^{L-1}$ with $||\mathbf{x}_{-1}^* - \mathbf{y}_{-1}|| < \delta$. Equivalently, $\left\{\mathbf{y}_{-1} \in \mathbb{R}_+^{L-1} : ||\mathbf{x}_{-1}^* - \mathbf{y}_{-1}|| < \delta\right\} \subseteq \mathrm{E}\left(\mathbf{x}_{-1}\right)$.

Since for every $\mathbf{x}_{-1}^* \in \mathrm{E}\left(\mathbf{x}_{-1}\right)$, there exists an open ball $B_{\mathbf{x}_{-1}^*}(\delta)$ which lies entirely within the set $\mathrm{E}\left(\mathbf{x}_{-1}\right)$, the claim is immediate. ∎

Claim 3 *For every* $\mathbf{x}_{-1} \in \mathbb{R}^{L-1}_{+}$, $E(\mathbf{x}_{-1}) = \mathbb{R}^{L-1}_{+}$.

Proof. Since $\mathrm{E}(\mathbf{x}_{-1}) \subseteq \mathbb{R}^{L-1}_{+}$, it suffices to show that $\mathbb{R}^{L-1}_{+} \setminus \mathrm{E}(\mathbf{x}_{-1}) = \varnothing$. Suppose otherwise and let $\mathbf{y}_{-1} \in \mathbb{R}^{L-1}_{+} \setminus \mathrm{E}(\mathbf{x}_{-1})$. Since $\mathrm{I}(\mathbf{y}_{-1}) \neq \mathrm{I}(\mathbf{x}_{-1})$, we have $\mathrm{I}(\mathbf{y}_{-1}) \cap \mathrm{I}(\mathbf{x}_{-1}) = \varnothing$ (Claim 1); consequently, $\mathrm{E}(\mathbf{y}_{-1}) \cap \mathrm{E}(\mathbf{x}_{-1}) = \varnothing$. That is, $\mathbb{R}^{L-1}_{+} \setminus \mathrm{E}(\mathbf{x}_{-1}) = \bigcup_{\mathbf{y}_{-1} \in \mathbb{R}^{L-1}_{+} \setminus \mathrm{E}(\mathbf{x}_{-1})} \mathrm{E}(\mathbf{y}_{-1})$,. Since arbitrary unions of open sets are open, the set $\mathbb{R}^{L-1}_{+} \setminus \mathrm{E}(\mathbf{x}_{-1})$ is open (Claim 2).

We have arrived now at a contradiction since $\mathrm{E}(\mathbf{x}_{-1}) \bigcup \left(\mathbb{R}^{L-1}_{+} \setminus \mathrm{E}(\mathbf{x}_{-1})\right)$ partitions the connected space $\mathbb{R}^{L-1}_{+}$ into two non-empty, open sets. ■

Take now any $\mathbf{x}_{-1} \in \mathbb{R}^{L-1}_{+}$. We have $\mathrm{E}(\mathbf{x}_{-1}) = \mathrm{E}(\mathbf{0})$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^{L-1}$. Equivalently, $\mathrm{I}(\mathbf{x}_{-1}) = \mathrm{I}(\mathbf{0})$ or $\{v(y_1, \mathbf{x}_{-1}) \in \mathbb{R} : y_1 \in \mathbb{R}_{+}\} = \{v(y_1, \mathbf{0}) \in \mathbb{R} : y_1 \in \mathbb{R}_{+}\}$. There exists, therefore, a unique $\alpha \in \mathbb{R}$: $(0, \mathbf{x}_{-1}) \sim \alpha \mathbf{e}_1$.[3] Define $\phi : \mathbb{R}^{L-1}_{+} \rightarrow \mathbb{R}$ by $\phi(\mathbf{x}_{-1}) \mathbf{e}_1 \sim (0, \mathbf{x}_{-1})$ and $u : \mathbb{R}^{L}_{+} \rightarrow \mathbb{R}$ by $u(x_1, \mathbf{x}_{-1}) = x_1 + \phi(\mathbf{x}_{-1})$.

Claim 4 *The function* $u(\cdot)$ *represents the preference relation* $\succsim$.

Proof. Take any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{L}_{+}$ and let $\mathbf{x} \succsim \mathbf{y}$. By quasi-linearity,

$$(x_1, \mathbf{x}_{-1}) \succsim (y_1, \mathbf{y}_{-1}) \;\Leftrightarrow\; (x_1 - y_1, \mathbf{x}_{-1}) \succsim (0, \mathbf{y}_{-1})$$

while, by the definition of $\phi(\cdot)$, $(0, \mathbf{y}_{-1}) \sim \phi(\mathbf{y}_{-1}) \mathbf{e}_1$. Again by quasi-linearity,

$$(x_1 - y_1, \mathbf{x}_{-1}) \succsim \phi(\mathbf{y}_{-1}) \mathbf{e}_1 \;\Leftrightarrow\; (0, \mathbf{x}_{-1}) \succsim [\phi(\mathbf{y}_{-1}) + y_1 - x_1] \mathbf{e}_1$$

whereas, again by the definition of $\phi(\cdot)$, $(0, \mathbf{x}_{-1}) \sim \phi(\mathbf{y}_{-1}) \mathbf{e}_1$.

Therefore, we have $\phi(\mathbf{y}_{-1}) \mathbf{e}_1 \succsim [\phi(\mathbf{y}_{-1}) + y_1 - x_1] \mathbf{e}_1$. Equivalently, by quasi-linearity, $\phi(\mathbf{y}_{-1}) \geq \phi(\mathbf{y}_{-1}) + y_1 - x_1$ or $u(\mathbf{x}) \geq u(\mathbf{y})$ as required. ■

[3] Existence is immediate: the utility index $v(0, \mathbf{x}_{-1})$ appears also in the set $\mathrm{I}(\mathbf{0})$ as $v(y_1', \mathbf{0})$, for some $y_1' \in \mathbb{R}_{+}$. Uniqueness follows from the fact that $\succsim$ is monotone along the dimension of the first commodity. The argument is essentially identical to that made in the proof of MWG Proposition 3.C.1.