Cass Transversality Condition and Sequential Asset Bubbles[*]

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Abstract

The objective of this paper is to illustrate the connection existing between the asymptotic value of a certain random series and the absence of asset pricing valuation bubbles in stochastic economies with sequential markets. This series, in turn, is closely related to the one proposed by Cass to characterize efficient accumulation paths in Solow models.

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1 Introduction

In a seminal paper Cass (see [7], [8]) provided a complete characterization of the efficient growth paths in Solow models in terms of the properties of supporting competitive prices. Such a transversality condition involves the divergence of a certain series. Cass' result has been later generalized by various authors who found similar characterizations for other competitive economies (see [19], [20], [9]). The reader will find a fairly complete list of references on this subject in the recent paper of Chattopadhyay and Gottardi [9], which extends Cass' criterion to stochastic OLG economies.

The present study addresses a different issue concerning infinite-horizon stochastic economies, not properly related to the efficiency, but to the existence of valuation bubbles for long-lived assets. We shall show that such an issue turns out to be closely related as well to the behavior of a random series like the one proposed by Cass. This is quite surprising since it is well-known that the connection between inefficiency and the existence of pricing bubbles is fairly loose.

The literature on rational bubbles arising in sequential economies is rather large (see [4], [6], [12], [14], [15], [16], [21], [11], [22], [23]). Here, our attempt will be narrow in scope, and we restrict our inquiry only to what has a close connection with the behavior of the series under investigation. We wish only to mention explicitly the papers that have more directly influenced the present research. We draw from Santos and Woodford [22] the distinction between ambiguous and non-ambiguous bubbles. As the equilibria are typically sequentially incomplete, the valuation of the stream of future wealths is not necessarily uniquely determined. This entails different definitions for the fundamental value of an asset, which, in turn, may imply the existence of a bubble component for some state prices and not so for some other. Unlike [22], along the lines of [15], [21] and [11], we consider a general probabilistic structure, while [22] assumes informational σ-algebras generated by finite partitions. Indeed, their description of uncertainty by a date-events tree allows a quite complete theory about potential bubbles arising in sequential economies. One of their most important results is that perpetual assets in non-zero net supply cannot give rise to unambiguous price bubbles, as long as the aggregate endowment of the economy is finite.

The paper is organized as follows. Section 2 presents the model and notation. Section 3 introduces the short-run pricing equilibria

$$a_t p_t = \mathbf{E}_t \left[a_{t+1} \left(p_{t+1} + d_{t+1} \right) \right] \tag{1}$$

where p_t is the spot prices vector of assets, d_t is their dividend vector, while a_t denotes a state-price process. A brief description of asset pricing bubbles according to a given state price processes is also discussed.

Section 4 studies the relation existing between arbitrage-free prices, obeying Eq. (1), and the behavior of the series

$$\sum_{t=0}^{\infty} \frac{|d_t|}{|p_t|} \tag{2}$$

where $|.|$ is the vector l_1 norm. Several results will illustrate the intimate connection between the divergence of (2) and the absence of bubbles. The main one is Theorem 2, which establishes the non-existence, unambiguously, of pricing bubbles, provided series (2) goes to infinity uniformly. This simple result can be established thanks to a supermartingale property of a related random process (see Lemma 12). All the remaining results of this section deliver weaker versions of this basic property.

Section 5 collects a few examples which illustrate the results of Section 4.

Section 6 faces the more difficult issue of relating asymptotic behavior of (2) to truly market pricing equilibrium, where some long-lived agent is assumed to exist. While our results are here less sharp, we prove that our series is forced to diverge or, at least, to behave in a quite unbounded way. Indeed, Theorem 8 of this section is the exact converse of Theorem 2 and it establishes that our series diverges uniformly under an assumption of impatience of some long-lived agent.

A final section collects all the proofs of theorems.

2 The set-up

Let us describe a class of infinite-horizon exchange economies, where asset trading takes place sequentially. The underlying model is not so general as those assumed by several authors (see [13], [18], [22]) and is somewhat similar to Lucas [17] model, but with heterogeneous agents.

The uncertainty will be modelled by means of a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mu)$ where Ω represents the states of the world, $\mathcal{F}$ is the σ-algebra of events, μ is a probability measure defined over events and $\mathbf{F}$ is a filtration of σ-algebras: $\mathcal{F}_0 \subset \mathcal{F}_1 \subset .... \subset \mathcal{F}_t \subset ... \subset \mathcal{F}$ representing the dynamics of information available to agents. $\mathcal{F}_0$ is assumed to be the trivial algebra.

All the variables introduced below will be random processes defined over $(\Omega, \mathcal{F}, \mathbf{F}, \mu)$. We recall that a random sequence X_t is said to be adapted, provided X_t is $\mathcal{F}_t$ measurable for all t. The X_t are said to be predictable, if each X_t is $\mathcal{F}_{t-1}$ measurable. Given a random variable Y, as usual, $\mathbf{E}_t [Y]$ denotes the conditional expectation $\mathbf{E} [Y \mid \mathcal{F}_t]$.

There is a single perishable consumption good that will be taken as the numeraire. In addition, there are k perpetual assets, like equity, characterized by the stream of their dividends $d_t (\omega) \in \mathbb{R}^k_+$, indicating the amount of consumption good payed by one unit of each asset. Therefore, the assets structure is characterized by the adapted process (d_t), $t = 0, 1, ...$ and by the initial asset supply $\overline{y}_0 \in \mathbb{R}^k_+$ which is owned by agents at the beginning of the economy.

There is a set I, not necessarily finite, of agents. Each one is characterized by his/her time-separable state dependent preferences $u^i_t (c, \omega)$, $i \in I$, $t \in T_i$, where the time interval T_i is her life-span. We do not require a complete participation of individuals at all dates, although a few results will need the presence of at least one infinitely long-lived agent (namely, T_i unbounded). The utilities $u^i_t (., \omega)$ will be assumed to be concave and increasing over $\mathbb{R}_{++}$, while $u^i_t (c, .)$ are adapted. At each trading date $t \in T_i$, agent i receives also an exogenous endowment $w^i_t \geq 0$ of the consumption good. The sequence (w^i_t) is an adapted processes as well.

Denote by $I (t)$ the set of agents living at that period, namely, $i \in I (t)$ $\Longleftrightarrow t \in T_i$. An asset holding strategy for agent i will be denoted by y^i_t and it will be a predictable process. At time $t = 0$, each agent $i \in I (0)$ owns a share $\overline{y}^i_0$ of the whole assets supply. That is,

$$\sum_{i \in I(0)} \overline{y}^i_0 = \overline{y}_0$$

As a rule, superscripts refer to the specific agent, while subscripts denote the trading time. Whenever we need to emphasize a certain vector component, we shall add one more subscript. For instance, y^i_{nt}, $n \in \{1, 2, ..., k\}$, is the amount of asset n held by agent i at trading time t.

Given an adapted asset price process $p_t \in \mathbb{R}^k_+$, $t \in \mathbb{N}$, each agent i trades assets, facing sequential constraints

$$c_t^i + p_t \cdot \left(y_{t+1}^i - y_t^i\right) \leq d_t \cdot y_t^i + w_t^i, \qquad t \in T_i \tag{3}$$

where consumptions $c_t^i \geq 0$ are an adapted processes and $y_0^i = \overline{y}_0^i$, when $0 \in T_i$.

As is well known, a further restriction on accumulation of debts is needed to prevent Ponzi schemes, otherwise equilibria may fail to exist. For example, one can assume $y_t^i \geq -k_t^i$ where the exogenous vectors k_t^i have non-negative entries that may depend on (i,t). Alternatively, one may set the restriction $p_t \cdot y_{t+1}^i \geq -\sigma_t^i$, with $\sigma_t^i \geq 0$, on borrowing limits. The various types of possible constraints have been extensively discussed in the literature ([13], [14], [16], [18], [22]) but here it is of little relevance.

Below we give the Arrow-Radner notion of sequential equilibrium.

Definition 1 *A collection* $\left(\overline{c}_t^i, \overline{y}_t^i, p_t\right)$, $i \in I$, $t = 0, 1, ...$ *is said to be an equilibrium if*
i) given the process p_t*, for every* i *the consumption plan* $\overline{c}_t^i$ *is optimal with respect to all feasible consumption plans (i.e., satisfying (3)).*
ii) $\sum_{i \in I(t)} \overline{y}_t^i \leq \overline{y}_0$ *and* $p_{t-1} \cdot \left[\overline{y}_0 - \sum_{i \in I(t)} \overline{y}_t^i\right] = 0$ *for all* $t \geq 1$.

Clearly, (ii) makes sense provided either $I(t)$ is finite or the infinite sum $\sum_{i \in I(t)} \overline{y}_t^i$ is well defined.

We have still to specify more closely the notion of optimality in order to make precise condition (i) in the above definition. No problem arises, as long as agent's life-span is finite. If T_i is unbounded and since we are planning to go through a rather general setting, including unbounded utility as well as general underlying probabilistic structures, following [15], [21] and [11], we shall adopt Brock's [5] concept of weak optimality, defined as follows.

Let $T_i = [\tau, \tau+1, ...]$. The consumption plan $\overline{c}_t^i$ is said to be *weakly optimal* if for any other feasible plan c_t^i one has:

a) $\mathbf{E}\left[u_t^i\left(\overline{c}_t^i\right) - u_t^i\left(c_t^i\right)\right]^- < \infty$ for all $t \in T_i$

b) $\limsup_{N \to \infty} \mathbf{E}_\tau \sum_{s=0}^{N-1} \left[u_{\tau+s}^i\left(\overline{c}_{\tau+s}^i\right) - u_{\tau+s}^i\left(c_{\tau+s}^i\right)\right] \geq 0$

Here the symbol X^- denotes the negative part of a random variable X, namely, $X^- = -\min(X, 0)$. Therefore, condition (a) ensures that the random variables $u_t^i\left(\overline{c}_t^i\right) - u_t^i\left(c_t^i\right)$ are quasi-integrable. Consequently, the expectation of finite truncations $\sum_{s=0}^{N-1}\left[u_{\tau+s}^i\left(\overline{c}_{\tau+s}^i\right) - u_{\tau+s}^i\left(c_{\tau+s}^i\right)\right]$ are always well-defined, taking values on $(-\infty, +\infty]$. Clearly, the above definition also encompasses the case in which the time interval $T_i = [\tau, ..., \tau_f]$ is finite, as long as conventionally one adds the assumption $u_t^i = 0$ when $t > \tau_f$.

3 Short-run equilibria and bubbles

Given an asset pricing equilibrium (p_t), an adapted sequence $a_t(\omega)$ of strictly positive functions will be called a *(pseudo) state-price sequence,* consistent with (p_t), if

$$a_t p_t = \mathbf{E}_t\left[a_{t+1}\left(p_{t+1} + d_{t+1}\right)\right] \tag{4}$$

for all $t \geq 0$[1].

The martingale relation (4) might deserve several comments, as it may have slightly different interpretations. Strictly speaking, the a_t's are not the usual state-prices of Finance, because they are distorted by the probability law. Whenever uncertainty is described through finite information nodes, (4) becomes

$$a\left(s^t\right) p\left(s^t\right) = \sum_{s^{t+1}|s^t} \pi\left(s^{t+1} \mid s^t\right) a\left(s^{t+1}\right)\left(p\left(s^{t+1}\right) + d\left(s^{t+1}\right)\right)$$

where $\pi\left(s^{t+1} \mid s^t\right)$ is the transition probability and s^t, s^{t+1} are adjacent nodes (we are using the notation of [22]). After multiplying by $\mu\left(s^t\right)$, we get

$$\overline{a}\left(s^t\right) p\left(s^t\right) = \sum_{s^{t+1}|s^t} \overline{a}\left(s^{t+1}\right)\left(p\left(s^{t+1}\right) + d\left(s^{t+1}\right)\right)$$

where $\overline{a}\left(s^t\right) = a\left(s^t\right)\mu\left(s^t\right)$. This is the traditional intertemporal no-arbitrage equation and the $\overline{a}\left(s^t\right)$ are the familiar state-prices and p_t have the usual feature of being arbitrage-free prices. Clearly, formulation (4) is more appropriate when the states of the world are not necessarily finite at any trading time. As the Fundamental Theorem of Asset Pricing may fail for infinite-dimensional economies, it seems better to take (4) as a definition rather than deriving it by no-arbitrage theory.

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[1] Alternative names are known in literature. For instance, (a_t) is called a pricing kernel by [10].

Eq. (4) may take another interpretation, related to agent's maximizing behavior. If there is some agent $i \in I$, who lives at periods $t-1$ and t, then the so-called stochastic Euler equation

$$Du_{t-1}^{i}\left(\overline{c}_{t-1}^{i}\right)p_{t-1} = \mathbf{E}_{t-1}\left[Du_{t}^{i}\left(\overline{c}_{t}^{i}\right)\left(p_{t}+d_{t}\right)\right] \tag{5}$$

should hold. Therefore, $a_t = Du_t^i\left(\overline{c}_t^i\right)$ assumes the marginal utility interpretation as "personalized" state-prices. However, this is not always true since the conditional Euler equation may fail, at least in our general setting (see [15], [21], [11]). We refer to [11, Prop. 3.2] for precise conditions assuring (5) to hold true.

As a matter of fact, our approach can be enlarged by encompassing a wider class of equilibria. A triple $\left(\overline{c}_t^i, \overline{y}_t^i, p_t\right)_{i\in I}$ will be called a *myopic equilibrium* (see [23]) if feasibility condition (3), and market clearing condition (ii) of Definition 1 are fulfilled, while prices p_t satisfy (4) for some state-prices process a_t.

Given a fixed state-price process a_t, the definition of the fundamental value and of the relative bubble is then quite standard. By iterating (4), we obtain the so-called forward-solution:

$$a_0 p_0 = \mathbf{E}\sum_{t=1}^{\infty} a_t d_t + \lim_{t\to\infty}\mathbf{E}\left[a_t p_t\right] \tag{6}$$

Setting $p_0 = f_0 + b_0$, with

$$\begin{aligned} f_0 &= a_0^{-1}\mathbf{E}\sum_{t=1}^{\infty} a_t d_t \\ b_0 &= a_0^{-1}\lim_{t\to\infty}\mathbf{E}_0\left[a_t p_t\right] \end{aligned}$$

the first term f_0 may be viewed as the fundamental solution, while b_0 is the bubble component. The same argument applies if we start at each fixed period. Hence, $p_t = f_t + b_t$ with

$$f_t = a_t^{-1}\mathbf{E}_t\sum_{s>t}^{\infty} a_s d_s$$

while the bubble component follows the martingale law

$$a_t b_t = \mathbf{E}_t\left[a_{t+1}b_{t+1}\right]$$

Clearly, the above argument holds in accordance with one particular choice of the state-prices process a_t. Different state prices a_t could give rise to a different decomposition. This is a consequence of the fact that the equilibrium is dynamically complete only under very particular circumstances.

According to [22], a pricing equilibrium (p_t) is said ambiguously to involve a bubble if one has $\lim_{t\to\infty} \mathbf{E}\left[a_t p_t\right] = 0$ for some state-price process a_t, while $\lim_{t\to\infty} \mathbf{E}\left[a'_t p_t\right] > 0$ for some other process a'_t. To the contrary, an equilibrium involves, unambiguously, no bubble, provided that $\lim_{t\to\infty} \mathbf{E}\left[a_t p_t\right] = 0$, regardless of state-price processes a_t which are chosen. Finally, if $\lim_{t\to\infty} \mathbf{E}\left[a_t p_t\right] > 0$ for all state-price processes, we shall say that the equilibrium unambiguously involves a bubble component. For instance, this is the case of equilibria with positively priced fiat-money, which clearly always enjoys this last property.

A piece of notation will be needed. The symbols $|p_t|$ and $|d_t|$ denote $\sum_{i=1}^{k} p_{it}$ and $\sum_{i=1}^{k} d_{it}$ respectively. Clearly, (4) implies

$$a_t\left|p_t\right| = \mathbf{E}_t\left[a_{t+1}\left(\left|p_{t+1}\right| + \left|d_{t+1}\right|\right)\right]. \tag{7}$$

Given an equilibrium, denote by $N_t = \{|p_t| = 0\}$, the zero-price event belonging to $\mathcal{F}_t$. From (7), it turns out that $N_t \subset N_{t+1}$ (up to a μ-negligible set). One can also define $N_\infty = \cup_{t=0}^{\infty} N_t$.

A final caveat is in order. We have not made explicit assumptions about the integrability of the random variables involved. As a_t, $|p_t|$, $|d_t|$ are non-negative, they are quasi-integrable and all the conditional expectations make sense. Hence, the variables $a_t\left|p_t\right|$, $a_t\left|d_t\right|$ and $a_t\left|b_t\right|$ are necessarily integrable for all t. This is easily seen by recurrence. For, by (7), $a_0\left|p_0\right| = \mathbf{E}\left[a_1\left(\left|p_1\right| + \left|d_1\right|\right)\right]$, and $a_1\left|p_1\right|$ and $a_1\left|d_1\right|$ are integrable. By $a_1\left|p_1\right| = \mathbf{E}_1\left[a_2\left(\left|p_2\right| + \left|d_2\right|\right)\right]$, we have $\mathbf{E}\left[a_1\left|p_1\right|\right] = \mathbf{E}\left[a_2\left(\left|p_2\right| + \left|d_2\right|\right)\right]$. Hence, $a_2\left|p_2\right|$ and $a_2\left|d_2\right|$ are integrable and so on.

4 Results

In this section we prove the existence of the already mentioned relationship between the non-existence of asset bubbles and the asymptotic behavior of a certain series. The focus here is on prices supporting an equilibrium. Indeed, everything remains unchanged, provided that the allocation is a myopic equilibrium.

The first assumption we shall present is the following one.

A.1) The limit

$$\lim_{T\to\infty}\sum_{t=0}^{T}\frac{|d_t|}{|p_t|}=+\infty \tag{8}$$

holds uniformly.

Condition (A.1) must be carefully understood. As $|p_t|$ may vanish with positive probability, we are here adopting the convention $\sum_{t=0}^{T}|d_t|\,/\,|p_t|=+\infty$ over N_T. Thanks to the inclusion $N_t\subset N_{t+1}$, if $\sum_{t=0}^{T}|d_t|\,/\,|p_t|=+\infty$ for some T, then $\sum_{t=0}^{T_1}|d_t|\,/\,|p_t|=+\infty$ for all $T_1\geq T$. The uniform limit in (8) means that for all scalar M, there is some T_M such that

$$\mu\left\{\sum_{t=0}^{T_M}\frac{|d_t|}{|p_t|}\geq M\right\}=1.$$

Theorem 2 *If for a (myopic) pricing equilibrium* (p_t) *condition A.1 is fulfilled, then it unambiguously involves no bubble. That is,*

$$\lim_{t\to\infty}\mathbf{E}\left[a_t p_t\right]=0$$

for all the compatible state-prices a_t*.*

In a general setting, the converse implication of this theorem does not hold and the examples of the next section show that assumption A.1 cannot be weakened without costs. All the subsequent statements of this section deliver weaker results by relaxing the limit condition (8) as well as some partial converse results.

A natural weaker condition on the series is that the limit in (8) just holds almost surely. Clearly, it suffices that $\sum_{t=0}^{\infty}|d_t|\,/\,|p_t|=+\infty$ a.s. over N_∞^c, as the value of this series is infinite over N_∞ by definition.

We recall that a sequence X_t of positive and integrable random variables is *equicontinuous* if, for all $\varepsilon>0$, there is some $\eta_\varepsilon>0$ such that $\mathbf{E}\left[1_A X_t\right]\leq\varepsilon$ for all t and all $A\in\mathcal{F}$ with $\mu(A)\leq\eta_\varepsilon$. Clearly, a necessary condition for $\mathbf{E}\left[a_t\,|p_t|\right]\to 0$ is that the sequence $a_t\,|p_t|$ be equicontinuous. We show that this condition turns out to be sufficient, as long as Cass' series diverges.

Proposition 3 *Suppose that*

$$\sum_{t=0}^{\infty}\frac{|d_t|}{|p_t|}=+\infty \tag{9}$$

holds a.s. and that the random sequence $\overline{a}_t |p_t|$ is equicontinuous for some state-price $\overline{a}_t$. Then

$$\lim_{t\to\infty} \mathbf{E}\left[\overline{a}_t p_t\right] = 0.$$

The use of the previous result is often difficult, since the equicontinuity property is not so simple to check. However, in a one-asset world we can state a sharper result. We must only add a further mild qualification, based on the following assumption. For all t, denote $B_t = N_{t-1}^c \cap N_t \equiv \{p_{t-1} > 0 \text{ and } p_t = 0\}$.

A.2) If $\mu(B_t) > 0$, then $\mathbf{E}\left[d_t 1_{B_t}\right] > 0$ for all t.

Proposition 4 *Suppose that there is a single asset and A.2 holds. If*

$$\sum_{t=0}^{\infty} \frac{d_t}{p_t} = +\infty \tag{10}$$

holds a.s., then there exists a compatible state-price sequence $\overline{a}_t$ such that

$$\lim_{t\to\infty} \mathbf{E}\left[\overline{a}_t p_t\right] = 0.$$

Conversely, if one equilibrium unambiguously involves no bubble, then (10) holds a.s.

This Proposition provides a rather good description of the relation between our series and the occurrence of bubbles, in an economy with a single asset. In fact, it provides also a converse implication of our main theorem. The next proposition establishes another result, based on different restrictions.

Proposition 5 *Suppose a pricing equilibrium p_t satisfies $|p_t| > 0$ a.s. for all $t \geq 0$ and that some state prices $\overline{a}_t$ exist such that $\mathbf{E}\left[\overline{a}_t p_t\right] \to 0$. Under one of the following conditions:*
i) there exists an integrable function ϕ such that $\overline{a}_t |d_t| \leq \mathbf{E}\left[\overline{a}_t |d_t|\right] \phi$ for all $t \geq 0$
ii) $\exists \rho > 0$, $\exists \sigma > 0$ such that $\mu\left\{\overline{a}_t |d_t| \geq \rho \mathbf{E}\left[\overline{a}_t |d_t|\right]\right\} \geq \sigma$ for all $t \geq 0$,
then we have

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \frac{|d_t|}{|p_t|}\right] = +\infty. \tag{11}$$

Clearly, (11) is sensibly weaker than (9). At any rate, it proves that the sum $\sum_{t=0}^{T} |d_t| / |p_t|$ grows so fast that, asymptotically, either it is infinite with strictly positive probability or its value is not integrable.

We close this section by presenting a rather simple result, which relies on a non-random series related to (8), but defined for a specific choice of state prices.

Proposition 6 *Let p_t be a pricing equilibrium and $\overline{a}_t$ be consistent with it. Then, $\mathbf{E}[\overline{a}_t p_t] \to 0$ if and only if the numerical series*

$$\sum_{t=0}^{\infty} \frac{\mathbf{E}[\overline{a}_t |d_t|]}{\mathbf{E}[\overline{a}_t |p_t|]} = \infty. \tag{12}$$

Eq. (12) reduces to $\sum_{t=0}^{\infty} |d_t| / |p_t| = \infty$, regardless of the choice of a_t, provided that there is no uncertainty. Consequently, in this case the series (8) provides a full characterization of the existence of bubbles. Along with Theorem 2, Proposition 6 delivers the following simple result.

Proposition 7 *In a deterministic economy, any myopic equilibrium p_t involves no ambiguous bubble. No bubble exists at all if and only if*

$$\sum_{t=0}^{\infty} \frac{|d_t|}{|p_t|} = \infty. \tag{13}$$

For, if (13) holds, it trivially goes to infinity uniformly. By Theorem 2, there is unambiguously no bubble. Conversely, if (13) fails, then for any state-prices a_t condition (12) fails as well. Proposition 6 implies that $a_t p_t$ does not vanish as $t \to \infty$.

5 Examples

We pause our presentation to illustrate by means of some examples our first set of results. In each of them, borrowing is prohibited.

Example 1. Consider a Lucas-type economy [17], with a single infinitely-lived agent and a unique asset which yields a constant dividend $d_t = 1$ at all the states of the world. The asset net supply is $y_0 = 1$. The agent's preferences are not uniform across states and are given by $u_t(c) = \beta_t(\omega) u(c)$, where $u(c)$ is concave and increasing function, differentiable at $c = 1$, while $\beta_t(\omega)$ is a discounted martingale $\beta_t = \rho^t m_t$. That is, $0 < \rho < 1$ and m_t is a strictly positive process satisfying $\mathbf{E}_{t-1}[m_t] = m_{t-1}$. There are no exogenous endowments, namely, $w_t = 0$ for all t. Of course, the equilibrium allocations are of no-trade type, i.e., $\overline{c}_t = d_t = 1$ and $\overline{y}_t = 1$. It is not difficult to prove that there is a unique equilibrium price supporting such an allocation. The reader is referred to [21] where a detailed analysis can be found (existence and uniqueness is a consequence of Theorems 1 and 2 of [21]). The pricing equilibrium is given by its fundamental value

$$\rho^t m_t u'(1)\, p_t = \mathbf{E}_t\left[\sum_{s>t}^{\infty} \rho^s m_s u'(1)\right]$$

that yields $p_t = \rho(1-\rho)^{-1}$. Clearly,

$$\sum_{t=0}^{\infty} \frac{1}{p_t} = \infty$$

and, consequently, the transversality condition (8) holds. Theorem 2 implies that unambiguously no bubble is involved. Notice that several state-prices are compatible with this equilibrium. As a matter of fact, any discounted martingale $a_t = \rho^{-1}\mathbf{E}_t[a_{t+1}]$ is a compatible state-price process. For example, $a_t = \rho^t$ as well as $a'_t = \rho^t m_t$ are two of these possibilities.

Example 2. We present an example of an ambiguous bubble arising in an economy with a single infinitely lived agent. It turns out to be a generalization of the binomial example studied in [22]. The economy is like that of Example 1, but agent's preferences $u_t(c) = \beta_t(\omega)\, u(c)$ are somewhat different. This will require a specification of the underlying probabilistic space $(\Omega, \mathcal{F}, \mathbf{F}, \mu)$.

Assume that every σ-algebra $\mathcal{F}_t$ of the filtration is atomic. That means that it is generated by a countable set of disjoint atoms of $\mathcal{F}_t$. We recall that an event $A \in \mathcal{F}_t$ is an atom in $\mathcal{F}_t$, if $B \subseteq A$ and $B \in \mathcal{F}_t$ implies $\mu(A) = \mu(B)$ or $\mu(B) = 0$.

Let us suppose that a sequence of atomic events $A_t \in \mathcal{F}_t$, $t \geq 0$, exists, so that $A_0 = \Omega$, $A_{t+1} \subset A_t$ and $0 < \mu(A_{t+1}) < \mu(A_t)$ for all t.

Denote $\mu(A_t) = \pi_t$ and $\mu(A_{t+1} \mid A_t) = q_t$. Clearly $\pi_{t+1} = q_t \pi_t$ and $0 < q_t < 1$. All these requirements can be summarized into the martingale property

$$\mathbf{E}_t\left[1_{A_{t+1}}\right] = q_t 1_{A_t}.$$

After having characterized the sequence A_t of events, fix three numbers $r > 0$, $\Delta > 0$ and $0 < \delta < (1+r)^{-1}$. The time-discounting process $\beta_t(\omega)$ will be defined as

$$\beta_t = \frac{\delta^t}{(1+r)^t} 1_{A_t} + X_t$$

where the process X_t will be specified later. The basic assumption is that X_t is disjoint from 1_{A_t}. Namely, $X_t 1_{A_t} = 0$. Let us write down the prices

$$p_t = \frac{1}{r} + \frac{\Delta (1+r)^t}{\pi_t} 1_{A_t} \tag{14}$$

which will be the candidates to be the equilibrium prevailing in this economy.

Next, determine X_t, such that p_t is a solution of the Euler equation (see (5)),

$$\beta_t p_t = \mathbf{E}_t\left[\beta_{t+1}(1+p_{t+1})\right]. \tag{15}$$

Tedious algebra shows that (15) is fulfilled, provided X_t satisfies

$$\mathbf{E}_t\left[X_{t+1}\right] = \frac{1}{1+r} X_t + m_t 1_{A_t} \tag{16}$$

where

$$m_t = \frac{\delta^t}{1+r}\left(\frac{1-\delta q_t}{(1+r)^t} + \frac{\Delta(1-\delta) r}{\pi_t}\right).$$

Taking the expected value in (16), we have

$$\mathbf{E}\left[X_{t+1}\right] = \frac{1}{1+r}\mathbf{E}\left[X_t\right] + \delta^t L_t$$

where the positive sequence L_t is bounded, i.e., $L_t \leq M$ for all $t \geq 0$. By iteration, it is then easy to check that $\mathbf{E}[X_t] \to 0$ as $t \to \infty$, provided $\delta < (1+r)^{-1}$. Since

$$\beta_t p_t = \delta^t \left[\frac{1}{r(1+r)^t} + \frac{\Delta}{\pi_t}\right] 1_{A_t} + \frac{1}{r} X_t,$$

it follows that $\mathbf{E}[\beta_t p_t] \to 0$. Therefore, the equilibrium allocation is supported by prices (14). More specifically, they are the fundamental values, according to the state-prices β_t. As in Example 1, prices (14) are an equilibrium in this economy. Of course, to complete our construction we have still to show that some process X_t, satisfying (16), does exist. A solution can be found in the class of step-functions. Denote

$$X_t = \sum_{i=0}^{t-1} \lambda_i^t 1_{A_i \setminus A_{i+1}}$$

for $t \geq 1$, where the parameters λ_i^t $(i = 0, 1, ..., t-1)$ must be specified to have (16). It is then easy to obtain the following solution, by the recursive system

$$\begin{aligned} \lambda_t^{t+1} &= m_t \left(1 - q_t\right)^{-1} \\ \lambda_i^{t+1} &= \left(1 + r\right)^{-1} \lambda_i^t \qquad 0 \leq i \leq t-1 \end{aligned}$$

with the initial condition $\lambda_0^1 = m_0 \left(1 - q_0\right)^{-1}$.

Finally, let us show that equilibrium (14) can (ambiguously) involve a bubble component. It suffices to choose $a_t = \left(1 + r\right)^{-t}$. Actually, it is readily seen that

$$a_t p_t = \mathbf{E}_t \left[a_{t+1} \left(p_{t+1} + 1\right)\right].$$

Through these a_t, which are consistent with the no-arbitrage condition, the fundamental values become $f_t = 1/r$, while

$$b_t = \frac{\Delta \left(1 + r\right)^t}{\pi_t} 1_{A_t}$$

is the bubble component. Note that this bubble may or may not burst almost surely, depending on whether the asymptotic event A_∞ (being $A_t \downarrow A_\infty$) has zero or positive probability.

In view of the results of Section 4, the random Cass' series $\sum_{t=0}^{\infty} 1/p_t$ never satisfies condition (8). Indeed, the value of this series is

$$\begin{aligned} \sum_{t=0}^{\infty} 1/p_t &= \infty \qquad \text{if } \omega \notin A_\infty \\ \sum_{t=0}^{\infty} 1/p_t &< \infty \qquad \text{if } \omega \in A_\infty . \end{aligned}$$

Even if we assume $\mu \left(A_\infty\right) = 0$, the series goes to infinity only point-wise, but never uniformly, as it is easily checked. This illustrates how the first statement of Proposition 4 cannot be improved. Namely, almost surely divergence does not imply unambiguously the absence of bubbles. Note that the random variables $(1+r)^{-t} p_t$ are not equicontinuous when $\mu(A_\infty) = 0$.

Example 3. In this last example, we provide a simple economy with a continuum of equilibria: none of them involving a bubble component. This underlines the well-known fact that indeterminacy has nothing to do with valuation bubbles. It is essentially Gilles and LeRoy's [12] example (see also [15]).

The economy is deterministic, $w_t = 0$ and the future dividends of a single asset are constant over time, $d_t = r$. The infinitely-lived representative agent has preferences

$$u_t(c) = \begin{cases} c - r & \text{if } c \leq r, \\ (1+r)^{-t}(c-r) & \text{if } c \geq r \end{cases}$$

which are not differentiable at the point $c = r$.

Denote by $\partial u_t(r) = \left[(1+r)^{-t}, 1\right]$ the set of supergradients of u_t at r. Let $a_t \in \partial u_t(r)$ be an arbitrary sequence such that $\sum_{t=0}^{\infty} a_t < +\infty$. The price sequence

$$p_t = \frac{r}{a_t} \sum_{s>t}^{\infty} a_s \tag{17}$$

is an equilibrium. To see this, it suffices to observe that it satisfies the short-run optimality conditions $a_t p_t = a_{t+1}(p_{t+1} + r)$ and $a_t p_t \to 0$. The short-run equilibrium is thus without bubbles (see Proposition 7). Note that by Proposition 7 necessarily we have

$$\sum_{t=0}^{\infty} \frac{r}{p_t} = \infty$$

along any equilibrium sequence (17).

Indeed, prices (17) are truly equilibria. This can be derived by a general theorem (see [11, Prop. 3.9]), but a direct method can be used. It suffices to observe that any feasible consumption stream c_t satisfies

$$\sum_{t=0}^{\infty} a_t c_t \leq r \sum_{t=0}^{\infty} a_t.$$

Thanks to concavity, $u_t(r) - u_t(c_t) \geq a_t(r - c_t)$. From these inequalities, the optimality condition (i) of Definition 1 is easily achieved.

6 Cass series and long-run equilibria

So far, we have not investigated to what extent the asymptotic value of Cass' series is consistent with the presence of individuals who maximize consumption welfare functions. Put differently, the results formulated in Section 4 have the feature of being related to short-run considerations. In this section we shall try to analyze the more difficult issue of the necessary conditions to be satisfied by our series along an equilibrium. Thanks to the examples of Section 5, we are already aware of the need of some further qualifications. Otherwise, Cass' series behavior may be of various nature.

The first statement provides the right qualification such that (8) holds necessarily. A similar result for an economy with one representative agent has been given by [21].

Most of the results below will require the presence in the economy of at least one infinitely-lived agent. There is no loss of generality in assuming $T_i = [0, 1, ...]$.

An equilibrium $\left(\overline{c}_t^i, \overline{y}_t^i, p_t\right)_{i\in I}$ will be called *uniformly interior* for an infinitely lived agent i, provided some scalar $\eta > 0$ exists such that $\overline{c}_t^i \geq \eta\left|d_t\right|$, for all $t \geq 1$, and the trading strategy $\overline{y}_t^i - \eta\mathbf{1}$ is still feasible, where $\mathbf{1} = (1, 1, ..., 1)$. It will be referred as a η-equilibrium, for short.

Theorem 8 *Let* $\left(\overline{c}_t^i, \overline{y}_t^i, p_t\right)_{i\in I}$ *be an* η*-equilibrium for some long-lived agent* i*. Suppose there exists a non-negative scalar sequence* $\{\sigma_t\}$ *having the following two properties:*
i) $\sum_{t=1}^{\infty}\sigma_t = +\infty$
ii) for every time $s \geq 1$ *and* $A \in \mathcal{F}_s$ *with* $\mu(A) > 0$ *and* $A \subset N_s^c$*, there exists a random variable* $\zeta = \zeta(s, A)$*, depending on* s *and* A*, such that:*
a) ζ *is* $\mathcal{F}_s$ *measurable,*
b) $0 < \zeta \leq \eta\sigma_s^{-1}$ *a.s.,*
c) the consumption stream $(\widetilde{c}_t^i)$*, defined as*

$$\widetilde{c}_t^i = \begin{cases} \overline{c}_t^i & \text{for } 0 \leq t \leq s-1, \\ \overline{c}_t^i + \zeta\left|d_t\right| 1_A & \text{for } t = s, \\ \overline{c}_t^i - \zeta\sigma_s\left|d_t\right| 1_A & \text{for } t \geq s+1 \end{cases}$$

overtakes $\left(\overline{c}_t^i\right)$*.*
Then, property (A.1) holds and, therefore, unambiguously no bubble is involved.

We recall that the above overtaking criterion means that the consumption stream $\widetilde{c}_t^i$ is strictly preferred by agent i, whenever event $A_s \in \mathcal{F}_s$ occurs. Namely,

$$\liminf_{N\to\infty} \mathbf{E} \sum_{t=0}^{N} \left[u_t^i\left(\widetilde{c}_t^i\right) - u_t^i\left(\overline{c}_t^i\right) \right] > 0.$$

The assumptions of Theorem 8 deserve some comments. The key one, that the equilibrium has to be uniformly interior for at least some agent, entails two well-known requirements. First, some agent is long-lived, avoiding OLG models. Second, the trading strategy must not be binding infinitely often. This eliminates Bewley's consumption-smoothing models with positively priced fiat-money (see [2], [16], [11], [22]).

The additional assumptions (i)-(ii) are loosely referred as an impatience property postulated on agent's preference. They are closely related to the assumption A.2 on agent's impatience postulated by Santos and Woodford [22] as well as to the uniform lower bound on impatience assumption of Magill and Quinzii [18]. However, our conditions are weaker than theirs.

Example 1 (continued). Let us illustrate Theorem 8 by means of Example 1. Clearly here the equilibrium is a η-equilibrium for any $\eta < 1$. For the sake of simplicity, we assume that $u(c)$ is differentiable at $c = 1$ and that the process m_t satisfies the uniform condition $0 < \underline{m} \leq m_t \leq \overline{m}$. It is easy to check that the impatience condition (ii) of Theorem 8 holds if

$$u(1+\zeta) C_s + u(1-\zeta\sigma_s) B_s > u(1)(C_s + B_s) \tag{19}$$

with $C_s = \mathbf{E}\left[1_{A_s}\beta_s\right]$ and $B_s = \mathbf{E}\left[\sum_{t>s} 1_{A_s}\beta_t\right]$. Here we have set ζ to be a constant function. By differentiating the scalar function

$$\rho(\zeta) = u(1+\zeta) C_s + u(1-\zeta\sigma_s) B_s$$

at $\zeta = 0$, it is easy to check that (19) is true for small ζ, as long as $\sigma_s < C_s B_s^{-1}$. Consequently, if $C_s B_s^{-1}$ is uniformly away from zero, we can pick a constant sequence for σ_s, clearly satisfying condition (i). On the other hand, under our uniform condition, we have

$$C_s B_s^{-1} \geq \frac{(1-\rho)\,\underline{m}}{\rho\overline{m}}$$

which gives the desired result.

Without resorting to some impatience condition, the existence of an infinitely-lived agent does not suffice to infer that the series goes to infinity, not even almost surely. It is of some interest to look at what happens under very mild hypotheses. Given any positive scalar N, define the event

$$S_N = \left\{ \sum_{t=0}^{\infty} \frac{|d_t|}{|p_t|} > N \right\}.$$

Proposition 9 *Let* $\left(\overline{c}_t^i, \overline{y}_t^i, p_t\right)_{i\in I}$ *be an equilibrium. Assume that, for at least one long-lived agent* i*, the equilibrium asset holding* $\overline{y}_t^i$ *is uniformly interior (i.e.,* $\overline{y}_t^i - \eta\mathbf{1}$ *is feasible for some* $\eta > 0$*) and that her utilities* $u_t^i(c)$ *are strictly increasing. Then the following property must hold:*

$$\mu\left(S_N \cap A\right) > 0 \tag{20}$$

for all N*,* A *and* t*, such that* $A \in \mathcal{F}_t$ *and* $\mu(A) > 0$*.*

This proposition gives some information on the limit behavior of our series. Clearly, (20) does not even imply that $\mu\left\{\sum_{t=0}^{\infty} |d_t| / |p_t| = \infty\right\}$ be strictly positive nor that $\mathbf{E}\left[\sum_{t=0}^{\infty} |d_t| / |p_t|\right] = \infty$. It asserts that at each informational node, there is a positive probability that the series gets arbitrarily large.

A sharper insight is achieved, at least for some models, as the next corollary shows.

Corollary 10 *Suppose the probability space* $(\Omega, \mathcal{F}, \mathbf{F}, \mu)$ *enjoys the following property: for all sequences* $A_t \downarrow A$*, with* $A_t \in \mathcal{F}_t$*,* $A_t \subset N_\infty^c$ *and* $\mu(A) > 0$ *there is a time* s *and an event* $B \subseteq A$ *such that* $\mu(B) > 0$ *and* $B \in \mathcal{F}_s$*. Under the hypotheses of Proposition 9, we have*

$$\sum_{t=0}^{\infty} \frac{|d_t|}{|p_t|} = \infty$$

a.s.

The probabilistic assumption postulated in this corollary is quite strong. Anyway, the deterministic case falls evidently into this class. This completes Proposition 7 and we can formalize it as follows:

Uniform interiority of asset holdings for some agent suffices to claim (unambiguously) the non-occurrence of bubbles in the deterministic setting.

Another class of models, enjoying the condition of Corollary 10, is described in the next example.

Example 5. One of the simplest models with uncertainty, is the one where there are countable many states of the world: $\Omega = \{1, ..., t, ...\}$. The σ-algebra $\mathcal{F}_t$ will be generated by the finite partition

$$\{1\}, ..., \{t\}, \; A_t = \{t+1, t+2, ...\}.$$

Assign the probability measure $\mu(\{t\}) = \mu_t = (1-q) q^{t-1}$, $t \geq 1$, over the states of the world, where $0 < q < 1$. Clearly, this probabilistic structure satisfies the assumption of Corollary 10. Nevertheless, this model is a source of bubble examples. For instance, the sequence A_t meets the conditions described in Example 2 and therefore a bubble can be constructed according to Example 2. Another different specification which gives rise to bubbles is described in [21] and [11, Sect. 5.3].

According to Proposition 9, one way to get some insight into the behavior of Cass' series is to add a topological structure to the probability space. For instance, the familiar event-tree model can be reformulated by means of the Cantor space $\Delta = \{1, 2, ..., n\}^{\mathbb{N}}$ of all the n symbols sequences, endowed with the product topology. The space Δ is compact and metrizable. The interpretation is that there is a realization of one random state $s_t \in \{1, 2, ..., n\}$ at each date, while a node s^t is identified with a sequence $\{s_1, s_2, ..., s_t\}$. The σ-algebra $\mathcal{F}_t$ is generated by the finite partitions of the clopen sets (i.e., sets which are both closed and open)

$$A_{s^t} = \{s_1\} \times \{s_2\} \times .... \times \{s_t\} \times \{1, 2, ..., n\} \times \{1, 2, ..., n\} \times .....$$

with $s_i \in \{1, 2, ..., n\}$ for $1 \leq i \leq t$. The next result shows that the region on which the series diverges is relevant.

Proposition 11 *Let (Δ, μ) be the n-states space with its natural filtration. Assume that $\mu(A_{s^t}) > 0$ for all A_{s^t}. Under the assumptions of Proposition 9, the event*

$$S_\infty = \left\{ \sum_{t=0}^{\infty} \frac{|d_t|}{|p_t|} = \infty \right\}$$

is a dense G_δ set.

A G_δ set is the countable intersection of open sets. In complete metrizable spaces, like Δ, a dense G_δ set is rather a large set, at least from the topological point of view (see for instance [1]).

7 Proofs

Theorem 2 will follow after the construction of a predictable and increasing process ζ_t, taking on, possibly, extended values. More specifically, define recursively

$$\zeta_{t+1} = \zeta_t \left(1 + \frac{|d_t|}{|p_t|}\right) \tag{21}$$

with $\zeta_0 = 1$, provided $\omega \notin N_t$. Otherwise, set $\zeta_{t+1}(\omega) = +\infty$. Of course, $\zeta_{t+1} \geq \zeta_t$. The inclusion $N_t \subset N_{t+1}$ entails that $\zeta_t(\omega) = +\infty \Longrightarrow \zeta_{t+1}(\omega) = +\infty$.

The key property of this process is enclosed in the next Lemma.

Lemma 12 *Given the scalar random process (21) and any state-price sequence a_t compatible with the equilibrium, the process $a_t |p_t| \zeta_{t+1}$ is a supermartingale, as long as one agrees upon setting $|p_t| \zeta_{t+1} = 0$ over N_t.*

Proof. Set $C_t = N_t^c$. Then

$$\mathbf{E}_{t-1}\left[a_t |p_t| \zeta_{t+1}\right] = \mathbf{E}_{t-1}\left[1_{C_t} a_t |p_t| \zeta_{t+1}\right].$$

According to (21),

$$\zeta_{t+1} |p_t| 1_{C_t} = (|p_t| + |d_t|) \zeta_t 1_{C_t}$$

that, by means of (4), leads to

$$\mathbf{E}_{t-1}\left[a_t |p_t| \zeta_{t+1}\right] = \mathbf{E}_{t-1}\left[1_{C_t} a_t |p_t| \zeta_{t+1}\right] =$$
$$\mathbf{E}_{t-1}\left[1_{C_t} a_t (|p_t| + |d_t|) \zeta_t\right] \leq \mathbf{E}_{t-1}\left[a_t (|p_t| + |d_t|)\right] \zeta_t = a_{t-1} |p_{t-1}| \zeta_t$$

which is the desired supermartingale property. ■

Theorem 2. According to (21), we have

$$\zeta_{t+1} = \prod_{s=0}^{t} \left(1 + \frac{|d_s|}{|p_s|}\right)$$

over N_t^c and $\zeta_{t+1} = +\infty$ if $\omega \in N_t$. It is well know that the following inequalities

$$1 + \sum_{k=0}^{t} \alpha_k \leq \prod_{k=0}^{t} (1 + \alpha_k) \leq \exp\left(\sum_{k=0}^{t} \alpha_k\right) \tag{22}$$

are true, for all sequences of scalars $\alpha_k \geq 0$ and every $t \geq 0$. Therefore, by condition (8) and the first of (22), it follows that $\zeta_t \to \infty$ uniformly as well.

Fix any state-prices sequence a_t. By Lemma 12, $a_t |p_t| \zeta_{t+1}$ is a supermartingale. Hence, $\mathbf{E}\left[a_t |p_t| \zeta_{t+1}\right] \leq a_0 |p_0| \zeta_1$. This means $\mathbf{E}\left[1_{C_t} a_t |p_t| \zeta_{t+1}\right] \leq a_0 |p_0| \zeta_1$ as well, where $C_t = N_t^c$. Since the sequence ζ_t diverges uniformly, for each N we can find a time T so that $\zeta_{t+1} \geq N$ for all $t \geq T$, a.s. Consequently, $\mathbf{E}\left[1_{C_t} a_t |p_t|\right] \leq N^{-1} a_0 |p_0| \zeta_1$. On the other hand, $\mathbf{E}\left[1_{N_t} a_t |p_t|\right] = 0$, which yields $\mathbf{E}\left[a_t |p_t|\right] \leq N^{-1} a_0 |p_0| \zeta_1$. Hence, $\mathbf{E}\left[a_t |p_t|\right] \to 0$ as $t \to \infty$. In view of (6), we deduce that the bubble vanishes, relatively to the selected state-prices sequence. ■

Proposition 3. From the condition $\mathbf{E}\left[a_t |p_t| \zeta_{t+1}\right] \leq a_0 |p_0| \zeta_1$, we infer $a_t |p_t| \zeta_{t+1} \to \lambda(\omega)$ almost surely. This is a consequence of Doob's supermartingale convergence theorem (see for example [3]). Recalling the convention made on $|p_t| \zeta_{t+1}$, we have that $\lambda(\omega) = 0$, if $\omega \in N_\infty$ and $|p_t(\omega)| \zeta_{t+1}(\omega) = 0$ for t large enough. Clearly, $\zeta_{t+1} \to \infty$, over the event N_∞^c. Hence, $a_t |p_t| \to 0$. We can conclude that $a_t |p_t| \to 0$, a.s. over Ω. Now, as $\mathbf{E}\left[a_t |p_t|\right] \leq a_0 p_0$, the equicontinuity property implies that the random variables $a_t |p_t|$ are uniformly integrable. It is known that almost surely convergence and uniform integrability together imply $\mathbf{E}\left[a_t |p_t|\right] \to 0$ [3, Th. 16.3]. This is the desired result. ■

Proposition 4. Define the following state-prices process

$$\begin{cases} a_t = 1/\left(\zeta_{t+1} p_t\right) & \omega \in N_t^c \\ a_t = b_t/\zeta_t & \omega \in B_t = N_t \cap N_{t-1}^c \\ a_t = 1 & \omega \in N_{t-1} \end{cases}$$

where ζ_t is defined in (21) and $b_t = 1/\mathbf{E}_{t-1}\left[d_t 1_{B_t}\right]$. Let us first check that

$$a_{t-1} p_{t-1} = \mathbf{E}_{t-1}\left[a_t \left(p_t + d_t\right)\right]. \tag{23}$$

Actually, $a_{t-1} p_{t-1} = a_{t-1} p_{t-1} 1_{N_{t-1}^c} = \zeta_t^{-1} 1_{N_{t-1}^c}$, while

$$\begin{aligned} a_t \left(p_t + d_t\right) &= a_t \left(p_t + d_t\right) 1_{N_{t-1}} + a_t \left(p_t + d_t\right) 1_{B_t} + a_t \left(p_t + d_t\right) 1_{N_t^c} \\ &= a_t d_t 1_{B_t} + a_t \left(p_t + d_t\right) 1_{N_t^c} = \zeta_t^{-1} b_t d_t 1_{B_t} + \zeta_t^{-1} 1_{N_t^c}. \end{aligned}$$

Plunging these into (23), we get

$$1_{N_{t-1}^c} = \mathbf{E}_{t-1}\left[b_t d_t 1_{B_t} + 1_{N_t^c}\right]. \tag{24}$$

Multiplying (24) by $1_{N_{t-1}^c}$ and $1_{N_{t-1}}$, respectively, we get that (23) is equivalent to

$$\begin{aligned} 1_{N_{t-1}^c} &= \mathbf{E}_{t-1}\left[d_t b_t 1_{B_t} + 1_{N_t^c}\right] \\ 0 &= \mathbf{E}_{t-1}\left[0\right]. \end{aligned}$$

Hence, (23) is true, provided

$$b_t = \frac{1_{N_{t-1}^c} - \mathbf{E}_{t-1}\left[1_{N_t^c}\right]}{\mathbf{E}_{t-1}\left[d_t 1_{B_t}\right]}.$$

On the other hand, it is easy to see that $1_{N_{t-1}^c} - \mathbf{E}_{t-1}\left[1_{N_t^c}\right] = 1_{B_t}$ and therefore (23) is proven.

Let us now demonstrate the first statement. If the series goes to infinity almost surely, then $\zeta_t \uparrow \infty$. This implies $a_t p_t \downarrow 0$ *a.s.* that, in turn, leads to $\mathbf{E}\left[a_t p_t\right] \downarrow 0$ and the first part of the theorem is proven.

Let us prove the second one. Suppose that (10) fails. There will exist an event $C_\infty \subset \cap_1^\infty C_t$ such that $\zeta_t \uparrow \zeta_\infty < \infty$ over C_∞, with $\mu\left(C_\infty\right) > 0$. Hence, $\mathbf{E}\left[a_t p_t\right] \geq \mathbf{E}\left[a_t p_t 1_{C_\infty}\right] = \mathbf{E}\left[\zeta_{t+1}^{-1} 1_{C_\infty}\right]$ and $\zeta_{t+1}^{-1} 1_{C_\infty} \downarrow \zeta_\infty^{-1} 1_{C_\infty}$. Consequently, $\mathbf{E}\left[a_t p_t\right]$ does not vanish as $t \to \infty$. Therefore, the bubble component does not vanish. ■

Proposition 5 requires a preliminary Lemma.

Lemma 13 *Let* X, Y *be non-negative random variables with* $0 < \mathbf{E}\left[Y\right] < \infty$ *and* $\mathbf{E}\left[X\right] < \infty$. *Then*

$$\mathbf{E}\left[XY^{-1}\right] \geq \frac{\mathbf{E}^2\left[X^{1/2}\right]}{\mathbf{E}\left[Y\right]} \tag{25}$$

If, in addition, there is a number $k > 0$ *such that* $\mathbf{E}^2\left[X^{1/2}\right] \geq k\mathbf{E}\left[X\right]$, *then*

$$\mathbf{E}\left[XY^{-1}\right] \geq k\frac{\mathbf{E}\left[X\right]}{\mathbf{E}\left[Y\right]} \tag{26}$$

Proof. We can assume $\mathbf{E}\left[XY^{-1}\right] < \infty$, otherwise (25) is trivially true. Thanks to Jensen inequality, we have $\mathbf{E}\left[\left(XY^{-1}\right)^{1/2}\right] < \infty$, $\mathbf{E}\left[X^{1/2}\right] < \infty$ and $\mathbf{E}\left[Y^{1/2}\right] < \infty$. By Cauchy-Schwartz

$$\mathbf{E}^2\left[X^{1/2}\right] = \mathbf{E}^2\left[\left(XY^{-1}\right)^{1/2} Y^{1/2}\right] \leq \mathbf{E}\left[XY^{-1}\right]\mathbf{E}\left[Y\right]$$

which is Eq. (25). Eq. (26) follows immediately. ■

Proposition 5. Assume $\mathbf{E}\left[\overline{a}_t\left|d_t\right|\right] > 0$, otherwise our argument is trivial. In view of (25), we have

$$\mathbf{E}\left[\frac{\left|d_t\right|}{\left|p_t\right|}\right] = \mathbf{E}\left[\frac{\overline{a}_t\left|d_t\right|}{\overline{a}_t\left|p_t\right|}\right] \geq \frac{\mathbf{E}^2\left[\left(\overline{a}_t\left|d_t\right|\right)^{1/2}\right]}{\mathbf{E}\left[\overline{a}_t\left|p_t\right|\right]}.$$

We claim that if one of the two conditions (i), (ii) holds, then $\mathbf{E}^2\left[\left(\overline{a}_t\left|d_t\right|\right)^{1/2}\right] \geq k\mathbf{E}\left[\overline{a}_t\left|d_t\right|\right]$, where the constant k can be chosen independently of t. Accordingly,

$$\mathbf{E}\left[\frac{\left|d_t\right|}{\left|p_t\right|}\right] \geq k\frac{\mathbf{E}\left[\overline{a}_t\left|d_t\right|\right]}{\mathbf{E}\left[\overline{a}_t\left|p_t\right|\right]}$$

and

$$\mathbf{E}\left[\sum_{t=0}^{\infty}\frac{\left|d_t\right|}{\left|p_t\right|}\right] \geq k\sum_{t=0}^{\infty}\frac{\mathbf{E}\left[\overline{a}_t\left|d_t\right|\right]}{\mathbf{E}\left[\overline{a}_t\left|p_t\right|\right]} = \infty.$$

Where the last equality is valid by Proposition 6. The theorem is thus proved, as long as our claim is correct. Denote $X = \overline{a}_t\left|d_t\right|$ for short. Given any constant M, we can write

$$\int X^{1/2}d\mu \geq \int_{X\leq M} X^{1/2}d\mu = M^{1/2}\int_{X\leq M}\left(XM^{-1}\right)^{1/2}d\mu \geq M^{-1/2}\int_{X\leq M} X d\mu$$

where we are using that $x^{1/2} \geq x$, if $x \in [0,1]$. Hence,

$$\int X^{1/2}d\mu \geq M^{-1/2}\left[\int X d\mu - \int_{X>M} X d\mu\right].$$

Setting $M = h\mathbf{E}\left[X\right]$, where h is a positive parameter, we have easily

$$\int X^{1/2}d\mu \geq \mathbf{E}^{1/2}\left(X\right)h^{-1/2}\left(1 - \int_{X/\mathbf{E}(X)>h}\frac{X}{\mathbf{E}\left(X\right)}d\mu\right).$$

If condition (i) is true, the random variables $\overline{a}_t\left|d_t\right|/\mathbf{E}\left[\overline{a}_t\left|d_t\right|\right]$ are dominated by the integrable function ϕ. Consequently, $\overline{a}_t\left|d_t\right|/\mathbf{E}\left[\overline{a}_t\left|p_t\right|\right]$ are uniformly integrable. This implies that there exists a sufficiently large h so that $\int_{X/\mathbf{E}(X)>h}\frac{X}{\mathbf{E}(X)}d\mu<\varepsilon$, independently of t. Thus, it turns out

$$\mathbf{E}^2\left[X^{1/2}\right]\geq k\mathbf{E}\left[X\right]$$

for some $k>0$ and this proves our claim. Suppose now that condition (ii) is fulfilled. By Markov inequality, we have

$$\int X^{1/2}d\mu\geq\int_{X\geq M}X^{1/2}d\mu\geq M^{1/2}\mu\left\{X\geq M\right\}.$$

Setting $M=\rho\mathbf{E}\left[X\right]$,

$$\mathbf{E}^2\left[X^{1/2}\right]\geq\rho\mathbf{E}\left[X\right]\mu^2\left\{X\geq\rho\mathbf{E}\left[X\right]\right\}\geq\rho\sigma^2\mathbf{E}\left[X\right]$$

and once again the desired result is obtained. ■

Proposition 6. Define inductively the scalar sequence

$$\eta_{t+1}=\eta_t\left(1+\frac{\mathbf{E}\left[a_t\left|d_t\right|\right]}{\mathbf{E}\left[a_t\left|p_t\right|\right]}\right),$$

with $\eta_0=1$. Thanks to (22), $\eta_t\uparrow\infty$ if and only if (12) holds. Since

$$\mathbf{E}\left[a_{t-1}\left|p_{t-1}\right|\right]=\mathbf{E}\left[a_t\left|p_t\right|+a_t\left|d_t\right|\right],$$

we have

$$\begin{aligned}\eta_{t+1}\mathbf{E}\left[a_t\left|p_t\right|\right]&=\eta_t\mathbf{E}\left[a_{t-1}\left|p_{t-1}\right|\right]\\ \frac{\eta_{t+1}}{\eta_t}&=\frac{\mathbf{E}\left[a_{t-1}\left|p_{t-1}\right|\right]}{\mathbf{E}\left[a_t\left|p_t\right|\right]}\end{aligned}$$

for $t\geq1$. By the decomposition

$$\eta_{t+1}=\left(\frac{\eta_{t+1}}{\eta_t}\right)\left(\frac{\eta_t}{\eta_{t-1}}\right)\cdots\cdot\left(\frac{\eta_2}{\eta_1}\right)\eta_1$$

it follows that

$$\eta_{t+1}=\frac{a_0\left|p_0\right|+a_0\left|d_0\right|}{\mathbf{E}\left[a_t\left|p_t\right|\right]}$$

and this concludes the proof. ■

Theorem 8. We claim that the events

$$A_t = \{|d_t| \,/\, |p_t| < \sigma_t\} \subseteq N_t^c$$

are μ-negligible for all t. Arguing by contradiction, suppose that $\mu(A_s) > 0$ for some $s \geq 1$. Picking the random variable ζ, relatively to A_s, as in (ii), one can rewrite this event as

$$A_s = \{\zeta\,|d_s| - \zeta\sigma_s\,|p_s| < 0\}. \tag{27}$$

We now construct an agent i's plan $(\widetilde{c}_t, \widetilde{y}_t)$ as follows (we ignore the superscript i):

$(\widetilde{c}_t, \widetilde{y}_t) = (\overline{c}_t, \overline{y}_t)$ for all $\omega \in \Omega$, if $t < s$, and for $\omega \notin A_s$ if $t \geq s$. If $\omega \in A_s$, then $(\widetilde{c}_s, \widetilde{y}_s) = (\overline{c}_s + \zeta\,|d_s|, \overline{y}_s)$ and $(\widetilde{c}_t, \widetilde{y}_t) = (\overline{c}_t - \zeta\sigma_s\,|d_t|, \overline{y}_t - \zeta\sigma_s\mathbf{1})$, for $t \geq s+1$.

Let us check that the plan $(\widetilde{c}_t, \widetilde{y}_t)$ is feasible. From the budget constraint (3)

$$\overline{c}_s + p_s \cdot \left(\overline{y}_{s+1} - \overline{y}_s\right) \leq d_s \cdot \overline{y}_s + w_s$$

we have

$$\overline{c}_s + p_s \cdot \left[\left(\overline{y}_{s+1} - \zeta\sigma_s\mathbf{1}\right) - \overline{y}_s\right] + \zeta\sigma_s\,|p_s| \leq d_s \cdot \overline{y}_s + w_s.$$

By virtue of (27), from this last inequality it follows

$$\overline{c}_s + \zeta\,|d_s| + p_s \cdot \left[\left(\overline{y}_{s+1} - \zeta\sigma_s\mathbf{1}\right) - \overline{y}_s\right] < d_s \cdot \overline{y}_s + w_s \tag{28}$$

which is valid for $\omega \in A_s$. Likewise, we have

$$\overline{c}_t - \zeta\sigma_s\,|d_t| + p_t \cdot (\widetilde{y}_{t+1} - \widetilde{y}_t) \leq d_t \cdot \widetilde{y}_t + w_t$$

for $t > s$ and $\omega \in A_s$. Hence, the plan $(\widetilde{c}_t, \widetilde{y}_t)$ is feasible. By (ii), $(\widetilde{c}_t, \widetilde{y}_t)$ would overtake $(\overline{c}_t, \overline{y}_t)$. Hence

$$\liminf_{N \to +\infty} \sum_{t=0}^{N-1} \mathbf{E}_0\left[u_t(\widetilde{c}_t) - u_t(\overline{c}_t)\right] > 0$$

which contradicts the weak optimality of plan $(\overline{c}_t, \overline{y}_t)$. To conclude, $\mu(A_t) = 0$, for all t and, consequently, $|d_t| \,/\, |p_t| \geq \sigma_t$ for almost all $\omega \in \Omega$. Condition (i) on the series $\sum_{t=1}^{\infty} \sigma_t$ implies that $\sum_{t=1}^{\infty} |d_t| \,/\, |p_t| = \infty$, uniformly. ■

Proposition 9. It will be proven by contradiction. Suppose some $A \in \mathcal{F}_s$ exists so that

$$\mu\left\{\omega \in A;\ \sum_{t=0}^{\infty} \frac{|d_t|}{|p_t|} \leq N\right\} = \mu(A) > 0$$

for some scalar N. In view of (22), we have

$$\prod_{t=0}^{\infty}\left(1 + \frac{|d_t|}{|p_t|}\right) \leq e^N \tag{29}$$

a.s. over A. Consider the following agent i's trading strategy (as usual, the superscripts are omitted)

$$\widetilde{y}_t = \overline{y}_t - g_t\mathbf{1}$$

where $\mathbf{1} = (1, 1, ..., 1)$ and $g_t = 0$ for $t \leq s$, while $g_t = \eta e^{-N}\zeta_t 1_A$ if $t > s$. Here ζ_t is the process (21). By virtue of (29) $\zeta_t \leq e^N$ and, consequently, $\widetilde{y}_t$ is a feasible trading plan. It finances the consumptions

$$\widetilde{c}_t = \overline{c}_t + g_{t+1}|p_t| - g_t(|p_t| + |d_t|)$$

According to the definition of g_t, we have $\widetilde{c}_t = \overline{c}_t$, except for $t = s$, where

$$\widetilde{c}_s = \overline{c}_s + \eta e^{-N}\zeta_{s+1}|p_s| 1_A$$

Since $|p_s| 1_A > 0$, the feasible plan $\widetilde{y}_t$ finances a higher level of consumption with positive probability, contradicting the optimality of agent i's plan $\overline{y}_t$. ■

Corollary 10. Set $S_t = \sum_{i=0}^{t} |d_i| / |p_i|$ and $S = \sum_{i=0}^{\infty} |d_i| / |p_i|$, defined over N_{∞}^c. If S_t does not go to infinity a.s., then there is some N such that $\mu\{S \leq N\} > 0$. Since $\{S_t \leq N\} \downarrow \{S \leq N\}$ as $t \to \infty$, and $\{S_t \leq N\} \in \mathcal{F}_t$, thanks to our assumptions there will exist some $B \in \mathcal{F}_s$ such that $S(\omega) \leq N$ for all $\omega \in B$. This contradicts Proposition 9. ■

Proposition 11. As $|d_t| / |p_t|$ is $\mathcal{F}_t$ measurable and $\mathcal{F}_t$ is generated by clopen sets, $|d_t| / |p_t|$ is continuous, provided $|p_t| \neq 0$. Therefore, the function $\sum_{i=0}^{t} |d_i| / |p_i|$ is continuous over N_t^c, while its value is infinity on N_t. Hence, we can view $\sum_{i=0}^{t} |d_i| / |p_i|$ as a lower semicontinuous function, taking values on $[0, +\infty]$. Since $\sum_{i=0}^{t} |d_i| / |p_i|$ increases as t increases, we have that $\sum_{i=0}^{\infty} |d_i| / |p_i|$ is lower semicontinuous. Hence, the sets

$$S_N = \left\{\sum_{i=0}^{\infty} |d_i| / |p_i| > N\right\}$$

are open. As $S_\infty = \cap_{N=1}^{\infty} S_N$, the event S_∞ is G_δ. Let us now prove that each S_N is dense. Actually, given a point $\omega = (s_1, s_2, ...) \in \Delta$, the collection $A_{s^1}, A_{s^2}, ...$ is a base of the neighborhoods of ω. In view of (20), $S_N \cap A_{s^t} \neq \varnothing$, which proves the density of S_N for all N. Thanks to the Baire property of the space Δ, which is a complete metric space, $S_\infty = \cap_{N=1}^{\infty} S_N$ is dense as well (see [1, Ths. 3.34-35]). $\blacksquare$

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