# Financial Economics

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Collegio Carlo Alberto, Fall 2007

Handout #2

# 1 Equilibrium and Pricing

## 1.1 Individual Agent Optimality

Let $(\mathbf{X}, \mathbf{p})$ be given. An *agent* is defined by a strictly increasing[1] utility function $U : \mathbb{R}_+^S \to \mathbb{R}$ and an endowment $\mathbf{e} \in \mathbb{R}_+^S$. The *budget feasible set* $B(\mathbf{p}, \mathbf{e})$ is the set

$$B(\mathbf{p}, \mathbf{e}) = \left\{ \mathbf{e} + \mathbf{X}\theta \in \mathbb{R}_+^S : \theta \in \mathbb{R}^N,\ \mathbf{p}'\theta \leq \mathbf{0} \right\}.$$

Suppose $\exists\ \theta^0 : \mathbf{X}\theta^0 > 0$. Then, since $U$ is strictly increasing, if $\mathbf{c}^* = \mathbf{e} + \mathbf{X}\theta^*$ solves

$$\sup_{\mathbf{c} \in B(\mathbf{p}, \mathbf{e})} U(\mathbf{c}), \tag{1}$$

then the solution is slack free, i.e., $\mathbf{p}'\theta = \mathbf{0}$. The following Proposition provides the link between NA and individual optimality.

**Proposition 1.** *i) if $\exists$ a solution to (1)$\Longrightarrow$NA; ii) if $U$ is continuous and there is NA$\Longrightarrow \exists$ a solution to (1).*

*Proof.* i) By contradiction, assume there is an arbitrage opportunity. Then, $\exists\ \overline{\theta} \in \mathbb{R}^N$ s.t. $\mathbf{Y}\overline{\theta} > \mathbf{0}$, i.e., either:

a) $\mathbf{p}'\overline{\theta} \leq \mathbf{0}$ and $\mathbf{X}\overline{\theta} > \mathbf{0}$.

b) $\mathbf{p}'\overline{\theta} < \mathbf{0}$ and $\mathbf{X}\overline{\theta} \geq \mathbf{0}$.

a) Let $\mathbf{c}^*$ solve (1). Then $\overline{\mathbf{c}} = \mathbf{c}^* + \mathbf{X}\overline{\theta} > \mathbf{c}^*$ and is still feasible. Since $U$ is strictly increasing, $U(\mathbf{c}^*) < U(\overline{\mathbf{c}})$, contradicting the fact that $\mathbf{c}^*$ solves (1).

b) In this case $p(\theta^* + \overline{\theta}) < 0$ and $\overline{\mathbf{c}} = \mathbf{c}^* + \mathbf{X}\overline{\theta} \geq \mathbf{c}^*$ so that $U(\overline{\mathbf{c}}) \geq U(\mathbf{c}^*)$, yielding a contradiction since the solution has to be slack free.

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[1] Meaning $U(\mathbf{x}) > U(\mathbf{x}')$ if $\mathbf{x} > \mathbf{x}'$.

ii) We know that NA$\Longleftrightarrow$ $\mathbf{q} \in \mathbb{R}^S_{++}$ s.t. $\mathbf{p}' = \mathbf{q}'\mathbf{X}$. Then we can rewrite the budget set as

$$B(\mathbf{p}, \mathbf{e}) = \left\{\mathbf{e} + \mathbf{X}\theta \in \mathbb{R}^S_+ : \theta \in \mathbb{R}^N,\ \mathbf{q}'\mathbf{X}\theta \leq \mathbf{0}\right\}.$$

Notice that

$$\mathbf{q}'\mathbf{X}\theta \leq 0 \Longleftrightarrow (q_1, ..., q_S) \begin{pmatrix} \sum\limits_{n=1}^{N} x_{1n}\theta_n \pm e_1 \\ ... \\ \sum\limits_{n=1}^{N} x_{Sn}\theta_n \pm e_s \end{pmatrix} \leq 0$$

$$\Longleftrightarrow \sum_{s=1}^{S} q_s c_s \leq \sum_{s=1}^{S} q_s e_s \equiv w > 0.$$

Then $c_s \geq 0$ by assumption and $c_s \leq \dfrac{w}{q_s}$ for all $s$ so that $B(\mathbf{p}, \mathbf{e})$ is closed and bounded, and hence is compact. Since $U$ is continuous, a continuous function on a compact set always achieves a maximum and a minimum $\Longrightarrow$ $\exists$ solution to (1). □

## 1.2 Individual optimality and Pricing

**Theorem 1.** *Suppose* $\mathbf{c}^* >> \mathbf{0}$ *is solution to (1), that* $U$ *is continuously differentiable at* $\mathbf{c}^*$*, and that the vector of partial derivatives* $\nabla\mathbf{U}$ *at* $\mathbf{c}^*$ *is strictly positive. Then, there is some scalar* $\lambda > 0$ *s.t.* $\lambda\nabla\mathbf{U}(\mathbf{c}^*)$ *is a state price vector.*

*Proof.* $\mathbf{c}^* >> \mathbf{0}$ implies that $\forall\theta\ \exists\ k \in \mathbb{R}_{++}$ s.t. $\mathbf{c}^* + \alpha\mathbf{X}\theta \geq 0$ for all $\alpha \in [-k, k]$. Let $g_\theta : [-k, k] \to \mathbb{R}$ defined by $g_\theta(\alpha) = U(\mathbf{c}^* + \alpha\mathbf{X}\theta)$. Suppose $\mathbf{p}'\theta = \mathbf{0}$, then optimality of $\mathbf{c}^*$ implies that $g_\theta$ is maximized at $\alpha = 0$ (i.e., the marginal utility of buying a feasible portfolio is zero). The f.o.c. for this is that

$$g'_\theta(0) = \nabla\mathbf{U}(\mathbf{c}^*)'\mathbf{X}\theta = \mathbf{0}.$$

Then, $\forall\theta \in \mathbb{R}^N$, if $\mathbf{p}'\theta = \mathbf{0}$ it must be $\nabla\mathbf{U}(\mathbf{c}^*)'\mathbf{X}\theta = \mathbf{0}$. This implies that the vectors $\mathbf{p}$ and $\mathbf{X}'\nabla\mathbf{U}(\mathbf{c}^*)$ are co-linear, i.e. $\exists\mu \in \mathbb{R}$ s.t. $\mathbf{X}'\nabla\mathbf{U}(\mathbf{c}^*) = \mu\mathbf{p}$. By assumption $\exists\theta^0 : \mathbf{X}\theta^0 > 0$. From the existence of a solution to (1), there is NA. Therefore $\mathbf{p}'\theta^{\mathbf{0}} > 0$, and we have $\mu\mathbf{p}'\theta^{\mathbf{0}} = \nabla\mathbf{U}(\mathbf{c}^*)'\mathbf{X}\theta^{\mathbf{0}} > 0$ because the vector $\nabla\mathbf{U}(\mathbf{c}^*)$ is assumed to be strictly positive. Hence, $\mu > 0$. Let $\lambda = \dfrac{1}{\mu}$ and obtain $\mathbf{p}' = \lambda\nabla\mathbf{U}(\mathbf{c}^*))\mathbf{X}$. □

**Remark:** assuming that $U$ is strictly increasing does not guarantee $\nabla \mathbf{U}(\mathbf{c}^*) >> 0$, but if $U$ is concave and strictly increasing, then $\nabla \mathbf{U}(\mathbf{c}^*) >> 0$.

**Corollary 2.** *Assume $U$ strictly increasing and concave, differentiable at $c^* >> 0$ with $p'\theta^* = 0$. Then $c^*$ is optimal iff $\lambda \nabla U(c^*)$ is a state price vector for some scalar $\lambda > 0$.*

The Corollary follows from sufficiency of f.o.c. for concave functions.

*Intuition*: state prices are therefore proportional to an investor's marginal utility. But what is the interpretation of $\lambda$? In order to answer this we need to introduce t=0 consumption.

**Aside**: matrix notation for derivatives.

The *gradient vector* $\nabla \mathbf{f}(\overline{\mathbf{x}}) \in \mathbb{R}^N$ is a vector where the n-th entry is the n-th partial derivative of the real valued function $f : \mathbb{R}^N \to \mathbb{R}$ evaluated at $\overline{\mathbf{x}} \in \mathbb{R}^N$. If $f : \mathbb{R}^N \to \mathbb{R}^M$ is differentiable, at any $\mathbf{x} \in \mathbb{R}^N$ denote $Df(\mathbf{x})$ the $M \times N$ matrix whose $mn$ entry is $\frac{\partial f_m(x)}{\partial x_n}$. If $M = 1$ then $Df(\mathbf{x}) = [\nabla f(\mathbf{x})]^T$ .

*Chain rule*: Let $g : \mathbb{R}^S \to \mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}^M$ be differentiable functions. The *composite function* $f(g(\cdot))$ is also differentiable. Let $\mathbf{x} \in \mathbb{R}^S$. Then

$$\underset{M\times S}{D_{\mathbf{x}} f(g(\mathbf{x}))} = \underset{M\times N}{Df(g(\mathbf{x}))} \underset{N\times S}{Dg(\mathbf{x})}$$

### 1.2.1 Adding t=0 consumption

An *agent* is now defined by a strictly increasing utility function $U : \mathbb{R}_+^{S+1} \to \mathbb{R}$ and an endowment $(e_0, \mathbf{e}) \in \mathbb{R}_+^{S+1}$. Let $(c_0, \mathbf{c}) \in \mathbb{R}_+^{S+1}$. The agent's problem is now:

$$\underset{(c_0,\mathbf{c})}{Max} U(c_0, \mathbf{c}), \tag{2}$$

subject to the constraints

$$\begin{aligned} \mathbf{c} &= \mathbf{e} + \mathbf{X}\theta, \\ c_0 + \mathbf{p}'\theta &\leq e_0. \end{aligned}$$

Let $\mathbf{z} = \begin{pmatrix} c_0 \\ \theta \end{pmatrix}$ and $f(\mathbf{z}) = \begin{pmatrix} c_0 \\ \mathbf{e} + \mathbf{X}\theta \end{pmatrix}$. Then (2) is equivalent to

$$\underset{\mathbf{z}}{Max}\ U(f(\mathbf{z})) \tag{3}$$

subject to

$$c_0 + \mathbf{p}'\theta \leq e_0.$$

In the above problem, $U : \mathbb{R}^{S+1} \to \mathbb{R}$ and $f : \mathbb{R}^{N+1} \to \mathbb{R}^{S+1}$. Applying the chain rule,

$$\begin{aligned} D_z U(f(\mathbf{z})) &= \underset{1\times(S+1)}{DU}\ \underset{(S+1)(N+1)}{Df} \\ &= \left(\frac{\partial U}{\partial c_0}, ..., \frac{\partial U}{\partial c_S}\right) \times \begin{pmatrix} 1 & \underbrace{0,..,0}_{N} \\ \left.\begin{array}{c} 0 \\ . \\ . \\ 0 \end{array}\right\} S & \mathbf{X} \end{pmatrix} = \left[\nabla_z U(f(\mathbf{z}))\right]^T. \end{aligned}$$

Then, by simply transposing the last expression we get the gradient

$$\nabla_z U(f(\mathbf{z})) = \begin{pmatrix} 1 & \underbrace{0,...0}_{S} \\ \left.\begin{array}{c} 0 \\ . \\ . \\ 0 \end{array}\right\} N & \mathbf{X}' \end{pmatrix} \begin{pmatrix} \frac{\partial U}{\partial c_0} \\ . \\ . \\ \frac{\partial U}{\partial c_S} \end{pmatrix} = \begin{pmatrix} \frac{\partial U}{\partial c_0} \\ \mathbf{X}' \begin{pmatrix} \frac{\partial U}{\partial c_1} \\ ... \\ \frac{\partial U}{\partial c_S} \end{pmatrix} \end{pmatrix}.$$

Let $\mathbf{z}^*$ solve (3). Denote $\mathbf{c}^* = \mathbf{e} + \mathbf{X}\theta^*$. Suppose $U$ is concave, then $\exists\ \mu \in \mathbb{R}_+$ s.t. $\mathbf{z}^*$ solves (2) iff

$$\begin{aligned} \nabla_z U(f(\mathbf{z}^*)) &= \mu \begin{pmatrix} 1 \\ \mathbf{p} \end{pmatrix}, \\ \mu(c_0^* + \mathbf{p}'\theta^* - e_0) &= 0. \end{aligned}$$

Since the solution is slack free, $\mu > 0$. Rewriting explicitly the system of first order conditions yields

$$\left(\begin{array}{c} \dfrac{\partial U(c_0^*,\mathbf{c}^*)}{\partial c_0} \\ \mathbf{X}'\left(\begin{array}{c} \dfrac{\partial U(c_0^*,\mathbf{c}^*)}{\partial c_1} \\ \dots \\ \dfrac{\partial U(c_0^*,\mathbf{c}^*)}{\partial c_S} \end{array}\right) \end{array}\right) = \mu \left(\begin{array}{c} 1 \\ \mathbf{p} \end{array}\right).$$

From the first equation, $\mu = \dfrac{\partial U(c_0^*,\mathbf{c}^*)}{\partial c_0}$, so the other $N$ equations can be written as

$$\frac{1}{\dfrac{\partial U(c_0^*,\mathbf{c}^*)}{\partial c_0}}\mathbf{X}'\nabla_c U(c_0^*,\mathbf{c}^*) = \mathbf{p},$$

or, equation by equation

$$p_n = \sum_{s=1}^{S} \frac{\dfrac{\partial U}{\partial c_s}}{\dfrac{\partial U}{\partial c_0}} x_{ns},$$

implying $q_s = \dfrac{\partial U}{\partial c_s}/\dfrac{\partial U}{\partial c_0}$, i.e., that the that price for state $s$ is the marginal rate of substitution between $c_0$ and $c_s$.

## 1.3 Expected Utility

We now specialize to the expected utility case, namely

$$U(c_0,\mathbf{c}) = u(c_0) + \sum_{s=1}^{S} \pi_s u(c_s).$$

Then the results in the previous paragraph imply that $q_s = \pi_s \dfrac{u'(c_s^*)}{u'(c_0^*)}$. Let $\varphi_s = \dfrac{u'(c_s^*)}{u'(c_0^*)}$. Then $p_n = \sum\limits_{s=1}^{S} \pi_s \varphi_s x_{ns} = E\left[\varphi x_n\right] \implies$ the vector of marginal rates of substitution is a valid pricing kernel or stochastic discount factor. Let

$$q_0 = \frac{\sum\limits_{s=1}^{S} \pi_s u'(c_s^*)}{u'(c_0^*)}.$$

Then, the vector in which each entry $s$ is given by

$$\pi_s^* = \frac{q_s}{q_0} = \frac{\pi_s u'(c_s^*)}{\sum\limits_{s=1}^{S} \pi_s u'(c_s^*)}$$

is a risk neutral probability vector.