Financial Economics

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Handout #4

1 Equilibrium with complete markets: properties

Assume markets are complete. Then we know that

$$\mathbf{X}'\nabla U_i = \lambda_i \mathbf{p} \Longrightarrow \frac{1}{\lambda_i}\nabla U_i = (\mathbf{X}')^{-1}\mathbf{p},$$

meaning that each agent vector of marginal utility is a valid state price vector. Hence,

$$\frac{1}{\lambda_i}\frac{\partial U_i(\mathbf{c}^i)}{\partial c_s} = \frac{1}{\lambda_j}\frac{\partial U_j(\mathbf{c}^j)}{\partial c_s} \text{ for all } i,j \in I, \text{ for all } s \in S. \quad (1)$$

The above expression can be more conveniently interpreted considering the case of time 0 consumption. In this case remember that $\lambda_i = \frac{\partial U_i(\mathbf{c}^i)}{\partial c_0}$, so that marginal rates of substitution across agents are equalized in equilibrium state by state. Remember that equilibrium allocations with complete markets are Pareto optimal (first welfare theorem), and that any Pareto optimal allocation can be represented as the solution to the optimization problem of a social planner. Condition (1) therefore follows by the f.o.c. of the planner's problem, setting the agent weights as the inverse of the lagrange multipliers associated with the individual budget constraints in the market equilibrium.

In the analysis that follows we consider the case of expected utility: $U_i(\mathbf{c}) = \sum_{s=1}^{S} \pi_s u_i(c_s)$, and assume u_i strictly concave. Then (1) becomes $\frac{u_i'(c_s)}{\lambda_i} = \frac{u_j'(c_s)}{\lambda_j}$. Strict concavity of u_i implies the *co-monotonicity* property of consumption plans: if one agent's optimal consumption is greater in one state (say, s) than another (say, s'), then this has to be the case for all other agents. Formally, if $c_s^i > c_{s'}^i$,by strict concavity of u_i it follows that $\frac{u_i'(c_s^i)}{\lambda_i} < \frac{u_i'(c_{s'}^i)}{\lambda_i}$, and because of (1) we have $\frac{u_j'(c_s^j)}{\lambda_j} < \frac{u_j'(c_{s'}^j)}{\lambda_j}$ for all j and again by by strict concavity of u_j this implies $c_s^j > c_{s'}^j$. for all j.

Moreover, across states,

$$\frac{\pi_s}{\pi_{s'}} \frac{u_i'(c_s^i)}{u_i'(c_{s'}^i)} = \frac{q_s}{q_{s'}}.$$

Therefore, by strict concavity of u_i, $\frac{q_s}{\pi_s} > \frac{q_{s'}}{\pi_{s'}}$ iff $c_s^i < c_{s'}^i$; but because of co-monotonicity, $c_s^i < c_{s'}^i \forall i \Longrightarrow \sum_{i=1}^{I} c_s^i < \sum_{i=1}^{I} c_{s'}^i \Longleftrightarrow e_s < e_{s'}$, meaning that the "marginal cost per unit of probability" $\frac{q_s}{\pi_s}$ of state s is greater than in state s' iff the economy is poorer in state s than in state s'.

1.1 Pareto Optimal sharing rules

Co-monotonicity implies that if $c_s^i > c_{s'}^i$ for some i, then $e_s > e_{s'}$. Also, if $e_s > e_{s'}$, then for at least one agent i it must be $c_s^i > c_{s'}^i$, but then $c_s^j > c_{s'}^j$ $\forall j$. Therefore there is a one to one relation between the aggregate consumption (endowment) for a state and individual consumption in that state. This means that there are functions f_i such that. $c_s^i = f_i(e_s)$ $\forall i, s$. These functions are called *Pareto Optimal sharing rules* (since $u_i's$ are state independent, these functions are state independent). Also, by the previous reasoning it follows that $f_i's$ are monotonically increasing.

In general, these functions can be nonlinear. If they are linear, however, any Pareto optimal allocation lies in the span of the risk-free payoff and the aggregate endowment. In this case we say that *two-fund spanning* holds: the social welfare problem can be reduced to assigning to agents claims on two mutual funds, one consisting of the risk-free payoff and the other is a claim on the aggregate endowment.

When are $f_i's$ linear?

Proposition 1. *$f_i's$ are linear if*

$$-\frac{u_i'(z)}{u_i''(z)} = A_i + Bz \tag{2}$$

i.e. if the inverse of Arrow-Pratt measure of absolute risk aversion is linear (agents' utilities exhibit linear risk tolerance) with the same slope B (cautiousness).

Proof. It can be easily verified by taking derivatives, that

$$\begin{aligned} B &\neq 0,\ (2) \iff u_i'(z) = \rho^i (A_i + Bz)^{-\frac{1}{B}} \\ B &= 0,\ (2) \iff u_i'(z) = \rho^i \exp\{-\frac{z}{A_i}\} \end{aligned}$$

In a Pareto optimal allocation, it follows from the f.o.c. of the social welfare problem that

$$\lambda_i u_i'(c_s^i) = \lambda_j u_j'(c_s^j).$$

Assume $B \neq 0$, then the above becomes

$$\frac{\lambda_i \rho^i}{\lambda_j \rho^j}(A_i + Bc_s^i)^{-\frac{1}{B}} = (A_j + Bc_s^j)^{-\frac{1}{B}},$$

or equivalently

$$\left(\frac{\lambda_i \rho^i}{\lambda_j \rho^j}\right)^{-B}(A_i + Bc_s^i) = A_j + Bc_s^j.$$

Taking the sum over j:

$$(A_i + Bc_s^i)(\lambda_i \rho_i)^{-B} \sum_{j=1}^{I} \frac{1}{(\lambda_j \rho_j)^{-B}} = \sum_{j=1}^{I} A_j + Be_s;$$

rearranging

$$A_i + Bc_s^i = \frac{Be_s + \sum_{j=1}^{I} A_j}{(\lambda_i \rho_i)^{-B} \sum_{j=1}^{I} (\lambda_j \rho_j)^B};$$

solving for c_s^i yields

$$c_s^i = \frac{Be_s + \sum_{j=1}^{I} A_j}{(\lambda_i \rho_i)^{-B} \sum_{j=1}^{I} (\lambda_j \rho_j)^B B} - \frac{A_i}{B},$$

which is linear in e_s. □

1.2 The aggregation property

Linear $f_i'$$s$ have important implications for pricing. We say an economy with heterogeneous agents satisfy the *aggregation property* if prices are determined independently of the distribution of initial endowments. A *sufficient condition* is the following: that individuals exhibit linear risk tolerance (i.e., $f_i's$ are linear) with identical cautiousness (i.e., $B_i = B \ \forall i$) and have same time preferences (i.e., $\rho_i = \rho \ \forall i$).

Solutions to (2) include power functions:

$$u_i(z) = \frac{1}{B-1}(A_i + Bz)^{1-\frac{1}{B}},$$

(which becomes $u_i(z) = \ln(A_i + Bz)$ if $B = 1$), and negative exponential functions

$$u_i(z) = -A_i \exp(-\frac{z}{A_i}).$$

In the following, we shall consider an example of an economy that satisfies the aggregation property. Suppose

$$U_i(\mathbf{c}) = \frac{1}{B-1}\left(A_i + Bc_0\right)^{1-\frac{1}{B}} + \rho \sum_{s=1}^{S} \pi_s \frac{1}{B-1}\left(A_i + Bc_s\right)^{1-\frac{1}{B}}$$

Remember that in this case the social welfare function can be written as $U_\lambda(\mathbf{e}) = \mathbb{E}[u_\lambda(e)]$, where

$$u_\lambda(e_s) = \max_{(c_s^1,\ldots,c_s^I)} \sum_{i=1}^{I} \lambda_i u_i(c_s^i) \quad s.t. \ \sum_{i=1}^{I} c_s^i = e_s. \tag{3}$$

Remember that the agent weights $\lambda' s$ are inversely related to the lagrange multipliers associated with individual budget constraints, and therefore depend on the distribution of initial wealth across agents in the economy. In the above example, utility functions satisfy the sufficient condition and we will prove that the social welfare function satisfies

$$u_\lambda(e_s) = \frac{1}{B-1} K\rho \left(\sum_{i=1}^{I} A_i + Be_s \right)^{1-\frac{1}{B}}, \tag{4}$$

where K is a function of $\lambda' s$. Notice that (4) implies

$$\pi_s \frac{u'_\lambda(e_s)}{u'_\lambda(e_0)} = \pi_s \frac{\left(\sum_{i=1}^{I} A_i + Be_s\right)^{-\frac{1}{B}}}{\left(\sum_{i=1}^{I} A_i + Be_0\right)^{-\frac{1}{B}}} = q_s, \qquad \forall s$$

so that state prices (and hence asset prices) are indeed independent of K and hence of the $\lambda' s$.

Proof. The f.o.c. for the problem (3) are

$$(A_i + Bc_s^i)^{-\frac{1}{B}} = \frac{\mu_s}{\lambda_i} \ \forall i \text{ and } s, \tag{5}$$

$$\sum_{i=1}^{I} c_s^i = e_s \ \forall s, \tag{6}$$

where μ_s is the multiplier associated with the resource constraint for state s. Raising to both sides of (5) to the power of $-B$ and summing over i:

$$\sum_{i=1}^{I} A_i + Be_s = (\mu_s)^{-B} \sum_{i=1}^{I} \lambda_i^B,$$

solving for μ_s, and plugging into (5) yields

$$\lambda_i (A_i + Bc_s^i)^{-\frac{1}{B}} = \left(\sum_{i=1}^{I} A_i + Be_s\right)^{-\frac{1}{B}} \left(\sum_{i=1}^{I} \lambda_i^B\right)^{\frac{1}{B}}.$$

Multiplying both sides by $(A_i + Bc_s^i)\frac{1}{B-1}$ we get

$$\frac{\lambda_i}{B-1} \left(A_i + Bc_s^i\right)^{1-\frac{1}{B}} = \left(\sum_{i=1}^{I} A_i + Be_s\right)^{-\frac{1}{B}} \left(\sum_{i=1}^{I} \lambda_i^B\right)^{\frac{1}{B}} (A_i + Bc_s^i)\frac{1}{B-1}.$$

Summing over i:

$$u_\lambda(e_s) = \frac{\left(\sum_{i=1}^{I} A_i + Be_s\right)^{1-\frac{1}{B}}}{B-1} \left(\sum_{i=1}^{I} \lambda_i^B\right)^{\frac{1}{B}},$$

defining the the last term, $\left(\sum_{i=1}^{I} \lambda_i^B\right)^{\frac{1}{B}} \equiv K$, we get (4). □