Financial Economics

Francesco Sangiorgi

Collegio Carlo Alberto, Fall 2007

Handout #6

1 Derivatives

- Nominal prices & payoffs
- Primitive assets:
  - stock: current price S at $t = 0$, future payoff x at $t = 1$.
  - bond: current price B at $t = 0$, future payoff 1 at $t = 1$.

The stock will be the *underlying asset* of the *derivative contracts*.

- Assume NA$\Longrightarrow \exists\, \mathbf{q} >> 0$ *s.t.* $\begin{pmatrix} S \\ B \end{pmatrix} = \sum_{s=1}^{S} q_s \begin{pmatrix} x_s \\ 1 \end{pmatrix}$.

  If there are only two states of the world, then $\mathbf{q}$ is unique and can be computed from the prices of existing assets S, B.

1.1 Definitions

We now introduce some derivative contracts.

Forwards: a *long position* in a forward contract on stock S with *delivery price* k delivers the payoff $x - k$. A *short position* delivers $k - x$.

Valuation: since the payoff of the forward is in the span of the stock and the bond, its value can be computed using state prices even if markets are incomplete. Denoting with f the price of the forward we must have

$$f = \sum_{s=1}^{S} q_s (x_s - k).$$

Usually, k is chosen such that $f = 0$. Think of forwards as bets (zero cost for long/short position when the positive is assumed).

Define $z^+ = \max(z, 0)$.

Options: a *call option* on stock S with *strike price* k is a contract that delivers the payoff $(x - k)^+$ at maturity. A *put option* on stock S with strike price k delivers the payoff $(k - x)^+$

Valuation: in general options payoff lie outside the span of stock and bond. If markets are complete, $\mathbf{q}$ is unique and can be used to price options as

$$\begin{pmatrix} c \\ p \end{pmatrix} = \sum_{s=1}^{S} q_s \begin{pmatrix} (x_s - k)^+ \\ (k - x_s)^+ \end{pmatrix},$$

where c and p denote call and put prices respectively. But if markets are incomplete, different $\mathbf{q}$'s might give different prices if options payoffs lie outside the span of existing securities. Moreover, this does not take into account that as options are effectively introduced, state prices might change.

1.2 Law of one price and arbitrage bounds

If derivatives are *redundant* assets (i.e., their payoffs lie inside the span of existing securities), then they can be priced by means of prices of existing securities only, as an application of the law of one price.

Forward Value Notice a portfolio which is long 1unit of stock and short k units of the bond replicates the payoff of the forward, hence

$$f = S - Bk;$$

k is usually chosen so that $f = 0$, so

$$k = \frac{S}{B} = R_0 S.$$

Remark: Options and spanning.

Notice options payoffs are nonlinear functions of the underlying asset. Hence, the introduction of options can be used to complete markets. Assume $\exists$ a traded portfolio with strictly positive payoffs that "separate states", i.e., $x_s \neq x'_s$ if $s \neq s'$. Assume $x_s < x'_s$ if $s < s\prime$ (re-labeling states). Consider $|S| - 1$ call options with strike prices equal to $x_1, ..., x_{S-1}$. The payoff matrix is

$$\mathbf{X}' = \begin{pmatrix} x_1 & x_2 & x_3 & ... & x_S \\ 0 & x_2 - x_1 & x_3 - x_1 & ... & x_{S-1} \\ 0 & 0 & x_3 - x_2 & ... & x_{S-2} \\ ... & ... & ... & ... & .. \\ 0 & 0 & 0 & ... & x_S - x_{S-1} \end{pmatrix},$$

and has full rank so that markets are complete.

Option value In general (i.e., incomplete markets) options cannot be replicated by a portfolio of primitive assets. Without additional information (only S, B), we can compute *arbitrage bounds.*

Proposition 1.

$$\begin{aligned} (S - Bk)^+ &\leq c \leq S, \\ (Bk - S)^+ &\leq p \leq Bk. \end{aligned}$$

Proof. (graphical). Draw a graph with assets' payoff on the vertical axis and the payoff of the stock (x) on the horizontal axis. The payoff of the call is below the payoff of the stock state by state, implying $c \leq S$. The payoff of the call is above the payoff of a portfolio long in the stock and short k units of the bond (the payoffs are equal for $x \geq k$), implying $S - Bk \leq c$. Moreover the payoff from the call is positive, so $c \geq 0 \Longrightarrow (S - Bk)^+ \leq c$. Therefore, $S = 0 \Longrightarrow c = 0$, and $k = 0 \Longrightarrow c = S$, hence $c \in [0, S]$. The proof for the put is left as an exercise. □

Proposition 2. *Put-call parity*

$$p + S = c + Bk$$

Proof. (graphical). A portfolio which is long 1 unit in a call and short 1 put (same strike k) replicates the payoff of forward with delivery price k. Therefore $c - p = f = S - Bk$ by the law of one price. □

1.3 Modigliani-Miller propositions

1.3.1 The irrelevance statement

Assume no arbitrage and perfect capital markets (i.e., no frictions of any type, including taxes). Consider a firm whose assets payoff is $x \geq 0$ at $t = 1$, and denote its current value V. The firm has two ways of raising capital, either through equity only or through equity together with debt. When the firm issues debt, the firm is said to be *levered*, otherwise is said to be *unlevered*. Debtholders (or creditors, or bondholders) buy a contract from the firm that gives them a claim to a face value k at $t = 1$. Equityholders (or shareholders), have the right to receive what is left from the value of the firm once debtholders are paid (the bond has higher seniority than equity, which is a residual claim). The firm's liabilities are, therefore:

- a risky bond that pays k if $x \geq k$, and x otherwise (i.e., the bond payoff at $t = 1$ is $x_B = k - (k - x)^+$),
- a stock that pays what is left of the payoff x after the bond is repaid (i.e., the stock payoff at $t = 1$ is $x_S = (x - k)^+$).

Denote S and B the current prices of the stock and the (risky) bond issued buy the firm. Notice that the bond payoff equals the payoff of a portfolio which is long in the risk free bond and short on a put on firm's assets with strike price k, while the stock payoff equals the payoff of a long position on a call on firm's assets with strike price k. Buying the bond and the stock is equivalent to holding the firm, since the payoff at $t = 1$ is the same in any state s:

$$x_{Bs} + x_{S_s} = k - (k - x_s)^+ + (x_x - k)^+ = x_s$$

Hence, by the law of one price, it must be that the value of the firm equals the sum of the prices of the stock and the bond,

$$V = S + B.$$

Denote V_u, V_l the values of the unlevered firm (i.e., $k = 0$) and of the levered firm (i.e., $k > 0$) respectively. Then,

$$V_l = \sum_{s=1}^{S} q_s(x_{Bs} + x_{S_s}) = \sum_{s=1}^{S} q_s \left(k - (k - x_s)^+ + (x_x - k)^+\right) = \sum_{s=1}^{S} q_s x_s = V_u.$$

The statement that $V_u = V_l$ means that the value of any firm is independent of its *capital structure* (i.e., of the structure of the financial sources used in the project), and implies that if equity is not sufficient to finance the firm, then debt may be used without affecting the value of the project.

Remark: there are several hidden assumptions behind the logic above:

- changing k has no effect on x,
- changing k does not affect state prices, which might not be true if markets are incomplete.

1.3.2 Cost of capital

The expected return on equity (or cost of equity, or cost of capital) for the levered firm $E[R^l_S]$ is equal to the expected return to a pure equity stream $E[R^u_S]$ plus a premium that is related to the financial risk equal to the debt/equity ratio times the spread between $E[R^u_S]$ and the expected return on debt (or cost of debt) $E[R^l_B]$, i.e.,

$$E[R^l_S] = E[R^u_S] + \left(E[R^u_S] - E[R^l_B]\right)\frac{B}{S}.$$

The proof is left as an exercise.

1.4 The binomial model

Law of one price is not enough to price options if markets are incomplete. We need to add more structure. In the following model, the bond and the stock are enough to complete markets , so that state prices are unique and the law of one price can be used. The setup we are going to introduce is too simple and restrictive in this static framework, but it will be very useful when we will deal with dynamic economies.

Assumption: only 2 different values for x at $t = 1$ one for state *up* and one for state *down*,. namely there is a stock whose return is $R_u > R_d$ in the two states and a bond whose return is R_0 in both states. We write the payoff matrix in terms of returns (so that prices equals 1 for both assets)

$$\mathbf{X} = \begin{pmatrix} R_0 & R_u \\ R_0 & R_d \end{pmatrix}.$$

Notice that markets are complete. State prices can be computed from $\mathbf{p}'= \mathbf{q}'\mathbf{X}$, i.e.,

$$(1, 1) = (q_u, q_d) \begin{pmatrix} R_0 & R_u \\ R_0 & R_d \end{pmatrix},$$

so that

$$(q_u, q_d) = \left(\frac{R_0 - R_d}{R_0 (R_u - R_d)}, \frac{R_u - R_0}{R_0 (R_u - R_d)} \right).$$

As a simple exercise, show that there is no arbitrage iff $R_d < R_0 < R_u$. Moreover, we know that risk neutral probabilities can be computed as $\pi_s^* = q_s/q_0$ so that

$$(\pi_u^*, \pi_d^*) = \left(\frac{R_0 - R_d}{R_n - R_d}, \frac{R_u - R_0}{(R_u - R_d)} \right).$$

1.4.1 Pricing in a binomial model

Any asset is redundant given the setup. If an asset has payoff z, the portfolio θ that replicates its payoff solves

$$\begin{pmatrix} BR_0 & SR_n \\ BR_0 & SR_d \end{pmatrix} \begin{pmatrix} \theta_B \\ \theta_S \end{pmatrix} = \begin{pmatrix} z_n \\ z_d \end{pmatrix},$$

so that

$$\begin{pmatrix} \theta_B \\ \theta_S \end{pmatrix} = \begin{pmatrix} (R_n z_d - R_d z_n)/(BR_0(R_n - R_d)) \\ (z_n - z_d)/(S(R_n - R_d)) \end{pmatrix}$$

By the law of one price, the value v of payoff z is

$$v = \theta_B B + \theta_S S.$$

Notice that by substituting the explicit values of θ and rearranging we can express v in terms of risk neutral pricing:

$$v = \frac{1}{R_0}\left[z_n\pi_n^* + z_d(1-\pi_n^*)\right] = \frac{E^*(Z)}{R_0}.$$

For an option, we would simply have

$$c = \frac{1}{R_0}E^*\left[(x-k)^+\right] = \frac{1}{R_0}\left[\pi_n^*\left(SR_n - k\right)^+ + (1-\pi_n^*)(SR_d - k)^+\right].$$