Financial Economics

Francesco Sangiorgi

Collegio Carlo Alberto, Fall 2007

Handout #9

1 Equilibrium and Pareto Optimality

Assume the economy is populated by I agents, defined by strictly increasing utility functions $U_i : L_+ \to \mathbb{R}$, and endowments $e^i \in L_+$. Given the dividend process δ for N securities, a *financial equilibrium* (FE) a collection $(\theta^1, ..., \theta^I, S)$, where S is a security price process, and for each i, θ^i is a trading strategy solving

$$\sup_{\theta \in \Theta} U_i(c) \qquad s.t. \quad c = e^i + \delta^\theta \in L_+, \tag{1}$$

and market clears:

$$\sum_{i=1}^{I} \theta^i = 0. \tag{2}$$

Proposition 1. *(First Welfare Theorem) Suppose $(\theta^1, ..., \theta^i, S)$ is a FE and markets are complete (CM).Then the associated consumption allocation is Pareto Optimal.*

Proof. Step 1. We show that the individual agent problem (1) under CM can be equivalently formulated as

$$\max_{c \in L_+} U_i(c) \qquad s.t. \quad \mathbb{E}\left[\sum_{t=0}^{T} c_t \pi_t\right] \leq \mathbb{E}\left[\sum_{t=0}^{T} e_t^i \pi_t\right] \tag{3}$$

where π is the SPD. We show it by showing that $c^i \in L_+$ is feasible under (1) iff it is feasible under (3). Let c^i be feasible in (1). Then,

$$\mathbb{E}\left[\sum_{t=1}^{T} \pi_t (e_t^i + \delta_t^\theta)\right] = \mathbb{E}\left[\sum_{t=0}^{T} \pi_t e_t^i\right] + \underbrace{\mathbb{E}\left[\sum_{t=0}^{T} \pi_t \delta_t^\theta\right]}_{=0 \text{ by } \pi \text{ SPD}}. \tag{4}$$

Let c^i be feasible in (3). Assume the constraint is tight (which follows by strictly increasing utility functions). Let δ^θ be s.t. $\delta_t^\theta = c_t^i - e_t^i$ for $t \geq 1$. By CM $\delta^\theta \in M$.

Then,

$$
\begin{aligned}
0 &= \mathbb{E}\left[\sum_{t=0}^{T} \pi_t (e_t^i - c_t^i)\right] \\
&= \pi_0 (e_0^i - c_0^i) + \mathbb{E}\left[\sum_{t=1}^{T} \pi_t (e_t^i - c_t^i)\right] + \underbrace{\mathbb{E}\left[\sum_{t=0}^{T} \pi_t \delta_t^\theta\right]}_{=0 \text{ by } \pi \text{ SPD}} \\
&= \pi_0 (e_0^i - c_0^i + \delta_0^\theta)
\end{aligned} \tag{5}
$$

since $\pi_0 > 0$, (5) implies $-\delta_0^\theta = \theta_0^{'} S_0 = e_0^i - c_0^i$ which is feasible. Then, c^i solves (1) iff it is feasible under (3).

Step 2. By contradiction, assume $\exists$ a feasible allocation c^* s.t. $U_i(c^{i*}) \geq U_i(c^i)$ for all i, $U_i(c^{i*}) > U_i(c^i)$ for some i. Then, it must be that

$$
\mathbb{E}\left[\sum_{t=0}^{T} \pi_t c_t^{i*}\right] \geq \mathbb{E}\left[\sum_{t=0}^{T} \pi_t e_t^i\right]
$$

for all i, with strict inequality for some i (otherwise, being U_i strictly increasing, this would violate c^i solving (3). Summing over i,

$$
\mathbb{E}\left[\sum_{t=0}^{T} \pi_t \sum_{i=1}^{I} c_t^{i*}\right] > \mathbb{E}\left[\sum_{t=0}^{T} \pi_t \sum_{i=1}^{I} e_t^i\right] = \mathbb{E}\left[\sum_{t=0}^{T} \pi_t e_t\right]; \tag{6}
$$

$\pi > 0$ implies that at least for some t we have $Prob\left(\sum_{i=1}^{I} c_t^{i*} > e_t\right) > 0$, violating the feasibility requirement $\sum_{i=1}^{I} c^{*i} \leq e$. $\square$

2 Equilibrium and Pricing

For each $\lambda \in \mathbb{R}_+^I$, define the utility function $U_\lambda : L_+ \to \mathbb{R}$ by

$$
U_\lambda(x) = \sup_{(c^1,\ldots,c^I)} \sum_{i=1}^{I} \lambda_i U_i(c^i) \qquad s.t. \quad \sum_{i=1}^{I} c^i \leq x \tag{7}
$$

Proposition 2. *Suppose U_i concave and strictly increasing for all i and $(\theta^1, \ldots, \theta^I, S)$ is* *a FE under CM.*

i) Then $\exists\ \lambda \in \mathbb{R}^I_+,\ \lambda \neq 0,$ *such that* $(0, S)$ *is a (no-trade) equilibrium for the one agent economy* $[(U_\lambda, e), \delta]$ *where* $e = \sum_{i=1}^{I} e^i$.

ii) With this λ *and* $x = e$*, problem (7) is solved by the FE consumption allocation.*

Proof. (sketch) ii) for each $i\ \exists\ \alpha_i > 0$ (since U_i strictly increasing) s.t. c^i solves

$$\sup_{c \in L_+} U_i(c) - \alpha_i \mathbb{E}\left[\sum_{t=0}^{T} \pi_t (c_t - e^i_t)\right]. \tag{8}$$

Let $\lambda_i = \alpha_i^{-1}$. Showing $\sum_{i=1}^{I} \lambda_i U_i(c^i) \geq \sum_{i=1}^{I} \lambda_i U_i(x^i)$ for $(x^1, ..x^I)$ feasible implies $(c^1, ..., c^I)$ solves (7) at e.

i) If not, $\exists\ x \in L_+$ s.t. $U_\lambda(x) > U_\lambda(e)$ and $\mathbb{E}\left[\sum_{t=0}^{T} \pi_t x_t\right] \leq \mathbb{E}\left[\sum_{t=0}^{T} \pi_t e_t\right]$. Then by definition of $U_\lambda \Longrightarrow \exists$ an allocation $(x^1, ..., x^I)$ s.t.

$$\sum_{i=1}^{I} \lambda_i U_i(x^i) \geq \sum_{i=1}^{I} \lambda_i U_i(c^i),$$

and

$$\sum_{i=1}^{I} \lambda_i \alpha_i \mathbb{E}\left[\sum_{t=0}^{T} \pi_t x^i_t\right] = \mathbb{E}\left[\sum_{t=0}^{T} \pi_t x_t\right] \leq \mathbb{E}\left[\sum_{t=0}^{T} \pi_t e_t\right] = \sum_{i=1}^{I} \lambda_i \alpha_i \mathbb{E}\left[\sum_{t=0}^{T} \pi_t e^i_t\right].$$

Hence,

$$\sum_{i=1}^{I} \lambda_i \left[U_i(x^i) - \alpha_i \mathbb{E}\left[\sum_{t=0}^{T} \pi_t (x^i_t - e^i_t)\right]\right] > \sum_{i=1}^{I} \lambda_i \left[U_i(c^i) - \alpha_i \mathbb{E}\left[\sum_{t=0}^{T} \pi_t (x^i_t - e^i_t)\right]\right],$$

contradicting c^i being optimal for all i. □

Corollary 1. *If* U_λ *differentiable at* e*,* λ *can be chosen s.t.* $\exists$ *SPD* π *equal to the Riesz representation of* $\nabla U_\lambda(e)$*.*

Corollary 2. *If for each* i U_i *is of the additive form*

$$U_i(c) = \mathbb{E}\left[\sum_{t=0}^{T} u_{it}(c^i_t)\right],$$

then U_λ is also additive with

$$U_\lambda(c) = \mathbb{E}\left[\sum_{t=0}^{T} u_{\lambda t}(c_t)\right],$$

and

$$u_{\lambda t}(y) = \sup_{x\in\mathbb{R}^I_+} \sum \lambda_i u_{it}(x^i) \qquad s.t. \ \sum_{i=1}^{I} x_i \leq y.$$

Then,

$$S_t = \frac{1}{u'_{\lambda t}(e_t)} \mathbb{E}_t\left[u'_\lambda(e_{t+1})(S_{t+1} + \delta_{t+1})\right].$$

3 Consumption based CAPM (CCAPM)

Assume $(c^1, ..., c^I)$ is a strictly positive consumption allocation and that utility functions are of the additive form:

$$U_i(c) = \mathbb{E}\left[\sum_{t=0}^{T} u_{it}(c_t)\right]. \tag{9}$$

Assume untility is quadratic in the relevant range: there is a constant $\bar{c}$ larger than c^i_t for all i and t such that

$$u_{it}(x) = a_{it}x - \frac{b_{it}}{2}x^2, \ x < \bar{c}, \tag{10}$$

where for all i and t, a_{it} and b_{it} are positive constants. Complete markets implies the existence of a representative agent with utility function

$$U_\lambda(e) = \mathbb{E}\left[\sum_{t=0}^{T} u_{\lambda t}(e_t)\right],$$

where

$$u_{\lambda t}(e_t) = \sup_{c^1_t,\ldots,c^I_t} \sum_{i=i}^{I} \lambda_i u_{it}(c^i_t) \qquad s.t. \ \sum_{i=i}^{I} c^i_t \leq e_t.$$

Therefore

$$\theta'_t S_t u'_{\lambda t}(e_t) = E_t\left[u'_{\lambda t+1}(e_{t+1})\left(\delta^\theta_{t+1} + \theta'_{t+1}S_{t+1}\right)\right]. \tag{11}$$

Given assumptions (9)-(10), let us now compute $u'_{\lambda t}(e_t)$: f.o.c. for optimality of the representative agent are

$$\begin{cases} \lambda_i(a_{it} - b_{it}c_t^i) = \mu_t \Longleftrightarrow c_t^i = \dfrac{a_{it}}{b_{it}} - \mu_t \dfrac{1}{\lambda_i b_{it}} \\ \sum_{i=i}^I c_t^i = e_t \end{cases} \tag{12}$$

Summing (12) over i and we get

$$e_t = \sum_{i=i}^I \frac{a_{it}}{b_{it}} - \mu_t \sum_{i=i}^I \frac{1}{\lambda_i b_{it}};$$

solving for μ_t gives

$$\mu_t = A_t - B_t e_t,$$

with

$$A_t = \left(\sum_{i=i}^I \frac{a_{it}}{b_{it}} \right) B_t; \qquad B_t = \left(\sum_{i=i}^I \frac{1}{\lambda_i b_{it}} \right)^{-1}.$$

Hence,

$$u'_{\lambda t}(e_t) = \sum_{i=i}^I \lambda_i u'_{it}(c_t^i) \frac{dc_t^i}{de_t} = \sum_{i=i}^I \mu_t \frac{dc_t^i}{d\mu_t} = \mu_t = A_t - B_t e_t,$$

and therefore (11) becomes

$$\theta'_t S_t \left(A_t - B_t e_t\right) = A_{t+1} \mathbb{E}_t \left[\delta_{t+1}^\theta + \theta'_{t+1} S_{t+1}\right] - B_{t+1} \mathbb{E}_t \left[\left(\delta_{t+1}^\theta + \theta'_{t+1} S_{t+1}\right) e_{t+1}\right]. \tag{13}$$

Define the return on a portfolio R^θ as

$$R_{t+1}^\theta = \frac{\theta'_t (S_{t+1} + \delta_{t+1})}{\theta'_t S_t}, \qquad \theta'_t S_t \neq 0;$$

rewrite (13) as

$$1 = \alpha_{t+1} \mathbb{E}_t [R_{t+1}^\theta] - \gamma_{t+1} \mathbb{E}_t [R_{t+1}^\theta e_{t+1}], \tag{14}$$

where

$$\alpha_{t+1} = \frac{A_{t+1}}{A_t - B_t e_t}; \qquad \gamma_{t+1} = \frac{B_{t+1}}{A_t - B_t e_t}$$

are known constants at time t.

For the *riskless rate* r_t we have

$$1 = \alpha_{t+1} r_t - \gamma_{t+1}\mathbb{E}_t[e_{t+1}] r_t \Longrightarrow r_t = \left(\alpha_{t+1} - \gamma_{t+1}\mathbb{E}_t[e_{t+1}]\right)^{-1} \tag{15}$$

Rewrite (14) as

$$1 = \mathbb{E}_t[R^\theta_{t+1}]\left(\alpha_{t+1} - \gamma_{t+1}\mathbb{E}_t[e_{t+1}]\right) - \gamma_{t+1}\mathbb{C}_t\left(R^\theta_{t+1}, e_{t+1}\right); \tag{16}$$

combining (15) and (16) we get

$$\mathbb{E}_t[R^\theta_{t+1}] - r_t = r_t\gamma_{t+1}\mathbb{C}_t\left(R^\theta_{t+1}, e_{t+1}\right). \tag{17}$$

CM implies that $\exists\ \theta^M$ s.t. $R^M_{t+1} = e_{t+1}$, for which (17) gives

$$\mathbb{E}_t[R^M_{t+1}] - r_t = r_t\gamma_{t+1}\mathbb{V}_t(R^M_{t+1}). \tag{18}$$

Combining (17) and (18) gives the CCAPM:

$$\mathbb{E}_t[R^\theta_{t+1}] - r_t = \beta^\theta_t\mathbb{E}_t\left[R^M_{t+1} - r_t\right],$$

where

$$\beta^\theta_t = \frac{\mathbb{C}_t(R^\theta, R^M)}{\mathbb{V}_t(R^M)}.$$

Remark 1. If markets are incomplete, then

$$\mathbb{E}_t[R^\theta_{t+1}] - r_t = \beta^\theta_t\mathbb{E}_t[R^*_{t+1} - r_t],$$

where

$$\beta^\theta_t = \frac{\mathbb{C}_t(R^*_{t+1}, R^\theta_{t+1})}{\mathbb{V}_t(R^*_{t+1})}$$

and R^*_{t+1} is the orthogonal projection of π_{t+1} onto the subspace

$$\overline{R}_{t+1} = \left\{\frac{\theta'_t(S_{t+1} + \delta_{t+1})}{\theta'_t S_t} \text{ s.t. } \theta'_t S_t \neq 0, \theta_t \in \Theta_t\right\}.$$

Remark 2. In a security market economy, in which agents are endowed with portfolios (assets are in positive supply), CCAPM holds without CM