

Hiring Mechanisms, Application Costs and Stability*

Matteo Triossi

Collegio Carlo Alberto[†]

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Abstract

This note considers an hiring mechanism with multiple applications and application costs, which encompasses the common features of many real-world procedures. Multiple applications impose serious coordination problems to colleges, but application costs restore stability. With zero application costs unstable allocations arise at equilibrium. We provide necessary and sufficient conditions that guarantee the stability of the outcomes.

1 Introduction

Many hiring and admission procedures share the same characteristics: candidates first apply to potential employers and then select the best offers they receive. Every Fall a large number of PhD candidates in Economics apply for positions at universities and other institutions. If the success of a market can be measured by its placement record, it surpasses other academic job markets that have developed different procedures (Carson and Navarro, 1988). Applications are costly (fees, information gathering costs, etc.) and they impose an heavy burden on PhD candidates.¹ Other markets have developed similar procedures like admissions to graduate schools and classified-based job markets.

This note develops a matching model aimed at identifying the basic forces at work in these settings. The game consists of three stages. First, a group of workers send costly applications to firms. Second, each firm makes offers to a subset of its applicants. Finally, each worker chooses one of the offers she receives, if any. In every stage agents play simultaneously. Since the mechanism

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[†]Via Real Collegio 30, 10024 Moncalieri (TO), Italy. E-mail: matteo.triossi@carloalberto.org, tel: +39 011 6705272, fax: +39 011 6705082.

¹See Cawley (2006) for a survey of the extensive empirical literature on the job market for economists .

reflects a sequential decision process we consider Subgame Perfect Equilibrium (*SPE* from now on) as equilibrium concept.

Although abstract, the model is nevertheless useful because it allows to isolate the common strategic options and provides a clear benchmark model against which one can assess the effects of the specific institutional details. It is widespread opinion that the markets producing stable outcomes are more successful than markets that do not produce stable outcomes (see Roth and Sotomayor, 1990 and Roth, 2002) so we focus on the stability of *SPE* outcomes.

First we consider the case of positive application costs and prove that, under this assumption, every worker applies to at most one firm at equilibrium and the mechanism implements the stable set in *SPE*. Then, we study the case of zero application costs. We observe that, in this case, the mechanism can produce unstable allocations due to multiple applications. We prove that if firms' preferences are acyclical (see Ergin, 2002) or if workers' preferences satisfy a weak form of alignment, then *SPE* outcomes are stable. The two conditions are also the minimal ones that guarantee stable *SPE* outcomes.

When information is incomplete the mechanism may produce unstable allocations even with positive application costs. The attempt of preventing informational inefficiencies might explain the particular arrangements that emerge in different markets, like the ASSA Meeting interviews and the GRE tests required to applicants to Graduate Programs. Further research should shed light on these aspects.

Different scholars have addressed the design of mechanisms able to implement the stable set.² In particular, Kara and Sönmez (1997) show that the stable correspondence is implementable in Nash Equilibrium. Alcalde (1996) approaches the design of “natural mechanisms” able to implement the stable set for the particular case of one-to-one matching. Our model extends the Workers-Propose-and-Firms-Choose Mechanism presented by Alcalde and Romero-Medina (2000). In this mechanism every worker sends a costless application to no more than one firm, then each firm selects the applicants to accept.³ In the single-application case there is never competition between firms because the sets of applicants to two different firms never intersect. Instead, in our model, firms select applicants and at the same times propose to them, so workers can potentially receive multiple offers.

The rest of the paper is organized as follows. Section 2 introduces the basic model and the hiring mechanism we analyze. Section 3 contains the main results and Section 4 concludes. The proofs are in the Appendix.

²Matching models have revealed useful in applied mechanism design, too. Roth and Sotomayor (1990) and Roth (2002) include a survey of earlier works, while Abdulkadiroğlu and Sönmez (2003), Niederle and Roth (2003) and Romero-Medina and Triossi (2007) are more recent applications.

³A similar model with monetary compensations is studied by Alcalde et al. (1998). In Alcalde and Romero-Medina (2005) offers and applications are made sequentially. See also Sotomayor (2003) and Haeringer and Wooders (2007).

2 The Model

In the hiring problem studied here there are a number of workers, each of whom is seeking a position at one of many firms. Let $F = \{f_1, \dots, f_k\}$ be the set of firms, let $W = \{w_1, \dots, w_t\}$ be the set of workers, let $P_F = (P_{f_1}, \dots, P_{f_k})$ be a list of firms' preferences over workers and let $P_W = (P_{w_1}, \dots, P_{w_t})$ be a list of workers' preferences over firms. The triple (F, W, P) , where $P = (P_F, P_W)$ is called a **Matching Market**. For every $f \in F$, P_f is a strict order defined on 2^W , the set of all subsets of W .⁴ Let $W' \subset W$ be a set of workers. The favorite group of workers for firm f among the ones belonging to W' is called the **choice set from W'** . It is denoted by $Ch_f(W', P_f)$ or by $Ch_f(W')$, when no ambiguity is possible. Formally, $Ch_f(W', P_f) = \arg \max_{P_f} \{W'' : W'' \subset W'\}$. If $\emptyset P_f W'$ firm f prefers not to employ any worker rather than jointly employing the workers in W' and W' is called **unacceptable to f** . Otherwise W' is **acceptable to f** . $A(f)$ denotes the set of workers who are individually acceptable to f . The maximum numbers of workers firm f is willing to hire is f 's **quota** and it is denoted by q_f , formally $q_f = \max \{\#W' : Ch_f(W', P_f) \neq \emptyset\}$. For every $w \in W$, P_w is a strict order defined on $F \cup \{w\}$. Any firm f such that $w P_w f$ is called **unacceptable to w** . Otherwise f is **acceptable to w** . $A(w)$ denotes the set of firms that are acceptable to w . For every agent $x \in F \cup W$, R_x denotes x 's weak preference relation.

In the paper an important role is played by application costs. For this reason a cardinal representation of workers' preferences is introduced. For every $w \in W$ let u_w be a function representing P_w , $u_w : F \cup \{w\} \rightarrow R$. Let $\delta \geq 0$ be the cost that every worker w pays for each application.⁵ Throughout the paper we assume that every worker is willing to apply to any acceptable firm. Formally $f \in A(w) \Rightarrow u_w(f) - \delta > u_w(w)$.

A matching assigns workers to firms. A **matching** on (F, W) is a function $\mu : F \cup W \rightarrow 2^W \cup F$, such that : (i) $\mu(f) \in 2^W$, (ii) $\mu(w) \in F \cup \{w\}$ and (iii) $\mu(w) = f \Leftrightarrow w \in \mu(f)$, for every $(f, w) \in F \times W$. We denote by \mathcal{M} the set of matchings on (F, W) . A matching μ is **individually rational** if it satisfies: (i) $Ch_f(\mu(f)) = \mu(f)$ for all $f \in F$ and (ii) $\mu(w) R_w w$ for all $w \in W$. A matching μ is **blocked by the pair** $(f, w) \in F \times W$ if (i) $f P_w \mu(w)$ and (ii) $w \in Ch_f(\mu(f) \cup \{w\})$. A matching μ is **stable in (F, W, P)** if it is individually rational and if no pair blocks it. Otherwise μ is **unstable**. $\Gamma(F, W, P)$ denotes the **stable set**, the set of matchings that are stable in market (F, W, P) .

The stable set may be empty. This is why the literature has focused on preference restrictions where workers are not seen as complements. A firm f has substitutable preferences if it wants to hire a worker even when other workers become unavailable. Formally, P_f are **substitutable** if, for every $W' \in 2^W$ and for all $w, w' \in W$, $w \neq w'$: $w \in Ch_f(W')$ \Rightarrow $w \in Ch_f(W' - \{w'\})$. Under this restriction, the deferred acceptance algorithm (Gale and Shapley, 1962) produces either the

⁴For all $w, w' \in S$ $w P_f w'$, $w P_f \emptyset$ and $\emptyset P_f w$ denote $\{w\} P_f \{w'\}$, $\{w\} P_f \emptyset$ and $\emptyset P_f \{w\}$, respectively.

⁵There is no loss of generality in assuming that the costs of applying to different firms are the same for all firms and all workers. Otherwise one can rerank firms taking in account application costs.

firm-optimal or the worker-optimal stable matching, denoted μ^W and μ^F respectively, depending on whether the firms or the workers make the offers (see Roth and Sotomayor, 1990). A stronger assumption is responsiveness. A firm f has responsive preferences if, for any two assignments that differ in one worker only, it prefers the assignment containing the most preferred worker. Formally, P_f are **responsive** if, for all $W' \subset W$ such that $\sharp W' \leq q_f - 1$ and for all $w, w' \in W$: (i) $W' \cup \{w\} P_f W' \cup \{w'\} \Leftrightarrow w P_f w'$ and (ii) $W' \cup \{w\} P_f W' \Leftrightarrow w \in A(f)$.

By P_F and P_W we denote the two following binary relations on the set of matchings. For every μ, ν matchings, let $\mu P_F \nu$ if and only if $\mu(f) R_f \nu(f)$ for all $f \in F$ and $\mu(f) P_f \nu(f)$ for at least one f . Let $\mu P_W \nu$ if and only if $\mu(w) R_w \nu(w)$ for all $w \in W$ and $\mu(w) P_w \nu(w)$ for at least one $w \in W$. An (extensive form) **matching mechanism** is an array $G = (F \cup W, H, S, g)$. $F \cup W$ is the set of players, H is the set of histories and S is the strategy space, $S = \prod_{x \in F \cup W} S_x$, where $S_x = \prod_{h \in H} S_x^h$ for every x . Set $S^h = \prod_{x \in F \cup W} S_x^h$. Histories and strategies are linked by the following property $S^h = \{s^h \mid (h, s^h) \in H\}$. There is an initial history $h^0 \in H$ and every history $h \in H$ is represented by a finite sequence $(h^0, s^1, \dots, s^{r-1}) = h^r$. If $h^{r+1} = (h^r, s^r)$ then history h^{r+1} proceeds history h^r . The set $Z = \{z \in H \mid \text{there is no } h \in H \text{ proceeding } z\}$ is the set of terminal histories. Given the initial history, every strategy profile $s \in S$ defines a unique terminal history z_s . The outcome function $g : Z \rightarrow \mathcal{M}$ specifies an outcome matching for each terminal history, and hence for each strategy profile s . With abuse of notation, we use $g(s)$ to denote $g(z_s)$. Given a preference profile P , (G, P) constitutes an extensive form game with simultaneous moves. Every $h \in H \setminus Z$ identifies a subgame $G(h) = (F \cup W, H(h), S(h), g_h, P)$, where h is the initial history, $H(h) = \{h' \in H \mid h' \text{ proceeds } h\}$ and $S(h) = \prod_{h' \in H(h)} S^{h'}$. Let $s \in S(h)$. Given the initial history h , strategy s specifies a unique terminal history, z_s . The outcome function is defined by $g_h(s) = g(z_s)$. Given $s \in S$ and $h \in H$, $s(h) \in S(h)$ be the strategy prescribed by s once h is reached. Formally if $s = (s^h)_{h \in H}$, then $s(h) = (s^h)_{h \in H(h)}$. A **Subgame Perfect Equilibrium** is a strategy that induces a Nash Equilibrium in every subgame. Formally, s^* is a SPE if for all $h \in H$ and for all $x \in F \cup W$: $g_h(s^*(h)) R_x g_h(s'_x, s^*_{-x}(h))$ for every $s'_x \in S_x(h)$. The matching $g(s^*)$ is called *SPE* outcome of (Γ, P) and the set of *SPE* outcomes of (G, P) is denoted by $SPE(G, P)$. Let \mathcal{S} be a set of matching markets and let $\Phi : \mathcal{S} \rightarrow \mathcal{M}$ be a correspondence. A **matching mechanism** G implements Φ in *SPE* if, for every $(F, W, P) \in \mathcal{S}$, $SPE(\Gamma, P) = \Phi(F, W, P)$, which is if every *SPE* outcome of (Γ, P) belongs to $\Phi(F, W, P)$ and for every matching $\mu \in \Phi(F, W, P)$ there exists a *SPE* of (Γ, P) yielding μ as outcome. Throughout the paper, only equilibria in pure strategies are considered.

2.1 The Hiring Mechanism

The **Sequential Hiring Mechanism** is described by the following procedure:

- **Applications.** Each candidate w sends applications to some firms. Let $F_1(w)$, be the set of

firms that receives one of w 's applications. If w does not apply to any firm then $F_1(w) = \emptyset$. Let $W_1(f) = \bigcup_{f \in F_1(w)} \{w\}$ be the set of workers who applies to firm f .

- **Offers.** Each firm f decides to make offers to some workers among the ones who have applied to it. Let $W_2(f) \subset W_1(f)$, be the set of workers who receive an offer from f . For each worker w let $F_2(w) = \bigcup_{w \in W_2(f)} \{f\}$ be the set of firms that make an offer to w .
- **Hiring.** Worker w decides which firm to join among the ones in $F_2(w)$.

Let μ be the matching resulting from such a procedure. The payoff of worker w , is $u_w(\mu(w)) - \delta \#F_1(w)$.

Let Z_2 be the set of subgames starting at the second stage. Each $z_2 \in Z_2$ is completely characterized by the family of sets of workers who applied to each firm $\{W_1(f, z_2)\}_{f \in F}$, or equivalently by the family of sets of firms each worker applied to, $\{F_1(w, z_2)\}_{w \in W}$. Let $W_2(f, z_2) \subset W_1(f, z_2)$ denote firm f 's strategy at z_2 .

Let $z_2 \in Z_2$ and let P be a profile of preferences and consider the following profile of preferences $P(z_2)$, called **revealed preferences**. In $P(z_2)$ firms' preferences are the same as in P . Each worker w deletes from her list of acceptable firm all the firms she did not apply to. Formally:

- (i) $P_F(z_2) = P_F$,
- (ii) for each $w \in W$: if $f \notin F_1(w, z_2)$ or if $wP_w f$ then $wP_w(z_2)f$. If $f, f' \in F_1(w, z_2)$ then $fP_w(z_2)f' \Leftrightarrow fP_w f'$.

For all $z_2 \in Z_2$, $\mu_{z_2}^W$ denotes the worker-optimal stable matching of $(W, F, P(z_2))$. The next assumption, weaker than substitutability, is crucial in the proof of our results.

Definition 1 *Agents' preferences satisfy **Assumption 1** if $\Gamma(F, W, P(z_2)) \neq \emptyset$ for every $z_2 \in Z_2$.*

Assumption 1 does not necessarily holds if $\Gamma(F, W, P) \neq \emptyset$ only.

Example 1 *Let $F = \{f_1, f_2, f_3\}$, $W = \{w_1, w_2, w_3\}$. Let $P_{w_1} : f_2 f_1 f_3$, $P_{w_2} : f_2 f_1 f_3$, $P_{w_3} : f_1 f_3 f_2$. Let $P_{f_1} : \{w_1, w_3\} \{w_1, w_2\} \{w_2, w_3\} \{w_1\} \{w_2\}$, $P_{f_2} : \{w_1, w_3\} \{w_2, w_3\} \{w_1, w_2\} \{w_3\} \{w_1\} \{w_2\}$, $P_{f_3} : \{w_3\} \{w_1\} \{w_2\}$. Firm f_1 has not substitutable preferences, but a stable matching exists:*

$$\mu = \begin{array}{ccc} f_1 & f_2 & f_3 \\ \emptyset & \{w_1, w_2\} & w_3 \end{array}$$

Let z_2 be the subgame induced by the following first stage strategies $F_1(w_i) = \{f_1, f_2\}$ for $i = 1, 2, 3$. It is easily seen that $\Gamma(F, W, P(z_2)) = \emptyset$.

3 The Results

In this model workers are allowed to apply to more than one firm. However, at equilibrium they apply to at most one firm when application costs are strictly positive. Intuitively every worker, given other agents' strategies can join any attainable firm by applying just to it. Doing so she saves in application costs.

Lemma 1 *When $\delta > 0$ no worker applies to more than one firm at any SPE.*

The result does not depend on any assumption on the preferences of the firms.

Remark 1 *At the third stage of the game, at equilibrium, every worker accepts the best offer she holds, if any.*

Second stage subgames are similar to the Firms-Propose-and-Workers-Choose mechanism in Alcalde and Romero-Medina (2000). In any subgame every firm has to make offers to some workers, but it can only propose to its applicants. Agents behave as the true preferences were the revealed ones. The outcome set coincide with the stable set with respect to such preferences.

Lemma 2 *Let $z_2 \in Z_2$. Then the set of pure strategy SPE outcomes of z_2 is $\Gamma(F, W, P(z_2))$.*

According to Lemma 2, if $\Gamma(F, W, P(z_2)) = \emptyset$, z_2 does not have pure strategy SPE. It follows that if Assumption 1 does not hold some subgame z_2 does not have a SPE in pure strategy, so the sequential hiring mechanism does not have a pure strategy SPE. The case in which every worker applies to all firms is Theorem 4.1 in Alcalde and Romero-Medina (2000).

Lemma 3 *Let $\delta \geq 0$. If assumption 1 holds any stable matching can be supported at a SPE.*

The result applies also to the case $\delta = 0$. The proof builds on Lemma 2 to construct equilibria at which each worker applies to exactly one firm. Hence, from Lemma 3 and Lemma 1 we have:

Proposition 1 *Let $\delta > 0$. Under Assumption 1 the Sequential Hiring Mechanism implements the stable set in SPE.*

Remark 2 *If the condition of small application costs is not met all the results hold in the smaller market (F, W, P') in which $P'_f = P_f$ for all f and $fP'_w f'P'_w w$ if and only if $u_w(f) > u_w(f') \geq u_w(w) + \delta$ and $wP'_w f$ if $u_w(f) < u_w(w) + \delta$. In this case the SPE outcome is stable in (F, W, P') . Hence, an increase in application costs can reduce the number of active participants.*

Costless Applications

In this section is assumed that firms have responsive preferences, unless stated otherwise.

When application costs are zero, Lemma 1 does not hold. There are equilibria where workers apply to more than one firm. Actually, there are equilibria where every worker applies to all firms.

Example 2 Let $\delta = 0$. Let $F_1(w) = F$ for all w , let $W_2(f, z_2) = \mu_{z_2}^W(f)$ for all f, z_2 and let every worker accept the best offer she holds at the third stage. It is easily seen that this strategy profile is a SPE yielding μ^W as outcome.

Acyclicity and weak alignment are sufficient to rule out unstable SPE allocations. Acyclicity was introduced by Ergin (2002) for priority structures.

Definition 2 Let P_F be a vector of firms' preferences. A **cycle** is constituted of distinct $f_1, f_2 \in F$ and $w_1, w_2, w_3 \in W$, $w_j P_{f_i} \emptyset$ for $j = 1, 2, 3$, $i = 1, 2$ and such that the following are satisfied:

(C) *Cycle condition:* $w_1 P_{f_1} w_2 P_{f_1} w_3 P_{f_2} w_1$.

(S) *Scarcity condition:* there exist (possibly empty) disjoint sets of agents $W_{f_1}, W_{f_2} \subset W \setminus \{w_1, w_2, w_3\}$ such that $\#W_{f_1} = q_{f_1} - 1$, $\#W_{f_2} = q_{f_2} - 1$ and $w P_{f_1} w_2$ for all $w \in W_{f_1}$, $w P_{f_2} w_1$ for all $w \in W_{f_2}$.

P_F is **acyclical** if it has no cycles.

Firms' preferences are acyclical if they never give rise to situation where a worker can block a settlement between any two other workers without affecting her own position. If all quotas are equal to one, then the cycle condition is sufficient to establish the existence of a cycle. For other quotas, the scarcity condition limits the definition of a cycle to cases where there exist enough workers competing for positions at f_1 and f_2 . Ergin (2002) proved that acyclicity is equivalent to efficiency, group strategy-proofness and consistency of the workers' optimal stable matching. The result straightforwardly extends to the case of responsive preferences.

Definition 3 Let P_W be a vector of workers' preferences. A **trading opportunity** is constituted of distinct $f_1, f_2 \in F$ and distinct $w_1, w_2, w_3 \in W$ such that the following are satisfied:

(T) *Trading condition:* $f_2 P_{w_1} f_1 P_{w_3} f_2$.

(A) *Acceptability Condition:* $f_1 \in A(w_1) \cap A(w_2)$ and $f_2 \in A(w_3)$.

P_W is **weakly aligned** if it has no trading opportunities.

Consider a situation where w_1 works for f_1 and w_3 works for f_2 . If the trading conditions holds w_1 and w_3 would be willing to swap their respective positions. The acceptability (A) condition requires that there is at least another worker who is competing with w_1 and w_3 for a position. If every two workers w, w' have the same preferences over the set of mutually acceptable firms $A(w) \cap A(w')$ then (T) does not hold so preferences are weakly aligned.

Proposition 2 *Let $\delta = 0$.*

(i) *If firms have acyclical preferences the Sequential Hiring Mechanism implements the stable set in SPE.*

(ii) *If workers have weakly aligned preferences the Sequential Hiring Mechanism implements the stable set in SPE.*

The proof of part (i) builds on Lemma 2 and on Ergin (2002).

If a cycle and a trading opportunity arise two workers can benefit from swapping their jobs and enforcing an unstable matching. This can be done through multiple applications.

Example 3 *Let $\delta = 0$. Let $F = \{f_1, f_2\}$, $W = \{w_1, w_2, w_3\}$. Let $P_{w_1} : f_2 f_1$, $P_{w_2} : f_1$, $P_{w_3} : f_1 f_2$ and let $P_{f_1} : \{w_1\} \{w_2\} \{w_3\}$, $P_{f_2} : \{w_3\} \{w_1\}$. There is a cycle because $w_1 P_{f_1} w_2 P_{f_1} w_3 P_{f_2} w_1$ and the scarcity condition is emptyly satisfied. There is also a trading opportunity because $f_2 P_{w_1} f_1 P_{w_3} f_2$, $f_1 P_{w_1} w_1$, $f_2 P_{w_3} w_3$ and $f_1 P_{w_2} w_2$. Market (F, W, P) has only one stable matching*

$$\mu = \begin{array}{ccc} w_1 & w_2 & w_3 \\ f_1 & w_2 & f_2 \end{array}$$

Let ν be the following matching:

$$\nu = \begin{array}{ccc} w_1 & w_2 & w_3 \\ f_2 & w_2 & f_1 \end{array}$$

The matching ν is blocked by (f_1, w_2) . We now prove it is a SPE outcome of the Sequential Hiring Mechanism. Consider the following strategy profile where w_1 and w_2 apply to $\{f_1, f_2\}$ and w_2 does not apply to any firm. In all subgames z_2 , firm f accepts only the workers in $\mu_{z_2}^W(f)$. In the last stage of the game every worker accepts the best offer she holds, if any. This strategy profile yields ν as outcome and it is a SPE. The stability of $\mu_{z_2}^W$ in $(F, W, P(z_2))$ implies subgame perfection in all subgames z_2 . In order to check that the profile is a SPE it suffices to check only deviations by w_2 , because w_1 and w_3 are matched to their favorite firms. Only deviations where w_2 applies to f_1 can be profitable. Let z'_2 a subgame reached by one of these deviations. The only stable matching in $(F, W, P(z'_2))$ is μ and $\mu(w_2) = w_2$. Then, from Lemma 2 no deviation is profitable to w_2 .

Also, responsiveness is necessary for Proposition 2 to hold.

Example 4 *Let $\delta = 0$. Let $F = \{f_1, f_2\}$, $W = \{w_1, w_2, w_3, w_4\}$. Let $P_{w_1} : f_2 f_1$, $P_{w_2} : f_1$, $P_{w_3} : f_1 f_2$, $P_{w_4} : f_1 f_2$ and let $P_{f_1} : \{w_1\} \{w_2\} \{w_3, w_4\} \{w_3\} \{w_4\}$, $P_{f_2} : \{w_3, w_4\} \{w_1\} \{w_2\} \{w_3\} \{w_4\}$. There are no cycles, firms have substitutable but not responsive preferences. Let ν be the following matching:*

$$\nu = \begin{array}{ccc} f_1 & f_2 & w_2 \\ \{w_3, w_4\} & w_1 & w_2 \end{array}$$

The matching ν is blocked by (f_1, w_2) . We now prove it is a SPE outcome of the Sequential Hiring Mechanism. Consider the following strategy profile where w_1, w_3 and w_4 apply to $\{f_1, f_2\}$ and w_2 does not apply to any firm. In all subgames z_2 , firm f accepts only the workers in $\mu_{z_2}^W(f)$. In the last stage of the game every worker accepts the best offer she holds, if any. This strategy profile yields ν as outcome and it is a SPE. The stability of $\mu_{z_2}^W$ in $(F, W, P(z_2))$ implies subgame perfection in all subgames z_2 . In order to check that the profile is a SPE it suffices to check only deviations by w_2 , because w_1, w_3 and w_4 are matched to their favorite firms. Only deviations where w_2 applies to f_1 can be profitable. Let z'_2 a subgame reached by one of these deviations. We have $\mu_{z'_2}^W(w_2) = w_2$ so no deviation is profitable to w_2 .

Generalizing Example 3, we prove that acyclicity and weak alignment are the minimal conditions that guarantee the stability of SPE outcomes.

Proposition 3 *Let $\delta = 0$.*

(i) *If P_F has a cycle, there exists a profile of workers' preferences P_W such that the Sequential Hiring Mechanism produces unstable allocations at (F, W, P) , where $P = (P_F, P_W)$.*

(ii) *If there is a trading opportunity, there exists a profile of firms' preferences P_F such that the Sequential Hiring Mechanism produces unstable allocations at (F, W, P) , where $P = (P_F, P_W)$.*

With incomplete information the mechanism can implement unstable allocations even when $\delta > 0$.

Example 5 *Let $F = \{f_1, f_2\}, W = \{w_1, w_2, w_3\}$. Let workers' preferences be public knowledge and coinciding with the ones defined in Example 3. Let firms' preferences be the following with probability 1/2: $\overline{P_{f_1}} = w_1, \overline{P_{f_2}} = w_3$. Let firms' preferences be as in Example 3 with probability 1/2. For any utility representation, if δ is small enough, there exists a sequential equilibrium of the Sequential Hiring Mechanism in which the workers apply like in (i), and in which each firm makes offers to $\mu_{z_2}^W$. The outcome is then unstable with probability 1/2.*

4 Conclusions

This note studies an hiring mechanism that features some characteristics common to many real-world procedures. It generalizes the model presented by Alcalde and Romero-Medina (2000). If application costs are positive the mechanism implements the stable set in SPE. The argument cannot be extended when application costs are zero, unless firms preferences are acyclic or workers preferences are weakly aligned. The result stems on the coordination problems that emerge among firms. Application costs pin down the number of equilibrium application to one preserving stability. The result is sensitive to the assumption of complete information. A detailed analysis of the incomplete information case would be a forward step toward a better understanding of hiring procedures. It would help to explain some of the particular features that specific markets have

developed to deal with this problem (like the recent institution of the so called “Job Scramble” by the AEA). An extension in a different direction could consider a model where firms act sequentially and make exploding offers to workers. The analysis would provide insights about the anomalies related with the timing of the markets (see Roth and Xing, 1994).

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Appendix

Proof of Lemma 1. By contradiction assume that there exists a *SPE* such that $\#F_1(w) > 1$ for some $w \in W$. Let μ the outcome matching. If $\mu(w) = w$ playing $F_1(w) = \emptyset$ is a profitable unilateral deviation because it increases worker w 's utility by $\delta\#F_1(w) > 0$. Thus, $\mu(w) = f$ for some firm $f \in F$. We prove that playing $F_1(w) = f$ in stage one and accepting the offer by firm f in stage three is a profitable deviation for w because it increases worker w 's utility by $\delta(\#F_1(w) - 1) > 0$. Only worker w 's strategy changes in stage one of the game. Thus at stage 2, firm f receives the same set of applications $W_1(f)$. Because we are considering a *SPE*, firm f will make offers to the same set of workers $W_2(f)$. Recall that $w \in W_2(f)$. Worker w receives only the offer from firm f and his strategy in this last stage of the game is to accept this offer. Hence, worker w is still matched with the same firm after his unilateral deviation, but by only applying to this firm could save the additional application costs he had when reporting $F_1(w)$. Hence, $\#F_1(w) > 1$ can never be part of a *SPE*. ■

Proof of Lemma 2. Let μ be stable in $(F, W, P(z_2))$. Consider the following strategy profile. Every firm f accepts only the applicants in $\mu(f)$. Workers accept the best offer they hold at the third stage. The stability of μ implies that no firm can profitably deviate.

Let μ be a *SPE* outcome of z_2 . We prove that if μ is not stable in $\Gamma(F, W, P(z_2))$ then some agent has a profitable deviation. First, μ is individually rational for firms and for workers. If μ was not individually rational for worker w , then she could profitably deviate by dropping all offers she holds. If μ was not individually rational for firm f , then accepting only the workers in $Ch_f(\mu(f))$ would be a profitable deviation for f . Let (f, w) be a firm-worker pair. If (f, w) blocks μ , then consider the following deviation for f : accept only the workers in $Ch_f(\mu(f) \cup \{w\})$. From Remark 1 it follows that the deviation would be profitable to f . In particular, if $\Gamma(F, W, P(z_2)) = \emptyset$ then z_2 has no pure strategy *SPE*. ■

Proof of Lemma 3. Let μ be a stable matching. For every $z_2 \in Z_2$ let $\mu_{z_2} \in \Gamma(F, W, P(z_2))$. Such a μ_{z_2} exists because P satisfies Assumption 1. Consider the following strategy profile. Each worker w applies to $\mu(w)$. At the second stage, for each subgame z_2 , every firm f accepts only the workers in $\mu_{z_2}(f)$. At the third stage the workers conform to the *SPE* strategies described in Remark 1. The profile yields μ as outcome. These strategies are a *SPE* for every second stage subgame (see Lemma 2). The stability of μ implies that no worker can profitably deviate. ■

Proof of Proposition 2. We show that if the sequential hiring mechanism has an unstable *SPE* outcome, then firms' preferences have a cycle and workers' preferences have a trading opportunity. Let μ be an unstable *SPE* equilibrium outcome and let (w, f) blocking μ . Lemma 2 implies that $f \notin F_1(w)$. Let z_2 be the second stage subgame reached on the equilibrium path. Consider the

following deviation $F'_1(w) = F_1(w) \cup \{f\}$ and let z'_2 be the subgame induced by such deviation. We first prove that $\mu P_W \mu_{z'_2}^W$, then we show that claims (i) and (ii) hold. Observe that $\mu \neq \mu_{z'_2}^W$ because μ is not stable in $\Gamma(F, W, P(z'_2))$. By contradiction, assume that $\mu_{z'_2}^W(w') P_W \mu(w')$ for some w' . We prove that the deviation $F'_1(w') = \mu_{z'_2}^W(w')$ is profitable for w' . Let z''_2 be the subgame induced by this deviation. Observe that $\mu_{z'_2}^W(w')$ is w' 's unique stable mate in $\Gamma(F, W, P(z''_2))$, because firms' preferences are responsive. From Lemma 2 the deviation matches w' with $\mu_{z'_2}^W(w')$, then it is profitable. As $\mu P_W \mu_{z'_2}^W, \mu_{z'_2}^W$ is not strongly Pareto optimal in market $(F, W, P(z'_2))$. From Theorem 1 in Ergin (2002) it follows that there is a cycle in firms' preferences so (i) holds⁶.

Define two sequences of workers and firms $\{w_t\}_{0 \leq t \leq T}$ and $\{f_t\}_{0 \leq t \leq T}$ as follows. Let $w_0 \in W$ such that $\mu(w_0) \neq \mu_{z'_2}^W(w_0)$. For all $t \geq 0$ set $f_{t+1} = \mu_{z'_2}^W(w_t)$. For all $t \geq 1$ let $w_t \in \mu(f_t) \setminus (\mu_{z'_2}^W(f_t) \cup \{w_1, \dots, w_{t-1}\})$. The sequence stops when it reaches some T such that $\mu(f_T) \subset (\mu_{z'_2}^W(f_T) \cup \{w_1, \dots, w_{T-1}\})$. Such a T exists because W and F are finite and all workers in the sequence are distinct. Set $K = \max\{t < T \mid f_t = f_T\}$. There is no loss of generality in assuming $K = 0$. The firms f_0, \dots, f_{T-1} are distinct and $T \geq 2$ because $f_t \neq f_{t+1}$ for every $t = 0, \dots, T-1$. The workers w_0, \dots, w_{T-1} are distinct by construction. For $0 \leq t \leq T-2$, $\mu_{z'_2}^W(w_t) = f_{t+1} = \mu(w_{t+1})$ and $\mu(w_0) = f_T$. We have $f_T P_{w_0} f_1 P_{w_1} \dots P_{w_{T-1}} f_T$. Consider the deferred acceptance algorithm where workers propose to firms in market $(F, W, P(z'_2))$. Without loss of generality assume w_0 is the last worker to be accepted by her mate under $\mu_{z'_2}^W, \mu_{z'_2}^W(w_0) = f_1$, say at step l . After step l no agent in $\{w_0, \dots, w_T\}$ is rejected. Since $f_1 P_{w_1} f_2$, before applying to f_2 worker w_1 was rejected by f_1 at step $l' \leq l-1$. So at the end of step $l-1$ the waiting list of f_1 is full and does not include any worker in $\{w_0, \dots, w_{T-1}\}$, otherwise some agent would be rejected after step l . Let w be the worker belonging to the waiting list of f_1 who has been rejected by f_1 in favor of w_0 and observe that $f_1 \in A(w)$. So we have proved that there are $T \geq 1, T+1$ distinct workers w, w_0, \dots, w_{T-1} and T distinct firms f_0, \dots, f_{T-1} , satisfying the two following properties (i) For $0 \leq t \leq T-1$, $\mu_{z'_2}^W(w_t) = f_{t+1}$ and (ii) $f_T P_{w_0} f_1 P_{w_1} \dots P_{w_{T-1}} f_T, f_1 \in A(w)$. Let T^* be the minimum of such T . If we show that $T^* = 2$ then there is a trading opportunity so claim (ii) holds. By contradiction assume $T^* > 2$. If $f_1 P_{w_1} f_T$ then $T^* = 2$ satisfies (i) and (ii) so we have a contradiction. So it must be the case that $f_T P_{w_1} f_2 \dots P_{w_{T-1}} f_0$. We also have $\mu_{z'_2}^W(w_t) = f_{t+1}$ for $1 \leq t \leq T-1$. If there exists $w \notin \{w_1, \dots, w_{T-1}\}$ such that $f_t \in A(w)$ for some $t = 0, 2, \dots, T-1$ then the minimality of T^* with respect to (i) and (ii) is contradicted. We now prove the existence of such a w . Consider again the deferred acceptance algorithm where workers propose to firms in market $(F, W, P(z'_2))$. Without loss of generality assume w_1 is the last of the workers in $\{w_1, \dots, w_{T-1}\}$ to be accepted by her mate under $\mu_{z'_2}^W, \mu_{z'_2}^W(w_1) = f_2$, say at step l . After step l no agent in $\{w_1, \dots, w_{T-1}\}$ is rejected. Since $f_T P_{w_1} f_2$, before applying to f_2 w_1 was rejected by f_0 at step $l' \leq l-1$. So at the end of step $l-1$ the waiting list of f_2 is full and does not include any worker in $\{w_1, \dots, w_{T-1}\}$, otherwise some

⁶An alternative proof is available on request from the author.

agent would be rejected reject after step l . Let w be the worker belonging to the waiting list of f_2 who has been rejected by f_2 in favor of w_1 , then $f_2 \in A(w)$. ■

Proof of Proposition 3. (i) Assume that there is a cycle in the preferences of the firms and let f_1, f_2, w_1, w_2, w_3 be like in Definition 2. We show that there exists a profile of workers' preferences P_W such that the Sequential Hiring Mechanism has an unstable *SPE* outcome at (F, W, P_F, P_W) . Let P_W any profile of preferences satisfying the following properties. First $P_{w_1} : f_2 f_1, P_{w_2} : f_1, P_{w_3} : f_1 f_2$. Then $A(w) = f_1$ for all $w \in W_{f_1}$, $A(w) = f_2$ for all $w \in W_{f_2}$ and $f_1, f_2 \notin A(w)$ for all other w . In every stable matching of this market $\mu(f_1) = W_{f_1} \cup \{w_1\}, \mu(f_2) = W_{f_2} \cup \{w_3\}$ and $\mu(w_2) = w_2$. Every matching where w_1 is unemployed is blocked by (f_1, w_1) and every matching where w_3 is unemployed is blocked by (f_2, w_3) . Every matching μ' where $\mu'(w_1) = f_2$ is blocked by (f_1, w_2) . Let μ^W be the worker optimal stable matching and consider the following matching ν : $\nu(f_1) = W_{f_1} \cup \{w_3\}, \nu(f_2) = W_{f_2} \cup \{w_1\}$ and $\nu(f) = \mu^W(f)$ for all $f \neq f_1, f_2$. The matching ν is blocked by the pair (f_1, w_2) , but it is supported at a *SPE*. Let $F_1(w_1) = F_2(w_3) = \{f_1, f_2\}, F_1(w) = \nu(w) = \mu^W(w)$ for all $w \neq w_1, w_3$. Let $W_2(f, z_2) = \mu_{z_2}^W$ for all f and for all z_2 . Finally let every worker accept the best offer she holds at the last stage. This profile of strategies yields ν as outcome. We now show it is a *SPE*. It suffices to prove that w_2 has no profitable deviation at which she applies to f_1 , because μ^W is stable and all workers in $W_{f_1} \cup W_{f_2} \cup \{w_1, w_3\}$ are matched to their favorite firm. By contradiction assume that w_2 has a profitable deviation and let z_2 be the subgame reached through through this deviation. In any matching μ' such that $\mu'(w_2) = f_1$, either w_1 or w_3 is unemployed. It follows that μ' is blocked in $(F, W, P(z_2))$ by (f_1, w_1) if w_1 is unemployed or by (f_2, w_3) if w_3 is unmatched, but this contradicts Lemma 2.

(ii) Assume that there is a trading opportunity in the preferences of the workers. Assume that f_1, f_2, w_1, w_2, w_3 are like in Definition 3. We show that there exists a profile of firms' preferences P_F such that the sequential hiring mechanism has an unstable *SPE* outcome at (F, W, P_F, P_W) . Let P_F be a profile of preferences such that $w_1 P_{f_1} w_2 P_{f_1} w_3 P_{f_2} w_1$ with $q_{f_1} = q_{f_2} = 1$. Let $A(f_1) = \{w_1, w_2, w_3\}, A(f_2) = \{w_1, w_3\}$. For every $f \in \{f_1, f_2\}$ let $w \notin A(f)$ for all $w \notin \{w_1, w_2, w_3\}$. For every $f \notin \{f_1, f_2\}$ let $w \notin A(f)$ for all $w \in \{w_1, w_2, w_3\}$. Let μ^W be the worker optimal stable matching and consider the following matching ν : $\nu(w_1) = f_1, \nu(w_3) = f_1$ and $\nu(w) = \mu^W(w)$ for all $f \neq w_1, w_3$. The matching ν is blocked by the pair (f_1, w_2) , but it is supported at a *SPE*. Let $F_1(w_1) = F_2(w_3) = \{f_1, f_2\}, F_1(w) = \nu(w) = \mu^W(w)$ for all $w \neq w_1, w_3$. Let $W_2(f, z_2) = \mu_{z_2}^W(f)$ for all f and for all z_2 . Finally let every worker accept the best offer she holds at the last stage. This profile of strategies yields ν as outcome. We now show it is a *SPE*. It suffices to prove that w_2 has no profitable deviation where she applies to f_1 , because μ^W is stable and w_1, w_2, w_3 are not acceptable to any firm different from f_1 or f_2 . In every matching where w_2 is hired by f_1 either w_1 or w_3 is unemployed so it is blocked by (f_1, w_1) if w_1 is unemployed or by (f_2, w_3) if w_3 is unmatched, so Lemma 2 implies that w_2 has no profitable deviations. ■