Equilibrium Corporate Finance and Intermediation *

Alberto Bisin — NYU
Gian Luca Clementi — NYU Stern
Piero Gottardi — EUI

November 17, 2016

Abstract

This paper analyzes a class of competitive economies with production, incomplete financial markets, and agency frictions. Firms take their production, financing, and contractual decisions so as to maximize their value under rational conjectures. We show that competitive equilibria exist and that shareholders always unanimously support firms' choices. In addition, equilibrium allocations have well-defined welfare properties: they are constrained efficient when information is symmetric, or when agency frictions satisfy certain specific conditions.

The Modigliani–Miller indeterminacy result does not apply. We illustrate our model's potential to generate appealing implications for investment in physical capital, leverage, corporate bond spreads, as well as excess equity returns. Firms choose their capital structure – and, to some extent, their investment – to satisfy investors' hedging needs. We also show that, when hedging demand is high, ex–ante identical firms end up choosing different strategies, catering to different investors. Less risky corporate securities, which are not traded when hedging demand is low, are featured in equilibrium, with the effect of increasing the spanning of financial markets.

Keywords: capital structure, competitive equilibria, incomplete markets, asymmetric information

*Thanks to Viral Acharya, Michele Boldrin, Marcus Brunnermeier, Douglas Gale, John Geanakoplos, Arvind Krishnamurty, David Levine, Larry Samuelson, Luigi Guiso, Enrico Perotti, Tom Sargent, Martin Schneider, Bill Zame and seminar audiences for comments. Guido Ruta contributed to an earlier version of the paper.
Bisin: alberto.bisin@nyu.edu. Clementi: clem@nyu.edu. Gottardi: piero.gottardi@EUI.eu.

1 Introduction

The notion of competitive equilibrium in incomplete market economies with production is considered problematic in economics. This is because, when financial markets are incomplete and equity is traded in asset markets, firms' production decisions may affect the set of insurance possibilities available to consumers, the asset span of the economy.[1] As a consequence, macro models with production and incomplete markets typically assume that firms' equity is not traded, or that firms operate with a backyard technology and are managed by households.[2]

Similarly, while agency frictions are at the core of corporate finance, they have been hardly studied in the context of equilibrium models. This is also arguably due to the conceptual difficulties involved in the definition of competitive equilibria with asymmetric information.[3]

In this paper we study a class of economies with production, incomplete financial markets, and agency frictions (for instance between the firm's manager and its shareholders, or between shareholders and bondholders). To highlight the foundational aspect of our analysis, we restrict attention to simple two period economies along the lines of classical general equilibrium theory[4] embedding the key features of macroeconomic models with production. At a competitive equilibrium - we postulate - price-taking firms take their production, financing, and contractual decisions so as to maximize their value defined on the basis of *rational conjectures*, as in Makowski (1983a,b). These conjectures guide firms' decisions when the value of production plans lies outside the asset span of the economy and the rationality condition can be interpreted as a consistency condition on firms' out-of-equilibrium beliefs. The analysis is first carried out in a set-up where short sales of assets are not allowed[5], but

[1] It is only in rather special environments, as pointed out by Diamond (1967), that the spanning condition holds and such issue does not arise; see also the more recent contribution by Carceles-Poveda and Coen-Pirani (2009).

[2] This is the case, for instance, in Bewley economies, the workhorse of macroeconomic model with incomplete markets; see e.g., Ljungqvist and Sargent (2004) and Heathcoate, Storesletten, and Violante (2010) for recent surveys.

[3] See, e.g., Bolton and Dewatripont (2005) and Tirole (2006). A few notable exceptions include Dow, Gorton, and Krishnamurthy (2005), Acharya and Bisin (2009), and Parlour and Walde (2011).

[4] In a complementary paper, Bisin and Gottardi (2012), we consider Bewley economies, that is, infinite horizon economies with incomplete markets but no agency frictions.

[5] This condition ensures the perfectly competitive nature of forms' decisions even when markets are incomplete.

it is then extended to incorporate financial intermediation and short sales.

We show that competitive equilibria, according to this definition, exist and have theoretically appealing properties. First of all, in the absence of agency frictions, or when such frictions satisfy appropriate conditions (satisfied, for instance, when the frictions concern the firm's manager and its shareholders) equilibrium allocations are constrained efficient. Equilibria may otherwise fail to be constrained efficient, with the source of the inefficiency lying in an externality generated by the agency friction, and we show that when this happens equilibria may display excessive aggregate risk. In addition, shareholders unanimously support value maximization and hence firms' choices, even when allocations are not efficient at equilibrium. We also identify conditions under which ex-ante identical firms might choose to specialize in equilibrium, that is to adopt different production, financial, and contractual decisions so as to optimally accommodate the demand of different consumers.

Last, but definitely not least, in the class of economies considered the Modigliani-Miller result does not hold in general. Firms' financial decisions are determinate at equilibrium and depend not only on the nature of the financial frictions but also on the consumers' demand for risk.

We take these findings to imply that the analysis of production economies with incomplete markets and agency frictions rests on solid theoretical foundations in general equilibrium, thereby providing some foundations to the integrated study of macroeconomics and corporate finance.

1.1 Related literature

Starting with the contributions of Dreze (1974), Grossman and Hart (1979) and Duffie and Shafer (1986), a large literature has dealt with the question of what is the appropriate objective function of the firm in economies with incomplete markets (under symmetric information, that is, with no agency frictions). Different objective functions have been proposed and results generally display unappealing theoretical properties, in particular the lack of unanimity of shareholders on the firms' decisions. This literature however seems to have somewhat overlooked an important contributions by Louis Makowski (1983a).[6] Indeed, Makowski showed

[6]For instance, Makowski is not cited in Dreze (1985) nor in the main later contributions to this literature, like DeMarzo (1993), Kelsey and Milne (1996), Dierker, Dierker and Grodal (2002), Bonnisseau and Lachiri (2004), Dreze, Lachiri and Minelli (2007), Carceles-Poveda and Coen-Pirani (2009). When it is cited, as in Duffie and Shafer (1986), it is to a large extent disregarded. Makowski is not even cited in the main surveys

that if firms operate on the basis of rational conjectures, under the condition that agents cannot short-sell equity and under symmetric information, value maximization is unanimously supported by shareholders as the firm's objective.[7]

In this paper we re-formulate and extend Makowski's notion of *rational conjectures* to economies with various forms of agency frictions under asymmetric information and we provide a systematic study of the properties of competitive equilibria for this general class of economies.Furthermore, we extend the analysis to financial intermediation under frictions, permitting short-sales on equity and general financial intermediation.

With regards to agency frictions and asymmetric information, most of the competitive equilibrium concepts which have been proposed build on the concept proposed by Prescott and Townsend (1984) for exchange economies, therefore exhibiting no traded equity.[8] While Prescott and Townsend's approach, rooted in mechanism design, is quite different from ours, which instead relies on the extension of *rational conjectures* to economies with asymmetric information, we show that our equilibrium concept is indeed equivalent to the one of Prescott and Townsend once this is extended to economies with incomplete markets where firms rather than consumers face agency frictions.[9] Nonetheless, interesting and important conceptual differences emerge when the analysis is extended from exchange to production economies, since we show there are natural environments where informational asymmetries in firms' decisions give rise to externalities while in consumers' problems they do not.

The class of economies considered is described in Section 2, where the equilibrium notion is also presented. Existence, the welfare properties of equilibria and unanimity are then established in Section 3, while additional properties of equilibria are derived in Section 4. In Section 5, we employ a numerical example to explore the potential our framework to generate appealing implications for capital structure and investment choices. In particular, we show that firms set choose their capital structure – and, some extent, their investment – in order to cater to the hedging needs of households whose endowment is riskier. We also show that when hedging demand is higher, ex–ante identical firms end up choosing different

of the GEI literature, as Geanakoplos (1990) and Magill and Shafer (1991).

[7]Under the same conditions, Makowski (1983b) shows that competitive equilibria are constrained Pareto optimal.

[8]See, e.g., Magill and Quinzii (2002), Prescott and Townsend (2006), and Zame (2007).

[9]We do not discuss economies with adverse selection in this paper. We conjecture that the equilibrium concepts studied by Bisin and Gottardi (2006) have an equivalent reformulation in terms of equilibria with *rational conjectures* in economies with production along similar lines to those considered in the present paper.

strategies. Some opt for a particularly safe technology, which provide risky investors with a new hedging instrument. Section 6 extends the analysis to allow for financial intermediation and short sales. Section 7 concludes. Proofs are collected in the Appendix.

2 Production economies with incomplete markets and agency frictions

In this section we introduce an abstract economy with production, incomplete financial markets and agency frictions. Various applications, examples, and extensions will be considered in later sections.

The economy lasts two periods, $t = 0, 1$, and at each date a single commodity is available. Uncertainty is described by a random variable s on the finite support $\mathcal{S} = \{1, ..., S\}$, which realizes at $t = 1$.[10] We assume the economy is populated by i) a continuum of consumers, of I different types, each of them of unit mass; and ii) a continuum of firms, of unit mass, for simplicity all identical. The economy is perfectly competitive and both firms and consumers take then prices as given.

Each consumer $i = 1, .., I$ has an endowment of w_0^i units of the single commodity at date 0 and $w_1^i(s)$ units at date 1, thus the agent's endowment is also subject to the shock affecting the economy at $t = 1$. He is also endowed with $\theta_0^i \geq 0$ units of equity of the representative firm. Consumer i has preferences over consumption in the two dates, represented by $\mathbb{E}u^i\left(c_0^i, c_1^i(s)\right)$, where $u^i(\cdot)$ is also continuously differentiable, increasing and concave.

Firms in the economy produce at date 1 using as physical input the single commodity invested as capital at time 0. Each firm's output depends on the investment k but is also in principle affected by agency frictions. At an abstract level, we model these frictions by assuming that the firm's output also depends on two other choices: ϕ, not observable to outside investors, and m, which is instead observable. For instance, ϕ could represent a technological or administrative choice and m could represent a - possibly costly - action undertaken to limit the effects of agency frictions.[11] The cost of this action might be born

[10] Any function of s, say $g(s)$ is then a random variable and we denote its mean by $\mathbb{E}[g(s)]$. Abusing notation we shall let s also denote the realization of the random variable when clear from the context.

[11] Some examples are presented in Section 2.2 to illustrate possible interpretations of these variables and applications to standard frictions considered in corporate finance concerning managers, shareholders, and outside investors.

both at time 0 and at time 1. Let $f(k,\phi,m;s)$ denote the time 1 output, net of costs, for $k\in K$, $\phi\in\Phi$, $m\in M$. We assume that Φ,K,M are closed, compact subsets of non-negative Euclidean spaces, with $K\subseteq\mathbb{R}_+$, convex and $0\in K$. Also, unless stated otherwise, Φ is a finite set.[12] Moreover, $f(k,\phi,m;s)$ is continuous in k,ϕ,m and continuously differentiable, increasing and concave in k, with $f(k,\phi,m;s)\geq 0$ for all k and some ϕ,m. The cost of the firm's actions at date $t=1$ is captured by the effect of m on the firm's net output $f(.)$, while the cost paid at time 0 is denoted by $W(k,\phi,m,B)$ and we allow them to depend also on the firms' financial decisions, described below.

Each firm takes both production and financial decisions. The outstanding amount of equity is normalized to 1: the initial distribution of equity among consumers satisfies then $\sum_i\theta_0^i=1$. We assume this amount of equity is kept constant and a firm can issue (non contingent) bonds. Hence the capital structure of a firm is only determined by its decision concerning the amount B of bonds issued. The total payment due to bondholders at $t=1$ equals B, but the actual payment may be smaller if the resources available for such payment - at most equal to the firm's net output $f(k,\phi,m;s)$ - are insufficient, in which case the firm defaults and these resources are divided pro-rata among all bondholders. As a consequence the unit return on bonds depends on the firm's production and financing choices, k,ϕ,m,B, as well as the date 1 shock, and is so denoted by $R^b\left(k,\phi,m,B;s\right)$. The rest of the firm's net output is then entirely distributed to shareholders, so that the unit return on equity $R^e\left(k,\phi,m,B;s\right)$ satisfies:

$$f(k,\phi,m;s)=R^e\left(k,\phi,m,B;s\right)+R^b\left(k,\phi,m,B;s\right)B \tag{1}$$

It is natural to assume that $R^e\left(k,\phi,m,B;s\right)$ and $R^b\left(k,\phi,m,B;s\right)$ are non negative, continuous and that the set of admissible debt levels $\mathcal{B}$ is also a closed and compact subset of $\mathbb{R}_+$ with $0\in\mathcal{B}$.

The firms' equity and debt are the only assets in the economy. If all firms make the same production and financial decisions there are then effectively two assets each consumer can trade and we write below his choice problem for that case. At $t=0$ each consumer i chooses his consumption plan $c^i(s)=(c_0^i,c_1^i(s))$, his portfolio of equity and bonds, θ^i and b^i respectively, so as to maximize his utility, taking as given bond and equity prices p,q, their

[12]The condition that the set of admissible values of k is bounded above is restrictive but by no means essential and is only introduced for simplicity. The concavity assumption can also be relaxed with no essential loss of generality.

returns $R^b(s), R^e(s)$, as well as the firms' initial market value, V. We assume that agents cannot short-sell the firm's equity nor its debt.[13] The problem of agent i is then:

$$\max_{\theta^i, b^i, c^i(s)} \mathbb{E}u^i(c_0^i, c_1^i(s)) \tag{2}$$

subject to

$$c_0^i \;= w_0^i \;+ V\theta_0^i - q\;\theta^i \;- p\;b^i \tag{3}$$

$$c_1^i(s) = w_1^i(s) + R^e(s)\theta^i \;+ R^b(s)b^i \tag{4}$$

$$b^i \geq 0,\;\; \theta^i \geq 0 \tag{5}$$

2.1 Equilibrium in a special case: no agency frictions

It is useful to introduce the competitive equilibrium notion we propose by considering first the special case where there are no agency frictions, that is, all the firm's decisions, including ϕ, are commonly observed by market participants. In this case the resources available to pay bondholders are always equal to all the firm's output $f(k, \phi, m; s)$, so that we have:

$$R^e\,(k, \phi, m, B; s) \;\;=\;\; \max\{f(k, \phi, m; s) - B, 0\} \tag{6}$$

$$R^b\,(k, \phi, m, B; s) \;\;=\;\; \min\{1, \frac{f(k, \phi, m; s)}{B}\}, \tag{7}$$

In evaluating alternative production and financing plans k, ϕ, m, B, firms operate on the basis of price conjectures $q(k, \phi, m, B)$ and $p(k, \phi, m, B)$, which specify the market valuation of the future yields of equity and debt for any possible choice of the firm that is observable by traders in the market.[14] Formally, the firm's optimization problem consists in the choice of k, ϕ, m, B that maximizes its initial market value, at time $t = 0$:

$$\max_{k,\phi,m,B} V(k, \phi, m, B) = -k + q(k, \phi, m, B) + p(k, \phi, m, B)B \tag{8}$$

At equilibrium we shall require conjectures to be *rational*, that is:

M) $$q(k, \phi, m, B) = \max_i \mathbb{E}\left[MRS^i(c^i(s))R^e(k, \phi, m, B; s)\right],$$
$$p(k, \phi, m, B) = \max_i \mathbb{E}\left[MRS^i(c^i(s))R^b(k, \phi, m, B; s)\right], \;\; \forall k, \phi, m, B;$$

[13] This is in line with Makowski (1983a, 1983b). In Section 6 we show how to introduce the possibility of short sales and financial intermediation more generally in our analysis.

[14] These conjectures are also referred to as *price perceptions*; see Grossman and Hart (1979), Kihlstrom and Matthews (1990) and Magill and Quinzii (1998).

where $MRS^i(c^i(s))$ denotes the marginal rate of substitution between consumption at date 0 and at date 1 for consumer i, evaluated at his equilibrium consumption level $c^i(s)$.[15]

Condition M) is the *Makowski criterion for rational conjectures* (after Makowski (1983a), (1983b)). It requires that for any k, ϕ, m, B the value of the equity and bond price conjectures $q(k, \phi, m, B)$ and $p(k, \phi, m, B)$ equals the highest marginal valuation - across all consumers in the economy - of the return on equity and bonds associated to k, ϕ, m, B. Consider for instance equity: the consumers with the highest marginal valuation for its yield $R^e(k, \phi, m, B; s)$ when the firm chooses k, ϕ, m, B are in fact those willing to pay the most for the firm's equity in that case and the only ones willing to buy equity - at the margin - at the price given by M). When financial markets are complete marginal rates of substitutions are equalized across all consumers at equilibrium and hence property M) holds whatever is the type i whose $MRS^i(c^i(s))$ is considered.[16] More generally, with incomplete markets it is easy to verify from the first order conditions of the consumers' choice problem (2) that property M) is satisfied by the prices q, p and returns $R^e(s)$, $R^b(s)$ faced by consumers. The rationality of conjectures requires that the same is true for any possible choice of the firm k, ϕ, m, B: the value attributed to equity equals the maximum any consumer is willing to pay for it, similarly for bonds.

Furthermore, we impose the following consistency condition between the values of prices and returns appearing in the consumers' choice problem and those conjectured by firms:

$$\textbf{C}) \qquad q = q(k, \phi, m, B), \quad p = p(k, \phi, m, B), \quad V = V(k, \phi, m, B),$$
$$R^e(s) = R^e(k, \phi, m, B; s), \quad R^b(s) = R^b(k, \phi, m, B; s)$$
for k, ϕ, m, B indicating the firms' equilibrium choice.

This condition requires that the prices of equity and bonds conjectured by firms in correspondence of the choice they make in equilibrium coincide with the prices at which these assets trade in the market. The same must then also be true for the returns on these assets and the firms' market value V.

Therefore at a competitive equilibrium k, ϕ, m, B solves the firms' problem (8), with conjectures satisfying the rationality criterion M); $\theta^i, b^i, c^i(s)$ solves the consumer's problem

[15]The marginal rates of substitution MRS^i are taken as given, independent of the firm's decision. To simplify the notation we avoid to make explicit the dependence of equity and bond price conjectures on agents' consumption levels $c^i(s)$, $i = 1, ..I$.

[16]As the property is readily implied by no-arbitrage in the case of complete markets, it is not usually explicitly imposed at equilibrium.

(2), subject to (3)-(5), for each i; prices, returns and conjectures satisfy the consistency condition C), and markets clear:

$$\begin{aligned} \textstyle\sum_i b^i &\leq B \\ \textstyle\sum_i \theta^i &\leq 1 \end{aligned} \tag{9}$$

2.2 Agency frictions

In the general case the choice of ϕ by the firm is not observable by outside investors and hence the firm faces some agency frictions. More specifically, outside investors can decide their portfolio on the basis of the firm's choice of k, B, m, which are observable, but will only have expectations, which in equilibrium will be assumed to be rational, about the level of ϕ that is chosen for any given k, B, m. Hence while k, m, B are freely chosen by the firm so as to maximize its market value (and we will show this is in the interest of all the firm's shareholders), the same is generally no longer the case for ϕ, whose choice is subject to implementability constraints. Such constraints reflect the fact that the choice of ϕ is the solution of an independent problem, which depends on the specific agency frictions present in the economy: for instance, the choice of ϕ might be delegated to a manager, or shareholders might choose ϕ to maximize the value of equity. Here we adopt an abstract specification, whereby the firm's choice of $\phi \in \Phi$ is subject to an abstract constraint described by the following map:

$$\phi \in \phi(k, m, B; c(s)), \tag{10}$$

where $c(s) = \{c^i(s)\}_{i=1}^I$. Thus the level of ϕ depends on the other decisions of the firm k, m, B and, possibly, also on other variables external to the firm, as the consumption allocation $c(s)$.[17]

In the analysis of competitive equilibria the map $\phi(.)$ is taken as exogenously given. All agents in the economy (outside investors as bondholders as well as shareholders) expect then the choice of ϕ to satisfy (10). The specific form of the map $\phi(.)$ depends on the nature of the agency frictions faced by firms and hence of the choice problem determining ϕ. In the next section we shall present some leading examples of agency frictions on which we shall build on in the rest of the paper, distinguishing between environments in which shareholders choose ϕ directly from others where such choice is instead delegated to a manager. In these cases we shall derive explicitly the form of the map $\phi(.)$.

[17] This is without loss of generality in our environment: other variables, as equilibrium prices, could be added with no change in the results.

2.2.1 Shareholders vs. bondholders

Suppose the firm's shareholders choose directly ϕ to maximize their benefit from holding equity (more precisely, the consumers' marginal valuation of the payoff of equity). In this case we have:

$$\phi(k, m, B; c(s)) \in \arg\max \left\{ \max_i \mathbb{E}\left[MRS^i(c^i(s)) R^e(k, \phi, m, B; s) \right] \right\} \tag{11}$$

As a consequence, even though ϕ affects both the returns on equity and debt, ϕ is chosen to maximize only the shareholders' valuation of the return on equity. This induces an agency problem between the firms' shareholders and bondholders: shareholders have in fact an incentive to choose values of ϕ for which the yield of equity is the highest, but at such values the yield of debt may be lower than what it could otherwise be. This is in turn anticipated by bondholders and hence reduces the value of debt. It is the *asset substitution* problem, as in Jensen and Meckling (1976). As a consequence, the firm's valuation is lower than if shareholders could commit to a different choice of ϕ, hence the agency problem.

Notice that in the situation considered here the map $\phi(.)$ depends not only on the firm's choices but also on the equilibrium consumption allocation $c(s)$.

2.2.2 Delegated management

Consider next the situation in which the choice of ϕ is delegated to a manager whose type and compensation are chosen by the firm. In this case m includes the choice of the type i of agent serving as the firm's manager as well as that of its compensation package, consisting of a net payment z_0, in units of the consumption good at date 0, and a net portfolio of ζ^m units of equity and b^m units of bonds.

An agent, if chosen as manager of a firm, will choose ϕ so as to maximize his utility, since the choice of ϕ is not observable. The choice of ϕ affects this agent's utility both because the agent may hold a portfolio whose return is affected by ϕ but also because the agent may incur some disutility costs (or benefits) associated to different choices of ϕ. Let these disutility costs be $v^i(\phi)$ for a type i consumer. Thus the map $\phi(.)$ describes the manager's optimal choice of ϕ, given his compensation package:

$$\phi(k,m,B) \in \begin{cases} \arg\max_\phi \mathbb{E}\left[u^i\left(c_0^i, c_1^i(s)\right)\right] - v^i(\phi) \\ \text{s.t.} \\ \mathbb{E}\left[u^i\left(c_0^i, c_1^i(s)\right)\right] - v^i(\phi) \geq \bar{U}^i \\ c_0^i = w_0^i \ + z_0^m \\ c_1^i(s) = w_1^i(s) + R^e(k,\phi,m,B;s)\left(\theta_0^i + \zeta^m\right) \ + R^b(k,\phi,m,B;s)b^m \end{cases} \tag{12}$$

The constraints in (12) say that, to be able to hire a type i agent as manager, an appropriate participation constraint must be satisfied: the compensation offered must be such that its utility is not lower than i's reservation utility $\bar{U}^i$ (endogenously determined in equilibrium as the utility that a type i agent, not hired as a manager, can attain by trading in the market). This is the *delegation* problem, as e.g. in Jensen (1986).

Note that in this case the choice of ϕ only depends on m, k, B (hence $c(s)$ does not appear among the arguments of the map $\phi(.)$), and both shareholders and bondholders expect ϕ to be chosen according to (12). Also, the cost of action m incurred at time 0 is given by the cost of the compensation package offered to a type i agent chosen as manager when the other firm's choices are k, B:[18]

$$W(k,m,B) = \frac{1}{1-\theta_0^i}\left[z_0^m + q(k,m,B)\zeta^m + p(k,m,B)b^m - \theta_0^i(-k + p(k,m,B)B)\right].$$

2.2.3 Two examples

We present here two specifications of the firms' technology that differ for the interpretation of ϕ and m and the characterization of their effects on the firm's net output and asset returns and correspond to cases often considered in the literature.

i) Suppose ϕ represents the loading on different aggregate factors affecting the firm's output, $(a_1(s), a_2(s))$, as in the following specification:

$$f(k,\phi;s) = \left[(1-\phi)a_1(s) + \phi a_2(s)\right]k^\alpha \tag{13}$$

Shareholders or managers, depending on the agency friction, choose then the loading $\phi \in$

[18]This expression is obtained by summing to the net payment z_0^m the value of the net portfolio of equity and bonds ς^m, b^m and subtracting the dividends due to this agent on account of his initial endowment θ_0^i of equity.

$\{0,1\}$ on the various risk components unbeknownst to outside investors.[19] The yields of equity and bonds are given by analogous expressions to (6) and (7) in Section 2.1.

ii) Consider an environment where funds can be distracted from the firm's cashflow available to pay bondholders at some cost, while (some component of) m represents a costly monitoring mechanism, e.g. some form of collateral. For instance, suppose $\phi - m$ are the funds distracted, not available to bondholders, so that default occurs whenever $f(k, \phi, m; s) - (\phi - m) < B$. The distraction might have a cost in terms of output and exerting monitoring m may also be costly, so that $f(k, \phi, m; s)$ is weakly decreasing both in ϕ and m.

In this case the returns on equity and bonds are:

$$R^e(k, \phi, m, B; s) = \phi - m + \max\{f(k, \phi, m; s) - (\phi - m) - B, 0\} \tag{14}$$

$$R^b(k, \phi, m, B; s) = \min\{1, \frac{f(k, \phi; s) - (\phi - m)}{B}\}. \tag{15}$$

This specification allows to describe a *costly monitoring* problem, as in Townsend (1979).

2.3 Equilibrium in the general case: agency frictions

The agency frictions faced by firms have no direct impact on the agents' choice problem, still described by (2) subject to (3)-(5), given q, p, V and $R^e(s), R^b(s)$.

Where the presence of agency frictions displays its main effects is in the formulation of the firms' choice problem and the role played by price conjectures. Firms solve a value maximization problem analogous to (8), but subject now to an implementability constraint:[20]

$$\begin{aligned} \max_{k,\phi,m,B} \quad & V(k, \phi, m, B) = -k - W(k, \phi, m, B) + q(k, \phi, m, B) + p(k, \phi, m, B)B \\ s.t. \quad & \phi \in \phi(k, m, B; c(s)) \end{aligned} \tag{16}$$

The Makowski criterion requires that the firm rationally anticipates its value, that is the market value of its equity and bonds, for any of its possible choices. With symmetric information, as we saw, these conjectures equal the highest marginal valuation across all consumers for the yield of equity and bonds, for any possible value of k, ϕ, m, B. With asymmetric information regarding ϕ the admissible choices of ϕ are restricted by constraint

[19] In this specification there is no action m to affect the value of the firm's output at date 1 and hence also no cost, $W = 0$. Also, all the firm's output at $t = 1$ is available to pay bondholders.

[20] The date 0 cost $W(k, \phi, m, B)$ of the actions undertaken to mitigate the agency frictions now appear explicitly in the expression of the firms' market value. This term was instead omitted for simplicity in (8).

(10). Hence the price conjectures reflect, for any given k, m, B, the correct anticipation of the level of ϕ 'induced' by k, m, B, that is, chosen according to the map $\phi(k, m, B; c(s))$. This is seen more clearly when ϕ is univocally determined by the constraint, that is the map $\phi(k, m, B; c(s))$ is single valued. In this case we could equivalently write the rational price conjectures in problem (16) as follows:

$$
\begin{aligned}
q(k, m, B) &= \max_i \mathbb{E}\left[MRS^i(c^i(s))R^e(k, \phi(k, m, B; c(s)), m, B; s)\right] \\
p(k, m, B) &= \max_i \mathbb{E}\left[MRS^i(c^i(s))R^b(k, \phi(k, m, B; c(s)), m, B; s)\right], \quad \forall k, m, B
\end{aligned}
$$

The presence of the map $\phi(.)$ in the specification of the price conjectures and the fact that B appears among its arguments generate an additional link between production and financing decisions, due to the agency frictions.

Summarizing, we have:

Competitive equilibrium: *At a competitive equilibrium of the economy*

i) *For all i, $(c^i(s), \theta^i, b^i)$ solve consumer i's problem, (2) s.t. (3)-(5), for given p, q, V and $R^e(s), R^b(s)$;*

ii) *k, ϕ, m, B solve the firm's problem, (16), given $q(k, \phi, m, B), p(k, \phi, m, B)$;*

iv) *Price conjectures $q(k, \phi, B, m)$ and $p(k, \phi, B, m)$ satisfy the rationality condition M);*

v) *Prices p, q, V and returns $R^e(s)$, $R^b(s)$ satisfy the consistency condition C)*

vi) *Markets clear: (9) holds.*

To simplify notation, the above definition and most of the presentation refers to the case of symmetric equilibria, where all firms choose the same production and financial plan. When price conjectures satisfy conditions C) and M), the firms' choice problem is however not convex. Asymmetric equilibria might therefore exist, where firms optimally choose to specialize and make different choices in equilibrium (in which case more than just two different assets would be available for trade to consumers). We shall discuss firms' specialization in Section 4.1.

2.4 A few remarks on the equilibrium concept

The key feature of the competitive equilibrium notion we propose consists in the formulation of the restriction imposed on firms' price conjectures, the Makowski rationality criterion M). As already noticed in Section 2.1 the consistency condition C) together with the consumers' first order conditions imply that this restriction is satisfied by the equilibrium choice k, ϕ, m, B. Hence the main bite of the rationality criterion is to require that the same property holds for any other admissible choice k', ϕ', m', B'. It should then be interpreted as a consistency condition for out of equilibrium conjectures.

Note that the notion of rational price conjectures as specified in M) is consistent with competitive (indeed Walrasian) markets: the consumers' marginal rate of substitution $MRS^i(c^i(s))$ used to determine the conjectures over the market valuation of debt and equity are taken as given, evaluated at the equilibrium consumption values and unaffected by the firm's choice of k, ϕ, m, B. In this sense each firm is price taker, is "small" relative to the market, and we can think of each consumer as holding a negligible amount of shares of any given firm.

We claim this equilibrium notion is natural in competitive production economies. Before discussing the properties of equilibria, we argue here that this notion is equivalent to two others adopted in the literature (in different environments).

All markets open at market clearing prices. Consider a specification where markets for all possible 'types' of equity and bonds are open: that is, equity and bonds corresponding to any possible value of k', ϕ', B', m' are available for trade to consumers at the prices $q(k', \phi', B', m')$, $p(k', \phi', B', m')$. It is immediate to see that all such markets - except the one corresponding to the firms' equilibrium choice k, ϕ, B, m - clear at zero trades. As a consequence, $q(k', \phi', B', m')$ and $p(k', \phi', B', m')$ correspond to the equilibrium prices of equity and bonds of a firm who were to "deviate" from the equilibrium choice and choose k', ϕ', B', m' instead. In this sense, we can say that rational conjectures impose a consistency condition on the out of equilibrium values of the equity and bonds price conjectures, that corresponds to a "refinement" somewhat analogous to subgame perfection.

Prescott and Townsend equilibria. Consider the equilibrium concept adopted by Prescott and Townsend (1984) for exchange economies with asymmetric information. In this concept prices depend both on unobservable as well as observable choices and this is sustained, drawing a parallel with mechanism design formulations of related problems relying on the Revelation Principle, by restricting admissible choices to those which are incentive compatible. In contrast, the equilibrium concept we propose relies on price conjectures that reflect the correct anticipation of unobservable choices. It is however straightforward to show that these two approaches are equivalent. The equilibrium notion proposed by Prescott and Townsend (1984), once extended to the environment under consideration, and hence to production economies and incomplete markets, features markets and prices for any possible value of k, ϕ, B, m and the presence of condition (10) as a constraint in the firm's problem (16). In light also of the equivalence result established in the previous paragraph, it is then easy to verify that these Prescott Townsend competitive equilibria are equivalent to competitive equilibria as defined in the previous section.

3 Equilibrium properties

The equilibrium notion we propose has several desirable properties: i) existence of an equilibrium is ensured, ii) equilibrium allocations have well-defined welfare properties, and iii) shareholders unanimously support firms' decisions. We present and discuss these properties in turn.

Proposition 1 (Existence) *A competitive equilibrium always exist.*

As noticed in Section 2.3, the firms' choice problem is not convex and to ensure the existence of an equilibrium we have to allow for asymmetric equilibria. The existence proof (in the Appendix) exploits the presence of a continuum of firms of the same type to convexify the firms' choice problem.[21]

The appropriate efficiency notion for our economy is constrained: attainable allocations are restricted not only by the limited set of financial assets that are available but also by the presence of agency frictions. More formally, a consumption allocation $c(s)$ is *admissible* if:

[21] Also, the existence proof requires for simplicity that Φ is a discrete set and a natural regularity condition for the implementability constraints $\phi \in \phi(k, m, B; c(s))$ (spelled out in the Appendix). But existence is also guaranteed when Φ is more generally a compact set if the *first order approach* is satisfied, that is, if the problem whose solution yields the map $\phi(k, m, B; c(s))$ has a unique solution, described by a continuous function.

1. it is *feasible*: there exists a production plan[22] k, m, ϕ of firms such that

$$\begin{aligned} \sum_i c_0^i + k &\leq \sum_i w_0^i \\ \sum_i c_1^i(s) &\leq \sum_i w_1^i(s) + f(k, \phi, m; s); \end{aligned} \tag{17}$$

2. it is *attainable with the existing asset structure*: that is, there exists B and, for each consumer's type i, a pair θ^i, b^i such that

$$c_1^i(s) = w_1^i(s) + R^e(k, \phi, m, B; s)\theta^i \; + R^b(k, \phi, m, B; s)b^i; \tag{18}$$

3. it is *incentive compatible*: given the observable component of the production plan k, m, the financing plan B and the consumption allocation $c(s)$, the unobservable component satisfies

$$\phi \in \phi(k, m, B; c(s)) \tag{19}$$

We then say that a competitive equilibrium allocation is *constrained Pareto efficient* if we cannot find another admissible allocation which is Pareto improving.

Proposition 2 (First Welfare Theorem) *Competitive equilibria are constrained Pareto efficient when no agency frictions are present or whenever the incentive compatibility map $\phi(.)$ only depends on the firm's choice variables k, m, B.*

Thus in the economy with no agency frictions described in Section 2.1, where ϕ is observable and its choice is unrestricted in Φ, constrained efficiency always holds. With agency frictions, considering the characterization introduced in Section 2.2, we find that constrained efficiency obtains when the friction is of the delegated management type, that is, when firms delegate the choice of the unobservable variable ϕ to a manager and m contains the manager's type as well as his compensation contract. In this case, as we noted, ϕ is determined by (12) and is independent of $c(s)$.[23] Note that a key feature for the specification of the incentive constraint in (12), and thus also for the efficiency result, is that the manager's trades

[22] Again production and financing plans could differ across firms but we state for simplicity the notion of admissible allocations for the case in which they don't.

[23] Under the stated conditions the First Welfare Theorem is established by an argument (see the Appendix) essentially analogous to the one used to establish the Pareto efficiency of competitive equilibria in Arrow-Debreu economies.

are observable, so that the manager cannot trade his way out of his compensation package. In other words, it is crucial that the manager's compensation contract is exclusive.[24]

On the other hand constrained efficiency may fail when the incentive constraint depends also on variables not directly chosen by the firm, like the consumption allocation $c(s)$, as we showed it happens in the shareholders/bondholders problem considered in Section 2.2.1. In this case in fact an externality arises, generated by the agency friction.

At the same time, we should point out that this is the only source of inefficiency in our economy. In all other respects, firms' decisions are efficient and, as we show next, unanimously supported by shareholders. In both the economies described in Sections 2.1 and 2.2 in fact all shareholders unanimously agree on the firm's production and financing decisions, that is on the choice of k, ϕ, m, B which maximizes the firm's market value, defined by rational conjectures (subject, when ϕ is unobservable, to the implementability constraint (10)):

Proposition 3 (Unanimity) *Let k, ϕ, m, B be the firms' choice at a competitive equilibrium and $c(s)$ be the consumption allocation. Then every agent i holding a positive initial amount θ_0^i of equity of a firm will be made - weakly - worse off by any other possible choice of the firm (k', ϕ', m', B') (with ϕ' satisfying (10) when there are agency frictions).*

The result follows from the fact that, as noticed in Section 2.4, the equilibrium allocation is the same as the one which would obtain if markets for all possible types of equity and bonds were open. Consequently, the unanimity result holds by the same argument as the one used to establish this property for Arrow-Debreu economies.

3.1 A few remarks on the relationship with the literature

The problems found in the literature and recalled in Section 1.1, concerning the specification of the firms' objective function, do not arise for the equilibrium notion we propose. As shown in the previous section, in the set-up typically considered in this literature (that is, with no agency frictions), both unanimity and constrained efficiency hold. The key difference between this paper and this literature lies in the specification of the firms' price conjectures.

[24]The inefficiency of economies where this assumption is not satisfied have been studied in the literature; see, Arnott and Stiglitz (1993) and, more recently, e.g., Acharya and Bisin (2009) and Bisin, Gottardi, and Rampini (2008).

It is useful then to compare the *Makoswki criterion* for rational conjectures to the two main alternative specifications in the literature, the *Dreze* and the *Grossman-Hart criterions*, in the context of an economy without agency friction, as in Section 2.1.

Dreze (1974) proposes the following criterion for equity price conjectures (a similar condition holds for bond prices):

$$q(k, \phi, m, B) = \mathbb{E} \sum_i \theta^i MRS^i(c^i(s)) R^e(k, \phi, m, B; s), \ \forall k, \phi, m, B \tag{20}$$

It requires the conjectured price of equity for any plan k, ϕ, m, B to equal - pro rata - the marginal valuation of the agents who in equilibrium are shareholders of the firm (that is, the agents who value the most the plan chosen by the firm in equilibrium and hence choose to buy equity). It does not however require that the firm's shareholders are those who value the most any possible plan of the firm. Intuitively, the choice of a plan which maximizes the firm's value with $q(k, \phi, m, B)$ as in (20) corresponds to a situation in which the firm's shareholders choose the plan which is optimal for them without contemplating the possibility of selling the firm in the market, to allow the buyers of equity to operate the plan they instead prefer. Equivalently, the value of equity for out of equilibrium production and financial plans is determined using the - possibly incorrect - conjecture that the agents who in equilibrium own the equity of a firm remain the firm's shareholders also for any alternative production and financial plan.[25]

Grossman and Hart (1979) propose an alternative criterion for price conjectures which, when applied to the price of equity, requires:

$$q(k, \phi, m, B) = \mathbb{E} \sum_i \theta_0^i MRS^i(c^i(s)) R^e(k, \phi, m, B; s), \ \forall k, \phi, m, B$$

We can interpret this specification as describing a situation where the firm's plan is chosen by the initial shareholders (i.e., those with some predetermined equity holdings at the beginning of date 0) so as to maximize their welfare, again without contemplating the possibility of selling the equity to other consumers who value it more. Equivalently, the value of equity

[25]It is then easy to see that any allocation constituting an equilibrium with rational conjectures according to the criterion is also an equilibrium under the Dreze criterion: all shareholders of a firm have in fact the same valuation for the firm's production and financial plan and their marginal utility for any other possible plan is lower, hence a fortiori the chosen plan maximizes the weighted average of the shareholders' valuations. But the reverse implication is not true, i.e., an equilibrium under the Dreze criterion is not in general an equilibrium under rational conjectures.

for out of equilibrium production and financial plans is derived using the conjecture that the firm's initial shareholders stay in control of the firm whatever is the plan that is chosen.

In summary, according to the Makowski criterion for rational conjectures each firm evaluates different production and financial plans using possibly different marginal valuations (that is, possibly different pricing kernels, but all still consistent with the consumers' marginal rate of substitution at the equilibrium allocation). This is essential to ensure the unanimity of shareholders' decisions and is a key difference with respect to Dreze (1974) and Grossman and Hart (1979), both of whom rely on the use of a single pricing kernel.[26]

On a different note, our analysis also highlights an interesting and important difference between the properties of equilibria when agency frictions are faced by consumers, as e.g., in Prescott and Townsend's analysis of exchange economies with asymmetric information, and when instead such frictions are faced by firms. While competitive equilibria are always constrained efficient in the exchange economies considered by Prescott and Townsend, this is not necessarily the case in production economies, as we have shown in Proposition 2. The nature of the equilibrium concept adopted plays no role in this: as we discussed in Section 2.4, our equilibrium concept is equivalent to the one of Prescott and Townsend once this is extended to production economies. Rather, agency frictions and production may naturally interact to generate an externality.[27]

An important implication of the welfare properties of production economies with agency frictions is that in economies where equilibrium allocations are constrained inefficient, e.g. when the agency friction is between shareholders and bondholders, a Pareto improvement may be achieved with different types of agents owning equity than the ones who do in equilibrium. Since the unanimity result in Proposition 3 always holds, even when equilibrium allocations are not constrained efficient, this misallocation of equity ownership is not a consequence of lack of unanimity, as it might instead be the case in equilibrium concepts adopting

[26]This feature distinguishes also the equilibrium notion based on the Makowski rationality criterion from the several others proposed in the literature, including those applying elements from the theory of social choice and voting to model the control of shareholders over the firm's decisions; see for instance DeMarzo (1993), Boyarchenko (2004), Cres and Tvede (2005).

[27]Prescott and Townsend also assume that markets are complete, while we do not. But whether markets are complete or not, and hence whether $MRS^i(c^i(s))$ are equalized or not across i, is not crucial for the welfare result. What is crucial is that the agents' marginal rates of substitution enter the incentive constraint, so that a change in the consumption allocation may relax this constraint.

the Dreze or the Grossman-Hart criterion. It is rather a consequence of the externality affecting firms' incentive constraints, which may turn out to be more severe when some types of agents are shareholders than when others are.

4 Specialization and amplification

In this section we present two results concerning properties of equilibria with the aim of better illustrating some important aspects of equilibrium allocations. While we present these results in the context of specific examples, it should be clear that the underlying economic phenomena we characterize represent robust equilibrium properties.

4.1 Efficient firms' specialization

In Section 2.3 we defined for simplicity competitive equilibria for the case where firms' choices are symmetric, that is all firms choose the same production and financial plan, but we also acknowledged that asymmetric equilibria may exist. This is not just a technical issue, arising from the non concavity of the firms' objective problem, but reflects a fundamental implication of the rationality of firms' conjectures: firms may have an incentive to specialize their production and financial plans so as to cater to the different demands of different consumers. In this section we analyze an example where we illustrate the incentives of firms to specialize, so as to offer consumers different risk profiles for the yields of their equity and bonds, but also the possible costs which may hinder specialization.

Consider an environment with no agency frictions, two types of consumers, and a single type of firms. Both consumers have the same initial equity holdings and first period endowments, as well as identical preferences. They only differ in their second period endowment $w_1^1(s)$ and $w_1^2(s)$. The production technology of each firm is as in (13), with ϕ representing the loading on the two risk factors $a_1(s)$, $a_2(s)$: $f(k, \phi; s) = [(1-\phi)a_1(s) + \phi a_2(s)]\, k^\alpha$. Also, $\mathbb{E}a_1(s) = \mathbb{E}a_2(s)$. Let us also ignore here, for simplicity, the firms' financial choice, by setting $B = 0$, so that $R^e(k, \phi, m, B; s) = f(k, \phi; s)$.

Under some symmetry conditions (spelled out in the Appendix), we obtain the following result:

Proposition 4 (Specialization) *Suppose the factors $a_1(s)$ and $a_2(s)$ vary anti-comonotonically.*[28]

[28] That is, for any pair $s_1, s_2 \in \mathcal{S}$, $a_1(s_1) \geq a_1(s_2)$ if and only if $a_2(s_1) \leq a_2(s_2)$.

Then if $w_1^1(s)$ varies comonotonically with one factor and $w_1^2(s)$ with the other factor, the equilibrium displays production specialization: a fraction of firms choose $\phi = 0$ and the remainder $\phi = 1$. If instead both $w_1^1(s)$ and $w_1^2(s)$ vary comonotonically with the same factor, the equilibrium is symmetric: all firms choose the same value of ϕ.

The incentives of firms to specialize their production plans are larger when the different factors in the firms' production function are good hedges of the endowment risk of different types of agents. In this case specialization more easily allows to satisfy the consumers' demand for risk. At the same time, specialization also involves some cost since it reduces the demand for each firm's equity as this comes from only one type of consumer. There is so a trade-off and when the differences in hedging properties of different factors for different agents are less clearly marked specialization does not arise in equilibrium.

To illustrate and provide some intuition for the result in the above proposition it is useful to present the first steps of the proof.

Proof of Proposition 4. *Consider the following case: $w_1^1(s)$ is comonotonic with $a_1(s)$ and $w_1^2(s)$ with $a_2(s)$, hence $Cov(w_1^1(s), a_2(s)) < 0$ and $Cov(w_1^2(s), a_1(s)) < 0$. In this situation factor 2 is a clearly a good hedge for type 1 agents while factor 1 is a good hedge for type 2 agents and the claim is that specialization obtains.*

Suppose, by contradiction, that we have an equilibrium where all firms choose the same factor, say $a_1(s)$, that is, choose $\phi = 0$. Then from the first order conditions for the consumers' optimal choice we have, for each $i = 1, 2$:

$$\mathbb{E}\left[MRS(c^i(s))a_1(s)k^\alpha\right] \leq q, \tag{21}$$

where $MRS(c^i(s))$ is evaluated at the equilibrium consumption level $c_0^i = w_0 \ + V0.5 - q\theta^i$, $c_1^i(s) = w_1^i(s) + a_1(s)k^\alpha\theta^i$. Note that (21) must hold as equality for at least one i. Furthermore, for the choice $\phi = 0$ to be optimal for all firms the following relationship must hold:

$$\max_i \mathbb{E}\left[MRS(c^i(s))a_1(s)k^\alpha\right] \geq \max_i \mathbb{E}\left[MRS(c^i(s))a_2(s)k^\alpha\right], \tag{22}$$

where we used the rationality of price conjectures to determine the value of a firm corresponding to the alternative choice $\phi = 1$. For any type i for whom (21) holds as equality, the firm's optimality condition (22) reduces to

$$Cov\left[MRS(c^i(s)), a_1(s)k^\alpha\right] \geq Cov\left[MRS(c^i(s)), a_2(s)k^\alpha\right]$$

But this is clearly impossible for $i = 1$: given that $\mathbb{E}a_1(s) = \mathbb{E}a_2(s)$, the comonotonicity conditions imply that $Cov\left[MRS(c^1(s)), a_1(s)k^\alpha\right] < 0$ while $Cov\left[MRS(c^1(s)), a_2(s)k^\alpha\right] > 0$, since $a_2(s)$ is clearly a better risk hedge than $a_1(s)$ when $c_1^1(s) = w_1^1(s) + a_1(s)k^\alpha\theta^1$. In this situation a firm could increase its value by switching to factor $a_2(s)$, hence the contradiction. It remains then to establish the claim when (21) holds as equality only for type $i = 2$. The proof of this and the second part of the claim in the proposition are in the Appendix. □

As shown above, production specialization may arise in equilibrium to satisfy the agents' demand for hedging their endowment risk. Given the constrained efficiency of competitive equilibria with no agency frictions, established in Proposition 2, when the equilibrium exhibits specialization this is also efficient. Efficiency requires to evaluate the alternative production and financing plans by firms on the basis of the different preferences of consumers for such plans. When the profitability of all possible plans is assessed on the basis of rational price conjectures as in condition M), firms do indeed this, taking into account which type of agent will hold the firms' assets for each possible plan. This implies, as we noticed, a non convexity of the firm's choice problem so that indeed specialization may emerge. In contrast, with the Dreze or the Grossman-Hart criteria firms evaluate any (equilibrium and out of equilibrium) production and financial plan according to the preferences of a given subset of agents. As a consequence the firm's objective function is linear and its choice problem convex, equilibria will generally be symmetric and may turn out to be constrained inefficient.

To illustrate this point further we take up the well-known example studied by Dierker, Dierker, and Grodal (2002). In this example, uncertainty is represented by $\mathcal{S} = \{s_1, s_2\}$. There are two types of consumers. The technology of the representative firm is described by $f(k, \phi; s) = \phi k$ for $s = s_1$ and $(1 - \phi)k$ for $s = s_2$, with $\phi \in \Phi = [2/3, 0.99]$. Again abstracting from the firms' financial decisions and setting $B = 0$, the problem faced by firms in this environment is to choose (k, ϕ) so as to maximize $-k + q(k, \phi)$, where $q(k, \phi) = \max\left\{\mathbb{E}MRS^1(c^1(s))f(k, \phi; s); \mathbb{E}MRS^2(c^2(s))f(k, \phi; s)\right\}.$

For specific functional forms of the agents' preferences and parameter values, in this economy Dierker, Dierker and Grodal (2002) find a unique symmetric equilibrium, according to the Dreze criterion, where all firms choose the same value of k and ϕ and such equilibrium is constrained inefficient.[29] In contrast, a symmetric competitive equilibrium according to the Makowski rational conjecture criterion does not exist: at a symmetric allocation the

[29]The remaining description of the environment considered in this example, together with the derivation

firms' optimality condition with respect to ϕ in fact can never hold. This is because, for the specification of agents' preferences and endowments considered, type 1 consumers strongly prefer assets paying when $s = s_1$ (and hence $\phi = 0.99$) while type 2 consumers strongly prefer assets paying when $s = s_2$ ($\phi = 2/3$). On the other hand, allowing for production specialization, a unique equilibrium exists, which is constrained Pareto optimal.

4.2 Aggregate risk amplification

Proposition 2 identifies conditions on the financial frictions faced by firms under which equilibrium allocations might be constrained inefficient. In this section we characterize the form the inefficiency takes in an economy where the agency frictions are of the *asset substitution* type, as in Section 2.2.1. Interestingly, in this case the constrained inefficiency might manifest itself in terms of excessive aggregate risk at equilibrium.

Consider the following small variant of the firms' technology in (13):

$$f(k, \phi, m; s) = [a_1(s) + C(\phi)\,(a_2(s) - a_1(s))]k^\alpha$$

with[30] $\phi \in \Phi = [0, 1]$ and the loading on $a_2(s)$ described by the function $C(\phi)$, increasing and concave and such that $C(0) = 0$, $C(1) = 1$. Suppose that $\mathcal{S} = \{s_1, s_2\}$ and that $a_1(s)$ is a mean preserving spread of $a_2(s)$ with

$$a_1(s_1) > a_2(s_1) > a_2(s_2) > a_1(s_2). \tag{23}$$

Postulate that parameters are such that at a competitive equilibrium of the economy with no agency frictions (with observable ϕ, as in Section 2.1) the firm's choice of ϕ is interior and B is such that the firm defaults in state s_2 but not in state s_1. This is clearly a robust property in the class of economies we are considering. As a consequence, the yields of equity and bonds are:

$$R^e(k, \phi, B; s) = \begin{cases} [a_1(s) + C(\phi)\,(a_2(s) - a_1(s))]\,k^\alpha - B & \text{for } s = s_1 \\ 0 & \text{for } s = s_2 \end{cases}$$

$$R^b(k, \phi, B; s) = \begin{cases} 1 & \text{for } s = s_1 \\ \frac{[a_1(s) + C(\phi)(a_2(s) - a_1(s))]k^\alpha}{B} & \text{for } s = s_2 \end{cases}$$

---

of the competitive equilibria according to the Dreze and the Makowski rational conjecture criteria can be found in the Appendix.

[30]Here (and at the end of the previous section) we leave our standard environment where Φ is discrete. Hence the implementability constraints are characterized in terms of derivatives.

Since the value of ϕ only affects the return on equity in state s_1 and the return on bonds only in state s_2, it follows from (23) that

$$\begin{aligned}\partial R^e(k,\phi,B;s_1)/\partial\phi &= C'(\phi)\left(a_2(s_1)-a_1(s_1)\right)k^\alpha < 0,\\ \partial R^b(k,\phi,B;s_2)/\partial\phi &= C'(\phi)\left(a_2(s_2)-a_1(s_2)\right)k^\alpha > 0.\end{aligned}$$

Letting i_e denote any of the (agents who in equilibrium are) shareholders and i_b any of the bondholders, the firms' first order condition with respect to the choice of ϕ at this equilibrium requires:

$$\mathbb{E}\left[MRS^{i_e}(c^{i_e}(s))\frac{\partial}{\partial\phi}R^e(k,\phi,B;s) + MRS^{i_b}(c^{i_b}(s))\frac{\partial}{\partial\phi}R^b(k,\phi,B;s)B\right] = 0. \tag{24}$$

As shown above, the first term in the above expression is negative and the second one positive.

Now consider the case with agency frictions in which ϕ is not observable and chosen by shareholders, as in Section 2.2.1. Suppose at a competitive equilibrium the debt level is still such that the firm defaults in state s_2. The first order condition with respect to ϕ of problem (11), stating the incentive constraint in the environment under consideration, is then in this case:

$$\mathbb{E}\left[MRS^{i_e}(c^{i_e}(s))\frac{\partial}{\partial\phi}R^e(k,\phi,m,B;s)\right] = MRS^{i_e}(c^{i_e}(s_1))C'(\phi)\left(a_2(s_1)-a_1(s_1)\right)k^\alpha \leq 0, \tag{25}$$

which is satisfied as a strict inequality at a corner solution[31] $\phi = 0$. This is because, as argued above, $R^e(k,\phi,m,B;s_1)$ is decreasing in ϕ, the loading factor on $a_2(s)$. At a competitive equilibrium with agency frictions, therefore, shareholders will choose to load on the riskier factor $a_1(s)$ more than at the equilibrium without agency frictions.

We show next that a marginal increase in ϕ by all firms, with respect to its level at a competitive equilibrium with agency frictions, may be feasible, that is satisfy the incentive constraint (11).[32] This is because the marginal effect of a change in ϕ on the objective

[31] It is straightforward to see that at the equilibrium allocation with no agency frictions the term on the left hand side of (25) is always negative. Hence the incentive constraint (11) is violated, since shareholders would like to decrease ϕ, that is, to increase the loading on (the riskier) factor $a_1(s)$. Thus we cannot have an interior solution for ϕ.

[32] We limit our analysis here to a local argument, following on this the literature on economies with financial frictions, e.g., Gromb and Vayanos (2002), Krishnamurty (2000, 2010) and Lorenzoni (2005).

function of problem (11), when ϕ is modified by all firms so that also the consumers' MRS is modified, is given by:

$$\mathbb{E}\left[MRS^{i_e}(c^{i_e}(s))\frac{\partial}{\partial\phi}R^e(k,\phi,m,B;s)+\frac{\partial}{\partial\phi}MRS^{i_e}(c^{i_e}(s))R^e(k,\phi,m,B;s)\right] \tag{26}$$

The first term is the same as in the equilibrium condition (25), and has a negative sign, but there is now a second term, $\frac{\partial}{\partial\phi}MRS^{i_e}(c^{i_e}(s_1))R^e(k,\phi,m,B;s_1)$, which has a strictly positive sign: an increase in ϕ in fact reduces the consumption of shareholders in state s_1, $c^{i_e}(s_1)$, and this in turn increases these agents' marginal utility, as this is decreasing in consumption, and also $MRS^{i_e}(c^{i_e}(s_1))$[33]. The overall expression in (26) may then have a strictly positive sign, in which case an increase in ϕ (by all firms) is feasible, in the sense that it satisfies the incentive constraint (11). In other words, by internalizing the effect of ϕ on the shareholders' consumption, a higher level of ϕ can be supported, that is a value closer to the one obtained at the equilibrium without agency frictions.[34]

In the environment considered we can thus say that a competitive equilibrium with agency frictions displays excessive aggregate risk, as ϕ is set at the level 0, with full loading on the riskier factor, while a higher level of ϕ, with less loading on this factor, could be feasible and allow to reduce the aggregate risk in the economy. By not taking into account the effects of their choice of ϕ on the agents' marginal rates of substitution, shareholders will tend to choose a too low level of ϕ, increasing the amount of risk in the economy at equilibrium.

5 A Numerical Example

In this section we introduce a specialized case of the economy described above, we parameterize it, and solve numerically for the equilibrium allocation. The objective is to showcase the role of the incomplete markets assumption in an otherwise plain–vanilla general equilibrium competitive economy.

The production function displays decreasing returns to scale: $f(k) = e^{z_1}k^{\alpha}$, with $\alpha \in (0,1)$. With z_1, we denote a random variable distributed according to

$$z_1 = \rho z_0 + \varepsilon, \;\; \varepsilon \sim N(\mu, \sigma^2), \;\; \sigma > 0.$$

[33] $c_0^{i_e}$ is in fact not affected by the change in ϕ.

[34] The marginal increase in ϕ generates an increase in the surplus from the firms' production activity, which can then be distributed among agents with appropriate transfers at date 0 so as to generate a welfare improvement.

It follows that the payoffs of the firm's financial assets are

$$d^e(k, B; \varepsilon) = \max\left[e^{z_1}k^\alpha - B, 0\right],$$
$$d^b(k, B; \varepsilon) = \min\left[1, \frac{e^{z_1}k^\alpha}{B}\right].$$

We assume that $I = 2$, i.e. there are only two types of households, with preferences described by

$$U(c_0^i, c_1^i(\varepsilon)) = u(c_0^i) + \beta \int u(c_1^i(\varepsilon))g(\varepsilon)d\varepsilon,\ \beta > 0, u' > 0, u'' < 0.$$

Initial equity ownership is uniformly distributed. Households differ in their endowments at both dates. At time $t = 0$, agents of type i have deterministic endowments Ω^i, with $\Omega^i \geq 0$. At time $t = 1$, endowments are random and given by

$$w_1^i(\varepsilon) = e^{\rho z_0 - \chi_i \mu - \frac{1}{2}\chi_i^2\sigma^2 + \chi_i\varepsilon},\quad \chi_i \in [0, 1].$$

This implies the following moments for the endowment processes:

$$E[w_1^i] = e^{\rho z_0},$$
$$Var[w_1^i] = e^{2\rho z_0}[e^{\chi_i^2\sigma^2} - 1],$$
$$Cov(w_1^i, e^{z_1}) = e^{2\rho z_0 + \mu + \frac{1}{2}\sigma^2}\left(e^{\chi_i\sigma^2} - 1\right).$$

The rationale for this structure is to allow households to differ in their wealth and in their exposure to risk. For $\chi_i = 0$, the endowment is riskless. As χ_i increases, so do the variance of the endowment and its covariance with the common shock to firms' productivity – the aggregate shock.

The households' optimization problem writes as

$$\max_{c_0^i, \theta_1^i, b^i, c_1^i(\cdot)} u(c_0^i) + \beta \int u[c_1^i(\varepsilon)]g(\varepsilon)d\varepsilon,$$
$$\text{s.t.}\quad c_0^i = [\Omega^i + \theta_0 V] - q\theta_1^i - pb^i,$$
$$c_1^i(\varepsilon) = w_1^i(\varepsilon) + \theta_1^i d^e(\varepsilon) + b^i d^b(\varepsilon),$$
$$\theta_1^i \geq 0,\ \ b^i \geq 0.$$

Firm value is

$$V \equiv \max_{k,B} -k + q(k,B) + p(k,B)B,$$

where the price conjectures satisfy the rationality requirement

$$q(k,B) = \max_i \ \beta \int \frac{u'(c_1^i(\varepsilon))}{u'(c_0^i)} d^e(k,B;\varepsilon)g(\varepsilon)d\varepsilon,$$
$$p(k,B) = \max_i \ \beta \int \frac{u'(c_1^i(\varepsilon))}{u'(c_0^i)} d^b(k,B;\varepsilon)g(\varepsilon)d\varepsilon,$$

as well as consistency (asterisks denote equilibrium choices),

$$q = q(k^*, B^*),$$
$$p = p(k^*, B^*),$$
$$V = V(k^*, B^*),$$
$$d^e(\varepsilon) = \max[e^{z_1}Ak^{*\alpha} - B^*, 0],$$
$$d^b(\varepsilon) = \min[1, e^{z_1}Ak^{*\alpha}/B^*],$$

and markets clear:

$$b^1 + b^2 \leq B^*,$$
$$\theta_1^1 + \theta_1^2 \leq 1.$$

5.1 Parameterization and Characterization

We assume CRRA preferences with a relative risk aversion coefficient of 3. The span of control parameter is $\alpha = 0.6$, while the parameters of the normal distribution are $\mu = 0$ and $\sigma = 0.2$.

The key parameters of our exercise are the individual loadings on the aggregate shock χ_i, $i = 1, 2$. Since we want households to differ in their demands for hedging, we select $\chi_1 = 0$ and $\chi_2 = 0.9$, respectively. Figure 1 illustrates the variation of endowments and firm productivity at $t = 1$ over the (truncated) set of realizations for the innovation ε, along with the density function for the latter. We begin by assuming that initial endowments are the same, i.e. $\Omega^1 = \Omega^2$.

Our calculations how that in equilibrium both households hold equity, but type–2 households hold all debt. The reason is rather intuitive, as type–1 agents endowment profile is riskless, while type–2 agents have non-trivial hedging demand.

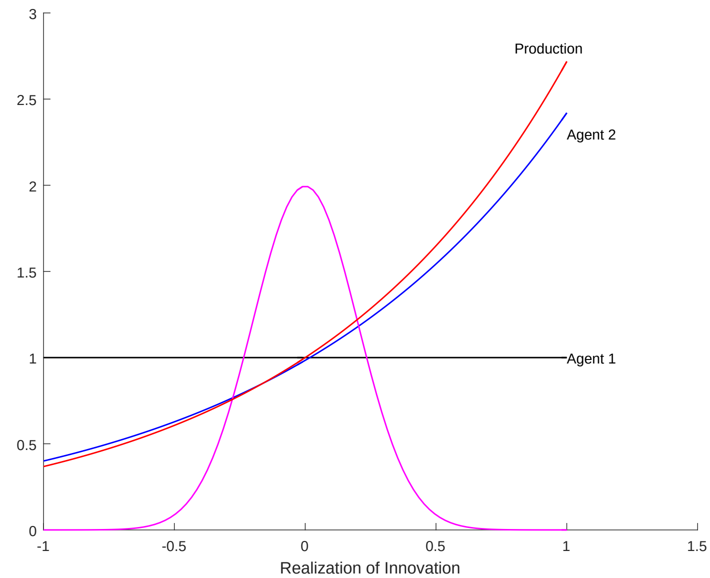

Figure 1: Household Endowments and Firm Productivity.

Because of the incomplete market assumption, the Modigliani-Miller indeterminacy result does not arise. Let g denote the density of the normal distribution and let $\varepsilon^*(k,B) \equiv \log\left(\frac{B}{k^\alpha}\right) - \rho z_t$ be lowest realization of the innovation ε consistent with solvency. Then, the necessary, although not sufficient condition for debt optimization reads as

$$\int_{\varepsilon^*(k,B)}^{+\infty} \frac{u'(c_1^1)}{u'(c_0^1)} g(\varepsilon) d\varepsilon = \int_{\varepsilon^*(k,B)}^{+\infty} \frac{u'(c_1^2)}{u'(c_0^2)} g(\varepsilon) d\varepsilon.$$

At the margin, raising debt issuance transfers resources from shareholders to bondholders in the states of nature where the firm is solvent. The choice of leverage is determinate because the marginal rates of substitution of the two types of investors are not equal. In other words, the choice of leverage is dictated by the desire to cater to agent 2's hedging needs.

Raising capital positively affects cash flows to shareholders in solvency states, and to bondholders in default states. The necessary condition for optimality of the capital choice is

$$1 = \beta e^{\rho z_0} \alpha A k^{\alpha-1} \left[\int_{\varepsilon^*(k)}^{+\infty} \frac{u'(c_1^2)}{u'(c_0^2)} e^\varepsilon g(\varepsilon) d\varepsilon + \int_{-\infty}^{\varepsilon^*(k)} \frac{u'(c_1^2)}{u'(c_0^2)} e^\varepsilon g(\varepsilon) d\varepsilon \right].$$

A little algebra allows to rewrite it as

$$1 = \beta e^{\rho z_0} \alpha A k^{\alpha-1} \left[cov\left(\frac{u'(c_1^2)}{u'(c_0^2)}, e^{\varepsilon} \right) + E\left(\frac{u'(c_1^2)}{u'(c_0^2)} \right) E\left(e^{\varepsilon}\right) \right]. \tag{27}$$

In the square brackets on the right-hand side are two terms familiar to financial economists. The first is the covariance between household 2's marginal rate of substitution and the innovation. The second is the inverse of the (virtual) risk-free rate. This expression makes it transparent that the investment policy is also shaped by household 2's hedging needs. We further elaborate on this issue below, when we consider a comparative statics exercise with respect to the distribution of the endowment at $t = 0$.

5.2 Comparative statics with respect to the wealth distribution

In this section, we assess how the distribution of wealth at time $t = 0$ affects the equilibrium allocation. To this end, we compute equilibria in a number of scenarios that differ in the ratio $\Omega^2/(\Omega^1 + \Omega^2)$, for given total endowment. Such an exercise is relevant, because an increase in such ratio leads to a larger demand for hedging, since type-2 households' savings increase.

The red (solid) lines in Figure 2 outline how the choices for capital, debt issuance, and leverage vary, as a greater fraction of total endowment accrues to type-2 agents. Leverage is computed as $p_b B/k$. The black (dashed) and blue (dash-dot) lines refer to scenarios where markets are complete and equity is the only asset, respectively. We do not report either debt or leverage in the complete-markets scenario, as capital structure is indeterminate.

As type-2 households grow wealthier, they need to invest more in financial assets. Since their earnings are positively correlated with equity payoffs, their valuation of debt will also increase. It is then optimal for the the firm to cater to their hedging needs by issuing more debt.

When debt issuance is not allowed, equilibrium capital is uniformly higher. This is the case because, faced with higher risk, type–2 agents engage in larger precautionary savings. The firm caters to such need by increasing its size. As the fraction of wealth held by agent 2 declines towards 1/2, the incomplete–markets allocation converges towards the complete–market one. This is not surprising, as the aggregate demand for hedging declines, since a larger fraction of wealth is in the hands of agents with safe endowments.

Figure 3 further corroborates our narrative. When equity is the only asset, type-2 agents' consumption growth is uniformly higher, reflecting greater precautionary saving. When

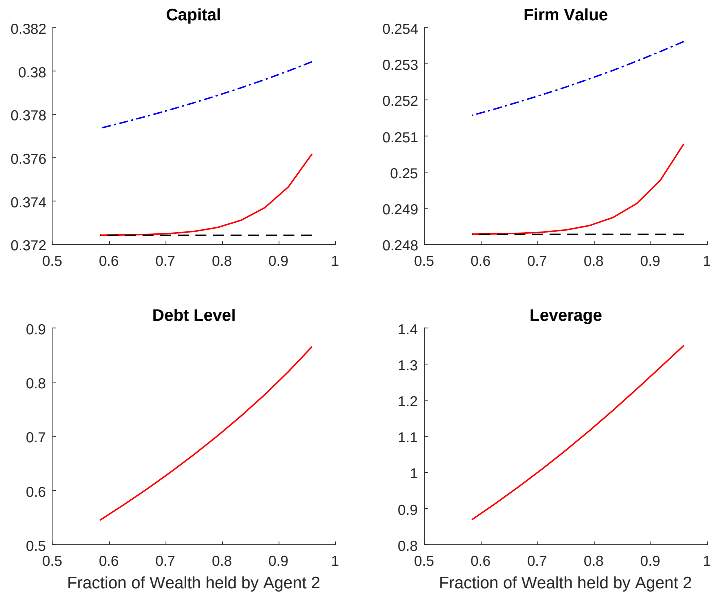

Figure 2: Firms' Choices.

hedging demand is relatively low (i.e. the fraction of wealth held by type-2 agents is low), the additon of debt is enough to bring consumption profiles close to those implied by complete markets. However, when the fraction of wealth held by type-2 agents approaches 100%, both the mean and variance of consumption growth for such agents exceed the values obtained with complete markets.

Figure 4 illustrates the implications for assets returns. The risk–free rate is defined as the reciprocal of the valuation expressed by type-2 agents for the risk–free asset, evaluated at zero supply. Type-2 agents' evaluation is the only one that matters, since type-1 agents have a uniformly strictly lower valuation for the same security. In turn, this means that any marginal increase (from zero) in the supply of that asset would go to the benefit of type–2 agents only. The risk–free rate declines as the type–2 agents becomes richer, thanks to an increase in the precautionary motive.

The behavior of excess equity returns under incomplete markets can be rationalized by appealing to the role of leverage. In the scenario without debt – the blue, dash-dot line

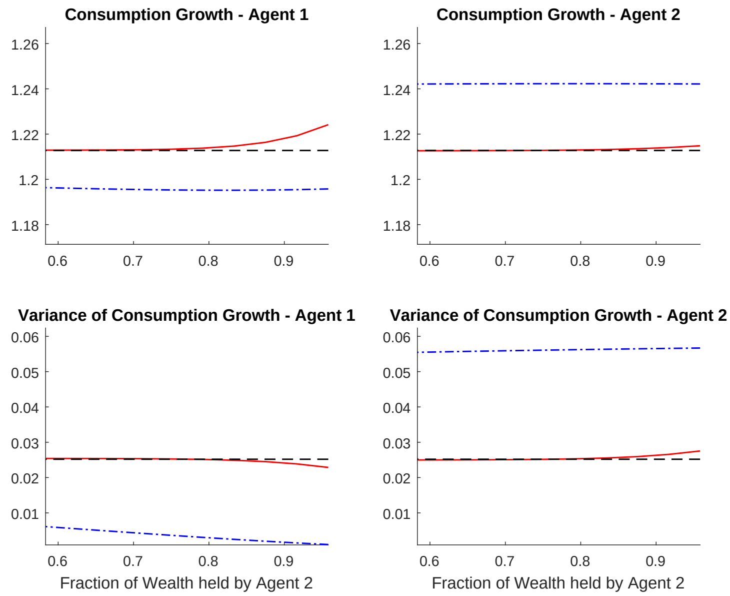

Figure 3: Consumption profiles.

– equity is less risky simply because it is unlevered. Recall also that leverage increases as type-2 agents' wealth increases, which contributes to widening the difference between excess returns in the two scenarios. Finally, corporate bond spreads increase, as the probability of default rises with leverage.

5.3 Specialization

Assume now that firms can choose between the production function introduced above and another, safer technology. For simplicity, let the alternative be entirely deterministic: $y = Ak^{\alpha}$, with $A > 0$. We show that there exist circumstances in which firms will find it optimal to randomize between the two technologies, resulting in a scenario where a non-degenerate fraction of them chooses the safe technology.

We remind the reader that this is possible in our framework, because the problem is not convex. With complete markets, it simply cannot happen.

Figure 5 indicates that specialization occurs only when the demand for hedging is high.

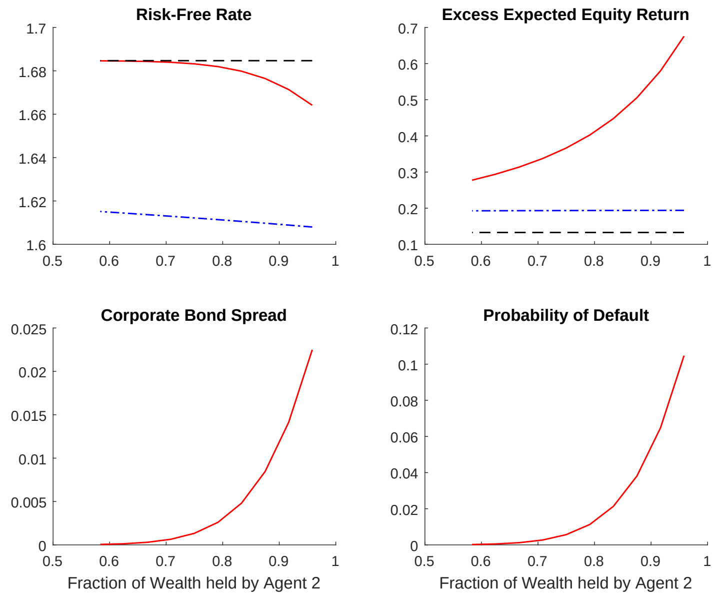

Figure 4: Financial Assets.

The red, solid line reproduces the equilibrium allocations obtained under incomplete markets in the previous section. When the fraction of wealth held by type–2 agents is relatively low, the economy under consideration here produces exactly the same allocation. Opting for the risky technology is the optimal choice for all firms.

As wealth is redistributed towards type–2 households, however, the equilibrium allocation features mixed strategies. Such allocations are illustrated by the blue dash-dot line. A non–zero fraction of firms find it optimal to cater to type–2 agents by providing them with risk-free asset. The top-right panel indicates that, as a result, type–2 agents decrease their holdings of equity.

Our simple exercise shows that under certain circumstances markets get more complete, as a new asset becomes available to investors. Figure **??** shows that in such scenarios risky firms shrink in size and value, and reduce their leverage.

Figure 7 illustrates the impact that the emergence of the new security has on asset returns. Hedging improves, thereby diminishing the precautionary motive. As result, the risk–free

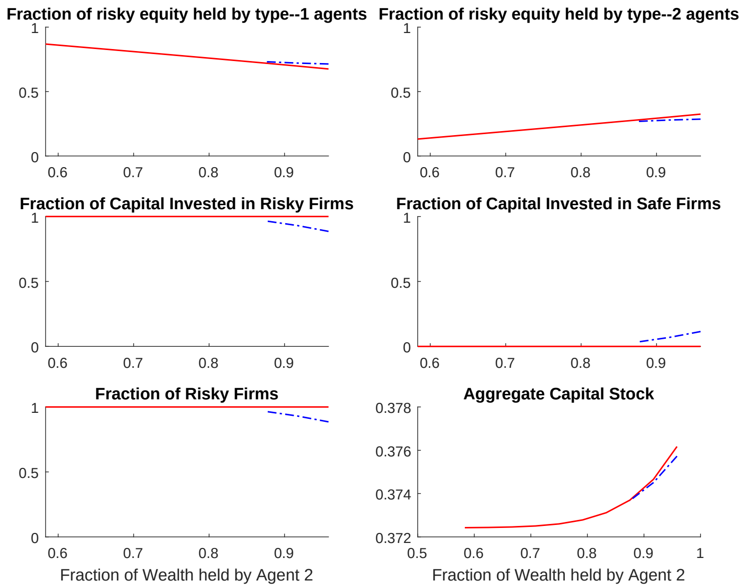

Figure 5: Specialization.

rate increases. Excess returns of risky equity, default probability, ad corporate bond spreads, all decline, since leverage is now lower.

6 Intermediation and short sales

In this section we introduce markets for derivatives on the firms' financial assets. This is important obviously because derivatives markets exist and we claim their modeling fits naturally into our set-up with interesting implications for financial economics. They are also a natural way to model short sales of the existing assets. Indeed in this section we focus our attention on the case where derivatives are simply given by short and long positions on the firms' equity.[35] A short position on a firm's equity is in fact, both conceptually and in the practice of financial markets, different from a simple negative holding of equity: it is a

[35]It should be clear that the analysis could be extended to short sales of the bond as well as other forms of derivatives, at only a notational cost.

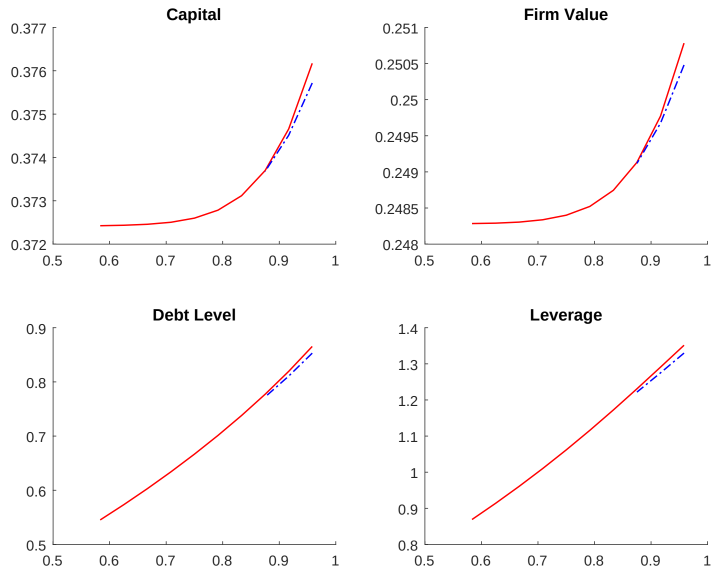

Figure 6: Specialization – The Risky Firm.

loan contract with a promise to repay an amount equal to the future value of equity. To model short sales it is then natural to introduce financial intermediaries who can issue claims corresponding to short positions on the firm's equity.

We assume that intermediation is subject to frictions, e.g. default or transaction costs. This ensures that the notion of competitive equilibrium is well-defined, even if such frictions are arbitrarily small (and hence short sales are "essentially unlimited").[36]

Consider an environment where intermediaries bear no cost to issue derivative claims, but

[36]The analysis of equilibria with intermediated short sales is also important from a theoretical standpoint. It is evident from our analysis in the previous sections that the unlimited short sales paradigm adopted by the GEI literature cited in the Introduction, while elegant and convenient, is incompatible with competitive equilibrium modeling in economies with production and incomplete markets. With infinite short sales, e.g., of equity, a *small* firm can in fact have a *large* effect on the economy by choosing a production plan with cash flows which, when traded as equity, change the asset span and hence the admissible trades of all consumers, allocations and equilibrium prices. In this section we show how not only limited but also "essentially unlimited" short sales can be consistently introduced in our competitive economy with production.

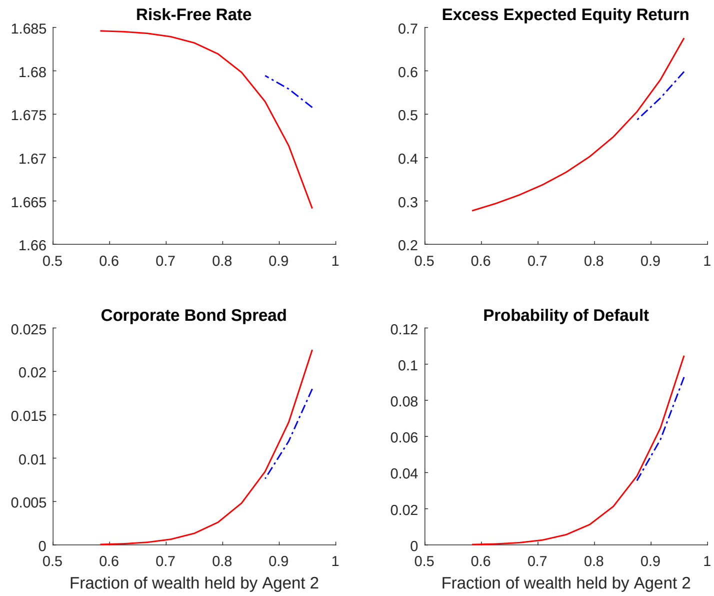

Figure 7: Specialization – Asset Returns.

face the possibility of default by consumers on the short positions they issue (e.g., on the loans induced by the sale of such positions to consumers), this is the friction on intermediation. Assume for simplicity that i) the default rate on the short positions issued is exogenously given and constant in every state, for all consumers;[37] ii) a solvency constraint is imposed on intermediaries' portfolio, to ensure they are never insolvent.

More specifically, an intermediary who is intermediating H units of the derivative on the firm's equity (that is, issuing H long and short positions) is repaid only a fraction $(1-\delta) \in (0,1)$ of the amount due on each short position issued.[38] To ensure its own solvency, the intermediary must hold an appropriate portfolio of claims, as a form of collateral, whose yield can cover the shortfalls in its revenue from the intermediation activity due to consumers' defaults. The best hedge against the risk of consumers' default on short positions on equity

[37] In Appendix A we show how the analysis and results extend to the general case where default rates are endogenously chosen by consumers.

[38] The analysis in this section holds for all $\delta > 0$, even arbitrarily small (hence even when the friction introduced is of negligible amount).

is clearly equity itself. The intermediary's solvency constraint then requires that it holds an amount γ of equity of the firm to ensure its ability to meet all its future obligations:

$$H \leq H(1-\delta)+\gamma, \tag{28}$$

To cover the cost of this collateral, intermediaries may charge a different price for long and short positions in the derivative issued. Let q^+ (resp. q^-) be the price at which long (resp. short) positions in the derivative issued by the intermediary are traded, while q is still the price at which equity trades in the market. The intermediary chooses then the amount H issued of long and short positions in the derivative and the amount γ of equity held as a hedge, so as to maximize its total revenue at date 0:

$$\max_{H,\gamma \in R^2_+} \left[(q^+ - q^-)H - q\gamma\right] \tag{29}$$

subject to the solvency constraint (28).

A solution to the intermediary's choice problem exists provided

$$q \geq \frac{q^+ - q^-}{\delta}; \tag{30}$$

and is characterized by $\gamma = \delta H$ and $H > 0$ only if $q = \frac{q^+ - q^-}{\delta}$.

Let $h^i_+ \in R_+$ denote consumer i's holdings of long positions in the derivative issued by intermediaries, and $h^i_- \in R_+$ his holdings of short positions. The consumer's choice problem consists in maximizing his expected utility subject to the budget constraints

$$c^i_0 \ = w^i_0 \ + V\theta^i_0 - q\ \theta^i \ - p\ b^i - q^+ h^i_+ + q^- h^i_- \tag{31}$$

$$c^i_1(s) = w^i_1(s) + R^e(s)(\theta^i + h^i_+ - h^i_-) + R^b(s) b^i \tag{32}$$

and $\left(\theta^i, b^i, h^i_+, h^i_-\right) \geq 0$.

The asset market clearing conditions are now, for equity

$$\gamma + \sum_{i \in I} \theta^i = 1,$$

and for the derivative security

$$\sum_{i \in I} h^i_+ = \sum_{i \in I} h^i_- = H.$$

Furthermore, the firm's choice problem is unchanged, still given - if we consider for simplicity the case where there is no agency friction, as in Section 2.1 - by (8). However, the condition specifying the criterion for *rational conjectures* for equity, $q(k,\phi,m,B)$, has to be properly adjusted to reflect the fact that now intermediaries may also demand equity in the market:

$$q(k,\phi,m,B) = \max \left\{ \begin{array}{c} \max_i \mathbb{E}\left[MRS^i(c^i(s))R^e(k,\phi,m,B;s)\right], \\ \frac{\max_i \mathbb{E}\left[MRS^i(c^i(s))R^e(k,\phi,m,B;s)\right] - \min_i \mathbb{E}\left[MRS^i(c^i(s))R^e(k,\phi,m,B;s)\right]}{\delta} \end{array} \right\} \tag{33}$$

for all k,ϕ,m,B.

The above expression states that the conjecture of a firm over the price of its equity when the firm chooses the plan k,ϕ,m,B equals the maximal valuation, at the margin, *among intermediaries as well as* consumers, of the equity's cash flow corresponding to k,ϕ,m,B. The second term on the right hand side of the above expression is in fact the intermediaries' marginal valuation for equity and can be interpreted as the *value of intermediation.* Since an appropriate amount of equity is needed, to be retained as collateral, in order to issue the corresponding derivative claims, the intermediary's willingness to pay for equity with yield $R^e(k,\phi,m,B;s)$ is determined by the consumers' marginal valuation for the corresponding derivative claims which can be issued[39]. Hence the above specification of the firms' equity price conjectures allows firms to take into account the effects of their decisions on the value of intermediation.

In all other respects, a competitive equilibrium of the economy with intermediation and short sales is defined along similar lines to Section 2.1. By a similar argument as in Propositions 1, 2 and 3 we can show that a competitive equilibrium of an economy with intermediated short sales exists; moreover, any equilibrium allocation is constrained Pareto efficient and shareholders unanimously support the production and financial decisions of the firms.

The model of intermediation proposed in this section is admittedly quite stylized. We believe however it allows to capture in a simple way the relationship between the financial claims issued by firms and the intermediation process. The key feature is that the derivatives issues by intermediaries are backed by the claims issued by firms in two ways. First, the yields of these derivatives are pegged to the yield of the claims issued by firms; second, the intermediaries must hold some amount of these claims to back the derivatives issued. Hence part of the demand fior the firms' claims now also comes from intermediaries (as such claims enter as some sort of input in the intermediation technology).

[39]More precisely, the first term on the numerator of the second expression in (33) equals the consumers' valuation for long positions in the derivative, the second one their valuation for short positions; dividing by δ yields the profits of intermediation, per unit of equity purchased.

It is interesting to compare this optimality result with Theorem 5 in Allen and Gale (1991), where it is shown that the competitive equilibria of an economy where consumers face a finite, exogenous bound $\bar{K}$ on short sales are constrained inefficient. In their set-up, long and short positions trade at the same price, i.e., the bid ask spread is zero. The inefficiency result in Allen and Gale (1991) then follows from the fact that firms maximize a conjecture over their market value which ignores the effect of their decisions on the value of intermediation. In other words, a firm does not take into account the possible gains arising from the demand for short positions in the firm's equity. In contrast in our economy, when a firm makes its production and financial decisions the firm considers the value of its equity not only for the consumers but also for the intermediaries who use equity as an input in the intermediation process. The gains from trade due to intermediation are so taken into account by firms.

It is also useful to contrast our findings with the inefficiency result in Pesendorfer (1995). Example 2 in Pesendorfer (1995) shows that a competitive economy where financial intermediaries can introduce complementary innovations in the market may get stuck at an equilibrium in which no intermediary innovates, even though welfare would be higher if all innovations were traded in the market.[40] The source of the inefficiency arising in the environment considered by Pesendorfer (1995) is analogous to the one of the result of Allen and Gale (1991) just discussed: each intermediary is implicitly restricted not to trade with other intermediaries. Equivalently, equilibrium prices for non-traded innovations do not include their effect on the value of intermediation. If instead prices for non-traded innovations were specified so as to equal the maximum between the consumers' and the intermediaries' marginal valuation, as in (33) above, constrained efficiency would obtain at equilibrium.

Finally, we can provide the following simple characterization of the intermediation levels at equilibrium, which follows from (30):

Proposition 5 (Intermediation) *In the economy with financial intermediation and short sales, at an equilibrium, either (i) $q = (q^+ - q^-)/\delta > q^+$ and intermediation is full (the whole amount of outstanding equity is purchased by intermediaries) or (ii) $q = q^+$ and intermediation is partial (some if not all the amount of outstanding equity is held by consumers).*

At an equilibrium where intermediation is full, equity sells at a premium over the long

[40] This finding is related to similar ones obtained in competitive equilibrium models with differentiated goods; notably Hart (1980) and Makowski (1980).

positions on the derivative claim issued by the intermediary, due to its additional value as input in the intermediation technology. Intermediaries in turn recoup the higher cost of equity through a sufficiently high spread $q^+ - q^-$ between the price of long and short positions on the derivative. When on the contrary intermediation is partial, equity and long positions in the derivative trade at the same price, intermediaries may not be active in equilibrium and the bid ask spread $q^+ - q^-$ is low (in particular, less or equal than δq).[41]

7 Conclusions

In this paper we have provided an equilibrium foundation to the study of corporate finance by showing how a consistent definition of competitive equilibria can be provided in environments with production, incomplete financial markets, and agency frictions. We have shown that, once firms are postulated to operate under rational conjectures, along the lines of Makowski (1983a,b), equilibria exist and have natural and appealing properties (in terms, e.g. of welfare and unanimity).

We have considered various classes of economies and examples to illustrate how the equilibrium concept we introduced allows to study simple finance and macroeconomic issues, from the firms' capital structure, to firms' specialization, corporate default, and financial intermediation.[42]

The next step, which we leave for future work, consists in adapting the equilibrium concept and extending the analysis to dynamic economies, e.g., Bewley economies, as the ones typically considered in macroeconomics and finance.

[41] Interestingly, we see from (33) that the same two situations arise for equity price conjectures.

[42] Acharya and Bisin (2013) extend the analysis of this equilibrium concept to a class of financial intermediation economies with strategic default to capture counterparty risk.

References

Acharya, V. V. and A. Bisin (2009): "Managerial hedging, equity ownership, and firm value," *RAND Journal of Economics*, 40 (1), 47-77.

Allen, F. and D. Gale (1988): "Optimal security design," *Review of Financial Studies*, 1 (3), 229-263.

Allen, F. and D. Gale (1991): "Arbitrage, short sales, and financial innovation," *Econometrica*, 59 (4), 1041-68.

Bisin, A. and P. Gottardi (1999): "Competitive equilibria with asymmetric information," *Journal of Economic Theory*, 87 (1), 1-48.

Bisin, A. and P. Gottardi (2006): "Efficient competitive equilibria with adverse selection," *Journal of Political Economy*, 114 (3), 485-516.

Bisin, A., P. Gottardi, and A. A. Rampini (2008): "Managerial hedging and portfolio monitoring," *Journal of the European Economic Association*, 6 (1), 158-209.

Bisin, A. and P. Gottardi (2014): "Production in Bewley models," NYU, mimeo.

Bonnisseau, J. M. and O. Lachiri (2004): "Dreze's criterion in a multi-period economy with stock markets," *Royal Economic Society Annual Conference 2004*, 88, Royal Economic Society.

Boyarchenko, S. (2004): "Arrow's equivalency theorem in a model with neoclassical firms," *Economic Theory*, 23 (4), 739-775.

Carceles-Poveda, E. and D. Coen-Pirani (2009): "Shareholders' unanimity with incomplete markets," *International Economic Review*, 50 (2), 577-606.

Cres, H. and M. Tvede (2005): "Voting in assemblies of shareholders and incomplete markets," *Economic Theory*, 26 (4), 887-906.

DeMarzo, P. M. (1993): "Majority voting and corporate control: The rule of the dominant shareholder," *Review of Economic Studies* 60 (3), 713-34.

Diamond, P. A. (1967): "The role of a stock market in a general equilibrium model with technological uncertainty," *American Economic Review*, 57 (4), 759-776.

Dierker, E., H. Dierker, and B. Grodal (2002): "Nonexistence of constrained efficient equilibria when markets are incomplete," *Econometrica* 70 (3), 1245-1251.

Dow, J., G. Gorton, and A. Krishnamurthy (2005): "Equilibrium investment and asset prices under imperfect corporate control," *American Economic Review*, 95 (3), 659-681.

Dreze, J., E. Minelli, and M. Tirelli (2008): "Production and financial policies under asymmetric information," *Economic Theory* 35 (2), 217-231.

Dreze, J. H. (1974): "Investment under private ownership: Optimality, equilibrium and stability," in J. H. Dreze (ed.), *Allocation Under Uncertainty: Equilibrium and Optimality*, New York, Macmillan.

Dreze, J. H. (1985): "(Uncertainty and) the firm in general equilibrium theory," *Economic Journal*, 95 (380a), 1-20.

Dreze, J. H., O. Lachiri, and E. Minelli (2007): "Shareholder-efficient production plans in multi-period economy," Discussion Papers 2007-042, Catholic University of Louvain.

Dubey, P., J. Geanakoplos, and M. Shubik (2005): "Default and punishment in general equilibrium, *Econometrica*, 73 (1), 1-37.

Duffie, D. and W. Shafer (1986): "Equilibrium and the role of the firm in incomplete markets," Graduate School of Business, Stanford University, mimeo.

Geanakoplos, J. (1990): "An introduction to general equilibrium with incomplete asset mar- kets," *Journal of Mathematical Economics*, 19 (1-2), 1-38.

Geanakoplos, J. (2004): "Liquidity, default and crashes: Endogenous contracts in general equilibrium," Econometric Society 2004 North American Summer Meetings, Econometric Society.

Grossman, S. J. and O. D. Hart (1979): "A theory of competitive equilibrium in stock market economies," *Econometrica*, 47 (2), 293-329.

Hart, O. D. (1979): "On shareholder unanimity in large stock market economies," *Econometrica* 47 (5), 1057-83.

Hart, O. D. (1980): "Perfect competition and optimal product differentiation," *Journal of Economic Theory*, 22 (2), 279-312.

Heathcote, J., K. Storesletten, and G. L. Violante (2009): "Quantitative macroeconomics with heterogeneous households," *Annual Review of Economics*, 1 (1), 319-354.

Jensen, M.C. (1986): "Agency costs of free cash flow, corporate finance and takeovers", *American Economic Review,* 76 (2), 323-329.

Jensen, M. C. and W. H. Meckling (1976): "Theory of the firm: Managerial behavior, agency costs and ownership structure," *Journal of Financial Economics*, 3 (4), 305-360.

Jerez, B. (2005): "Incentive compatibility and pricing under moral hazard," *Review of Economic Dynamics*, 8 (1), 28-47.

Kelsey, D. and F. Milne (1996): "The existence of equilibrium in incomplete markets and the objective function of the firm," *Journal of Mathematical Economics*, 25 (2), 229-245.

Kihlstrom, R. E. and S. A. Matthews (1990): "Managerial incentives in an entrepreneurial stock market model," *Journal of Financial Intermediation*, 1 (1), 57-79.

Magill, M. and M. Quinzii (1998): "Incentives and risk sharing in a stock market equilibrium," in A. Alkan and N. Yannelis (eds.), *Studies in Economic Theory, Volume 8, Current Trends in Economics: Theory and Applications*, Berlin, Springer Verlag.

Magill, M. and M. Quinzii (2002): "Capital market equilibrium with moral hazard," *Journal of Mathematical Economics*, 38 (1-2), 149-190.

Magill, M. and W. Shafer (1991): "Incomplete markets," in W. Hildenbrand and H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, Volume 4, Chapter 30, pp. 1523-1614, Amsterdam, Elsevier.

Makowski, L. (1980): "A characterization of perfectly competitive economies with production," *Journal of Economic Theory*, 22 (2), 208-221.

Makowski, L. (1983a): "Competition and unanimity revisited," *American Economic Review*, 73 (3), 329-39.

Makowski, L. (1983b): "Competitive stock markets," *Review of Economic Studies*, 50 (2), 305-30.

Ostroy, J. M. (1980): "The no-surplus condition as a characterization of perfectly competitive equilibrium," *Journal of Economic Theory*, 22 (2), 183-207.

Ostroy, J. M. (1984): "A reformulation of the marginal productivity theory of distribution," *Econometrica*, 52 (3), 599-630.

Parlour, C. A. and J. Walden (2011): "General equilibrium returns to human and investment capital under moral hazard," *Review of Economic Studies*, 78 (1), 394-428.

Pesendorfer, W. (1995): "Financial innovation in a general equilibrium model," *Journal of Economic Theory*, 65 (1), 79-116.

Prescott, E. C. and R. M. Townsend (1984): "Pareto optima and competitive equilibria with adverse selection and moral hazard," *Econometrica*, 52 (1), 21-45.

Prescott, E. S. and R. M. Townsend (2006): "Firms as clubs in Walrasian markets with private information," *Journal of Political Economy*, 114 (4), 644-671.

Stiglitz, J. E. (1969): "A re-examination of the Modigliani-Miller theorem," *American Economic Review*, 59 (5), 784-93.

Tirole, J. (2006): *The Theory of Corporate Finance*, Princeton, Princeton University Press.

Zame, W. R. (2007): "Incentives, contracts, and markets: A general equilibrium theory of firms," *Econometrica*, 75 (5), 1453-1500.

Appendix

Proof of Proposition 1

We only provide here an outline of the main steps. Since the firms' choice problem is non convex, we allow for the possibility that firms undertake different production and financial plans in equilibrium. By Caratheodory's Theorem, given the finite dimensionality of the sets where these variables lie, it is enough to consider the case where firms make at most a finite number N of different choices k^n, ϕ^n, m^n, B^n. As a consequence, we extend the consumers' budget constraints (3)-(5) to allow for the possibility that they trade N different types of equity and bonds, with prices q^n, p^n and returns $R^{e,n}(s), R^{b,n}(s)$. Since short sales are not allowed, the consumers' budget set is non empty, compact and convex for all $p^n, q^n \gg 0$, all $R^{e,n}(s)$,$R^{b,n}(s) \geq 0$ and all $V^n \geq 0$, for $n = 1, .., N$. Under the assumptions made on individual preferences, consumers' net demand (for the consumption good and the different types of bonds and equity) are then well behaved, continuous functions.

Let us turn then our attention to the firms' problem (16). Whenever the first order approach is not satisfied and the map $\phi(k, m, B; c(s))$ is not single-valued and continuous, it is convenient to write the implementability constraint (10) in terms of the inverse map:

$$k, m, B \in \phi^{-1}(\phi; c(s)).$$

We also impose here the following regularity condition, requiring that the above inverse map can be described by a set of functions

$$k, m, B \in \phi^{-1}(\phi; c(s)) \Leftrightarrow G(k, m, B; c(s), \phi) \leq 0, \tag{34}$$

with $G(.)$ assumed to be continuous in $k, m, B, c(s)$ for all $\phi \in \Phi$. Note that this condition is satisfied in natural environments, as for instance in the case of (11) and (12).

Let us partition the set $N \equiv \{1, .., N\}$ into equal-sized subsets $N(\phi)$ for each $\phi \in \Phi$. The firms' choice problem can then be rewritten as

$$\max_{(k^n,m^n,B^n,\gamma^n)_{n\in N(\phi),\phi\in\Phi}} \left[\begin{array}{c} \sum_{\phi\in\Phi} \sum_{n\in N(\phi)} \gamma^n \left(-k^n + \mathbb{E}\left[\max_i MRS^i(c^i(s))R^e(k^n, \phi, m^n, B^n; s)\right]\right. \\ \left. +\mathbb{E}\left[\max_i MRS^i(c^i(s))R^b(k^n, \phi, m^n, B^n; s)\right] B^n\right) \end{array} \right]$$
$$s.t. \left\{ \begin{array}{c} \gamma \in \Delta^{N-1} \\ G(k^n, m^n, B^n; c(s), \phi) \leq 0 \text{ for all } n \in N(\phi) \text{ and all } \phi \end{array} \right. \tag{35}$$

where $\gamma \in \Delta^{N-1}$ can be equivalently interpreted as the fraction of firms choosing each of the N plans, or the probability weights of the lottery over production and financial plans describing the choice of each firm[43]. In the above expression of the firms' problem we have also used condition M) to substitute for the equity and bond price conjectures and used (34) to rewrite the incentive constraint (10).

The objective function and the constraints of the firms' problem (35) are continuous w.r.t. $(k^n, m^n, B^n, \gamma^n)_{n\in N(\phi),\phi\in\Phi}$ and $c(s)$. Since the sets $K, M, \mathcal{B}$ are compact, the correspondence describing the solution of the firm's problem (35) above is then non empty and upper hemicontinuous, for all $c_0^i \in (0, \max\{\sum_i w_0^i\}]$, $c_1^i(s) \in (0, \max \sum_i w^i(s)]$.

By a standard fixed point argument there exists so a value $\bar{\phi}^n, \bar{k}^n, \bar{m}^n, \bar{B}^n, \bar{p}^n, \bar{q}^n, \bar{\gamma}^n, \bar{R}^{e,n}(s)$, $\bar{R}^{b,n}(s)$ for $n = 1,..,N$ and $\bar{c}(s)$ such that: (a) $\bar{k}^n, \bar{m}^n, \bar{B}^n, \bar{\gamma}^n$ for $n = 1,..,N$ solve the firms' optimal choice problem (35) when the terms MRS^i appearing in the equity and bond price conjecture maps above are evaluated at $\bar{c}(s)$, and $n \in N(\phi)$ implies $\bar{\phi}^n = \phi$, (b) for each $i = 1,..,I$, $\bar{c}^i(s)$ is a solution of the choice problem of type i consumers at prices and returns $\bar{p}^n, \bar{q}^n, \bar{V}^n, \bar{R}^{e,n}(s), \bar{R}^{b,n}(s)$, $n = 1,..,N$, satisfying the consistency condition C), (c) the market clearing conditions hold (for each type n of equity and bonds, the supply $\bar{\gamma}^n$ equals consumers' demand).■

Proof of Proposition 2

Suppose $\hat{c}(s)$ is admissible and Pareto dominates the competitive equilibrium allocation $\bar{c}(s)$. By the definition of admissibility a collection $\hat{k}, \hat{m}, \hat{\phi}, \hat{B}$ and $\left(\hat{\theta}^i, \hat{b}^i\right)_{i=1}^{I}$ exists such that $\hat{c}(s)$ satisfies (17), (18) and (19). The equilibrium consumption level $\bar{c}^i(s)$ is the optimal choice of a type i consumer at the equilibrium prices $\bar{q}, \bar{p}$ and returns $\bar{R}^e(s) = R^e(\bar{k}, \bar{\phi}, \bar{m}, \bar{B}; s)$, $\bar{R}^b(s) = R^b(\bar{k}, \bar{\phi}, \bar{m}, \bar{B}; s)$. As argued in Section 2.4, the consumer's choice problem is analogous to one where any possible type of equity and bonds are available for trade, at the prices $q(k, \phi, B, m)$, $p(k, \phi, B, m)$ satisfying the Makowski criterion M) with $\phi \in \phi(k, m, B; \bar{c}(s))$. When the map $\phi(.)$ only depends on k, m, B, we have $\hat{\phi} \in \phi(\hat{k}, \hat{m}, \hat{B})$ and so we get:

$$\hat{c}_0^i + \hat{q}\hat{\theta}^i \ + \hat{p}\ \hat{b}^i \geq \bar{c}_0^i + \bar{q}\ \bar{\theta}^i + \bar{p}\ \bar{b}^i\ ,$$

where $\hat{q} = q(\hat{k}, \hat{\phi}, \hat{m}, \hat{B})$, $\hat{p} = p(\hat{k}, \hat{\phi}, \hat{m}, \hat{B})$. Or, equivalently,

$$\left[-\hat{k}\ + \hat{q}\ + \hat{p}\ \hat{B}\ \right] \theta_0^i + \tau^i \geq \left[-\bar{k}\ + \bar{q}\ + \bar{p}\ \bar{B}\ \right] \theta_0^i, \tag{36}$$

[43] With the realizations of the lottery observed by consumers when choosing their portfolios.

for $\tau^i \equiv \hat{c}_0^i + \hat{q}\hat{\theta}^i \quad + \hat{p}\ \hat{b}^i - \left[-\hat{k}\ \ + \hat{q}\ \ + \hat{p}\ \hat{B}\ \right]\theta_0^i$. Since (36) holds for all i, strictly for some i, summing over i yields:

$$\left[-\hat{k}\ \ + \hat{q}\ \ + \hat{p}\ \hat{B}\ \right] + \sum_i \tau^i > \left[-\bar{k}\ \ + \bar{q}\ \ + \bar{p}\ \bar{B}\ \right] \tag{37}$$

The fact that $\bar{k}, \bar{m}, \bar{B}$ solves the firms' optimization problem (8) in turn implies that:

$$-\bar{k}\ \ + \bar{q}\ \ + \bar{p}\ \bar{B} \geq -\hat{k}\ \ + \hat{q}\ \ + \bar{p}\ \hat{B},$$

which, together with (37), yields:

$$\sum_i \tau^i > 0,$$

or equivalently:

$$\sum_i \hat{c}_0^i + \hat{k} > \sum_i w_0^i,$$

a contradiction to (17) at date 0. ■

Proof of Proposition 3

Note that we can always consider a situation where, in equilibrium, each consumer holds at most a negligible amount of equity of any individual firm and so the effects on a consumer's utility of alternative choices by a firm can then be evaluated using the consumer's marginal utility. Let $c(s)$ be the equilibrium consumption allocation. For any possible choice k', ϕ', m', B' by a firm, with $\phi' \in \phi(k', m', B'; c(s))$, the (marginal) utility of a type j consumer if he holds the firm's equity and debt is

$$-k' - W(k', \phi', m', B') + \mathbb{E}\left[MRS^j(c^j(s))R^e(k', \phi', m', B'; s)\right] + \mathbb{E}\left[MRS^j(c^j(s))R^b(k', \phi', m', B'; s)\right] B',$$

But this is always lower or equal than the agent's utility if instead he sells the firm's equity and bonds at the market price, evaluated on the basis of price conjectures satisfying M),

$$\begin{aligned} &-k' - W(k', \phi', m', B') + \max_i \mathbb{E}\left[MRS^i(c^i(s))R^e(k', \phi', m', B'; s)\right] \\ &+ \max_i \mathbb{E}\left[MRS^i(c^i(s))R^b(k', \phi', m', B'; s)\right] B', \end{aligned}$$

and the latter is in turn lower than the corresponding expression if the firm adopts the equilibrium choice k, ϕ, m, B, since this choice solves problem (16). ■

Further details of the proof of Proposition 4

When (21) holds as equality only for consumer $i = 2$ we have $c_1^2(s) = w_1^2(s) + a_1(s)k^\alpha > c_1^1(s) = w_1^1(s)$, $c_0^2 = w_0 + V0.5 - q < c_0^1 = w_0 + V0.5$. For simplicity we assume here that the following symmetry condition also holds: $\mathbb{E}\left[MRS(c^2(s))a_1(s)k^\alpha\right] = \mathbb{E}\left[MRS(\hat{c}^1(s))a_2(s)k^\alpha\right]$, for $\widehat{c}_0^1 = w_0 + V0.5 - q$, $\widehat{c}_1^1(s) = w_1^1(s) + a_2(s)k^\alpha$ for all $k, q, V > 0$.

For $\phi = 0$ to be an optimal choice for the firms, we must have in this case:

$$q = \mathbb{E}\left[MRS(c^2(s))a_1(s)k^\alpha\right] \geq \mathbb{E}\left[MRS(c^1(s))a_2(s)k^\alpha\right]$$

which contradicts the assumed symmetry condition, since

$$\mathbb{E}\left[MRS(c^1(s))a_2(s)k^\alpha\right] > \mathbb{E}\left[MRS(\hat{c}^1(s))a_2(s)k^\alpha\right].$$

Consider next the case where $w_1^1(s) + a_2(s)k^\alpha$ and $w_1^2(s) + a_2(s)k^\alpha$ varies comonotonically with $a_1(s)$ for all $k \in K$ (a slightly stronger condition than the comononicity of $w_1^1(s)$, $w_1^2(s)$ and $a_1(s)$). In this case we have

$$\mathbb{E}\left[MRS(c(s))a_2(s)k^\alpha\right] > \mathbb{E}\left[MRS(c(s))a_1(s)k^\alpha\right]$$

for all $k \in K, c_0$ and $c_1(s) = w_1^i(s) + \theta a_2(s)k^\alpha$, $i = 1, 2$, $\theta \in [0, 1]$, since $Cov(MRS(c(s)), a_2(s)) > 0 > Cov(MRS(c(s)), a_1(s))$. Hence in equilibrium both consumers' types are only willing to buy equity of firms with full loading on factor $a_2(s)$.

Details on the Dierker, Dierker, and Grodal (2002) example

There are two types of consumers, with type 2 having twice the mass of type 1, and (non expected utility) preferences, respectively, $u^1(c_0^1, c_1^1(s_1), c_1^1(s_2)) = c_1^1(s_1)/\left(1 - (c_0^1)^{\frac{9}{10}}\right)^{\frac{10}{9}}$ and $u^2(c_0^2, c_1^2(s_1), c_1^2(s_2)) = c_0^2 + (c_1^2(s_2))^{1/2}$, endowments $w_0^1 = .95$, $w_0^2 = 1$ and $w_1^1(s) = w_1^2(s) = 0$ for all $s \in \mathcal{S}$.

In this economy Dierker, Dierker and Grodal (2002) find a unique, symmetric Dreze equilibrium where all firms choose the same value of k and $\phi \approx 0.7$[44] and this equilibrium is constrained inefficient. We show next that a symmetric competitive equilibrium, according to

[44]The notion of Dreze criterion used by Dierker, Dierker, and Grodal to specify price conjectures differs from the Makowski criterion M) in two main respects: i) only the MRS of the consumers who in equilibrium are shareholders of the firms are considered to evaluate alternative production plans, and ii) these MRS are not constant but vary to take into account the effect of each plan on the agents' consumption.

our definition in Section 2.3, does not exist. Given the agents' endowments and preferences, both types of consumers buy equity in equilibrium. It is then easy to see that the firms' optimality condition with respect to ϕ can never hold for an interior value of ϕ nor for a corner solution.[45] On the other hand, an asymmetric equilibrium exists, where a fraction 1/3 of the firms choose $\phi^1 = 0.99$ and $k^1 = 0.3513$ and the remaining fraction chooses $\phi^2 = 2/3$ and $k^2 = 0.1667$, type 1 consumers hold only equity of the firms choosing ϕ^1, k^1 and type 2 consumers only equity of the other firms. At this allocation, we have $\frac{\partial u^1/\partial c_1^1(s_1)}{\partial u^1/\partial c_0^1} = 1.0101$, $\frac{\partial u^2/\partial c_1^2(s_2)}{\partial u^2/\partial c_0^2} = 3$. Also, the marginal valuation of type 1 agents for the equity of firms choosing ϕ^2, k^2 is 0.1122, thus smaller than the market value of these firms' equity, equal to 0.1667, while the marginal valuation of type 2 agents for the equity of the firms choosing ϕ^1, k^1 is 0.0105, smaller than the market value of these firms' equity, equal to 0.3513. Therefore, at these values the firms' optimality conditions are satisfied. It can then be easily verified that this constitutes a competitive equilibrium according to our definition and that the equilibrium allocation is constrained optimal.

A Parametric Example

Consumers have identifical preferences described by $\mathbb{E}u(c_0, c_1(s)) = u(c_0) + \mathbb{E}u(c_1(s))$, with $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, for $\gamma = 2$. The state space is $\mathcal{S} = \{1, 2, 3\}$ with $\pi(1) = \pi(2) = \pi(3) = \frac{1}{3}$. The production technology is as in (13), with $\alpha = .75$ and productivity shocks $a_1(s)$ and $a_2(s)$ taking values, respectively, $\{1, 2, 3\}$ and $\{1.1, 2, 2.9\}$. The second period endowments of type 1 and type 2 agents take values, respectively, $\{1, 2, 3\}$ and $\{1.1, 2, 2.9\}$, while in the first period they are endowed with $w_0^i = w_1^i(2)$, $i = 1, 2$, units of the good and the same amount $\theta_0 = .5$ units of equity. Also, the utility cost of different choices of ϕ is $v^i(1) = -.006$ and $v^i(0) = 0$, for all i.

The equilibrium values with and without the agency friction are reported in the following table.

In order to implement the same choice $\phi = 0$ the firm modifies its production and financial decisions together with the portfolio of the agent selected as manager (in particular, the manager's compensation exhibits a higher amount of equity, (.6456), a lower one of debt (0)

[45] Consider for instance $\phi = 0.99$. To have an equilibrium at this value the marginal valuation of equity for both consumers must be the same at $\phi = 0.99$, and higher than at any other values of ϕ, but this second property clearly cannot hold for type 2 consumers.

	Without agency friction	With agency frictions
ϕ	0	0
k	.4888	.4896
B	[.1828,.6431]	.2160
θ^1	.3877	.3544
b^1	[.1828,.3613]	.2160
q	[.5108,.1559]	.4870
p	.7712	.7689
$-k+q+pB-W$	.1629	.1633
U^1	-1.0372	-1.0371
U^2	-1.0217	-1.0219

Table 1: Equilibrium values with and without moral hazard.

and also a lower consumption at date 0).

Appendix A: Additional material

Characterization of the firms' optimal capital structure conditions

Let I^e (resp. I^d) denote the set of shareholders (resp. bondholders) of a firm and consider for simplicity the case where capital is the only input, that is the technology is given by $f(k,s)$.

Proposition A. 1 *If the optimal production and financing decisions of a firm are obtained* [46] *at a level B such that bonds are risk free, that is, $f(k;s) \geq B$ with probability 1, then all equity holders are also bond holders (while the reverse may not be true: $I^e \subseteq I^d$):*

$$\max_{i\in I^e} \mathbb{E}MRS^i(c^i(s)) = \min_{i\in I^e} \mathbb{E}MRS^i(c^i(s)) = p = \max_i \mathbb{E}MRS^i(c^i(s)) \tag{38}$$

and

$$\max_{i\in I^e} \mathbb{E}\left[MRS^i(c^i(s))f_k(s)\right] = \min_{i\in I^e} \mathbb{E}\left[MRS^i(c^i(s))f_k(s)\right] = 1; \tag{39}$$

In the situation described above all shareholders value equally the effect on the payoff of equity of an infinitesimal increase in the investment level k, and such value is always equal to the marginal cost of the investment.

Proof of Proposition A. 1 *Note first that*

$$q(k, B+dB) = \max_i \mathbb{E}MRS^i(c^i(s))\ \ [f(k;s) - B - dB].$$

Since for all $i \notin I^e$, $\mathbb{E}MRS^i(c^i(s))\ [f(k;s) - B] < q(k,B)$, the max in the above expression is attained for some $i \in I^e$ and hence

$$q(k, B+dB) = q(k,B) + \max_{i\in I^e} \mathbb{E}MRS^i(c^i(s))\ \ [-dB].$$

The right and left derivative of $q(k,B)$ with respect to B are then given by:

$$\frac{\partial q}{\partial B_+} = -\min_{i\in I^e} \mathbb{E}MRS^i(c^i(s))\ ;\quad \frac{\partial q}{\partial B_-} = -\max_{i\in I^e} \mathbb{E}MRS^i(c^i(s)) \tag{40}$$

and may differ. Similarly the derivatives with respect to k are:

$$\frac{\partial q}{\partial k_+} = \max_{i\in I^e} \mathbb{E}\left[MRS^i(c^i(s))\ f_k(s)\right];\ \frac{\partial q}{\partial k_-} = \min_{i\in I^e} \mathbb{E}\left[MRS^i(c^i(s))\ f_k(s)\right] \tag{41}$$

[46] We focus here on the conditions concerning the investment level k and capital structure B, ignoring those regarding ϕ, which are straightforward.

where f_k denotes the derivative of f with respect to k.

The first order conditions when $f(k, \phi, m; s) \geq B$ with probability 1 are:

$$\begin{aligned} \frac{\partial V}{\partial B}_+ &= \frac{\partial q}{\partial B}_+ + p \leq 0, \ \frac{\partial V}{\partial k}_+ = \frac{\partial q}{\partial k}_+ - 1 \leq 0, \\ \frac{\partial V}{\partial B}_- &= \frac{\partial q}{\partial B}_- + p \geq 0, \ \frac{\partial V}{\partial k}_- = \frac{\partial q}{\partial k}_- - 1 \geq 0; \end{aligned} \tag{42}$$

Since (40) implies that $\frac{\partial q}{\partial B}_+ \geq \frac{\partial q}{\partial B}_-$, the above conditions (with respect to B) are equivalent to:

$$\frac{\partial V}{\partial B}_+ = \frac{\partial q}{\partial B}_+ + p = \frac{\partial V}{\partial B}_- = \frac{\partial q}{\partial B}_- + p = 0,$$

that is:

$$\max_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) \ = \min_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) \ = p = \max_i \mathbb{E} MRS^i(c^i(s))$$

or (38) holds. Similarly, from (41) we see that $\frac{\partial q}{\partial k}_+ \geq \frac{\partial q}{\partial k}_-$, the above conditions (with respect to k) are equivalent to:

$$\frac{\partial q}{\partial k}_+ - 1 = \frac{\partial q}{\partial k}_- - 1 = 0,$$

that is,

$$\max_{i \in I^e} \mathbb{E}\left[MRS^i(c^i(s)) \ f_k(s)\right] = \min_{i \in I^e} \mathbb{E}\left[MRS^i(c^i(s)) \ f_k(s)\right] = 1$$

or (39) holds, thus completing the proof of the proposition. ■

We study next the case where firms can default on their debt obligations, hence corporate debt is risky. Before stating the conditions for an optimum of the firms' decision problem in the presence of risky debt, it is useful to introduce some further notation. Given a face value of debt equal to B, let S^{nd} denote the collection of states in $t = 1$ for which $f(k; s) \geq B$ and by $\underline{s}^{nd}$ the lowest state in S^{nd}, that is the state with the lowest realization of the technology shock for which the firm does not default. Conversely, denote S^d the collection of states in $t = 1$ for which $f(k; s) < B$, i.e. the firm (partially) defaults on its debt.

Proposition A. 2 *If the optimal production and financing decisions of a firm are obtained at a level B such that bonds are risk free, the optimal investment and debt levels obtain either* *at an interior solution, where* $f(k;\underline{s}^{nd}) > B$*, with:*

$$p = \min_{i\in I^d} \mathbb{E}\left(MRS^i(c^i(s)) \left[\frac{f(k;s)}{B}\right] \mid s\in S^d \right) \Pr\{s\in S^d\} +$$

$$\min_{i\in I^e} \mathbb{E}(MRS^i(c^i(s)) \big| s\in S^{nd}) \Pr\{s\in S^{nd}\} = \tag{43}$$

$$= \max_{i\in I^d} \mathbb{E}\left(MRS^i(c^i(s)) \left[\frac{f(k;s)}{B}\right] \mid s\in S^d \right) \Pr\{s\in S^d\} +$$

$$\max_{i\in I^e} \mathbb{E}\left(MRS^i(c^i(s)) \big| s\in S^{nd}\right) \Pr\{s\in S^{nd}\}.$$

and

$$1 = \max_{i\in I^e} \mathbb{E}\left\{ MRS^i(c^i(s)) f_k(k,s) \big| s\in S^{nd}\right\} \Pr\{s\in S^{nd}\} + \tag{44}$$

$$\max_{i\in I^d} \mathbb{E}(MRS^i(c^i(s)) f_k(k;s) \big| s\in S^d) \Pr\{s\in S^d\}$$

$$= \min_{i\in I^e} \mathbb{E}\left\{ MRS^i(c^i(s_1)) f_k(k,s) \big| s\in S^{nd}\right\} \Pr\{s\in S^{nd}\} +$$

$$\min_{i\in I^d} \mathbb{E}(MRS^i(c^i(s)) f_k(k;s) \big| s\in S^d) \Pr\{s\in S^d\}$$

or at a corner solution, $f(k;\underline{s}^{nd}) = B$*.*

Proof of Proposition A. 2 *We first proceed to characterize the conditions for corner solutions.*

Claim 1 *The conditions for an optimum at a corner,* $f(k;\underline{s}_1^{nd}) = B$*, are:*

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) \big| s_1\in S^{nd\prime}\right\} \Pr\{s_1\in S^{nd\prime}\} + \tag{45}$$

$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right]\bigg| s_1\in S^{d\prime}) \Pr\{s_1\in S^{d\prime}\} \geq p \geq$$

$$\geq \max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) \big| s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\} +$$

$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right]\bigg| s_1\in S^{d}) \Pr\{s_1\in S^{d}\}$$

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1) \big| s_1\in S^{nd\prime}\right\} \Pr\{s_1\in S^{nd\prime}\} + \tag{46}$$

$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B}\bigg| s_1\in S^{d\prime}) \Pr\{s_1\in S^{d\prime}\} \geq 1 \geq$$

$$\geq \max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1) \big| s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\} +$$

$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B}\bigg| s_1\in S^{d}) \Pr\{s_1\in S^{d}\}$$

$$1-\max_{i\in I^e} E_{s_0}\left\{ MRS^i(s_1) f_k(k,s_1)\right| s_1\in S^{nd}\left.\right\} \Pr\{s_1\in S^{nd}\} - \quad (47)$$
$$\max_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B}\right| s_1\in S^d\left.\right) \Pr\{s_1\in S^d\} =$$
$$\left[-\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1\in S^{nd}\left.\right\} \Pr\{s_1\in S^{nd}\}\right.$$
$$\left. -\min_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right]\right| s_1\in S^d\left.\right) \Pr\{s_1\in S^d\} + p\right] f_k(\underline{s}_1^{nd}) =$$
$$\left[-\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1\in S^{nd}\left.\right\} \Pr\{s_1\in S^{nd}\}\right.$$
$$\left. -\max_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right]\right| s_1\in S^d\left.\right) \Pr\{s_1\in S^d\} + p\right] f_k(\underline{s}_1^{nd}) =$$
$$1-\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1)\right| s_1\in S^{nd}\left.\right\} \Pr\{s_1\in S^{nd}\}$$
$$-\min_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B}\right| s_1\in S^d\left.\right) \Pr\{s_1\in S^d\}$$

Proof of Claim 1 *Note first that, in this case, $f(k;\underline{s}_1^{nd}) = B$. Denote by $S_1^{nd\prime} \subset S_1^{nd}$ the collection of states in $t=1$ for which the firm does not default, after marginal deviations $dB>0$ and/or $dk<0$ (and similarly $S^{d\prime} \supset S^d$). Evidently, for marginal deviations $dB>0$ and/or $dk<0$ the collection of such states is still given by S_1^{nd}.*

The partials of the price maps wrt to B are[47]

$$\frac{\partial q}{\partial B_+} = -\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1\in S^{nd\prime}\left.\right\} \Pr\{s_1\in S^{nd\prime}\}$$
$$\frac{\partial q}{\partial B_-} = -\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1\in S^{nd}\left.\right\} \Pr\{s_1\in S^{nd}\}$$

and

$$\frac{\partial p}{\partial B_+} = -\min_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B^2}\right]\right| s_1\in S^{d\prime}\left.\right) \Pr\{s_1\in S^{d\prime}\}$$
$$\frac{\partial p}{\partial B_-} = -\max_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B^2}\right]\right| s_1\in S^{d}\left.\right) \Pr\{s_1\in S^{d}\}$$

Analogously, the partials wrt to k are[48]

$$\frac{\partial q}{\partial k_+} = \max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1)\right| s_1\in S^{nd}\left.\right\} \Pr\{s_1\in S^{nd}\}$$
$$\frac{\partial q}{\partial k_-} = \min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1)\right| s_1\in S^{nd\prime}\left.\right\} \Pr\{s_1\in S^{nd\prime}\}$$

[47] Obviously, if S^{nd} is a singleton, the right derivative is equal to 0.

[48] Obviously, if $S^{nd}=\{s_1\}$ - is a singleton - the left derivative is equal to 0.

and

$$
\begin{aligned}
\frac{\partial p}{\partial k_+} &= \max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\bigg| s_1\in S^d)\Pr\{s_1\in S^d\} \\
\frac{\partial p}{\partial k_-} &= \min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\bigg| s_1\in S^{d\prime})\Pr\{s_1\in S^{d\prime}\}
\end{aligned}
$$

So, if $f(k;\underline{s}_1^{nd})=B$, the FOCs wrt B are:

$$
\begin{aligned}
&\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + \left(\frac{\partial p}{\partial B_+}B + p\right) = \\
&-\min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big| s_1\in S^{nd\prime}\right\}\Pr\{s_1\in S^{nd\prime}\}+ \\
&\left(-\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B^2}\right]\bigg| s_1\in S^{d\prime})\Pr\{s_1\in S^{d\prime}\}B + p\right) \le 0 \\
&\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + \left(\frac{\partial p}{\partial B_-}B + p\right) = \\
&-\max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}+ \\
&\left(-\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B^2}\right]\bigg| s_1\in S^{d})\Pr\{s_1\in S^{d}\}B + p\right) \ge 0
\end{aligned}
$$

which implies (45). Finally, the FOCs wrt k are:

$$
\begin{aligned}
&\frac{\partial V}{\partial k_+} = -1 + \frac{\partial q}{\partial k_+} + \left(\frac{\partial p}{\partial k_+}B\right) = \\
&-1 + \max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}+ \\
&\left(\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\bigg| s_1\in S^{d})\Pr\{s_1\in S^{d}\}B\right) \le 0 \\
&\frac{\partial V}{\partial k_-} = -1 + \frac{\partial q}{\partial k_-} + \left(\frac{\partial p}{\partial k_-}B\right) = \\
&-1 + \min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big| s_1\in S^{nd\prime}\right\}\Pr\{s_1\in S^{nd\prime}\}+ \\
&\left(\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\bigg| s_1\in S^{d\prime})\Pr\{s_1\in S^{d\prime}\}B\right) \ge 0
\end{aligned}
$$

which implies (46). Since now expectations in the terms on the two sides of the inequality are taken over different sets, such condition is a little harder to interpret. In particular we can no longer say that all equity holders have the same valuation for the marginal productivity of capital in the no default states. Rather the condition imposes some relationship between the difference among equity holders and bond holders' valuation for the marginal productivity of capital in the two situations (S^d and $S^{d\prime}$).

We also have to check in this case the optimality of k, B wrt joint deviations of B and k. As before, without loss of generality, we can restrict our attention to changes of B and k such that $f(k; \underline{s}_1^{nd}) = B$ keeps holding (the set of states for which default occurs does not change).

$$\frac{\partial V}{\partial B}_{+} dB + \frac{\partial V}{\partial k}_{+} dk = \left[\frac{\partial q}{\partial B}_{+} + \left(\frac{\partial p}{\partial B}_{+} B + p\right)\right] dB + \left[-1 + \frac{\partial q}{\partial k}_{+} + \left(\frac{\partial p}{\partial k}_{+} B\right)\right] dk \leq 0,$$

for $dB = f_k(\underline{s}_1^{nd})dk > 0$; also,

$$\frac{\partial V}{\partial B}_{-} dB + \frac{\partial V}{\partial k}_{-} dk = \left[\frac{\partial q}{\partial B}_{-} + \left(\frac{\partial p}{\partial B}_{-} B + p\right)\right] dB + \left[-1 + \frac{\partial q}{\partial k}_{-} + \left(\frac{\partial p}{\partial k}_{-} B\right)\right] dk \geq 0$$

for $dB = f_k(\underline{s}_1^{nd})dk < 0$. Substituting the expressions for the partials obtained above, we get

$$\Big[- \min_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\}$$
$$- \min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B}\right] |s_1 \in S^d) \Pr\{s_1 \in S^d\} + p\Big] f_k(\underline{s}_1^{nd})$$
$$- 1 + \max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} +$$
$$\max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \Big| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} \leq 0$$

or

$$1 - \max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} - \tag{48}$$
$$\max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \Big| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq$$
$$\Big[- \min_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| \, s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\}$$
$$- \min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B}\right] \Big| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} +$$
$$\max_i \Big\{ E_{s_0}(MRS^i(c^i(s_1)) \, |s_1 \in S^{nd}) \Pr\{s_1 \in S^{nd}\} +$$
$$E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B}\right] \Big| \, s_1 \in S^d) \Pr\{s_1 \in S^d\} \Big\} \Big] f_k(\underline{s}_1^{nd})$$

where the term on the lhs is nonnegative because of (46) and the one on the rhs is also nonnegative by construction. Analogously, substituting the expressions for the partial derivatives *into the FOC for* $dB = f_k(\underline{s}_1^{nd})dk < 0$ *yields:*

$$\Big[-\max_{i \in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) \middle| s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\}$$
$$-\max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right] \Big| s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \Big] f_k(\underline{s}_1^{nd}) +$$
$$-1 + \min_{i \in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1) \middle| s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} +$$
$$\min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B} \Big| s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq 0$$

or

$$\begin{aligned} &\Big[-\max_{i \in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) \middle| s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\ &-\max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right] \Big| s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \Big] f_k(\underline{s}_1^{nd}) \\ \geq\ & 1 - \min_{i \in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1) \middle| s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} - \\ &\min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B} \Big| s_1 \in S^d) \Pr\{s_1 \in S^d\} \end{aligned} \tag{49}$$

where the term on the lhs is nonnegative because of (45) and the one on the rhs is also nonnegative as it immediately follows from (46). Putting (48) and (49) together,

$$1 - \max_{i \in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1) \middle| s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\}$$
$$-\max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B} \Big| s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq$$
$$\Big[-\min_{i \in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) \middle| s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\}$$
$$-\min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right] \Big| s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \Big] f_k(\underline{s}_1^{nd}) \geq$$
$$\Big[-\max_{i \in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) \middle| s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} -$$
$$\max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right] \Big| s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \Big] f_k(\underline{s}_1^{nd}) \geq$$
$$1 - \min_{i \in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1) \middle| s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} -$$
$$\min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B} \Big| s_1 \in S^d) \Pr\{s_1 \in S^d\}$$

Since

$$
\begin{aligned}
&-\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\
&-\min_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right]\middle|\, s_1 \in S^d\right) \Pr\{s_1 \in S^d\} \geq \\
&-\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\
&-\max_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right]\middle|\, s_1 \in S^d\right) \Pr\{s_1 \in S^d\}
\end{aligned}
$$

and

$$
\begin{aligned}
&-\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1)\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\
&-\min_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B}\middle|\, s_1 \in S^d\right) \Pr\{s_1 \in S^d\} \geq \\
&-\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1)) f_k(k,s_1)\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\
&-\max_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \frac{f_k(k;s_1)}{B}\middle|\, s_1 \in S^d\right) \Pr\{s_1 \in S^d\}
\end{aligned}
$$

it must be that (47) holds, where recall that

$$
\begin{aligned}
p = \max_i \Bigg\{ & E_{s_0}\left(MRS^i(c^i(s_1))\middle|\, s_1 \in S^{nd}\right) \Pr\{s_1 \in S^{nd}\} + \\
& E_{s_0}\left(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right]\middle|\, s_1 \in S^d\right) \Pr\{s_1 \in S^d\} \Bigg\}
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} = \\
&\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\middle|\, s_1 \in S^{nd}\right\} \Pr\{s_1 \in S^{nd}\} \\
&\min_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right]\middle|\, s_1 \in S^d\right) \Pr\{s_1 \in S^d\} = \\
&\max_{i\in I^d} E_{s_0}\left(MRS^i(c^i(s_1)) \left[\frac{f(k;s_1)}{B}\right]\middle|\, s_1 \in S^d\right) \Pr\{s_1 \in S^d\}
\end{aligned}
$$

and

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\right| s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\} =$$
$$\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\right| s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\}$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\Big| s_1\in S^{d}) \Pr\{s_1\in S^{d}\}$$
$$= \max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\Big| s_1\in S^{d}) \Pr\{s_1\in S^{d}\}$$

Note that conditions (45), (46) and (47) can be alternatively stated as:

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1\in S^{nd\prime}\right\} \Pr\{s_1\in S^{nd\prime}\} +$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big| s_1\in S^{d\prime}) \Pr\{s_1\in S^{d\prime}\} \geq p \geq$$
$$\geq \max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\} +$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big| s_1\in S^{d}) \Pr\{s_1\in S^{d}\}.$$

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\right| s_1\in S^{nd\prime}\right\} \Pr\{s_1\in S^{nd\prime}\} +$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\Big| s_1\in S^{d\prime}) \Pr\{s_1\in S^{d\prime}\} \geq 1 \geq$$
$$\geq \max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\right| s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\} +$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\Big| s_1\in S^{d}) \Pr\{s_1\in S^{d}\}$$

and

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\} =$$
$$\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\right| s_1\in S^{nd}\right\} \Pr\{s_1\in S^{nd}\}$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big| s_1\in S^{d}) \Pr\{s_1\in S^{d}\} =$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big| s_1\in S^{d}) \Pr\{s_1\in S^{d}\}.$$

$$\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\middle|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\} =$$
$$\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\middle|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}$$
$$+\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\bigg|\, s_1\in S^{d})\Pr\{s_1\in S^{d}\} =$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\frac{f_k(k;s_1)}{B}\bigg|\, s_1\in S^{d})\Pr\{s_1\in S^{d}\}$$

This completes the proof of the claim. ■

We are now ready to complete the proof of the proposition. The equity price map in the presence of risky debt is giiven n by

$$q(k,B)=\max_i E_{s_0}\left\{ MRS^i(c^i(s_1))\,[f(k;s_1)-B]\middle|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}$$

The debt price map is

$$p(k,B)=\max_i \Big\{E_{s_0}(MRS^i(c^i(s_1))\big|\, s_1\in S^{nd})\Pr\{s_1\in S^{nd}\}+$$
$$+\,E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\bigg|\, s_1\in S^{d})\Pr\{s_1\in S^{d}\}\Big\}$$

The statement only refers to the interior case: $f(k;\underline{s}_1^{nd})>B$*. Here, the partials of the price maps with respect to* B *are*

$$\begin{aligned}\frac{\partial q}{\partial B_+} &= -\min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\middle|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}\\ \frac{\partial q}{\partial B_-} &= -\max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))\middle|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial p}{\partial B_+} &= -\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B^2}\right]\bigg|\, s_1\in S^{d})\Pr\{s_1\in S^{d}\}\\ \frac{\partial p}{\partial B_-} &= -\max_{i\in I^d}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B^2}\right]\bigg|\, s_1\in S^{d})\Pr\{s_1\in S^{d}\}\end{aligned}$$

Analogously, the partials with respect to k *are*

$$\begin{aligned}\frac{\partial q}{\partial k_+} &= \max_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\middle|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}\\ \frac{\partial q}{\partial k_-} &= \min_{i\in I^e} E_{s_0}\left\{ MRS^i(c^i(s_1))f_k(k,s_1)\middle|\, s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}\end{aligned}$$

and

$$\frac{\partial p}{\partial k_+} = \max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^d)\Pr\{s_1\in S^d\}$$
$$\frac{\partial p}{\partial k_-} = \min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^d)\Pr\{s_1\in S^d\}$$

So, if $f(k;\underline{s}_1^{nd}) > B$, the FOCs with respect to B are:

$$\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + \left(\frac{\partial p}{\partial B_+}B + p\right) =$$
$$-\min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}$$
$$+\left(-\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big| s_1\in S^d)\Pr\{s_1\in S^d\} + p\right)\le 0$$
$$\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + \left(\frac{\partial p}{\partial B_-}B + p\right) =$$
$$-\max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\}$$
$$+\left(-\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big| s_1\in S^d)\Pr\{s_1\in S^d\} + p\right)\ge 0$$

which implies

$$p = \max_i E_{s_0}\left(MRS^i(c^i(s_1))\big| s_1\in S^{nd}\right)\Pr\{s_1\in S^{nd}\} +$$
$$E_{s_0}\left(MRS^i(c^i(s_1))\left[\frac{f(k;s_1)}{B}\right]\Big| s_1\in S^d\right)\Pr\{s_1\in S^d\}$$

and (43). On the other hand, the FOCs with respect to k give:

$$\frac{\partial V}{\partial k_+} = -1 + \frac{\partial q}{\partial k_+} + \left(\frac{\partial p}{\partial k_+}B\right) =$$
$$= -1 + \max_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\} +$$
$$\max_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^d)B\Pr\{s_1\in S^d\}\le 0$$
$$\frac{\partial V}{\partial k_-} = -1 + \frac{\partial q}{\partial k_-} + \left(\frac{\partial p}{\partial k_-}B\right) =$$
$$= -1 + \min_{i\in I^e} E_{s_0}\left\{MRS^i(c^i(s_1))f_k(k,s_1)\big| s_1\in S^{nd}\right\}\Pr\{s_1\in S^{nd}\} +$$
$$\min_{i\in I^d} E_{s_0}(MRS^i(c^i(s_1))\left[\frac{f_k(k;s_1)}{B}\right]\Big| s_1\in S^d)B\Pr\{s_1\in S^d\}\ge 0$$

which implies (44). This completes the proof of Proposition 2. ■

Equilibria with Short sales when intermediation costs are negligible

In Section 6 we established the existence of an equilibrium with intermediated short sales for all levels $\delta > 0$ of the intermediation cost (capturing the default rate on short positions). It is then of interest to investigate the properties of these equilibria as we let δ go to 0. Clearly the spread $\max_i \mathbb{E}[MRS^i(c^i(s))R^e(k, \phi, m, B; s)] - \min_i \mathbb{E}[MRS^i(c^i(s))R^e(k, \phi, m, B; s)]$ must go to zero, since $q(k, \phi, m, B)$ is bounded above for all k, ϕ, m, B and all $\delta > 0$, total resources being finite. We conjecture therefore that the limit of the competitive equilibria with short sales as $\delta \to 0$ exists, as all variables lie in a compact set.

The previous observation also implies that the marginal valuation for all possible production and financial plans is equalized across all consumers, as in an environment where unlimited short sales are allowed and markets are complete (or a spanning property holds for all admissible production and financial plans of firms). In the limit as $\delta \to 0$ not only all possible markets, corresponding to all possible choices k, ϕ, m, B, are open, as in the case without short sales, but a òarger set of markets are open and active, to ensure the equalization of agents' marginal rates of substitution.

Short sales with endogenous default

We extend here the analysis of Section 6 by examining the case where the consumers' default rate, rather than being exogenous and state and type invariant, is optimally chosen by consumers, and may depend therefore on the state s as well as the type i of the consumer. We show in what follows the required changes in the model. The specification of the intermediation activity and the structure of markets is clearly more complicated, still the main results on unanimity and optimality remain valid.

Since consumers' loans are non-collateralized, we follow Dubey, Geanakoplos and Shubik (2005) in introducing a utility penalty ξ^i for a type i consumer per unit defaulted in any state s, for all i, s. It is convenient to assume here that preferences are additively separable over time, so that they take the following form:

$$u_0^i(c_0^i) + \mathbb{E}\left[u_1^i\left(c^i(s)\right) - \xi^i \delta_s^i \left[\lambda_-^i\left(f(k, \phi; s) - B\right)\right]\right] \tag{50}$$

where δ_s^i is the default rate of consumer i in state s. Given this feature of consumers' preferences, the optimal default level in each state s for consumer i is obtained by maximizing (50) with respect to $(\delta_s^i)_s$ subject to the date 1 budget constraint (32), where δ is replaced by δ_s^i. It is immediate to see that the solution is a well defined map $\delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)$ for all s and $\theta^i, \lambda_+^i, b^i, \lambda_-^i$, and for any given k, ϕ, B.

Thus the default rate in any state s on the loans granted to consumers via the sale of short positions depends not only on the type i of the consumer but also on his overall portfolio holdings. We consider then the case where both the consumer's type and his portfolio holdings are observable by his trading partners. The loan contract offered by intermediaries is so an exclusive contract and the price depends both on the consumer's type and portfolio, $q^-_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}$ as well as, obviously, on the return structure of the underlying equity. Hence the budget constraint faced by consumers at date 0 is now

$$c_0^i \;= w_0^i \;+ [-k \;+ q \;+ p\; B\;]\,\theta_0^i - q\;\theta^i \;- p\; b^i - q^+\lambda_+^i + q^-_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}\,\lambda_-^i \tag{51}$$

An intermediary who is intermediating m units of the derivative by selling the short positions to consumers of type i, with portfolio $(\theta^i, \lambda_+^i, b^i, \lambda_-^i)$, faces a default rate on its loans equal to $\delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)$. As a consequence, the shortfall in its revenue at date 1 is:

$$\left[(f(k,\phi;s) - B)\;\; \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)\right] m. \tag{52}$$

We consider still the case where only equity, an asset that is 'safe' as it is in positive net supply and backed by real claims, is used to hedge the consumers' default risk. To be able to fully meet the shortfall in (52) due to consumers' default, the intermediary must hold at least

$$\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i) m$$

units of equity. The total date 0 revenue of the intermediary is then:

$$\max_m \left[q^+ - q^-_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i} - q\left(\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)\right)\right] m \tag{53}$$

The intermediary's choice problem consists in the choice of the amount m to issue of each type $i, \theta^i, \lambda_+^i, b^i, \lambda_-^i$ of derivative so as to maximize its profits, that is its revenue at date 0. Note that the intermediation technology still exhibits constant returns to scale, hence a solution exists provided

$$q \geq \frac{q^+ - q^-_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}}{\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)};$$

and is characterized by $m(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i) > 0$ only if $q = \frac{q^+ - q^-_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}}{\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)}$.

The main difference with respect to the reduced form model is then the fact that the market for derivative claims is differentiated according to consumers' types and portfolio choices. This has the following implications for the consumers' optimization problem and the market clearing conditions.

Consumer i chooses his portfolio $\theta^i, \lambda_+^i, b^i, \lambda_-^i$ so as to maximize

$$u_0^i(c_0^i) + \mathbb{E}\left\{ \begin{array}{c} u_1^i\left[w^i(s) + b^i + (f(k,\phi;s) - B)(\theta^i + \lambda_+^i - \lambda_-^i(1 - \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i))\right] \\ -\xi^i \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i) \end{array} \right\}$$

subject to the budget constraint (51), given the asset prices q, q^+, p and $q_{i,\cdot}^-$ and the default map $\delta_s^i(\cdot)$ obtained as above. Let $\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i$ denote the consumer's optimal choice in equilibrium. The asset market clearing conditions are then

$$\sum_i m(i, \bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i)\left[\max_s \delta_s^i(\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i)\right] + \sum_{i \in I} \bar{\theta}^i = 1$$

for equity, and

$$\begin{aligned} \bar{\lambda}_-^i &= m(i, \bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) \text{ for each } i \\ 0 &= m(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i) \text{ for each } i,\ (\theta^i, \lambda_+^i, b^i, \lambda_-^i) \neq (\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) \\ \sum_i m(i, \bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) &= \sum_i \bar{\lambda}_+^i \end{aligned}$$

for the derivative security.

The consistency condition M) on the firms' equity conjectures must also be properly modified to reflect the different specification of the value of intermediation in the present context:

$$M'\text{)}\ \ q(k,\phi,B) = \max\left\{ \begin{array}{c} \max_i \mathbb{E}\left[MRS^i(c^i(s))\,(f(k,\phi;s) - B)\right], \\ \max_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i} \frac{\max_i \mathbb{E}\left[MRS^i(c^i(s))(f(k,\phi;s)-B)\right] - q^-(i,\theta^i,\lambda_+^i,b^i,\lambda_-^i;k,\phi,\bar{U}^i)}{\max_s \delta_s^i(\theta^i,\lambda_+^i,b^i,\lambda_-^i;k,\phi)} \end{array} \right\}, \forall k, \phi, B$$

where $q^-(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B, \bar{U}^i)$ is constructed as follows. For any k, ϕ, B and $i, \theta^i, \lambda_+^i, b^i, \lambda_-^i$, set $q^-(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B, \bar{U}^i)$ as the value of q^- that satisfies the following equation:

$$\begin{aligned} \bar{U}^i &= u_0^i(w_0^i + \left[-\bar{k} + \bar{q} + \bar{p}\,\bar{B}\right]\theta_0^i - \bar{q}\,\theta^i - \bar{p}\,b^i - \bar{q}^+\lambda_+^i + q^-\lambda_-^i) + \\ &\quad \mathbb{E}\left[\begin{array}{c} u_1^i\left[w^i(s) + b^i + (f(\bar{k},\bar{\phi};s) - \bar{B})(\theta^i + \lambda_+^i) - \lambda_-^i\left[f(k,\phi;s) - B\right]\left(1 - \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B)\right)\right] \\ -\xi^i \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B)\left[\lambda_-^i\left(f(k,\phi;s) - B\right)\right] \end{array} \right. \\ &= \bar{U}^i \end{aligned}$$

where $\bar{U}^i$ denotes the utility level of type i consumers at the equilibrium choices $\bar{\theta}^i, \bar{\lambda}^i_+, \bar{b}^i, \bar{\lambda}^i_-$ and the map $\delta^i_s(\theta^i, \lambda^i_+, b^i, \lambda^i_-; k, \phi, B)$ is similarly obtained by maximizing the expected utility term on the right hand side of the above expression with respect to δ^i_s. That is, $q^-(i, \theta^i, \lambda^i_+, b^i, \lambda^i_-; k, \phi, B, \bar{U}^i)$ identifies the maximal willingness to pay in equilibrium of consumer i for a short position equal to λ^i_- in the firm with plan k, ϕ, B when the rest of his portfolio is given by $\theta^i, \lambda^i_+, b^i$.[49] At these prices consumers are indifferent between choosing the equilibrium portfolio $\bar{\theta}^i, \bar{\lambda}^i_+, \bar{b}^i, \bar{\lambda}^i_-$ and any other portfolio with a short position λ^i_- in the equity of a firm with plan k, ϕ, B.

An important difference with respect to the previous analysis is the fact that here the price of short positions is no longer defined at the margin. This is due to the exclusive nature of loan contracts corresponding to short positions. Also, at the same prices intermediaries are indifferent between issuing the derivatives traded in equilibrium and any other derivative on equity of firms with plan k, ϕ, B such that $q = \frac{\max_i \mathbb{E}\left[MRS^i(c^i(s))(f(k,\phi;s)-B)\right] - q^-(i,\theta^i,\lambda^i_+,b^i,\lambda^i_-;k,\phi,B,\bar{U}^i)}{\max_s \delta^i_s(\theta^i,\lambda^i_+,b^i,\lambda^i_-;k,\phi,B)}$.

The unanimity and constrained optimality properties still hold in this environment. The argument again is very similar and relies on the the fact that, given the above specification of the intermediation technology and the price conjectures, the model is equivalent to a setup where the markets for all types of equity and all types of corresponding derivatives are available for trade. The notion of completeness here also requires the exclusivity of the loan contracts associated to short positions, so that the market for all types of derivatives can also be differentiated according to the type of a consumer and the level of his trades.

[49] In the specification of $q^-(i, \theta^i, \lambda^i_+, b^i, \lambda^i_-; k, \phi, B, \bar{U}^i)$ we have implicitly assumed that all the long positions of a consumer are in the assets corresponding to the firms' equilibrium choices. This is with no loss of generality and to avoid excessive notational complexities.