

# Dynamic Information Acquisition and Strategic Trading

Snehal Banerjee and Bradyn Breon-Drish\*

September 2017

## Abstract

We study dynamic information acquisition in a strategic trading model. We allow the strategic trader to optimally choose when to acquire costly information about an asset's payoff, instead of endowing her with this information. We show that whether the market maker observes the acquisition decision plays a crucial role. With observability, there exists an equilibrium in which the optimal acquisition decision follows a pure strategy and exhibits delay relative to a naive NPV rule. In contrast, when the acquisition decision is not observable, we show that an equilibrium with smooth trading and a pure acquisition strategy cannot exist. We also rule out the existence of a natural class of equilibria with smooth trading in which the trader mixes between acquiring information and not.

JEL: D82, D84, G12, G14

Keywords: Dynamic information acquisition, Strategic trading, Observability

---

\*Banerjee ([snehalb@ucsd.edu](mailto:snehalb@ucsd.edu)) and Breon-Drish ([bbreondrigh@ucsd.edu](mailto:bbreondrigh@ucsd.edu)) are at the University of California, San Diego. All errors are our own. We thank Brett Green for numerous, invaluable discussions during an early stage of this project. We also thank Stathi Avdis, Kerry Back, Joey Engelberg, Mariana Khapko, Igor Makarov, Sophie Moinas, Dmitry Orlov, Christine Parlour, Chris Parsons, Uday Rajan, Allan Timmermann, Dimitri Vayanos, and participants at UC Davis, LSE, UC Riverside, the University of Minnesota Junior Finance Faculty Conference, the FIRS 2017 Meeting, Barcelona GSE Summer Forum, and the Western Finance Association (2017) Meeting for helpful suggestions.

# 1 Introduction

Investors' incentives to acquire private information change over time and with current economic conditions. For instance, rising oil prices can trigger research into whether airlines are hedged against fuel price increases. A falling real estate market can lead investors to acquire loan-level data on their mortgage-backed securities in order to revalue their positions. A consolidation wave in a particular industry can lead market participants to investigate remaining firms as potential targets. Following [Grossman and Stiglitz \(1980\)](#), a large literature has studied how investors choose to acquire information, and what their decisions imply for financial markets. However, despite the inherently dynamic nature of the information acquisition decision, the existing literature has treated it as a static problem by requiring that investors make their information choices before the start of trading.

We study the dynamic information acquisition decision of a strategic trader. In contrast to prior work, we allow her to choose the timing of information acquisition in response to the evolution of a public signal. We find that whether or not the market maker observes the trader's acquisition decision plays a crucial role. When the trader's acquisition decision is observable, there is an equilibrium in which acquisition is a pure strategy and the optimal decision exhibits delay beyond what would be predicted by a naive "NPV" rule. In sharp contrast, we show that such pure strategy equilibria do not exist when information acquisition is not observable by the market maker. Moreover, we rule out existence of a class of mixed strategy equilibria in which the strategic trader follows a smooth trading strategy and a value matching condition characterizes points at which the trader acquires information with strictly positive probability. These results suggest that the standard strategic trading equilibrium may be inconsistent with costly information acquisition when information can be acquired dynamically and the acquisition decision is unobservable.

Our model builds on the continuous-time [Kyle \(1985\)](#) model in [Back and Baruch \(2004\)](#). There is a single risky asset, traded by a risk-neutral, strategic trader and a mass of noise traders. We introduce a publicly observable signal, which may or may not be payoff relevant, that evolves stochastically over time. For example, suppose an airline's fuel hedging positions are not publicly known. In this case, the market faces uncertainty about whether fluctuations in oil prices are relevant for the airline's stock price. A risk-neutral market maker competitively sets the asset's price, conditional on the public signal and aggregate order flow. The asset payoff (and consequently, the relevance of the signal) is publicly revealed at a random time, e.g., when the airline's derivative positions are made public.<sup>1</sup> The traders

---

<sup>1</sup>The assumption of a random horizon is largely for tractability and is not qualitatively important for our primary results. When appropriate, we indicate where the conclusions from a fixed-horizon model would differ.

and the market maker share a common prior about the payoff relevance of the public signal.

In contrast to much of the literature, we do not assume the strategic trader is endowed with private information. Instead, she can pay a cost to determine whether the public signal is payoff relevant. The value of this information varies over time as the signal evolves, e.g., whether or not the airline is hedged has a larger impact on its profits when oil prices have risen.<sup>2</sup> The key feature of our analysis is that the decision to become informed need not be made at the initial date before trading begins. Rather, the trader can acquire information at a time of her choosing.

We find that whether or not the market maker observes the trader’s information acquisition plays a key role. First, we follow the literature by considering the case with observable acquisition.<sup>3</sup> In this case, we show there exists an equilibrium where information acquisition follows a pure strategy. Appealing to standard results on optimal stopping, we characterize the trader’s optimal strategy and show that it follows a cutoff rule: she chooses to acquire information only when the public signal reaches a threshold. Intuitively, the ability to decide *when* to acquire information endows the trader with a call option on the expected profits from being privately informed, and she chooses to exercise the option only when the uncertainty about the asset payoff is sufficiently high. Moreover, we show that optimal information acquisition exhibits delay — the strategic trader chooses to wait beyond the threshold that would be prescribed by an “NPV” rule. As such, the standard assumption that the trader can only choose to acquire information at the initial date is restrictive if she can condition her acquisition decision on the evolution of public news.

Consistent with the intuition from real option decisions, we show that the benefit from waiting to acquire information increases in the cost of information and the volatility of the public signal, but decreases in the prior uncertainty about the payoff relevance of the signal. We show that the likelihood of information acquisition need not always increase with volatility of the public signal. While higher signal volatility increases the likelihood that the option to acquire information ends up “in the money,” it also increases the value of waiting. In fact, we show that when acquisition costs are sufficiently low, the likelihood of acquisition decreases with news volatility.

We also find that the likelihood of information acquisition need not be higher when the

---

<sup>2</sup>As we discuss further in Section 2, the public signal with uncertain payoff-relevance allows us to introduce time-varying uncertainty in the asset payoff in a tractable manner, which is a necessary feature for the trader’s timing decision to be nontrivial. In our setting, learning about the payoff relevance of the public signal is equivalent to learning about the payoff itself. When the public signal has zero volatility (i.e., there is no public news) our setting maps to the binary-payoff model of [Back and Baruch \(2004\)](#); however, we permit the strategic trader to endogenously choose to learn about the payoff.

<sup>3</sup>This is consistent with the standard framework, where the market maker knows with certainty whether there is an informed, strategic trader in the market.

trading horizon is longer. When the payoff is expected to be revealed quickly, the value from being informed is very low since there is little time over which to profit at the expense of noise traders, and so the acquisition boundary is high. However, as the expected trading horizon increases, there are two offsetting effects. On the one hand, the value from being informed increases with the horizon since the trader expects her information advantage to last longer. On the other hand, the cost of waiting decreases with the horizon, since the likelihood that the payoff is revealed before acquisition is low. We show that initially the first effect dominates, while eventually the second one does. As a result, the trader is less likely to acquire information when the trading horizon is very long or very short.

We then consider the case in which the strategic trader's decision to acquire information is not observable by the market maker. First, we explore whether there exist equilibria in which information acquisition follows a pure strategy. One can immediately rule out any equilibrium in which the strategic trader delays acquisition (or does not acquire at all). In any such equilibrium, the market maker should rationally set the price impact to zero before acquisition. But since acquisition is not observable, the uninformed strategic trader should deviate by acquiring information, trading against the unresponsive market maker, and making unbounded profits.

We also rule out pure-strategy equilibria in which the strategic trader acquires information immediately. In such an equilibrium the strategic trader could profitably deviate by not acquiring, thereby avoiding the cost of information, and trading optimally against an incorrect pricing rule. In the conjectured equilibrium, the market maker treats the order flow as informative, but an uninformed trader knows that it is not. In our setting, this informational advantage ensures that an uninformed strategic trader can generate the same value from trading as the expected value from becoming informed, but without paying the cost of acquisition.<sup>4</sup>

Given the absence of pure strategy equilibria, we consider the possibility of equilibria in which the strategic trader follows a mixed information acquisition strategy. We show that there cannot exist such an equilibrium if the value function of the strategic trader satisfies the usual HJB equation of dynamic programming, and there exists a point at which she mixes that satisfies standard indifference conditions of optimal stopping.<sup>5</sup> As we show, this is because there is a profitable deviation for the uninformed trader at the hypothesized point

---

<sup>4</sup>The source of this informational advantage is that in continuous time, noise trading is effectively observable by the strategic trader. By contrast, an uninformed strategic trader cannot observe noise trading contemporaneously in the standard, discrete-time setting, and so optimally chooses to trade zero and earns zero profit.

<sup>5</sup>This also rules out equilibria in which the strategic trader mixes continuously (over an interval of public news realizations) with a given intensity.

of acquisition. Instead of acquiring information with positive probability, she can deviate by refraining from trade for the next instant, and then re-evaluating her decision to become informed.<sup>6</sup> Her overall expected trading profit from deviating in that instant is at least as high as her expected profit from becoming informed and *then* trading in that instant, but since the trading game ends with positive probability immediately after she trades, she is better off by delaying the cost of acquiring information to the future.

Our paper relates to the large literature on asymmetric information models with endogenous information acquisition that was initiated by [Grossman and Stiglitz \(1980\)](#). While a number of papers extend this basic setting to allow for dynamic trading (e.g., [Mendelson and Tunca \(2004\)](#), [Avdis \(2016\)](#)), to allow traders to condition their information acquisition decision on a public signal (e.g., [Foster and Viswanathan \(1993\)](#)), to allow traders to pre-commit to receiving signals at particular dates (e.g., [Back and Pedersen \(1998\)](#), [Holden and Subrahmanyam \(2002\)](#)), or to incorporate a sequence of one-period information acquisition decisions ([Veldkamp \(2006\)](#)), the information acquisition decision remains essentially static — investors make their information acquisition decision before the start of trade. The unobservable acquisition case of our model is related to a recent literature that studies markets in which some participants face uncertainty about the existence or informedness of others (e.g., [Li \(2013\)](#), [Banerjee and Green \(2015\)](#), [Back, Crotty, and Li \(2016\)](#), [Wang and Yang \(2016\)](#)). To the best of our knowledge, however, our model is the first to allow for dynamic information acquisition in that the strategic trader can choose to become privately informed at any point of time. Our analysis implies that allowing for dynamic information acquisition has economically important consequences and highlights the fact that observability of the acquisition decision plays a critical role.

## 2 Model

Our framework is based on the continuous-time, [Kyle \(1985\)](#) model in [Back and Baruch \(2004\)](#). Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  on which is defined the standard Brownian motion  $(W_Z, W_N)$  and independent random variables  $\xi$  and  $T$ . Let  $\mathcal{F}_t$  denote the augmentation of the filtration  $\sigma(\{W_{Zt}, W_{Nt}\})$ . The random variable  $\xi \in \{0, 1\}$  is binomial with probability  $\alpha = \Pr(\xi = 1)$ , and  $T$  is exponentially distributed with rate  $r$ . There are two assets: a risky asset and a risk-free asset with interest rate normalized to zero. The risky asset pays off  $v$

---

<sup>6</sup>This deviation is made possible by a feature of the Kyle model in continuous time. As highlighted by [Back \(1992\)](#), the maximum value attainable by the strategic trader in equilibrium can be attained by not trading for an interval of time and then following the optimal trading strategy from that point.

at random time  $T$ , where

$$v = \xi N_T. \tag{1}$$

The public news process  $N_t$  is a geometric Brownian motion

$$dN_t = \sigma_N N_t dW_{Nt} \tag{2}$$

where  $\sigma_N > 0$  and the initial value  $N_0 > 0$  is constant.<sup>7</sup> We interpret  $\xi$  as the payoff-relevance of the news process. In particular, the news process is only informative about the payoff of the risky asset if  $\xi = 1$ . There is a single, risk-neutral strategic trader who can pay a fixed cost  $c$  at any time  $\tau$  to determine whether the public news process is payoff relevant (i.e., to observe the realization of  $\xi$ ).

To fix ideas, consider the example from the introduction. Suppose the market faces uncertainty about whether an airline is hedged against fuel price increases. The strategic trader must pay a cost (i.e.,  $c$ ) to investigate whether the airline is exposed (i.e.,  $\xi \in \{0, 1\}$ ), and optimally chooses when to do so. When the price of oil is relatively low, the incremental impact on firm value (i.e.,  $v$ ) of hedging is low. In contrast, when the price of oil is high, whether or not the airline is hedged affects its value much more. As such, one expects the value of learning about exposure varies over time with the publicly observable news about fuel prices (i.e., changes in  $N_t$ ).

Let  $X_t$  denote the cumulative holdings of the trader, and suppose the initial position  $X_0 = 0$ . Further, suppose  $X_t$  is absolutely continuous and let  $\theta(\cdot)$  be the trading rate (so  $dX_t = \theta(\cdot)dt$ ).<sup>8</sup> There are noise traders who hold  $Z_t$  shares of the asset at time  $t$ , where

$$dZ_t = \sigma_Z dW_{Zt}, \tag{3}$$

with  $\sigma_Z > 0$  a constant.

Finally, there is a competitive, risk neutral market maker who sets the price of the risky asset. This market maker observes the order flow  $Y_t = X_t + Z_t$  and sets the price equal to the conditional expected payoff given the public information set. Let  $\{\mathcal{F}_t^P\}$  denote the public information filtration, which we separately describe below for the observable and

---

<sup>7</sup>The assumptions that the public signal is perfectly informative about  $N_t$  and that  $N_t$  has zero drift are without loss of generality. In the more general case,  $N_t$  is replaced with  $\mathbb{E}[N_T | \mathcal{F}_t^P]$  in the pricing rule and trading strategy and the rest of the analysis is essentially unchanged. It is also straightforward to generalize to a general continuous, positive martingale for the news process, but at the expense of closed-form solutions to the optimal acquisition problem in most cases.

<sup>8</sup>Back (1992) shows that it is optimal for the trader to follow strategies of this form.

unobservable cases. The price at time  $t < T$  is then given by

$$P_t = \mathbb{E} [v | \mathcal{F}_t^P]. \quad (4)$$

Let  $\mathcal{T}$  denote the set of  $\mathcal{F}_t^P$  stopping times. We require that the trader's information acquisition time  $\tau \in \mathcal{T}$ . That is, we require acquisition to depend only on public information up to that point. Let  $\mathcal{F}_t^I$  denote the augmentation of the filtration  $\sigma(\mathcal{F}_t^P \cup \sigma(\xi))$ . Thus,  $\mathcal{F}_t^I$  represents the trader's information set, post-information acquisition. We require the trader's pre-acquisition trading strategy to be adapted to  $\mathcal{F}_t^P$  and her post-acquisition strategy to be adapted to  $\mathcal{F}_t^I$ .

To ensure that the trader's expected profit is well-defined, we must rule out trading strategies that first incur infinite losses by driving the price to zero or  $N_t$  and then reap infinite profits. Given a price process  $P_t$  (which will in general depend on  $\theta$  through the order flow) a trading strategy  $\theta$  is *admissible* if it satisfies the measurability restrictions given above (i.e., does not depend on  $\xi$  before the moment of information acquisition) and

$$\mathbb{E} \int_0^T (\theta_u(N_T\xi - P_u))^- du < \infty, \quad (5)$$

where  $x^- = \max\{0, -x\}$ . Note that this admissibility condition is identical to that of [Back and Baruch \(2004\)](#) in the case that  $\tau = 0$  and  $N_t \equiv 1$ .

Our definition of equilibrium is standard and follows [Back and Baruch \(2004\)](#), modified to account for information acquisition.<sup>9</sup>

**Definition 1.** *A pure strategy equilibrium is an information acquisition time  $\tau \in \mathcal{T}$  and admissible trading strategy  $\theta$  for the trader and a price process  $P_t$  such that, given the trader's strategy the price process satisfies (4) and, given the price process, the trading strategy and acquisition time maximize the expected profit*

$$\mathbb{E} \left[ \int_0^T \theta(N_T\xi - P_u) du \right].$$

**Remark:** The specification of the public news process allows us to introduce stochastic volatility in a parsimonious and tractable manner. Without variation in public news (e.g., if  $N_t \equiv 1$ ), the above setting reduces to the one analyzed by [Back and Baruch \(2004\)](#) but with endogenous information acquisition. In this case, however, the trader's acquisition decision is effectively static since the value of information is constant over time. With a stochastic news

---

<sup>9</sup>For now, we focus on pure acquisition strategies. We go into further detail on the interpretation of mixing when we consider unobservable acquisition below.

process, the value of information evolves over time, which introduces dynamic considerations to the acquisition decision. We expect alternative specifications that generate time-variation in uncertainty about fundamentals would generate similar predictions, although at the expense of tractability or a less natural economic interpretation.<sup>10</sup>

### 3 Observable information acquisition

In this section, we characterize equilibria where the strategic trader's decision to acquire information is observable by the market maker. In this case, the information acquisition decision by the strategic trader resembles the exercise of a real option. We show that optimal information acquisition exhibits delay, and derive predictions on the likelihood of information acquisition and price dynamics in this case.

Let  $I_t$  denote the indicator for whether the strategic trader has acquired information at time  $t$  or before. Because the market maker observes the news and order flow processes, the time of the asset payoff, and the acquisition status of the strategic trader, the public information filtration  $\mathcal{F}_t^P$  is the augmentation of the filtration  $\sigma(\{N_t, Y_t, \mathbf{1}_{\{T \leq t\}}, I_t\})$ .<sup>11</sup>

#### 3.1 Financial market equilibrium

We can construct equilibrium by working backwards. We begin by characterizing the financial market equilibrium, conditional on an acquisition time, and then find the optimal acquisition time given the financial market equilibrium.

**Proposition 1.** *Fix an information acquisition time  $\tau \in \mathcal{T}$ . There exists an equilibrium in the trading game in which the price of the risky asset is given by  $P_t = N_t \alpha$  prior to acquisition and  $P_t = N_t p_t$  after acquisition, where*

$$p_t \equiv \mathbb{E} [\xi I_t | \mathcal{F}_t^P] = \begin{cases} 0 & 0 \leq t < \tau \\ \Phi \left( \Phi^{-1}(\alpha) e^{r(t-\tau)} + \sqrt{\frac{2r}{\sigma^2}} \int_{\tau}^t e^{r(t-s)} dY_s \right) & \tau \leq t < T \end{cases}. \quad (6)$$

---

<sup>10</sup>A perhaps more standard specification of the model would be one in which the value  $v$  is normally distributed with stochastic volatility (e.g., variance  $\Sigma_t$ ). In order for this volatility to impact the acquisition decision, it must be publicly observable. However, this poses a difficulty: how does one interpret a setting in which the value of an asset is unobservable, but exhibits observable stochastic volatility? An alternative specification, in which there is a public signal with an error that exhibits stochastic volatility (e.g.,  $N_t = v + \epsilon_t$ , where  $\epsilon_t$  exhibits stochastic volatility  $\sigma_t$ ), necessitates the introduction of two state variables (i.e., the signal  $N_t$  and the conditional variance of  $v$  under the public information set,  $\Sigma_{P,t}$ ), which limits tractability.

<sup>11</sup>To reduce clutter, we abuse notation somewhat by using  $\mathcal{F}_t^P$  to denote both the market maker's information set, which includes the acquisition indicator  $I_t$  in this case, as well as the public information filtration, which includes only the news process and order flow variables, and defines the admissible class of stopping times for acquisition.



Prior to information acquisition, the trader does not trade (i.e.,  $\theta^U \equiv 0$ ), and conditional on information acquisition, her strategy depends only on  $p$  and is given by

$$\theta^1(p) = \frac{\sigma_Z^2 \lambda(p)}{p}, \text{ and } \theta^0(p) = -\frac{\sigma_Z^2 \lambda(p)}{1-p},$$

where  $\theta^i$ ,  $i \in \{U, 1, 0\}$ , denotes the trading strategy corresponding to the uninformed, informed of  $\xi = 1$ , and informed of  $\xi = 0$  types. In this equilibrium, conditional on becoming informed, the trader's value function is given by

$$J^1(p_t, N_t) = N_t \int_{p_t}^1 \frac{1-a}{\lambda(a)} da, \text{ and } J^0(p_t, N_t) = N_t \int_0^{p_t} \frac{a}{\lambda(a)} da, \quad (7)$$

where  $\lambda(p) = \sqrt{\frac{2r}{\sigma_Z^2}} \phi(\Phi^{-1}(1-p))$ .

Our equilibrium characterization naturally extends the equilibrium in [Back and Baruch \(2004\)](#) to (i) accommodate the news process  $N_t$  and (ii) account for the possibility that the strategic trader is uninformed before  $\tau$ . Before information acquisition, the strategic trader does not trade,<sup>12</sup> and consequently, the order-flow is uninformative and the market-maker does not update his beliefs about  $\xi$ . As a result, before  $\tau$  the price  $P_t = \alpha N_t$  evolves linearly with  $N_t$ . Conditional on information acquisition, the strategic trader optimally trades according to  $\theta^\xi$  characterized in the proposition. Since  $\theta^1 \neq \theta^0$ , the order flow provides a noisy signal about  $\xi$  to the market maker. The market maker's conditional expectation about  $\xi$ , given by  $p_t$ , depends on the cumulative (weighted) order-flow since the acquisition date (i.e.,  $\int_\tau^t e^{r(t-s)} dY_s$ ), and consequently, so does the price  $P_t$ .

### 3.2 Optimal information acquisition

Given the value function in Proposition 1, we can characterize the optimal information acquisition decision.

**Proposition 2.** *The strategic trader optimally acquires information the first time  $N_t$  hits the optimal exercise boundary  $N^* = \frac{\beta}{\beta-1} \frac{c}{K}$ , where*

$$K = \sqrt{\frac{\sigma_Z^2}{2r}} \phi(\Phi^{-1}(1-\alpha)), \text{ and } \beta = \frac{1+\sqrt{1+8r/\sigma_N^2}}{2}. \quad (8)$$

---

<sup>12</sup>Under the posited price function, the pre-acquisition trading strategy is indeterminate. Any strategy that uses only public information earns zero expected profit in this region. Given such a trading strategy, it also remains optimal for the market maker to set  $P_t = N_t \alpha$ . Without loss of generality, we focus on the case in which the trader does not trade before time  $\tau$ . In the presence of transaction costs, this would be the optimal strategy.

Moreover, the optimal exercise boundary  $N^*$  increases in  $c$  and  $\sigma_N$ , decreases in  $\sigma_Z$ , is U-shaped in  $\alpha$  (minimized at  $\alpha = 0.5$ ), and is U-shaped in  $r$ .

As we show in the proof of the above, the expected profit immediately prior to acquiring information at any date  $t$  (i.e., the value function the instant before  $\xi$  is observed) is given by

$$\bar{J}(N_t) \equiv \mathbb{E}_t [\alpha J^1(\alpha, N_t) + (1 - \alpha) J^0(\alpha, N_t)] = KN_t. \quad (9)$$

Note that the value function given information acquisition at date  $t$  is higher when there is more noise in the order flow (i.e., higher  $\sigma_Z$ ), when there is more prior uncertainty about whether  $N_t$  is informative (i.e., when  $\alpha$  is closer to 0.5), and when the information advantage is expected to be longer lived (i.e., when  $r$  is smaller).

The standard approach in the literature restricts the strategic trader to make her information choices before trading begins. In this case, she follows a naive “NPV” rule — she only acquires information if the value from becoming informed is higher than the cost i.e.,  $\bar{J}(N_0) \geq c$ . As the following corollary highlights, the resulting information acquisition decision is effectively a static one.

**Corollary 1.** *If the strategic trader is restricted to acquiring information at  $t = 0$ , she optimally acquires information only if  $N_0 \geq N_0^*$ , where  $N_0^* = \frac{c}{K}$ . Moreover, the optimal exercise boundary  $N_0^*$  increases in  $c$ , decreases in  $\sigma_Z$ , is U-shaped in  $\alpha$  (minimized at  $\alpha = 0.5$ ), and increases in  $r$ .*

With dynamic information acquisition, the optimal time to acquire information is characterized by the following problem:

$$J^U(n) \equiv \sup_{\tau \in \mathcal{T}} \mathbb{E} [1_{\{\tau < T\}} (\bar{J}(N_\tau) - c) | N_t = n] = \sup_{\tau \in \mathcal{T}} \mathbb{E} [e^{-r\tau} (KN_\tau - c)^+ | N_t = n]. \quad (10)$$

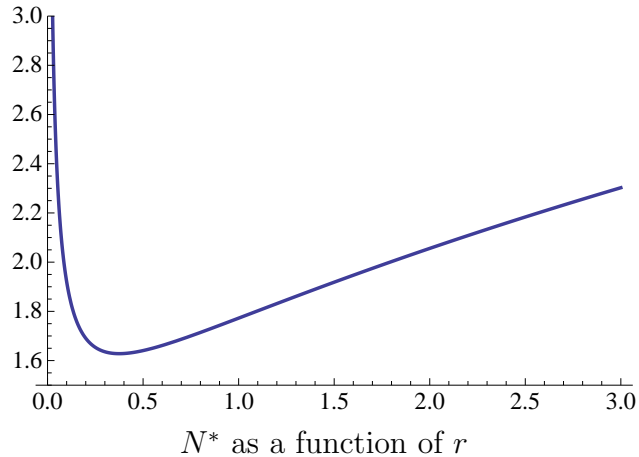
This problem is analogous to characterizing the optimal exercise time for a perpetual American call option.<sup>13</sup> Notably, the optimal information acquisition decision exhibits delay: information is not acquired when  $KN_t = c$ , as would be implied by the static NPV rule. The intuition for this effect is analogous to that for investment delay in a real options problem. At any point in time, the trader can exercise her “option” to acquire information and use that information to profit at the expense of the noise traders. However, by waiting and observing the news process she learns additional information about the asset payoff (and therefore her ultimate profits) on which she can condition her decision. Since acquiring information irreversibly sacrifices the ability to wait, it is optimal to acquire only when doing

---

<sup>13</sup>Hence, appealing to standard results, we establish that the optimal stopping time is a first hitting time for the  $N_t$  process and show that the given  $N^*$  is a solution to this problem.

Figure 1: Exercise Boundary  $N^*$

Unless otherwise specified, parameters are set to  $\sigma_Z = \sigma_N = 1$ ,  $c = 0.25$  and  $\alpha = 0.5$ .



so is sufficiently profitable to overcome this opportunity cost. Moreover, the option to wait is more valuable (and hence  $N^*$  is higher) when the volatility of the news process (i.e.,  $\sigma_N$ ) is higher.

A key difference between the static acquisition boundary of Corollary 1 and the dynamic acquisition boundary of Proposition 2 is how they respond to the expected trading horizon. In the static case, the exercise boundary is increasing in  $r$ . Recall that increasing  $r$  increases the likelihood that the payoff is revealed sooner i.e., it decreases the expected trading horizon. This naturally decreases the value from acquiring information, since the trader has a shorter window over which to exploit her informational advantage.

With dynamic information acquisition, the trader also accounts for the cost of waiting to acquire information. Specifically, as the trading horizon increases (i.e.,  $r$  decreases), the expected value from acquiring information at any date (i.e.,  $J^U(N_t)$ ) increases. However, she is also willing to wait longer to acquire this information, since the cost of waiting (the probability the value will be revealed before she acquires information) also decreases. Initially, the first effect dominates, which leads the exercise boundary to decrease as the trading horizon increases. Eventually, however, the second effect dominates, and the exercise boundary increases with the horizon. As Figure 1 illustrates, this implies that the exercise boundary is non-monotonic in  $r$  with dynamic information acquisition: the trader is less likely to acquire information when the asset payoff is expected to be revealed too quickly or too slowly.

### 3.3 Likelihood of information acquisition

The likelihood of information acquisition depends on two forces. First, the cost of information may be too high relative to the value of acquiring it: given  $c$ , the trader might never find it optimal to acquire the information. Second, even if the (relative) cost of acquisition is not too high, the asset payoff may be revealed before the strategic trader chooses to acquire information. The following results characterize how these effects interact to determine the likelihood of information acquisition.

In what follows, it is useful to define  $T_N$  as the first time  $N_t \geq N^*$ . Then, the time at which information is acquired can be expressed as

$$\tau = T_N 1_{\{T_N \leq T\}} + \infty \times 1_{\{T_N > T\}}, \quad (11)$$

where, as before,  $\tau = \infty$  corresponds to no information acquisition. To avoid the trivial case, assume  $N_0 < N^*$ . We begin with the following observation.

**Lemma 1.** *Suppose  $N_0 < N^*$ . For  $0 \leq t < \infty$ , the probability that  $T_N \in [t, t + dt]$  is given by*

$$\Pr(T_N \in [t, t + dt]) = \frac{\left(\log\left(\frac{N^*}{N_0}\right)\right)}{\sigma_N \sqrt{2\pi t^3}} \exp\left\{-\frac{\left(\frac{1}{\sigma_N} \log\left(\frac{N^*}{N_0}\right) + \frac{1}{2}\sigma_N^2 t\right)^2}{2t}\right\} dt. \quad (12)$$

The probability that  $T_N$  is not finite is given by  $\Pr(T_N = \infty) = 1 - \frac{N_0}{N^*}$ .

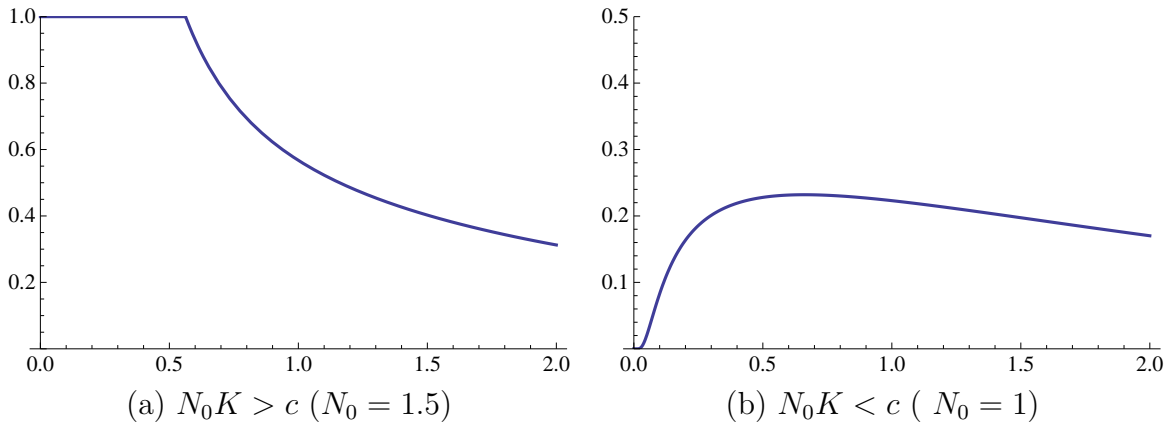
The result follows from applying results on the first hitting time of a Brownian motion with drift. Since information acquisition is costly and the news process is a martingale, there is a positive probability that the boundary is never hit, even if  $T \equiv \infty$ . Since the above expression is increasing in the boundary  $N^*$ , the probability of information acquisition decreases in the cost  $c$  and volatility  $\sigma_N$ , increases in volatility of noise trading  $\sigma_Z$  and uncertainty about  $\xi$  (i.e., is hump-shaped in  $\alpha$ ), and is hump-shaped in  $r$ .

The next result accounts for the possibility that the payoff is revealed before the information is acquired (i.e,  $T_N > T$ ).

**Proposition 3.** *Suppose  $N_0 < N^*$ . The probability that information is acquired is  $\Pr(\tau < \infty) = \left(\frac{N_0}{N^*}\right)^\beta$ . The probability is decreasing in  $c$ , increasing in  $N_0$  and  $\sigma_Z$ , hump-shaped in  $\alpha$  (around  $\frac{1}{2}$ ), and hump-shaped in  $r$ . When  $c \leq N_0 K$ , the probability is decreasing in  $\sigma_N$ ; when  $c > N_0 K$ , it is hump-shaped in  $\sigma_N$ .*

Not surprisingly, accounting for the possibility that the payoff is revealed before  $N_t$  hits  $N^*$  reduces the likelihood of information acquisition (i.e.,  $\Pr(\tau < \infty) < \Pr(T_N < \infty)$ ), since

Figure 2: Probability that information is acquired  $\Pr(\tau < \infty)$  versus  $\sigma_N$ . Unless otherwise specified, parameters are set to  $\sigma_Z = 1$ ,  $c = 0.25$ ,  $r = 1.5$ ,  $\alpha = 0.5$ .



$N_0 < N^*$  and  $\beta > 1$ ). More interestingly, it also changes the effect of the volatility  $\sigma_N$  of the news process on the likelihood of acquisition. Increasing  $\sigma_N$  has two effects: (i) it increases the acquisition boundary (i.e.,  $N^*$  increases in  $\sigma_N$ ), and (ii) fixing the boundary, it increases the likelihood that  $N_t$  will hit the boundary by any given time (i.e.,  $N_t$  is more volatile). Appealing to the analogy with an American call option, the above result highlights that when the option starts in the money (i.e.,  $c \leq N_0K$ ), the first effect dominates and the probability of acquisition (i.e., the probability the option is exercised) decreases in  $\sigma_N$ . However, when the option is initially out of the money (i.e.,  $c > N_0K$ ), then for low values of  $\sigma_N$ , the second effect dominates the first and the probability of acquisition initially increases in  $\sigma_N$ .

Figure 2 presents an example of this non-monotonic effect of  $\sigma_N$  on the probability of information acquisition. In panel (a),  $N_0$  is sufficiently high so that  $N_0K \geq c$ , and so the probability of information acquisition is decreasing in  $\sigma_N$ . In panel (b),  $N_0$  is low enough so that the probability of information acquisition initially increases and then decreases in  $\sigma_N$ .

In Appendix B we explore some additional properties of the equilibrium with observability. First, the dynamic nature of the trader's information acquisition decision leads to novel price dynamics: information acquisition triggers a jump in instantaneous volatility and price impact, and following acquisition, both evolve stochastically. Notably, these results are not driven by stochastic volatility of fundamentals or noise trading, but arise endogenously due to the trader's acquisition decision and the market maker's learning problem.<sup>14</sup>

Second, we characterize the average absolute price change at the time the asset payoff is

<sup>14</sup>Although not the focus of their analysis, a similar result on stochastic volatility and price impact arises in Back and Baruch (2004). However, our result differs from Collin-Dufresne and Fos (2016), where stochastic volatility and price impact are driven by stochastic volatility in noise trading.

publicly announced. Intuitively, one might expect that this announcement effect is smaller when the strategic trader is informed, since the order flow is more informative about the asset payoff.<sup>15</sup> We show that this need not be the case when the timing of information acquisition is endogenous, because the strategic trader only chooses to acquire information when uncertainty is sufficiently high. In fact, when the cost of information acquisition is sufficiently high, the public signal volatility is sufficiently high, or the expected trading horizon is sufficiently extreme (i.e., sufficiently short or sufficiently long), the expected announcement effect is larger when there is information acquisition.

## 4 Unobservable information acquisition

In this section, we study the case in which the strategic trader's decision to acquire information is not observable by the market maker. In contrast to the case in which acquisition is observable, we show that there cannot exist pure strategy equilibria. Moreover, we find that existence of a class of mixed strategy equilibria is also ruled out.

To analyze this scenario, we need to introduce some additional notation. As before, denote the types of strategic trader by  $i \in \{U, 1, 0\}$ , corresponding to uninformed, informed of  $\xi = 1$ , and informed of  $\xi = 0$ . In this case, the public information filtration  $\mathcal{F}_t^P$  is the augmentation of the filtration  $\sigma(\{N_t, Y_t, \mathbf{1}_{\{T \leq t\}}\})$ , which is analogous to the observable case except that the market maker no longer observes the indicator  $I_t$ . Let  $p_t = \mathbb{E}[I_t \xi | \mathcal{F}_t^P]$  and  $q_t = \mathbb{E}[I_t(1 - \xi) | \mathcal{F}_t^P]$  denote the market maker's conditional probabilities that the trader is informed and has observed  $\xi = 1$  and  $\xi = 0$ , respectively. As before, the public news follows (2). Hence, the state space is  $\mathcal{S} = \{(p, q, N) \in [0, 1]^2 \times [0, \infty) : 0 \leq p + q \leq 1\}$ . As updating on the presence of an informed trader ceases if  $p + q$  reaches zero or one, if  $(p, q, N)$  reaches the boundary  $\{(p, q, N) : p + q = 1\}$  the evolution of  $p, q$  is such that the state remains in this set. Furthermore, the edges of this boundary,  $(1, 0, \mathbb{R}_+)$  and  $(0, 1, \mathbb{R}_+)$ , are absorbing for  $(p, q)$ .

We first consider the possibility of equilibria in which information acquisition is a pure strategy. Note that never acquiring information cannot be an equilibrium. If it were, the market maker should rationally be insensitive to order flow. But in this case, the strategic trader can profitably deviate by acquiring information and trading arbitrarily large quantities at the unresponsive price.

---

<sup>15</sup>For instance, as [Back \(1992\)](#) establishes, conditional on the strategic trader being informed the announcement effect must be zero in the analogous, finite horizon model where the announcement is perfectly anticipated. When the announcement date is stochastic, but the strategic trader is exogenously endowed with information, as in [Back and Baruch \(2004\)](#), the announcement effect is smaller on average when the strategic trader is informed.

Next, consider an equilibrium where the strategic trader does not acquire information until some stopping time  $\tau > 0$  (as in the observable case). In any such pure-strategy equilibrium, the market maker should rationally set the price impact to zero before time  $\tau$ . But since information acquisition is not observable, this permits the strategic trader to deviate by acquiring information preemptively and trading with zero price impact, generating unbounded profits.

The final candidate for an equilibrium with pure-strategy acquisition is one in which the trader acquires information immediately. This type of equilibrium is only possible when the cost of information is sufficiently small. In this case, the expected payoff from acquiring information immediately is  $\bar{J}(N_0) - c$ , as described by (9) above. Anticipating that the strategic trader acquires information, the market maker rationally sets the price response to  $\lambda(p)$  from Proposition 1. However, given this price response, we show that the value from remaining uninformed is  $J^U(N_0) = \bar{J}(N_0)$ . This implies that the strategic trader can deviate by not acquiring information, save the cost of information acquisition  $c$ , and obtain the same expected trading profits going forward.

Intuitively, the uninformed strategic trader exploits “mis-pricing” in the risky asset: while the market maker treats the order flow as informative in the conjectured equilibrium, the uninformed strategic trader knows that it is not. Since the noise trader demand is effectively observable to the strategic trader in a continuous time setting, she possesses an informational advantage which she can profitably exploit.<sup>16</sup> In our setting, we show that this informational advantage generates the same value as the ex-ante expected value from acquiring information, gross of cost  $c$ .

The following result summarizes these arguments.

**Proposition 4.** *There does not exist an equilibrium in which the trader follows a smooth trading strategy and a pure information acquisition strategy.*

Next, we entertain the possibility that information acquisition follows a mixed strategy. A mixed information acquisition strategy is a probability distribution over stopping times in  $\mathcal{T}$ . That is, at the beginning of the game, the trader randomly chooses a (pure) stopping time according to some probability distribution, and follows the realized strategy for the duration of the game.<sup>17</sup> Because the trader must be indifferent between any stopping time

---

<sup>16</sup>In contrast, an uninformed strategic trader cannot infer the current realization of the noise trade in the standard, discrete-time model, and so optimally trades zero.

<sup>17</sup>There are multiple, equivalent ways of defining randomization over stopping times. [Aumann \(1964\)](#) introduced the notion of randomizing by choosing a stopping time according to some probability distribution at the start of the game. [Touzi and Vieille \(2002\)](#) treat randomization by identifying the stopping strategy with an adapted, non-decreasing, right-continuous processes on  $[0, 1]$  that represents the cdf of the time that stopping occurs. [Shmaya and Solan \(2014\)](#) show, under weak conditions, that these definitions are equivalent.

$\tau$  over which she mixes, each such  $\tau$  must also achieve the maximum in her optimization problem

$$\max_{\{\theta^i(\cdot)\}_{i \in \{U, 1, 0\}}, \tau \in \mathcal{T}} \mathbb{E} \left[ \int_0^\tau \theta^U (N_T \xi - P_s) ds + \int_\tau^T (\alpha \theta^1 + (1 - \alpha) \theta^0) (N_T \xi - P_s) ds \right].$$

Given a trading strategy  $\theta$ , let  $L^\theta$  denote the generator of the process  $\mathcal{X} \equiv (p, q, N)$  i.e., let

$$L^\theta f(x) \equiv \lim_{t \downarrow 0} \frac{\mathbb{E}^x [f(\mathcal{X}_t)] - f(x)}{t}.$$

As before, given a stopping time  $\tau$  over which the trader mixes, let  $J^i$ ,  $i \in \{U, 1, 0\}$  denote the value functions of the three types, and let  $\bar{J} = \alpha J^1 + (1 - \alpha) J^0$  reflect the uninformed trader's conditional expected value of the informed type value function at any point  $(p, q, N)$ . Let  $C = \{(p, q, N) \in \mathcal{S} : J^U > \bar{J} - c\}$ . Suppose that  $J^i$  are continuous on  $\mathcal{S}$ ,  $J^U$  is a twice continuously differentiable solution of the HJB equation in  $C$ , and  $J^i$  for  $i \in \{0, 1\}$  are twice continuously differentiable solutions of the HJB equation in the interior of  $\mathcal{S}$ . i.e., for  $i \in \{0, 1, U\}$ ,

$$rJ^i = \sup_{\theta \in \mathbb{R}} \{ \theta (Nv_i - P) + L^\theta J^i \}, \quad (13)$$

where  $v_i$  is the conditional expectation of  $\xi$  for type  $i$ .

The following Proposition establishes that there does not exist an overall equilibrium in the model in which the value functions of the three types solve the HJB equations in the relevant regions and satisfy the value matching condition at at least one point at which information is acquired with positive probability.<sup>18</sup>

**Proposition 5.** *There does not exist an overall equilibrium in which*

1. *The trader follows a smooth trading strategy and a mixed acquisition strategy,*
2. *The value functions of the three types satisfy the smoothness restrictions and are solutions to the associated HJB equations as stated above, and*
3. *There exists a stopping time  $\tau$  over which the trader mixes, and a point on the boundary of the corresponding continuation region, at which the trader is indifferent between*

---

<sup>18</sup>Note that in any mixed strategy equilibrium, there must be information acquisition at date zero with positive probability. If not, the price sensitivity to order flow should be zero, but this implies the uninformed strategic trader can deviate by acquiring information preemptively. As a result, in any mixed equilibrium,  $0 < p_0 + q_0 < 1$ .



*acquiring and not acquiring information i.e., at which the value matching condition  $J^U = \bar{J} - c$  holds.*

The result relies on a feature of the continuous-time Kyle model that has been highlighted by [Back \(1992\)](#). Importantly, the optimality of the trading strategy  $\theta^i$  for any strategic trader implies that

$$rJ^i = \sup_{\theta \in \mathbb{R}} \{ \theta(N\xi - P) + L^\theta J^i \} = L^0 J^i, \quad (14)$$

i.e., the strategic trader's maximum attainable value at date  $t$  can be attained by not trading until some later date  $s$  and then trading optimally going forward. But this implies a profitable deviation for the trader at the point of information acquisition. Instead of paying cost  $c$  and (with positive probability) irreversibly becoming an informed type for the remainder of the game, she can remain uninformed, refrain from trading, and re-evaluate her acquisition decision after the next instant. Owing to the effect of discounting (since the value of the asset is revealed with positive probability at every instant), waiting to become informed saves her at least  $crdt$ . Hence, she strictly prefers to remain uninformed over the next instant. But this deviation rules out indifference at any conjectured acquisition point.

The result also rules out the class of equilibria in which the strategic trader mixes within an open set of  $(p, q, N)$ , by acquiring with a given intensity. In fact, when the strategic trader mixes in such a way, value matching must hold identically over the set. However, analogous arguments to those in the proof of [Proposition 5](#) establish that these are inconsistent with optimal trading, i.e., condition [\(13\)](#).

Note that the above deviations are not feasible when information acquisition is observable, since the market maker could immediately respond to any decision to preempt / delay acquisition. As such, this analysis highlights the key role that observability plays when the strategic trader can both trade and choose when to acquire information dynamically. Moreover, while the above arguments are developed in the context of our benchmark model, the key arguments may be applicable to other settings.

## 5 Conclusions

The canonical Kyle-type framework, in which the market maker sets prices in response to strategic trading by an informed trader, provides an important benchmark for understanding how markets aggregate private information. A key limitation of the standard setup is that the strategic trader is endowed with private information before trading begins, instead of acquiring it endogenously. To explore the implications of endogenous information acquisition,

we consider a continuous-time strategic trading model in which the trader can choose *when* to acquire information about the asset payoff in response to the evolution of a public signal.

We show that the existence and nature of equilibrium depends crucially on whether the information acquisition decision is observable by the market maker. When acquisition is observable, we show there exists a unique equilibrium in pure (acquisition) strategies. Moreover, the equilibrium features delay in information acquisition. In contrast, when acquisition is not observable, we find there cannot exist pure strategy equilibria. Moreover, a class of mixed strategy equilibria satisfying natural conditions is also ruled out.

Our analysis suggests that key features of the standard, strategic trading framework may be difficult to reconcile with costly dynamic information acquisition. Exploring the robustness of these results to different distributional assumptions, information acquisition technologies (e.g., costs dependent upon the precision of information), and competition among traders are natural next steps. It would also be interesting to study how our analysis changes when the public signal is endogenized (e.g., in the form of strategic disclosure by firms or regulators).

## References

- Aumann, R., 1964. Mixed and behavior strategies in infinite extensive games. In: Dresher, M., Shapley, L., Tucker, A. (Eds.), *Advances in Game Theory*. No. 52. Princeton, pp. 627–650.
- Avdis, E., 2016. Information tradeoffs in dynamic financial markets. *forthcoming in Journal of Financial Economics*.
- Back, K., 1992. Insider trading in continuous time. *Review of financial Studies* 5 (3), 387–409.
- Back, K., Baruch, S., 2004. Information in securities markets: Kyle meets glosten and milgrom. *Econometrica* 72 (2), 433–465.
- Back, K., Crotty, K., Li, T., 2016. Identifying information asymmetry in securities markets. Available at SSRN 2565216.
- Back, K., Pedersen, H., 1998. Long-lived information and intraday patterns. *Journal of Financial Markets* 1 (3), 385–402.
- Banerjee, S., Green, B., 2015. Signal or noise? uncertainty and learning about whether other traders are informed. *Journal of Financial Economics* 117 (2), 398–423.
- Black, F., 1976. Studies of stock price volatility changes. *Proceedings of the 1976 Meetings of the American Statistical Association, Business and Economic Statistics Section*, 177–181.
- Collin-Dufresne, P., Fos, V., 2016. Insider trading, stochastic liquidity and equilibrium prices. *Econometrica*, Forthcoming.
- Foster, F. D., Viswanathan, S., 1993. The effect of public information and competition on trading volume and price volatility. *Review of Financial Studies* 6 (1), 23–56.
- Grossman, S. J., Stiglitz, J. E., 1980. On the impossibility of informationally efficient markets. *American Economic Review* 70 (3), 393–408.
- Holden, C. W., Subrahmanyam, A., January 2002. News events, information acquisition, and serial correlation. *Journal of Business* 75 (1), 1–32.
- Karatzas, I., Shreve, S. E., 1998. *Brownian Motion and Stochastic Calculus*, 2nd Edition. Vol. 113 of *Graduate Texts in Mathematics*. Springer, New York.
- Kyle, A. S., 1985. Continuous auctions and insider trading. *Econometrica*, 1315–1335.

- Li, T., 2013. Insider trading with uncertain informed trading. Available at SSRN 946324.
- Mendelson, H., Tunca, T. I., 2004. Strategic trading, liquidity, and information acquisition. *Review of Financial Studies* 17 (2), 295–337.
- Øksendal, B., Sulem, A., 2007. *Applied Stochastic Control of Jump Diffusions*, 2nd Edition. Springer, New York.
- Owen, D., 1980. A table of normal integrals: A table. *Communications in Statistics-Simulation and Computation* 9 (4), 389–419.
- Peskir, G., Shiryaev, A., 2006. *Optimal Stopping and Free-Boundary Problems*. Lectures in Mathematics, ETH Zürich. Birkhäuser, Boston.
- Protter, P. E., 2003. *Stochastic Integration and Differential Equations*, 2nd Edition. Vol. 21 of *Applications of Mathematics*. Springer, New York.
- Shmaya, E., Solan, E., 2014. Equivalence between random stopping times in continuous time, working paper.
- Touzi, N., Vieille, N., 2002. Continuous-time dynkin games with mixed strategies. *SIAM Journal on Control and Optimization* 41 (4), 1073–1088.
- Veldkamp, L. L., June 2006. Media frenzies in markets for financial information. *American Economic Review* 96 (3), 577–601.
- Wang, Y., Yang, M., 2016. Insider trading when there may not be an insider. Working paper (<https://ssrn.com/abstract=2720736>).

# A Proofs of Main Results

**Proof of Proposition 1.** To establish the equilibrium in the Proposition, we need to show: (i) the proposed price function is rational, and (ii) the informed trader's strategy is optimal. Fix any  $\tau \in \mathcal{T}$ .

## Rationality of pricing function

Consider the set  $\{t : t < \tau\}$  on which the trader has not acquired information. Then, because  $\{N_t\}$ ,  $\{Z_t\}$  and  $\xi$  are independent, and under the proposed trading strategy  $Y_t = Z_t$  for  $t < \tau$ , it is immediate that

$$\mathbb{E}[\xi N_T | \mathcal{F}_t^P] = \mathbb{E}[\xi | \mathcal{F}_t^P] \mathbb{E}[N_T | \mathcal{F}_t^P] = \alpha \mathbb{E}[N_T | \mathcal{F}_t^P].$$

Since  $T$  is almost surely finite and is independent of the process  $N_t$  we have  $\mathbb{E}[N_T | \mathcal{F}_t^P] = N_t$ , and so  $\mathbb{E}[\xi N_T | \mathcal{F}_t^P] = \alpha N_t$ .

Now, consider the set  $\{t : \tau \leq t < T\}$  on which the trader has acquired information and the asset payoff has not yet occurred. Up to the addition of the news process, the problem now resembles that considered in [Back and Baruch \(2004\)](#), and we can adapt the proof offered there. Specifically, consider the pricing rule from [Back and Baruch \(2004\)](#), adapted for the fact that information is acquired at time  $\tau$ ,

$$dp_t = \lambda(p) dY_t, \quad p_\tau = \alpha,$$

where  $\lambda(p)$  is given in the statement of the Proposition. (Later we will show that this pricing rule can be written in the explicit form in eq. (6).) Note that the proposed trading strategy depends only on  $\xi$  and  $p$ , the process  $p$  depends only on the order flow, and  $\{N_t\}$  is independent of  $\xi$  and  $\{Z_t\}$ , so  $(\xi, \{p_t\})$  is conditionally independent of  $\{N_t\}$ , and therefore

$$\mathbb{E}[\xi N_T | \mathcal{F}_t^P] = \mathbb{E}[\xi | \mathcal{F}_t^P] \mathbb{E}[N_T | \mathcal{F}_t^P] = \mathbb{E}[\xi | \{Y_s\}_{s \leq t}] N_t,$$

where the final equality follows since  $\mathbb{E}[N_T | \mathcal{F}_t^P] = N_t$ . Furthermore, since  $Y_t = Z_t$  for  $t < \tau$  under the proposed trading strategy and  $\xi$  is independent of  $\{Z_t\}$  it follows that  $\mathbb{E}[\xi | \{Y_s\}_{s \leq t}] = \mathbb{E}[\xi | \{Y_s\}_{\tau \leq s \leq t}]$ .

Recall that as of time  $\tau$ , the informed trader begins trading according to the strategy  $\theta^\xi(p)$  and the order flow becomes informative. The market maker's conditional expectation is simply equal to her prior  $\alpha$  since before this time only noise traders have been active. It follows that starting at time  $\tau$  the market maker's filtering problem becomes identical to

that of the market maker in [Back and Baruch \(2004\)](#). Hence, their Theorem 1 implies that for  $t \geq \tau$  the pricing rule

$$dp_t = \lambda(p)dY_t, \quad p_\tau = \alpha,$$

satisfies  $p_t = \mathbb{E}[\xi | \{Y_s\}_{s \geq \tau}]$ .

To complete the proof of the rationality of the proposed price, it suffices to show that the explicit form of  $p(\cdot)$  for  $\tau \leq t < T$  in eq. (6) satisfies  $dp_t = \lambda(p)dY_t$ . Applying Ito's Lemma to the function  $f(p) = \sqrt{\frac{\sigma_Z^2}{2r}} \Phi^{-1}(p)$  to the above process for  $p_t$  gives

$$\begin{aligned} df(p_t) &= \frac{1}{2} \sigma_Z^2 \lambda^2(p_t) \frac{\frac{2r}{\sigma_Z^2} f(p_t)}{\lambda^2(p_t)} dt + \frac{1}{\lambda(p_t)} \lambda(p_t) dY_t \\ &= r f(p_t) dt + dY_t. \end{aligned}$$

Now applying Ito's lemma to the function  $e^{-rt} f(p_t)$  and integrating allows one to express

$$f(p_t) = f(p_\tau) e^{rt} + \int_\tau^t e^{r(t-s)} dY_s.$$

Note that  $f(p_\tau) = \sqrt{\frac{\sigma_Z^2}{2r}} \Phi^{-1}(\alpha)$ , so returning to the explicit form of the function  $f(p)$  and inverting it follows that

$$p_t = \Phi \left( \Phi^{-1}(\alpha) e^{r(t-\tau)} + \sqrt{\frac{2r}{\sigma_Z^2}} \int_\tau^t e^{r(t-s)} dY_s \right).$$

### Optimality of trading strategy

Next, we demonstrate the optimality of the proposed trading strategy, taking as given the acquisition time  $\tau$ . This analysis closely follows the proof in [Back and Baruch \(2004\)](#). Define  $V(p) \equiv \int_p^1 \frac{1-a}{\lambda(a)} da$  and consider the proposed post-acquisition value function for the case  $\xi = 1$  (the case for  $\xi = 0$  is analogous)

$$J^1(p_t, N_t) = N_t V(p_t).$$

We begin by showing that the given  $J$  characterizes the value function for  $t \geq \tau$ . Consider  $\{t : \tau \leq t < T\}$  and suppose  $\xi = 1$ . Direct calculation on the function  $V$  yields

$$V' = \frac{p-1}{\lambda} \tag{15}$$

$$rV = \frac{1}{2}\sigma_Z^2\lambda^2V'', \quad (16)$$

which coincides with eq. (1.15) and (1.16) in [Back and Baruch \(2004\)](#).

Let  $\theta_t$  denote an arbitrary admissible trading strategy. Following [Back and Baruch \(2004\)](#), let  $\hat{p}_t$  denote the process defined by  $\hat{p}_s = \alpha$  for  $s \leq \tau$  and  $d\hat{p}_t = \lambda(\hat{p})dY_t$  for  $t > \tau$  and  $0 < \hat{p}_t < 1$ , with  $Y_t$  generated when the trader follows the given arbitrary trading strategy. In order to condense notation, in this section, we denote  $\mathbb{E}[\cdot | \mathcal{F}_t^P] = \mathbb{E}_t[\cdot]$ . Since  $\theta$  is admissible, we know that

$$\mathbb{E}_\tau \left[ \int_\tau^T N_u(1 - p_u)\theta_u^- du \right] = \mathbb{E}_\tau \left[ \int_\tau^\infty e^{-r(u-\tau)} N_u(1 - \hat{p}_u)\theta_u^- du \right] < \infty,$$

from which it follows that

$$\int_\tau^\infty e^{-r(u-\tau)} N_u(1 - \hat{p}_u)\theta_u^- du < \infty$$

almost surely, and therefore that the integral

$$\int_\tau^\infty e^{-r(u-\tau)} N_u(1 - \hat{p}_u)\theta_u du$$

is well-defined, though is possibly infinite.

Let  $\hat{T} = \inf\{t \geq \tau : \hat{p} \in \{0, 1\}\}$ . Applying Ito's lemma to  $e^{-r(t-\tau)}J$  yields

$$\begin{aligned} & e^{-r(t\wedge\hat{T}-\tau)}J^1(\hat{p}_{t\wedge\hat{T}}, N_{t\wedge\hat{T}}) - J^1(\hat{p}_\tau, N_\tau) \\ &= \int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}N \left( -rV(\hat{p}_u) + \lambda\theta V'(\hat{p}_u) + \frac{1}{2}\sigma_Z^2\lambda^2V'' \right) du \\ & \quad + \sigma_Z \int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}N\lambda V'(\hat{p}_u)dW_{Zu} + \sigma_N \int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}NV(\hat{p}_u)dW_{Nu} \\ &= - \int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}N_u\theta_u(1 - \hat{p}_u)du - \sigma_Z \int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}N_u(1 - \hat{p}_u)dW_{Zu} \\ & \quad + \sigma_N \int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}N_uV(\hat{p}_u)dW_{Nu} \end{aligned} \quad (17)$$

where the last equality uses eq. (15) and (38). Since  $V \geq 0$ , the above implies

$$\int_\tau^{t\wedge\hat{T}} e^{-r(u-\tau)}N_u\theta_u(1 - \hat{p}_u)du \leq N_\tau V(\alpha) + x(t), \quad (18)$$

where we define  $x(t) = \sigma_N \int_{\tau}^{t \wedge \hat{T}} e^{-r(u-\tau)} N_u V(\hat{p}_u) dW_{N_u} - \sigma_Z \int_{\tau}^{t \wedge \hat{T}} e^{-r(u-\tau)} N_u (1 - \hat{p}_u) dW_{Z_u}$ . The integrands in the stochastic integrals are locally bounded and hence the integrals are local martingales (Thm. 29, Ch. 4, Protter (2003)). It follows that  $x(t)$  is itself a local martingale (Thm. 48, Ch. 1, Protter (2003)).

Let  $\hat{\tau}_n$  be a localizing sequence of stopping times for  $x(t)$ . That is,  $\hat{\tau}_{n+1} \geq \hat{\tau}_n$ ,  $\hat{\tau}_n \rightarrow \infty$ , and  $x(t \wedge \hat{\tau}_n)$  is a martingale for each  $n$ . Because  $x(t)$  is a local martingale such a sequence exists (e.g., because  $x(t)$  is continuous we can take  $\hat{\tau}_n = \inf\{t : |x(t)| \geq n\}$ ). Further considering the sequence  $n \wedge \hat{\tau}_n$ , eq. (18) implies

$$\int_{\tau}^{n \wedge \hat{\tau}_n \wedge \hat{T}} e^{-r(u-\tau)} N_u \theta_u (1 - \hat{p}_u) du \leq N_{\tau} V(\alpha) + x(n \wedge \hat{\tau}_n).$$

Applying Fatou's lemma,<sup>19</sup> along with this inequality, yields

$$\begin{aligned} \mathbb{E}_{\tau} \left[ \int_{\tau}^{\hat{T}} e^{-r(u-\tau)} N_u \theta_u (1 - \hat{p}_u) du \right] &\leq \liminf_{n \rightarrow \infty} \mathbb{E}_{\tau} \left[ \int_{\tau}^{n \wedge \hat{\tau}_n \wedge \hat{T}} e^{-r(u-\tau)} N_u \theta_u (1 - \hat{p}_u) du \right] \\ &\leq N_{\tau} V(\alpha) + \liminf_{n \rightarrow \infty} \mathbb{E}_{\tau} [x(n \wedge \hat{\tau}_n)] \\ &\leq N_{\tau} V(\alpha). \end{aligned}$$

Note that for  $\hat{T} < \infty$  we have  $\hat{p}_{\hat{T}} = 1$  since  $\hat{p}_{\hat{T}} = 0$  would imply a violation of the admissibility condition. To establish this, note that eq. (17) implies

$$-\mathbb{E}_{\tau} \left[ \int_{\tau}^{n \wedge \hat{\tau}_n \wedge \hat{T}} e^{-r(u-\tau)} N_u \theta_u (1 - \hat{p}_u) du \right] = \mathbb{E}_{\tau} \left[ e^{-r(t \wedge \hat{T} - \tau)} N_{t \wedge \hat{T}} V(\hat{p}_{t \wedge \hat{T}}) - N_{\tau} V(\alpha) \right] - J^1(\hat{p}_{\tau}, N_{\tau}),$$

and therefore

$$\begin{aligned} &-\mathbb{E}_{\tau} \left[ \int_{\tau}^{\hat{T}} e^{-r(u-\tau)} N_u \theta_u (1 - \hat{p}_u) du \right] \\ &\geq \limsup_{n \rightarrow \infty} \mathbb{E}_{\tau} \left[ - \int_{\tau}^{n \wedge \hat{\tau}_n \wedge \hat{T}} e^{-r(u-\tau)} N_u \theta_u (1 - \hat{p}_u) du \right] \\ &= \limsup_{n \rightarrow \infty} \mathbb{E}_{\tau} \left[ e^{-r(n \wedge \hat{\tau}_n \wedge \hat{T} - \tau)} N_{n \wedge \hat{\tau}_n \wedge \hat{T}} V(\hat{p}_{n \wedge \hat{\tau}_n \wedge \hat{T}}) - N_{\tau} V(\alpha) \right] - J^1(\hat{p}_{\tau}, N_{\tau}) \\ &\geq \mathbb{E}_{\tau} \left[ e^{-r(\hat{T} - \tau)} N_{\hat{T}} V(\hat{p}_{\hat{T}}) \right] - J^1(\hat{p}_{\tau}, N_{\tau}) \\ &= \infty, \end{aligned}$$

---

<sup>19</sup>The typical formulation of Fatou's Lemma requires that the integrands  $f_n$  be weakly positive. However, if  $f_n^-$  is bounded above by an integrable function  $g$ , considering  $f_n + g$  in Fatou's lemma delivers the result. Here, due to the admissibility condition we can take  $g = N_u(1 - p_u)\theta_u^-$ .



where the first line applies the ‘reverse’ Fatou’s Lemma, the second line uses the equality in the previous displayed equation, the third line applies Fatou’s Lemma and the final line follows because  $V(0) = \infty$ . Furthermore,  $\hat{p}_u = \hat{p}_{\hat{T}} = 1$  for all  $u \geq \hat{T}$  since 1 is an absorbing state. It follows that

$$\mathbb{E}_\tau \left[ \int_\tau^\infty e^{-r(u-\tau)} N_u \theta_u (1 - \hat{p}_u) du \right] = \mathbb{E}_\tau \left[ \int_\tau^{\hat{T}} e^{-r(u-\tau)} N_u \theta_u (1 - \hat{p}_u) du \right] \leq N_\tau V(\alpha). \quad (19)$$

Furthermore, this inequality is trivially true for  $\hat{T} = \infty$ , so it holds regardless of the behavior of  $\hat{T}$ . It follows that

$$N_\tau V(\alpha) \geq \mathbb{E}_\tau \left[ \int_\tau^\infty e^{-r(u-\tau)} N_u \theta_u (1 - \hat{p}_u) du \right] = \mathbb{E}_\tau \left[ \int_\tau^T N_u \theta_u (1 - p_u) du \right],$$

since  $\hat{p} = p$  for  $t \leq T$ . Hence  $N_\tau V(\alpha)$  is an upper bound on the post-acquisition value function.

To establish the optimality of the trader’s post-acquisition strategy and the expression for the value function, it remains to show that the expected profits generated by the strategy attain the bound  $N_\tau V(\alpha)$ . (We show below that the trader’s overall trading strategy is admissible.) Compute the trader’s expected profit at time  $\tau$ . We have

$$\begin{aligned} \mathbb{E}_\tau \left[ \int_\tau^T \theta^1(p_u) N_u (1 - p_u) du \right] &= \int_\tau^\infty \mathbb{E}_\tau \left[ \mathbf{1}_{\{t \leq T\}} \theta^1(p_u) N_u (1 - p_u) \right] du \\ &= \int_\tau^\infty \mathbb{E}_\tau [N_u] \mathbb{E}_\tau \left[ \mathbf{1}_{\{t \leq T\}} \theta^1(p_u) (1 - p_u) \right] du \\ &= N_\tau \int_\tau^\infty \mathbb{E}_\tau \left[ \mathbf{1}_{\{t \leq T\}} \theta^1(p_u) (1 - p_u) \right] du \\ &= N_\tau \mathbb{E}_\tau \left[ \int_\tau^T \theta^1(p_u) (1 - p_u) du \right], \end{aligned}$$

where the first equality applies Fubini’s theorem which is permissible because the integrand is positive, the second equality uses the fact that  $N$  is independent of  $T$  and  $\{p_u\}$ , the next-to-last equality follows because  $N$  is a martingale, and the final equality applies Fubini’s theorem again. [Back and Baruch \(2004\)](#) establish that under the given trading strategy and pricing rule,  $V(\alpha) = \mathbb{E}_\tau \left[ \int_\tau^T \theta^1(p_u) (1 - p_u) du \right]$ . Hence,

$$N_\tau V(\alpha) = \mathbb{E}_\tau \left[ \int_\tau^T \theta^1(p_u) N_u (1 - p_u) du \right],$$

which establishes the optimality of the post-acquisition trading strategy.

Let  $J^U(N)$  denote the pre-acquisition value function (i.e., the value function for an uninformed trader). Note that because  $p \equiv 0$  for  $t < \tau$ ,  $J^U$  effectively depends only on the news process in this case. We need to characterize this function and establish that the overall posited trading strategy, involving no trade prior to acquisition, is optimal. Under the given trading strategy, we have

$$\begin{aligned} J^U(N) &= \mathbb{E} \left[ \mathbf{1}_{\{\tau < T\}} \int_{\tau}^T \theta^\xi(p_u) N_u(\xi - p_u) du \right] \\ &= \mathbb{E} \left[ \mathbf{1}_{\{\tau < T\}} J^\xi(p_\tau, N_\tau) \right] \end{aligned}$$

Let  $\check{\theta}$  be any admissible trading strategy that is adapted to  $\mathcal{F}_t^P$  and  $\hat{\theta}$  any admissible strategy that is adapted to  $\mathcal{F}_t^I$ . Then  $\theta = \mathbf{1}_{\{t < \tau\}} \check{\theta} + \mathbf{1}_{\{t \geq \tau\}} \hat{\theta}$  is an arbitrary admissible strategy that obeys the restriction that the trader does not observe  $\xi$  until time  $\tau$ . The expected profits from following this strategy are

$$\begin{aligned} &\mathbb{E}_0 \left[ \mathbf{1}_{\{\tau < T\}} \int_0^\tau \check{\theta}_u N_u(\xi - \alpha) du + \mathbf{1}_{\{\tau < T\}} \int_\tau^T \hat{\theta}_u N_u(\xi - p_u) du + \mathbf{1}_{\{\tau \geq T\}} \int_0^T \check{\theta}_u N_u(\xi - \alpha) du \right] \\ &= \mathbb{E}_0 \left[ \mathbf{1}_{\{\tau < T\}} \int_\tau^T \hat{\theta}_u N_u(\xi - p_u) du \right] \\ &= \mathbb{E}_0 \left[ \mathbf{1}_{\{\tau < T\}} \mathbb{E} \left[ \int_\tau^T \hat{\theta}_u N_u(\xi - p_u) du \middle| \mathcal{F}_\tau^I \right] \right] \\ &\leq \mathbb{E}_0 \left[ \mathbf{1}_{\{\tau < T\}} J^\xi(p_\tau, N_\tau) \right] \\ &= J^U(N), \end{aligned}$$

where the first equality takes expectations over  $\xi$ , the second equality uses the law of iterated expectations, and the inequality follows since it was shown above that as of time  $\tau$ , our posited trading strategy achieves higher expected profit than any other admissible strategy.  $\square$

**Proof of Proposition 2.** Let  $\bar{J}(N_t)$  denote the value of acquiring information when the news process is equal to  $N_t$ . Using the expression for the post-acquisition value function in Proposition 1, we have

$$\bar{J}(N_t) = N_t \left( \alpha \int_\alpha^1 \frac{1-a}{\lambda(a)} da + (1-\alpha) \int_0^\alpha \frac{a}{\lambda(a)} da \right) \equiv N_t K.$$

Make the change of variables  $x = \Phi^{-1}(1-a)$  in the integrals in the expression for  $J^U(N_t)$

$$K = \alpha \sqrt{\frac{\sigma_Z^2}{2r}} \int_\alpha^1 (1-a) \frac{1}{\phi(\Phi^{-1}(1-a))} da + (1-\alpha) \sqrt{\frac{\sigma_Z^2}{2r}} \int_0^\alpha a \frac{1}{\phi(\Phi^{-1}(1-a))} da$$

$$\begin{aligned}
&= -\alpha\sqrt{\frac{\sigma_Z^2}{2r}} \int_{\Phi^{-1}(1-\alpha)}^{-\infty} \Phi(x)dx - (1-\alpha)\sqrt{\frac{\sigma_Z^2}{2r}} \int_{\infty}^{\Phi^{-1}(1-\alpha)} (1-\Phi(x)) dx \\
&= \alpha\sqrt{\frac{\sigma_Z^2}{2r}} \int_{-\infty}^{\Phi^{-1}(1-\alpha)} \Phi(x)dx + (1-\alpha)\sqrt{\frac{\sigma_Z^2}{2r}} \int_{\Phi^{-1}(1-\alpha)}^{\infty} (1-\Phi(x)) dx.
\end{aligned}$$

Now integrate by parts

$$\begin{aligned}
K &= \alpha\sqrt{\frac{\sigma_Z^2}{2r}} \int_{-\infty}^{\Phi^{-1}(1-\alpha)} \Phi(x)dx + (1-\alpha)\sqrt{\frac{\sigma_Z^2}{2r}} \int_{\Phi^{-1}(1-\alpha)}^{\infty} (1-\Phi(x)) dx \\
&= \alpha\sqrt{\frac{\sigma_Z^2}{2r}} \left( - \int_{-\infty}^{\Phi^{-1}(1-\alpha)} x\phi(x) dx + x\Phi(x) \Big|_{-\infty}^{\Phi^{-1}(1-\alpha)} \right) \\
&\quad + (1-\alpha)\sqrt{\frac{\sigma_Z^2}{2r}} \left( \int_{\Phi^{-1}(1-\alpha)}^{\infty} x\phi(x) dx + x(1-\Phi(x)) \Big|_{\Phi^{-1}(1-\alpha)}^{\infty} \right) \\
&= \alpha\sqrt{\frac{\sigma_Z^2}{2r}} \left( - \int_{-\infty}^{\Phi^{-1}(1-\alpha)} x\phi(x) dx + (1-\alpha)\Phi^{-1}(1-\alpha) \right) \\
&\quad + (1-\alpha)\sqrt{\frac{\sigma_Z^2}{2r}} \left( \int_{\Phi^{-1}(1-\alpha)}^{\infty} x\phi(x) dx - \alpha\Phi^{-1}(1-\alpha) \right) \\
&= \sqrt{\frac{\sigma_Z^2}{2r}} \int_{-\infty}^{\Phi^{-1}(1-\alpha)} -x\phi(x) dx = \sqrt{\frac{\sigma_Z^2}{2r}} \phi(\Phi^{-1}(1-\alpha)),
\end{aligned}$$

since  $\int -x\phi(x)dx = \int \phi'(x)dx = \phi(x)$ .

The pre-acquisition value function under optimal stopping is

$$J^U(n) \equiv \sup_{\tau \in \mathcal{T}} \mathbb{E} [\mathbf{1}_{\{\tau < T\}}(KN_\tau - c) \mid N_t = n] = \sup_{\tau \in \mathcal{T}} \mathbb{E} [e^{-r\tau}(KN_\tau - c)^+ \mid N_t = n],$$

where the second equality follows because  $T$  is independently exponentially distributed and it suffices to consider only the positive part of  $KN_\tau - c$  since the trader can always guarantee herself zero profit by not acquiring. Note that this problem is similar to pricing a perpetual American call option on an asset with price process  $KN_t$  that follows a geometric Brownian motion and with strike price  $c$ . Hence, standard results ([Peskir and Shiryaev \(2006\)](#), Chapter 4) imply that the optimal stopping time is a first hitting time of the  $N_t$  process,

$$T_N = \inf\{t > 0 : N_t \geq N^*\},$$

where  $N^* > 0$  is a constant to be determined.

The value function and optimal  $N^*$  solve the following free boundary problem

$$\begin{aligned}
rJ^U &= \frac{1}{2}\sigma_N^2 N_t^2 J_{NN}^U && \text{for } n < N^* \\
J^U(N^*) &= KN^* - c && \text{'value matching'} \\
J_N^U(N^*) &= K && \text{'smooth pasting'} \\
J^U(n) &> (n - c)^+ && \text{for } n < N^* \\
J^U(n) &= (n - c)^+ && \text{for } n > N^* \\
J^U(0) &= 0.
\end{aligned}$$

To determine the solution in the continuation region  $n < N^*$ , consider a trial solution of the form  $J^U(n) = An^\beta$ . Substituting and matching terms in the differential equation yields

$$r = \frac{1}{2}\sigma_N^2\beta(\beta - 1), \quad \beta = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{8r}{\sigma_N^2}}$$

and the boundary condition at  $N = 0$  requires that one take the positive root

$$\beta = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8r}{\sigma_N^2}}.$$

Applying the above conjecture to the value-matching and smooth pasting conditions implies:

$$N^* = \frac{\beta}{\beta - 1} \frac{c}{K}, \quad A = \frac{K}{\beta} \left( \frac{\beta}{\beta - 1} \frac{c}{K} \right)^{1-\beta} = \frac{c}{\beta - 1} \frac{1}{(N^*)^\beta},$$

and the resulting function satisfies  $J^U(n) > n - c$  in the continuation region, which establishes the result. The comparative statics with respect to  $c$ ,  $\sigma_N$ ,  $\sigma_Z$ , and  $\alpha$  are immediate from the explicit expression for  $N^*$ . Moreover, since

$$\frac{\partial}{\partial r} N^* = \frac{c}{\sigma_Z^2 \phi(\Phi^{-1}(1 - \alpha))} \frac{4\sqrt{2} \left( \sqrt{r} - 2 \sqrt{\frac{r}{\frac{8r}{\sigma_N^2} + 1}} \right)}{(\sigma_N - \sqrt{\sigma_N^2 + 8r})^2} \quad (20)$$

we know that  $N^*$  is decreasing in  $r$  when  $r < \frac{3}{8}\sigma_N^2$ , but increasing otherwise.  $\square$

**Proof of Lemma 1.** Note that

$$\begin{aligned}
N_t \geq N^* &\iff \log(N_t) \geq \log(N^*) \\
&\iff -\frac{1}{2}\sigma_N t + W_{Nt} \geq \frac{1}{\sigma_N}(\log(N^*/N_0)),
\end{aligned}$$

so that the first time that  $N_t$  hits  $N^*$  is the first time that a Brownian motion with drift  $-\frac{1}{2}\sigma_N$  hits  $\frac{1}{\sigma_N}(\log(\frac{N^*}{N_0}))$ . It follows from [Karatzas and Shreve \(1998\)](#) (Chapter 3.5, Part C, p.196-197) that for  $N_0 < N^*$  the density of  $T_N$  is

$$\Pr(T_N \in [t, t + dt]) = \frac{\left(\log\left(\frac{N^*}{N_0}\right)\right)}{\sigma_N \sqrt{2\pi t^3}} \exp\left\{-\frac{\left(\frac{1}{\sigma_N} \log\left(\frac{N^*}{N_0}\right) + \frac{1}{2}\sigma_N t\right)^2}{2t}\right\} dt.$$

Moreover, since  $\frac{1}{\sigma_N}(\log(\frac{N^*}{N_0})) > 0$  but the drift of the Brownian motion is  $-\frac{1}{2}\sigma_N < 0$ , it follows from [Karatzas and Shreve \(1998\)](#) (p.197) that  $\Pr(T_N = \infty) > 0$ . Specifically, note that

$$\Pr(T_N < \infty) = \int_0^\infty \frac{\left(\log\left(\frac{N^*}{N_0}\right)\right)}{\sigma_N \sqrt{2\pi t^3}} \exp\left\{-\frac{\left(\frac{1}{\sigma_N} \log\left(\frac{N^*}{N_0}\right) + \frac{1}{2}\sigma_N t\right)^2}{2t}\right\} dt = \frac{N_0}{N^*}, \quad (21)$$

which implies  $\Pr(T_N = \infty) = 1 - \frac{N_0}{N^*}$ .  $\square$

**Proof of Proposition 3.** Given the definition of  $\tau$ , we have that for  $0 \leq t < \infty$ ,

$$\Pr(\tau \in [t, t + dt]) = \Pr(\tau \in [t, t + dt] | T_N \leq T) \Pr(T_N \leq T) + \Pr(\tau \in [t, t + dt] | T_N > T) \Pr(T_N > T) \quad (22)$$

$$= \Pr(T_N \in [t, t + dt] | T_N \leq T) \Pr(T_N \leq T) \quad (23)$$

$$= \Pr(T_N \in [t, t + dt]) \Pr(T \geq t) \quad (24)$$

$$= e^{-rt} \Pr(T_N \in [t, t + dt]). \quad (25)$$

Integrating gives us

$$\Pr(\tau < \infty) = \int_0^\infty e^{-rt} \frac{\left(\log\left(\frac{N^*}{N_0}\right)\right)}{\sigma_N \sqrt{2\pi t^3}} \exp\left\{-\frac{\left(\frac{1}{\sigma_N} \log\left(\frac{N^*}{N_0}\right) + \frac{1}{2}\sigma_N t\right)^2}{2t}\right\} dt \quad (26)$$

$$= \frac{e^{-\frac{\log(N^*/N_0)\sqrt{8r+\sigma_N^2}}{2\sigma_N}}}{\sqrt{N^*/N_0}} = \left(\frac{N_0}{N^*}\right)^\beta \quad (27)$$

The comparative statics for  $c$ ,  $N_0$ ,  $\sigma_Z$  and  $\alpha$  follow from plugging in the expressions for  $N^*$  and  $\beta$ . To establish the comparative statics for  $\sigma_N$ , first note that since  $\lim_{\sigma_N \rightarrow 0} \beta = \infty$ ,

$\lim_{\sigma_N \rightarrow \infty} \beta = 1$ , and  $N^* = \frac{\beta}{\beta-1} \frac{c}{K}$ ,

$$\lim_{\sigma_N \rightarrow \infty} \Pr(\tau < \infty) = 0 \quad (28)$$

$$\lim_{\sigma_N \rightarrow 0} \Pr(\tau < \infty) = \begin{cases} 0 & \text{if } c > N_0 K \\ 1 & \text{if } c \leq N_0 K \end{cases}. \quad (29)$$

Let

$$\zeta \equiv \frac{\partial}{\partial \beta} (\log(\Pr(\tau < \infty))) = \frac{\partial}{\partial \beta} \left( \frac{N_0}{N^*} \right)^\beta = \log\left(\frac{N_0}{N^*}\right) + \frac{1}{\beta-1} \quad (30)$$

which implies  $\lim_{\sigma_N \rightarrow 0} \zeta = \lim_{\beta \rightarrow \infty} \zeta = \log\left(\frac{N_0 K}{c}\right)$ ,  $\lim_{\sigma_N \rightarrow \infty} \zeta = \lim_{\beta \rightarrow 1} \zeta = \infty$ , and

$$\frac{\partial}{\partial \sigma_N} \zeta = \frac{\partial \zeta}{\partial \beta} \frac{\partial \beta}{\partial \sigma_N} = -\frac{1}{\beta(1-\beta)^2} \frac{\partial \beta}{\partial \sigma_N} > 0. \quad (31)$$

Since  $\frac{\partial}{\partial \sigma_N} \log(\Pr(\tau < \infty)) = \zeta \frac{\partial \beta}{\partial \sigma_N}$ , we have the following results:

- When  $c \leq N_0 K$ , since  $\zeta \geq 0$  for  $\sigma_N \rightarrow 0$  and  $\frac{\partial}{\partial \sigma_N} \zeta > 0$  we have  $\zeta > 0$  for all  $\sigma_N$ , which in turn implies  $\frac{\partial}{\partial \sigma_N} \log(\Pr(\tau < \infty)) < 0$  for all  $\sigma_N$ .
- When  $c > N_0 K$ ,  $\zeta$  crosses zero once, from below, as  $\sigma_N$  increases, which implies  $\frac{\partial}{\partial \sigma_N} \log(\Pr(\tau < \infty)) = 0$  at exactly this one point. In this case,  $\Pr(\tau < \infty)$  is hump-shaped.

Similarly, for  $r$ ,  $\frac{\partial}{\partial r} \log(\Pr(\tau < \infty)) = \zeta \frac{\partial}{\partial r} \beta - \frac{\beta}{2r}$ . We have  $\frac{\partial}{\partial r} \zeta = -\frac{1}{\beta(\beta-1)^2} \frac{\partial}{\partial r} \beta - \frac{1}{2r} < 0$ . Since  $\frac{\partial}{\partial r} \beta = \frac{1}{\sigma_N^2 (\beta - \frac{1}{2})} > 0$  this implies  $\frac{\partial}{\partial r} \log(\Pr(\tau < \infty))$  crosses zero as most once as  $r$  increases and from above if it does so. Consider the limit as  $r$  tends to zero,

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{\partial}{\partial r} \log(\Pr(\tau < \infty)) &= \lim_{r \rightarrow 0} \left( \zeta \frac{\partial}{\partial r} \beta - \frac{\beta}{2r} \right) \\ &= \lim_{r \rightarrow 0} \frac{2r\zeta - \sigma_N^2 \beta \left( \beta - \frac{1}{2} \right)}{2\sigma_N^2 r \left( \beta - \frac{1}{2} \right)}. \end{aligned} \quad (32)$$

If it can be shown that the numerator in eq. (32) has a finite, positive limit it will follow that the overall limit is  $\infty$ . Considering the numerator, we have

$$\begin{aligned} \lim_{r \rightarrow 0} \left( 2r\zeta - \sigma_N^2 \beta \left( \beta - \frac{1}{2} \right) \right) &= 2 \lim_{r \rightarrow 0} r \left( \frac{1}{\beta-1} - \log \frac{\beta}{\beta-1} - \log \sqrt{2r} \right) - \frac{1}{2} \sigma_N^2 \\ &= \sigma_N^2 - 2 \lim_{r \rightarrow 0} \frac{\frac{1}{\beta(\beta-1)}}{\frac{1}{r^2}} - \frac{1}{2} \sigma_N^2 \\ &= \frac{1}{2} \sigma_N^2 - 2 \lim_{r \rightarrow 0} \frac{2r}{(2\beta-1) \frac{\partial}{\partial r} \beta} = \frac{1}{2} \sigma_N^2 \end{aligned}$$

where the second equality applies l'Hôpital's rule to the three different terms and uses the fact  $\frac{\partial}{\partial r}\beta \rightarrow \frac{2}{\sigma_N^2}$  as  $\beta \rightarrow 1$ . The third equality rearranges the expression in the remaining limit to place  $r^2$  in the numerator and uses l'Hôpital's rule again. Returning to eq. (32), this implies  $\lim_{r \rightarrow 0} \frac{\partial}{\partial r} \log(\mathbb{P}(\tau < \infty)) = \infty$ .

Now, consider  $\lim_{r \rightarrow \infty} \frac{\partial}{\partial r} \log(\Pr(\tau < \infty))$ . We have

$$\lim_{r \rightarrow \infty} \zeta = \lim_{r \rightarrow \infty} \left( \frac{1}{\beta-1} - \log \frac{\beta}{\beta-1} - \log \sqrt{2r} \right) = \lim_{\beta \rightarrow \infty} \left( \frac{1}{\beta-1} - \log \frac{\beta}{\beta-1} \right) - \lim_{r \rightarrow \infty} \log \sqrt{2r} = -\infty.$$

Because  $\frac{\partial}{\partial r}\beta > 0$ , it follows that  $\lim_{r \rightarrow \infty} \frac{\partial}{\partial r} \log(\Pr(\tau < \infty)) = -\infty$ , which completes the proof.  $\square$

**Proof of Proposition 4.** First, note that with unobservable acquisition, if the market maker sets the price impact to zero, the strategic trader should acquire information and trade with zero price impact to generate unbounded profits. This rules out equilibria in which information is not immediately acquired. Now consider equilibria in which information is acquired at some stopping time  $\tau$ . Then, based on the analysis of the observable case, in any such equilibrium the price and value functions of the informed strategic traders are given by the expressions in Proposition 1. Consider the value function  $J^U$  of an uninformed strategic trader in this setting. By analogous arguments to those in Proposition 1, we can show that it is of the form:  $J^U(p_t, N_t) = N_t V^U(p_t)$ , where

$$V_p^U = \frac{p - \alpha}{\lambda}, \quad rV^U = \frac{1}{2}\sigma_z^2\lambda^2 V_{pp}^U. \quad (33)$$

hold identically for  $p \in (0, 1)$ . But appealing to the analogous equations for  $V^0$  and  $V^1$ , we can write

$$V_p^U = \alpha V_p^1 + (1 - \alpha)V_p^0 \quad (34)$$

$$\Rightarrow V_{pp}^U = \alpha V_{pp}^1 + (1 - \alpha)V_{pp}^0 \quad (35)$$

$$\Rightarrow V^U = \alpha V^1 + (1 - \alpha)V^0 \quad (36)$$

$$\Rightarrow J^U(\alpha, N_t) = \bar{J}(\alpha, N_t). \quad (37)$$

This implies that the value to being uninformed is equal to the (ex-ante) expected value from being informed. Hence, the trader can profitably deviate and avoid the cost of information acquisition without affecting her continuation value.  $\square$

**Proof of Proposition 5.** Suppose that an equilibrium exists that satisfies the three points in the Proposition. There are two cases to consider: (i) the uninformed trader acquires

information with positive probability on  $[0, T]$  (i.e., the set of stopping times that she mixes over in her acquisition strategy is nonempty) and (ii) the uninformed trader never acquires information (the set of stopping times over which she mixes is empty). We begin with case (i). Consider an arbitrary stopping time  $\tau$  over which the trader mixes in her acquisition strategy. This accommodates pure strategies, which are simply mixed strategies that place probability 1 on a particular stopping time.

Let  $C = \{(p, q, N) \in \mathcal{S} : J^U > \bar{J} - c\}$  denote the continuation region and  $B = \{(p, q, N) \in \mathcal{S} : J^U = \bar{J} - c\}$  the stopping region for the uninformed trader. Since the value functions are assumed to be continuous, it follows that  $C$  is open and  $B$  is closed. Pick a point  $(p^*, q^*, N^*)$  on the boundary of  $C$ . Suppose that in some nonempty open neighborhood of this point  $A = \{\|(p, q, N) - (p^*, q^*, N^*)\| < \varepsilon\} \cap C$ ,  $p_t$  and  $q_t$  follow

$$\begin{aligned} dp_t &= \mu_p dt + \lambda(\theta dt + dZ_t) + p_J dG \\ dq_t &= \mu_q dt + \kappa(\theta dt + dZ_t) + q_J dG, \end{aligned}$$

where  $\mu, \nu, \lambda$ , and  $\kappa$  are functions of  $p, q$ , and  $N$ ,  $G$  is a jump at the boundary of size 1, and  $p_J, q_J \in \mathbb{R}$  are jump sizes of the respective processes.<sup>20</sup> Let  $L^\theta$  be the generator of the  $(p, q, N)$  process under trading strategy  $\theta$ . Under the posited processes,  $L$  operates on  $f \in \mathcal{C}^2$  as

$$\begin{aligned} L^\theta f(p, q, N) &= (\mu_p + \theta\lambda)f_p + (\mu_q + \theta\kappa)f_q + \frac{1}{2}f_{pp}\lambda^2\sigma_Z^2 + \frac{1}{2}f_{qq}\kappa^2\sigma_Z^2 + f_{pq}\lambda\kappa\sigma_Z^2 + \frac{1}{2}\sigma_N^2 N^2 f_{NN} \\ &\quad + \{f(p^* + p_J, q^* + q_J, N) - f(p^*, q^*, N)\} \end{aligned}$$

For the informed types, the value function satisfies the HJB equation. Moreover, given the linearity in  $\theta$ , the terms multiply  $\theta$  in the HJB equations must equal zero. This implies the remaining terms must also equal zero i.e., for  $i \in \{0, 1\}$ .

$$0 = \sup_{\theta \in \mathbb{R}} \{L^\theta J^i + \theta(N\xi - P)\} - rJ^i = L^0 J^i - rJ^i, \quad (38)$$

where  $L^0$  corresponds to the generator when  $\theta = 0$ . Taking an  $\alpha$ -weighted average of eq. (38) over  $i \in \{0, 1\}$  and using the linearity of the right-hand side in  $\theta$  implies that  $\bar{J}$  also satisfies

$$0 = \sup_{\theta \in \mathbb{R}} \{L^\theta \bar{J} + \theta(N\alpha - P)\} - r\bar{J} = L^0 \bar{J} - r\bar{J} \quad (39)$$

---

<sup>20</sup>In equilibrium, the pricing rule responds optimally to the possibility that information is acquired at the boundary, so in the conjectured equilibrium  $p$  and  $q$  will jump at any point that information is acquired with positive probability.



The uninformed trader faces a combined optimal control and stopping problem. Hence, her value function must solve the following variational inequality (see Øksendal and Sulem (2007), Ch. 4) in the interior of  $\mathcal{S}$

$$\max \left\{ \sup_{\theta} (L^{\theta} J^U + \theta(N\alpha - P)) - rJ^U, \bar{J} - c - J^U \right\} = 0,$$

or equivalently,

$$\max \{ L^0 J^U - rJ^U, \bar{J} - c - J^U \} = 0.$$

In particular, the value function must be a (viscosity) supersolution of the variational inequality (Øksendal and Sulem (2007), Ch. 9). That is, given a function  $h$  that is twice continuously differentiable in the interior of  $\mathcal{S}$  and satisfies  $h \leq J^U$ , for any point  $(p, q, N)$  at which  $h = J^U$  we have

$$\max \{ L^0 h - rh, \bar{J} - c - J^U \} \leq 0.$$

Pick any point on the boundary of the continuation region at which the value matching holds and take  $h = \bar{J} - c$ . This function is twice continuously differentiable since we have assumed that  $J^1$  and  $J^0$  are. Owing to the value matching condition, at the chosen boundary point we have  $J^U = \bar{J} - c$  and therefore we have

$$\max \{ L^0 \bar{J} - r(\bar{J} - c), \bar{J} - c - J^U \} = \max \{ rc, 0 \} > 0,$$

where the equality follows from eq. (39) and value matching. This contradicts  $J^U$  being a supersolution of the variational inequality. This completes the proof of case (i).

For case (ii), note that if, with probability one, the uninformed trader does not acquire information, then the optimal pricing rule is unresponsive to order flows. In that case, the trader has a profitable deviation in which she acquires information at  $t = 0$ , trades against the unresponsive market maker, and makes unbounded profits. We conclude that there can be no equilibrium in which she remains uninformed with probability one.  $\square$

## B Additional Predictions from the Observable Case

### Price dynamics

The expression for the price in Proposition 1 immediately implies that price impact of order flow before information acquisition is zero, but jumps to  $\lambda(p_\tau)$  when information is acquired. Moreover, price impact evolves stochastically post-acquisition, since it is driven by the evolution of the market maker’s beliefs  $p_t$ .

The following result characterizes return volatility in our model.

**Proposition 6.** *The instantaneous variance of returns is*

$$\nu_t \equiv \begin{cases} \sigma_N^2 & 0 \leq t < \tau \\ \sigma_N^2 + \left(\frac{\lambda^*(p_t)}{p_t}\right)^2 \sigma_Z^2 & \tau \leq t < T \end{cases}$$

*Conditional on information acquisition, volatility is stochastic and exhibits the “leverage” effect i.e., the instantaneous covariance between returns and variance of returns is negative ( $\text{cov}\left(\nu_t, \frac{dp_t}{p_t}\right) \leq 0$ ).*

The above result highlights that return volatility is higher conditional on information acquisition. Conditional on no acquisition, price changes are driven purely by changes in the news process. However, conditional on the strategic trader being informed, the market maker also conditions on order flow to update his beliefs about the asset payoff, and as a result, return volatility is driven by two sources of variation.

In contrast to the standard Kyle (1985) model, our model generates stochastic return volatility and price impact, even though fundamentals (i.e.,  $N_t$ ) and noise trading (i.e.,  $Z_t$ ) are homoskedastic. This is a consequence of the non-linearity in the filtering problem of the market maker, and is in contrast to models where the (conditionally linear) filtering problem amplifies stochastic volatility in an underlying process (e.g., in Collin-Dufresne and Fos (2016), return volatility amplifies stochastic volatility in noise trading).<sup>21</sup> Moreover, conditional on information acquisition, return volatility also exhibits the “leverage effect” (see Black (1976) and the subsequent literature) — the instantaneous variance increases when returns are negative, and vice versa — even though there is no leverage (debt) in the underlying risky asset.

Despite the large empirical literature documenting the importance of stochastic volatility and jumps in volatility, there are relatively few theoretical explanations for how these

---

<sup>21</sup>Similar results obtain in the continuous-time models of Back and Baruch (2004), Li (2013), Back et al. (2016), and the discrete time model of Banerjee and Green (2015).

patterns arise. Our model provides an explanation for both, but it does not rely on jumps or stochastic volatility in fundamentals. Instead, volatility jumps (and becomes stochastic) when the public news process triggers private information acquisition by the strategic trader. Our analysis suggests that further understanding the interaction between public news and private information can provide new insights into what drives empirically observed patterns in volatility.

### Announcement effects

Next, we turn to the absolute price change at the time the payoff of the risky asset is announced. In finite horizon models where the announcement is perfectly anticipated (e.g., [Back \(1992\)](#)), the informed trader's optimal strategy ensures that the price change at announcement is zero. While this is no longer the case with a stochastic announcement date, the intuition from these models would suggest that the announcement effect is smaller on average if information is acquired than if it is not. However, as the next result highlights, this is not always the case.

**Proposition 7.** *The expected absolute price jump on announcement, conditional on information acquisition is*

$$\mathbb{E} [|\xi N_T - P_{T-}| | \tau < \infty] = 2N^* h(\alpha), \quad (40)$$

where  $h(\alpha)$  is characterized in the Appendix, and fully illustrated by the plot in [Figure 3](#). The expected absolute price jump on announcement, conditional on no information acquisition is

$$\mathbb{E} [|\xi N_T - P_{T-}| | \tau = \infty] = 2\alpha(1 - \alpha) N^* \frac{\frac{N_0}{N^*} - \left(\frac{N_0}{N^*}\right)^\beta}{1 - \left(\frac{N_0}{N^*}\right)^\beta}.$$

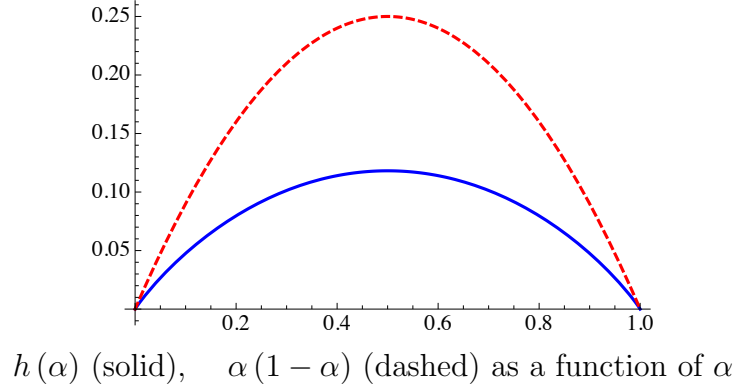
Fixing  $\alpha \in (0, 1)$  and the other parameters, the announcement effect is larger with information acquisition when:  $N_0$  is sufficiently small,  $c$  is sufficiently high,  $\sigma_N^2$  is sufficiently high,  $\sigma_Z^2$  is sufficiently low, or  $r$  is sufficiently extreme (i.e., sufficiently low, or sufficiently high).

The proposition characterizes conditions under which a potentially surprising result holds: the announcement effect is larger with information acquisition than without. In a setting where the strategic trader is exogenously endowed with information, the standard intuition holds — the announcement effect conditional on an informed trading is smaller than the announcement effect conditional on no informed trading. To see why, note that in this case, the announcement effect can be expressed as

$$\mathbb{E} [|\xi N_T - P_{T-}|] = N_0 \mathbb{E} [|\xi - \pi_T|] = 2N_0 \mathbb{E} [\pi_T(1 - \pi_T)], \quad (41)$$

Figure 3:  $h(\alpha)$  and  $\alpha(1 - \alpha)$

The figure plots  $h(\alpha)$  (solid) and  $\alpha(1 - \alpha)$  (dashed) as a function of  $\alpha$ .



where  $\pi_t = \mathbb{E}[\xi | \mathcal{F}_t^P]$ . When the strategic trader is not informed,  $\pi_T = \alpha$ . When the strategic trader is informed,  $\pi_T = p_T$ , and so Jensen's inequality implies that  $\mathbb{E}[p_T(1 - p_T)] \leq \alpha(1 - \alpha)$ . Intuitively, the market-maker's posterior beliefs are more precise when the strategic trader is informed, and as a result, the price reflects the asset payoff more accurately.

When information acquisition is endogenous, however, there is an offsetting effect at work. Recall that the strategic trader only acquires information when the news process is sufficiently high ( $N_t \geq N^*$ ). This implies that the expected level of  $N_T$ , conditional on information acquisition, is higher since  $\mathbb{E}[N_T | \tau < \infty] = N^* \geq N_0$ . Intuitively, the strategic trader only chooses to acquire information when the prior uncertainty about fundamentals is sufficiently high. This offsetting effect dominates when the initial news level  $N_0$  is sufficiently small or the optimal exercise boundary  $N^*$  is sufficiently large, and as a result, the announcement effect conditional on information acquisition is higher in these cases.

## Proofs

**Proof of Proposition 6.** Using the expression for the asset price in Proposition 1,

$$dP_t = \begin{cases} \alpha \sigma_N N_t dW_{Nt} & 0 \leq t < \tau \\ \sigma_N N_t p(Y_t) dW_{Nt} + N_t \lambda^*(p_t) \sigma_Z dW_{Yt} & \tau \leq t < T \end{cases},$$

where  $W_{Yt} \equiv Y_t / \sigma_Z$  is a standard Brownian motion under the public filtration and is

independent of  $W_{Nt}$ . Hence,

$$\frac{dP_t}{P_t} = \begin{cases} \sigma_N dW_{Nt} & 0 \leq t < \tau \\ \sigma_N dW_{Nt} + \frac{\lambda^*(p_t)}{p_t} \sigma_Z dW_{Yt} & \tau \leq t < T \end{cases}.$$

Letting  $\nu_t$  denote the instantaneous variance of the return process gives:

$$\nu_t \equiv \begin{cases} \sigma_N^2 & 0 \leq t < \tau \\ \sigma_N^2 + \left(\frac{\lambda(p_t)}{p_t}\right)^2 \sigma_Z^2 = \sigma_N^2 + 2r \left(\frac{\phi(\Phi^{-1}(p_t))}{p_t}\right)^2 & \tau \leq t \end{cases} \quad (42)$$

Let  $f(p) \equiv \phi(\Phi^{-1}(p))$ , and note that  $f_p = -\Phi^{-1}(p)$  and  $f_{pp} = -\frac{1}{f}$ . Conditional on information acquisition, note that by Ito's Lemma, we have:

$$d\nu_t = \nu_p dp_t + \frac{1}{2} \nu_{pp} (\lambda(p_t))^2 \sigma_Z^2 dt = \nu_p dp_t + r f(p)^2 \nu_{pp} dt, \quad (43)$$

where  $\nu_p = 4r \left(\frac{f}{p}\right) \left(\frac{f_p p - f}{p^2}\right) < 0$ , and

$$\nu_{pp} = 4r \left(\frac{f_p p - f}{p^2}\right)^2 + 4r \left(\frac{f}{p}\right) \left(\frac{p^2(f_{pp} p + f_p - f_p) - 2p(f_p p - f)}{p^4}\right). \quad (44)$$

Since  $\nu_p < 0$ , the above implies that conditional on information acquisition, instantaneous return variance  $\nu_t$  and returns are negatively related i.e.,  $\text{cov}\left(\frac{dp_t}{p_t}, d\nu_t\right) < 0$ .  $\square$

**Proof of Proposition 7.** For the no acquisition case,

$$\mathbb{E} [|\xi N_T - P_{T-}| | T_N > T] = \mathbb{E} [N_T | \xi - \alpha | T_N > T] = 2\alpha(1 - \alpha) \mathbb{E} [N_T | T_N > T] \quad (45)$$

Next, note that

$$\mathbb{E} [N_T] = \Pr(T_N < T) \mathbb{E} [N_T | T_N < T] + \Pr(T_N \geq T) \mathbb{E} [N_T | T_N \geq T] \quad (46)$$

$$\Rightarrow \mathbb{E} [N_T | T_N > T] = \frac{N_0 - \Pr(T_N < T) N^*}{\Pr(T_N \geq T)} \quad (47)$$

$$= \frac{N_0 - \left(\frac{N_0}{N^*}\right)^\beta N^*}{1 - \left(\frac{N_0}{N^*}\right)^\beta} \quad (48)$$

since  $\mathbb{E} [N_T] = N_0$ ,  $\mathbb{E} [N_T | T_N < T] = N^*$  and  $\Pr(T_N < T) = \left(\frac{N_0}{N^*}\right)^\beta$ . This produces the desired expression.

Conditional on information acquisition, the expected announcement effect is

$$\mathbb{E} [\xi N_T - P_T | \tau < \infty] = \mathbb{E} [N_T | \xi - p(Y_T) | T_N < T] \quad (49)$$

$$= 2\mathbb{E} [N_T p(Y_T) (1 - p(Y_T)) | T_N < T] \quad (50)$$

$$= 2\mathbb{E} [\mathbb{E}_{T_N} [N_T p(Y_T) (1 - p(Y_T)) | T_N < T] | T_N < T] \quad (51)$$

$$= 2\mathbb{E} [N_{T_N} \mathbb{E}_{T_N} [p(Y_T) (1 - p(Y_T)) | T_N < T] | T_N < T] \quad (52)$$

$$= 2N^* \mathbb{E} [p(Y_T) (1 - p(Y_T)) | T_N < T] \quad (53)$$

the first and second equalities use the law of iterated expectations, the third equality uses the fact that conditional on  $\sigma(\mathcal{F}_{T_N}^P \cup \{T_N < T\})$ ,  $N_T - N_{T_N}$  and  $Y_T$  are independent, the fourth equality uses the fact that  $N$  is a martingale, and the final equality uses  $N_{T_N} = N^*$ .

Suppose  $\tau \in [t, t + dt]$ . Given the characterization of  $p_t$  in Proposition 1, we can express  $p_s$  for  $s \geq t$  as  $p_s = \Phi\left(\frac{\sqrt{2r}}{\sigma_Z} z_s\right)$ , where

$$z_s | \{\tau \in [t, t + dt]\} \sim N\left(\Phi^{-1}(\alpha) e^{r(s-t)}, \frac{\sigma_Z^2}{2r} (e^{2r(s-t)} - 1)\right). \quad (54)$$

Next, note that for  $w \sim N(0, 1)$ , we have

$$\mathbb{E} [\Phi(a + bw) [1 - \Phi(a + bw)]] = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right) - \left[\Phi\left(\frac{a}{\sqrt{1+b^2}}\right) - 2T^o\left(\frac{a}{\sqrt{1+b^2}}, \frac{1}{\sqrt{1+2b^2}}\right)\right] \quad (55)$$

from Owen (1980) 10,010.8 and 20,010.4, where  $T^o(a, b)$  is the Owen T function. Let  $\tilde{z}_s \equiv \frac{z_s - e^{r(s-t)} z_0}{\sqrt{\frac{\sigma_Z^2}{2r} (e^{2r(s-t)} - 1)}} \sim N(0, 1)$ , and note that  $p(z_s) = \Phi(a + b\tilde{z}_s)$ . This implies

$$G(t, s) \equiv \mathbb{E}_t [p_s (1 - p_s) | \tau \in [t, t + dt], s > t] = 2T^o\left(\Phi^{-1}(\alpha), \frac{1}{\sqrt{2e^{2r(s-t)} - 1}}\right). \quad (56)$$

Since the stopping time  $T$  is exponentially distributed, we have

$$\begin{aligned} & \mathbb{E}_t [p(Y_T) (1 - p(Y_T)) | T > t, \tau \in [t, t + dt]] \\ &= e^{-rt} \int_{s=t}^{\infty} r e^{-r(s-t)} \mathbb{E}_t [p(Y_s) (1 - p(Y_s)) | \tau \in [t, t + dt]] ds \end{aligned} \quad (57)$$

$$= \int_0^{\infty} r e^{-rs} G(0, s) ds \quad (58)$$

$$= 2 \int_0^{\infty} r e^{-rs} T^o\left(\Phi^{-1}(\alpha), \frac{1}{\sqrt{2e^{2rs} - 1}}\right) ds \quad (59)$$

$$= 2 \int_0^{\infty} e^{-x} T^o\left(\Phi^{-1}(\alpha), \frac{1}{\sqrt{2e^{2x} - 1}}\right) dx, \text{ where } x = rs \quad (60)$$

$$\equiv h(\alpha) \tag{61}$$

This implies that

$$\begin{aligned} & \mathbb{E} [p(Y_T) (1 - p(Y_T)) | \tau < T] \\ &= \int_0^\infty \mathbb{E}_t [p(Y_T) (1 - p(Y_T)) | T > t, \tau \in [t, t + dt]] \Pr(\tau \in [t, t + dt] | T > \tau) dt \end{aligned} \tag{62}$$

$$= h(\alpha) \int_0^\infty \Pr(\tau \in [t, t + dt] | T > \tau) dt = h(\alpha) \tag{63}$$

which implies  $\mathbb{E} [|\xi N_T - P_{T-}| | \tau < T] = 2N^*h(\alpha)$ .

Note that the announcement effect is bigger conditional on no acquisition if and only if:

$$2N^*h(\alpha) < 2\alpha(1-\alpha) N^* \frac{\frac{N_0}{N^*} - \left(\frac{N_0}{N^*}\right)^\beta}{1 - \left(\frac{N_0}{N^*}\right)^\beta} \Leftrightarrow \frac{h(\alpha)}{\alpha(1-\alpha)} < \frac{\frac{N_0}{N^*} - \left(\frac{N_0}{N^*}\right)^\beta}{1 - \left(\frac{N_0}{N^*}\right)^\beta} \tag{64}$$

$$\Leftrightarrow \frac{h(\alpha)}{\alpha(1-\alpha)} \left(1 - \left(\frac{N_0}{N^*}\right)^\beta\right) < \frac{N_0}{N^*} - \left(\frac{N_0}{N^*}\right)^\beta \tag{65}$$

$$\Leftrightarrow \frac{h(\alpha)}{\alpha(1-\alpha)} < \frac{N_0}{N^*} - \left(\frac{N_0}{N^*}\right)^\beta \left(1 - \frac{h(\alpha)}{\alpha(1-\alpha)}\right) \tag{66}$$

For a fixed  $\alpha$ , since

$$\frac{N_0}{N^*} = N_0^{\frac{\beta-1}{\beta c}} K = \frac{N_0^{\frac{1}{2}} \left(1 + \sqrt{1 + 8\frac{r}{\sigma_N^2}}\right) - 1}{\frac{1}{2} \left(1 + \sqrt{1 + 8\frac{r}{\sigma_N^2}}\right)} \frac{\sigma_Z}{\sqrt{2r}} \phi(\Phi^{-1}(\alpha)), \tag{67}$$

implies that  $\frac{N_0}{N^*} \rightarrow 0$  when  $r \rightarrow 0$ ,  $r \rightarrow \infty$ ,  $\sigma_N \rightarrow \infty$ ,  $c \rightarrow \infty$  or  $\sigma_Z \rightarrow 0$ . Moreover, since  $\beta > 1$  and  $\frac{N_0}{N^*} < 1$ , we have  $\left(\frac{N_0}{N^*}\right)^\beta \rightarrow 0$  when  $\left(\frac{N_0}{N^*}\right) \rightarrow 0$ . Now, fix  $\alpha$  and pick a  $\delta$  such that  $0 < \delta < \frac{h(\alpha)}{\alpha(1-\alpha)}$ . Then, the above implies that for sufficiently extreme  $r$ , sufficiently large  $\sigma_N$ , sufficiently large  $c$  or sufficiently small  $\sigma_Z$ ,  $\frac{N_0}{N^*} - \left(\frac{N_0}{N^*}\right)^\beta \left(1 - \frac{h(\alpha)}{\alpha(1-\alpha)}\right) < \delta$ , and so the announcement effect is bigger conditional on acquisition.  $\square$