Finitely Well-Positioned Sets

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Abstract

We introduce and study finitely well-positioned sets, a class of asymptotically “narrow” sets that generalize the well-positioned sets recently investigated by Adly, Ernst and Thera [1] and [3], as well as the plastering property of Krasnoselskii [11].

1 Introduction

The analysis of asymptotic properties of sets and functions plays an important role in the study of existence of optima for noncoercive functions on normed spaces, that is, functions with unbounded level sets. For instance, in [15] we introduced finitely well-positioned sets, a class of suitably asymptotically “narrow” sets that generalize the well-positioned sets recently investigated by Adly, Ernst and Thera [1] and [3]. Using this notion, in [15] we established necessary and sufficient conditions for optima of noncoercive functionals on reflexive spaces.

In the present paper we investigate in depth this class of sets and their relationships with well-positioned sets, as well as with Krasnoselskii [11]’s plastering property. We will also show that our notion is closely related to some notions of asymptotic compactness recently introduced in the literature. Specifically, following [11] say that a set $C$ of a normed space $V$ allows plastering if a uniformly positive continuous linear functional exists over $C$, that is, if there is $0 \neq x^* \in V^*$ such that $\langle x^*, x \rangle \geq \|x\|$ for all $x \in C$, where $V^*$ is the topological dual of $V$. By defining the Bishop-Phelps cone $K_{x^*} = \{ x : \langle x^*, x \rangle \geq \|x\| \}$, a set $C$ allows plastering if and only if $C \subseteq K_{x^*}$ for some $x^* \in V^*$. For example, the positive cone of $L^1$ allows plastering, while that of $L^p$ with $p > 1$ does not.

Though $C$ can be any set, closed convex cones are the natural domain for this concept. In fact, a set allows plastering if and only if its closed conical hull does.\(^1\) Recently, [1] and

\(^1\)We refer to [11] for other properties related to this notion.
[3] reconsidered this property under a different name. In particular, they call a set $C$ well-positioned if there are $x_0 \in V$ and $x^* \in V^*$ such that $\langle x^*, x - x_0 \rangle \geq \|x - x_0\|$ for all $x \in C$. This amounts to require that the translated set $C - \{x_0\}$ allows plastering. Equivalently, $C \subseteq x_0 + K_{x^*}$. Many nice properties of well-positioned convex sets in reflexive spaces are studied in [1] and [3].

In this paper we study the following natural generalization of well-positioned sets, whose interest will become apparent through the properties that we will establish in the present paper.

**Definition 1** A set $C$ is said to be finitely well-positioned if $C \subseteq \bigcup_{i=1}^{n} C_i$, where each $C_i$ is well-positioned.\(^2\)

A set $C$ is well-positioned if and only if its closed convex hull is, while $C$ is finitely well-positioned if and only its closure is. This is a first basic difference between well-positioned sets and our generalization that shows that while closed convex sets are the relevant class of sets for well-positionedness, this is no longer the case for finite well-positionedness, whose interest goes beyond convex sets. As the paper will show, this natural generalization actually turns out to extend substantially the scope of well-positionedness.

The class of finitely well-positioned sets is obviously wider than that of the sets that are merely well-positioned. For instance, we will see momentarily that finite-dimensional vector subspaces (and so all their subsets) are finitely well-positioned, but not well-positioned. Our extension thus becomes relevant in infinite dimensional spaces, as discussed at length in the paper.

The paper is organized as follows. Section 3 establishes some of the main properties of finite well-positionedness. In particular, Theorem 8 characterizes this property in reflexive spaces in terms of asymptotic directions by showing that a set $C$ is finitely well-positioned when there are no unbounded sequences $\{x_n\} \subseteq C$ such that $x_n/\|x_n\| \to 0$. This characterization implies that, as already mentioned, finite dimensional sets are finitely well-positioned. But, more importantly, it shows that finitely well-positioned sets feature in reflexive spaces a key convergence property of finite dimensional sets. This property will play an important role in many results of the paper and shows that finite well-positionedness can be viewed as an asymptotic notion of finite dimensionality for sets.

Section 4 relates finite well-positionedness with some notions of asymptotic compactness studied by several authors. We show their equivalence in reflexive normed spaces. Section 5 studies in more detail convex sets by extending earlier results that [1] and [3] proved for well-positioned sets.

\(^2\)Since $C = \bigcup_{i=1}^{n} (C_i \cap C)$ and each $C_i \cap C$ is well-positioned, in this definition we could have equivalently used the equality.
Scalar functions feature some well known sets, such as epigraphs and upper level sets. Through them in Section 6 we introduce and study finitely well-positioned functions and other related concepts. They extend the notion of well-positioned functions introduced by [1] and [3].

Finally, notice that some of our results can be generalized by considering dual pairs \( \langle V, W \rangle \), with \( V \) normed. Here the Bishop-Phelps cones are \( K_w = \{ x : \langle w, x \rangle \geq \|x\| \} \) with \( w \in W \). The two cases \( \langle V, V^* \rangle \) and \( \langle V^*, V \rangle \) give rise to interesting results. For convenience in the paper we focus on \( \langle V, V^* \rangle \), that is, on the weak topology. Though for brevity we omit details, most of the results that we establish in the paper hold also for the dual pair \( \langle V^*, V \rangle \), typically under the assumption that \( V \) is separable.

2 Notation and Preliminaries

Unless otherwise specified, throughout the paper we consider subsets of a normed vector space \( V \) with norm \( \| \cdot \| \). We denote by \( B_V \) its unit ball \( \{ x \in V : \|x\| \leq 1 \} \) and by \( S_V = \{ x \in V : \|x\| = 1 \} \) its unit sphere. The pairing of \( V \) with its topological dual \( V^* \) is usually denoted by \( \langle x^*, x \rangle \), with \( x \in V \) and \( x^* \in V^* \). Norm convergence of a sequence will be denoted by \( x_n \to x \), while the familiar notation \( x_n \rightharpoonup x \) indicates weak convergence.

Given a function \( f : V \to \overline{\mathbb{R}} \), we denote by \( (f \leq \lambda) \) the sublevel set \( \{ x \in V : f(x) \leq \lambda \} \). A \( f : V \to (-\infty, \infty] \) is:

(i) \textit{sw-lower semicontinuous} if all \( (f \leq \lambda) \) are sw-closed;\(^3\)

(ii) \textit{lower semicontinuous} if all \( (f \leq \lambda) \) are norm closed;

(iii) \textit{coercive} if there is a nonempty sublevel set \( (f \leq \lambda) \) that is norm bounded.\(^4\)

We denote by \( \text{ri} C \) the relative interior of a convex set \( C \), that is, the interior of \( C \) relative to \( \text{aff} C \), the closed affine hull of \( C \). In infinite dimensional spaces some other weaker notions may be adopted (see [7]). To avoid any ambiguity we will often write \( \text{ri}_H C \) to denote the interior of \( C \) relative to a linear space \( H \) containing \( C \).

Throughout the paper \( K_{x^*} \) will denote the cone \( \{ x : \langle x^*, x \rangle \geq \|x\| \} \). Since \( K_{\lambda x^*} \supseteq K_{x^*} \) for \( \lambda > 1 \), it is not restrictive to assume \( \|x^*\| > 1 \). This guarantees that \( K_{x^*} \) is sufficiently “large.” In this case \( K_{x^*} \) has nonempty interior because the open cone \( \{ x : \langle x^*, x \rangle > \|x\| \} \) is nonempty.

\(^3\)We use “sw” in place of sequentially weakly.

\(^4\)This notion is weaker than the standard one. Our definition implies that \( f + \chi_{(f \leq \lambda)} \) is coercive in the usual sense (at least in the reflexive case).
We will need a few asymptotic notions for sets. The (weak) asymptotic cone $C_\infty$ is defined by

$$C_\infty = \left\{ x \in V : \exists t_n \to \infty \text{ and } \{x_n\} \subseteq C \text{ such that } \frac{x_n}{t_n} \to x \right\}.$$ 

It is well-known that $C_\infty$ reduces to the recession cone

$$R_C = \{ y \in V : x + ty \in C \text{ for some } x \in C \text{ and all } t \geq 0 \}$$

when $C$ is closed and convex. The lineality space $L_C$ of $C$ is

$$L_C = \{ y \in V : x + ty \in C \text{ for some } x \in C \text{ and all } t \in \mathbb{R} \}.$$ 

Clearly, $L_C = R_C \cap (-R_C) = R_C \cap R_{-C}$.

The following modification of the asymptotic cone, with normalized asymptotic directions, will play an important role in the paper

$$B_C = \left\{ d \in V : \exists \{x_n\} \subseteq C \text{ with } \|x_n\| \to \infty \text{ and } \frac{x_n}{\|x_n\|} \to d \right\}.$$ 

The vectors $d \in B_C$ may not be normalized, with possibly $d = 0$.

Clearly, $B_C = \emptyset$ when $C$ is bounded, while $B_C \neq \emptyset$ when $C$ is an unbounded set of a reflexive space. Moreover, cone $B_C \subseteq C_\infty$ since $B_C \subseteq C_\infty$. Next we collect some other useful properties of asymptotic cones that we will need in the paper.

**Proposition 2** (i) cone $B_C = C_\infty$ if $B_C \neq \emptyset$ and $C_\infty = \{0\}$ if $B_C = \emptyset$;

(ii) $B_C$ is weakly closed if $V$ is either reflexive or with a separable dual;

(iii) $B_C$ and $C_\infty$ are $sw$-closed;

(iv) if $x_n/t_n \to d \neq 0$, with $t_n \to \infty$ and $\{x_n\} \subseteq C$, then there is an unbounded subsequence $\{x_{n_k}\}$ and a scalar $\lambda > 0$ such that $x_{n_k}/\|x_{n_k}\| \to \lambda d$.

**Proof** (i) Let $0 \neq d \in C_\infty$. This implies $x_n/t_n \to d$, with $x_n \in C$ and $t_n \to +\infty$. We know that $\lim \inf_n \|x_n\|/t_n \geq \|d\| > 0$. Hence, a subsequence $\{x_{n_k}\}$ exists such that $\lim_k \|x_{n_k}\|/t_{n_k} = \lambda > 0$. Notice that $\|x_{n_k}\| \to \infty$. Consequently,

$$\frac{x_{n_k}}{\|x_{n_k}\|} = \frac{x_{n_k}t_{n_k}}{t_{n_k}\|x_{n_k}\|} \to \frac{d}{\lambda} \in B_C.$$ (1)

Hence, $C_\infty \neq \{0\}$ implies $B_C \neq \emptyset$ and $B_C \neq \{0\}$; i.e., $B_C \subseteq \{0\}$ implies $C_\infty = \{0\}$. In particular, $B_C = \emptyset$ implies $C_\infty = \{0\}$, while $B_C = \{0\}$ implies $C_\infty = \text{cone } B_C = \{0\}$. Moreover, $B_C \neq \emptyset$ and $B_C \neq \{0\}$ imply $C_\infty \neq \{0\}$ since $C_\infty \supseteq B_C$. Then, cone $B_C = C_\infty$ by (1).
(ii) Let us first prove that $B_C$ is weakly closed if $V^*$ is separable. Let $d \in \overline{B}_C^w$. A metric $\delta$ on $V$ exists for which the weak topology on every norm bounded subset $D$ of $V$ coincides with the topology induced on $D$ by $\delta$ (see, e.g., [5, Theorem 3.35]). As $B_C \subseteq B_V$ and $d \in B_V$, this ensures the existence of a sequence $\{d_n\} \subseteq B_C$ such that $\delta(d_n, d) \to 0$. Passing to a subsequence if necessary, we can suppose $\delta(d_n, d) \leq n^{-1}$. Since $d_n \in B_C$, there is $x_n \in C$ for which $\delta(x_n/\|x_n\|, d_n) \leq n^{-1}$ and $\|x_n\| \geq n$. It follows

$$
\delta \left( \frac{x_n}{\|x_n\|}, d \right) \leq \delta \left( \frac{x_n}{\|x_n\|}, d_n \right) + \delta(d, d_n) \leq \frac{2}{n}.
$$

Namely, $\delta(x_n/\|x_n\|, d) \to 0$ and $x_n/\|x_n\| \to d \in B_C$ with $\|x_n\| \to \infty$. Hence, $B_C$ is weakly closed.

Suppose $V$ reflexive and $d \in \overline{B}_C^w$. By Day’s Lemma (see [16, Lemma 2.8.5 and Corollary 2.8.7]), a sequence $d_n \to d$ exists with $\{d_n\} \subseteq B_C$. Hence, sequences $\{x_m^m\} \subseteq C$ exist such that $x_m^m/\|x_m^m\| \to d_n$, and $\|x_m^m\| \to \infty$ as $m \to \infty$. Wlog, we can suppose that $\|x_m^m\| \geq n$ for all $m$. Consider the separable linear subspace $W = \overline{\text{span} \left\{ x_m^m/\|x_m^m\| \right\}}_{n,m}$. The $\sigma(W, W^*)$ convergence of sequences in $W$ is equivalent to their $\sigma(V, V^*)$ convergence (see, e.g., [16, Proposition 2.5.22]). Since $W^*$ is separable, the previous argument implies the existence of a sequence $x_{mk}^m/\|x_{mk}^m\| \to d$ as $k \to \infty$. Clearly, $\|x_{mk}^m\| \to \infty$ because $\|x_{mk}^m\| \geq n_k$, and so $d \in B_C$.

(iii) Similar arguments apply for spaces not necessarily reflexive. But, only the sequentially weakly closure of $B_C$ can be proved. We omit details. As to $C_\infty$, suppose that $V^*$ is separable and that $d_n \to d$, with $d_n \in C_\infty$. Consequently there are sequences $\{x_m^m/t_m^n\}_m$ for all $n$ such that $x_m^m/t_m^m \to d_n$ as $m \to \infty$. Fix $x^* \in B_{V^*}$. As $\langle x^*, d_n \rangle \to \langle x^*, d \rangle$, there is a scalar $A$ such that $|\langle x^*, d_n \rangle| \leq A$. Fix $n$. As $\langle x^*, x_m^m/t_m^n \rangle \to \langle x^*, x_m^m/t_m^n \rangle$ as $m \to \infty$, there is $m = m(n)$ so that $|\langle x^*, x_m^m/t_m^n \rangle| \leq 2A$ for all $n$ and $m \geq m(n)$. By the Banach-Steinhaus Theorem, there is a scalar $K$ such that $\|x_m^m/t_m^n\| \leq K$ for all $n$ and $m \geq m(n)$. Since $V^*$ is separable, the weak-topology is metrizable on the ball $\|x\| \leq K$. Let $\delta$ be such a metric. For all $k$, we pick $\delta(d, d_{nk}) < 1/k$. Moreover, we select a point $x_{mk}^m/t_{mk}^m$ for which $\delta(d_{nk}, x_{mk}^m/t_{mk}^m) < 1/k$, $m_k \geq m(n_k)$ and $t_{mk}^m \geq k$. This implies that $x_{mk}^m/t_{mk}^m \to d$ and thus $d \in C_\infty$. By the standard technique we can extend the to general spaces by considering the separable subspace $W = \overline{\text{span} \left\{ x_m^m/t_m^n \right\}}_{n,m}$.

(iv) This has been already proved in point (i).

Example 3 The hypotheses on $V$ in Proposition 2-(iv) are necessary. For, let $C = l_1$. Clearly, $C_\infty = l_1$ and $B_C = S_{l_1}$ thanks to the Schur property of $l_1$. On the other hand, $S_{l_1}$ is sequentially weakly closed, but not weakly closed. For, it is well-known that there is a net $\{x_\alpha\} \subseteq S_{l_1}$ such that $x_\alpha \to 0$. Notice that the space $l_1$ is not reflexive and its dual is not separable.
We close by observing that if \( C \) is finitely well-positioned, then its asymptotic cone \( C_\infty \) is finitely well-positioned. From \( C \subseteq \bigcup_{i=1}^n (K_{x_i} + x_i) \) it actually follows \( C_\infty \subseteq \bigcup_{i=1}^n K_{x_i}^* \), and so \( C_\infty \) is the finite union of cones that allow plastering.

## 3 General Results

Let us consider the closed convex sets

\[
K_{x^*}(m) = \{ x \in V : \langle x^*, x \rangle \geq \|x\| - m \}
\]

associated with \( K_{x^*} \) and \( m \in \mathbb{R} \). They are all well-positioned sets (see (iii) of the next result), with \( K_{x^*}(m) \supseteq K_{x^*} \) if \( m \geq 0 \) and \( K_{x^*}(m) \subseteq K_{x^*} \) if \( m \leq 0 \).

**Proposition 4**

(i) A cone \( K \) is well-positioned if and only if it allows plastering.

(ii) An unbounded set \( C \) is well-positioned if and only the set \( C \cap \{\|x\| \geq \rho\} \) allows plastering for \( \rho \) large enough.

(iii) \( C \) is well positioned if and only if \( C \subseteq K_{x^*}(m) \) for some \( x^* \in V \) and \( m \in \mathbb{R} \).

**Proof**

(i) We omit the simple proof. (ii) Let \( C \) be well-positioned, i.e., \( \langle x^*, x - x_0 \rangle \geq \|x - x_0\| \) for all \( x \in C \). Then,

\[
\langle x^*, x \rangle = \langle x^*, x - x_0 \rangle + \langle x^*, x_0 \rangle \geq \|x\| \left( \|x\| - \frac{x_0}{\|x\|} + \frac{x^*}{\|x\|} \right).
\]

As \( \|x\| \to \infty \), \( \langle x^*, x_0/\|x\| \rangle \to 0 \) and \( \|x/\|x\| - x_0/\|x\| \to 1 \). It follows \( \langle x^*, x \rangle \geq (1 - \eta) \|x\| \) for \( \|x\| \geq \rho \) large enough. Hence, \( C \cap \{\|x\| \geq \rho\} \) allows plastering.

As to the converse, suppose that \( C \cap \{\rho(B)^c \) allows plastering, where \( \rho B \) is the open ball of radius \( \rho \). That is, \( C \cap \{\rho B\}^c \subseteq K_{x^*} \). We can assume \( \|x^*\| > 1 \) so that \( K_{x^*} \) has a nonempty interior. Fix \( d \in \text{int} K_{x^*} \). Clearly, \( d \) is a recession direction of \( K_{x^*} \). Hence, \( K_{x^*} + \lambda d \subseteq K_{x^*} \) that implies \( K_{x^*} \subseteq K_{x^*} - \lambda d \). Hence, \( C \cap \{\rho B\}^c \subseteq K_{x^*} - \lambda d \) for all \( \lambda \geq 0 \). On the other hand, \( \rho B + \lambda d \subseteq K_{x^*} \) for \( \lambda \) large enough.\(^5\) It follows that \( \rho B \subseteq K_{x^*} - \lambda d \), and so \( C \cap \rho B \subseteq K_{x^*} - \lambda d \). As \( C \cap \{\rho B\}^c \subseteq K_{x^*} - \lambda d \), we conclude that \( C \subseteq K_{x^*} - \lambda d \) if \( \lambda \) is large enough. Namely, \( C \) is well-positioned.

(iii) If \( C \) is well positioned, \( C \subseteq K_{x^*} + x_0 \). It is easy to check that \( K_{x^*} + x_0 \subseteq K_{x^*}(m) \), with \( m = \langle x^*, x_0 \rangle - \|x_0\| \). Conversely, suppose \( C \subseteq K_{x^*}(m) \). If \( m \leq 0 \), then \( K_{x^*}(m) \subseteq K_{x^*} \) and so \( C \) allows plastering. Let \( C \subseteq K_{x^*}(m) \) with \( m > 0 \) and consider any point \( x \) of \( C \) with \( \|x\| \geq \alpha m \) and \( \alpha > 1 \). Then,

\[
\langle x^*, x \rangle \geq \|x\| - m \geq \|x\| - \alpha^{-1} \|x\| = \alpha^{-1} (\alpha - 1) \|x\|.
\]

\(^5\)This is actually equivalent to \( (\rho/\lambda) B + d \subseteq K_{x^*} \), which is true if \( d \in \text{int} K_{x^*} \).
By point (ii), $C$ is well-positioned.

As well known, bounded sets are well-positioned (see [1], [3], and [11]). For later reference we report this property, which is here derived from Proposition 4-(iii).

**Corollary 5** Bounded sets are well-positioned.

**Proof** Suppose $C$ is bounded, say $C \subseteq \rho B_V$ for some $\rho > 0$. Then, for any $x^* \in V^*$ it holds $\langle x^*, x \rangle \geq \alpha$ for all $x \in C$. If $x \in C$, then $\langle x^*, x \rangle \geq \alpha = \alpha - \rho + \rho \geq \alpha - \rho + \|x\|$ and so $C \subseteq K_{x^*}(\alpha - \rho)$.

By Proposition 4-(ii), a set $C$ is well-positioned if $C = C_0 \cup C_1$ where $C_0$ is bounded and $C_1$ allows plastering. Hence, a set $C$ is finitely well-positioned if $C = C_0 \cup (\bigcup_{i=1}^n C_i)$ where $C_0$ is bounded and each $C_i$ allows plastering. The next useful properties are other implications of Proposition 4 (more general results will be seen later in the paper).

**Proposition 6** (i) Let $C \subseteq H$, where $H$ is a linear subspace of $V$. If $C$ is well-positioned in $H$, then $C$ is well-positioned also as subset of $V$.

(ii) Let $T : V \rightarrow W$ be a linear isomorphism between two normed space. If $C \subseteq V$ is well-positioned, then its image $T(C)$ is well-positioned.

It is easy to see that the result still holds if we replace well-positioned sets with finitely well-positioned ones.

**Proof** (i) Use Proposition 4-(iii) and the Hahn-Banach Theorem. (ii) Use Proposition 4-(iii) and the fact that the image $T(K_{x^*})$ of the cone $K_{x^*}$ is a cone of $V_1$ that allows plastering. More specifically, $T(K_{x^*}) \subseteq \{ y \in V_1 : \langle (T^{-1})^* x^*, y \rangle \geq \|T\|^{-1} \|y\| \}$.

The next lemma is key for the present theory and will entail several important consequences. In reading it recall that in the applications that we have in mind $C$ will be a portion of the unit sphere.

**Lemma 7** Let $C$ be a bounded set with $0 \notin \overline{C}$. Then:

(i) $0 \notin \overline{C}^w$ if and only if $C = \bigcup_{i=1}^n C_i$, where each $C_i$ allows plastering;

(ii) $0 \notin \overline{C}$ if and only if $C$ allows plastering.

If, in addition, $V$ is reflexive or with separable dual, in (i) we can replace weak closure with sw-closure.
(i) Suppose that $C = \bigcup_{i=1}^n C_i$, where each $C_i$ allows plastering. Let $d \in \overline{C}^w$. As $\overline{C}^w = (\bigcup_{i=1}^n C_i)^w = \bigcap_{i=1}^n \overline{C}_i^w$, it follows that $d \in \overline{C}_i^w$ for some $i$. Let $x_\alpha \rightarrow d$ be a net with $\{x_\alpha\} \subseteq C_i$. Since $C_i$ allows plastering, $\langle u^*, x_\alpha \rangle \geq \|x_\alpha\|$ for some $u^* \in V^*$. As $0 \notin \overline{C}$, $\|x_\alpha\| \geq \eta > 0$. Taking limit we get $\langle u^*, d \rangle \geq \eta > 0$. Hence, $d \neq 0$ and so $0 \notin \overline{C}^w$.

Conversely, suppose that $0 \notin \overline{C}^w$. There will be a weak neighborhood of zero that does not meet $C$. In other words, there is $\varepsilon > 0$ and a finite sequence $\{x_i^*\}_{i=1}^n$ of elements of $V^*$ such that $|\langle x, x_i^* \rangle| < \varepsilon$ for each $i$ implies $x \notin C$. Equivalently, $x \notin C$ if $\langle x, \pm x_i^* \rangle < \varepsilon$ for each $i$. Consider the finite set $D = \{\pm x_i^* : i = 1, ..., n\}$. The above property can then be equivalently described as: for each $x \in C$ there is $u^* \in D$ such that $\langle u^*, x \rangle \geq \varepsilon$. Define the possibly empty sets $C_{u^*} = C \cap \{x : \langle u^*, x \rangle \geq \varepsilon\}$ for each $u^* \in D$. The above arguments imply $C = \bigcup_{u^* \in D} C_{u^*}$. It remains to check that every $C_{u^*} \neq \emptyset$ allows plastering. In fact, $\langle u^*, x \rangle \geq \varepsilon$ for all $x \in C_{u^*}$. Since $C$ is bounded, $\|x\| \leq N$ for $x \in C_{u^*}$. Hence, $\langle u^*, x \rangle \geq \varepsilon = (\varepsilon/N)N \geq (\varepsilon/N)\|x\|$, which shows that $C_{u^*}$ allows plastering.

(ii) Let $0 \notin \overline{C}C$. By a separation argument, there is $x^* \in V^*$ such that $\langle x, x^* \rangle \geq \varepsilon > 0$ for all $x \in \overline{C}C$. As $C$ is bounded, say $\|x\| \leq N$ for $x \in C$, then $\langle x, x^* \rangle \geq \varepsilon = (\varepsilon/N)N \geq (\varepsilon/N)\|x\|$ for all $x \in C$, and so $C$ allows plastering. Conversely, suppose $C$ allows plastering. Hence, $\langle x, x^* \rangle \geq \|x\| \geq \varepsilon$ for all $x \in C$. It follows $\langle x, x^* \rangle \geq \varepsilon$ for all $x \in \overline{C}C$. Therefore, $0 \notin \overline{C}C$.

The proof is completed by noticing that $\overline{C}^w = \overline{C}^w$ under our hypotheses. For, if $V^*$ is separable, the bounded sets of $V$ are weakly metrizable (see, e.g., [5, Theorem 3.35]) and thus $\overline{C}^w = \overline{C}^w$. If $V$ is reflexive $C$ is relatively weakly compact if is bounded. By Day’s Lemma, if $d \in \overline{C}^w$, there is a sequence in $C$ that converges weakly to $d$. Also in this case the desired property thus holds.

Remarks (i) Lemma 7-(i) is closely related to Kadets and Pelczynski’s criterion. They show that in reflexive spaces the condition $0 \notin \overline{C}^w$ is equivalent to the fact that $C$ fails to contain a basic sequence (see [4, Theorem 1.5.6] for details). (ii) It is well-known that $\overline{C}^w = B_V$ holds for any infinite dimensional normed spaces. Since $V = \text{cone} S_V$, infinite dimensional vector spaces are never finitely well-positioned.

The next key characterization of finitely well-positioned sets is a first notable consequence of Lemma 7. Observe that the condition $0 \notin B_C$ means that there is no unbounded sequence $\{x_n\} \subseteq C$ such that $x_n/\|x_n\| \rightarrow 0$.

**Theorem 8** Let $C \subseteq V$ that is either reflexive or with a separable dual. The following properties are equivalent:

(i) $0 \notin B_C$;
(ii) $C$ is a finitely well-positioned set.

When $V$ is reflexive, (i) and (ii) are equivalent to:

(iii) for any unbounded sequence $\{x_n\} \subseteq C$, there is a subsequence $\{x_{n_k}\}$ and a scalar sequence $\{t_k\}$ such that $t_k \to \infty$ and $x_{n_k}/t_k \to d \neq 0$.

A first important consequence of this result is that finite dimensional spaces, and so all their subsets, are finitely well-positioned. As remarked in the Introduction, this show that the relevance of finite well-positionedness is in infinite dimensional spaces.

**Proof** The result holds for a bounded set $C$ since $B_C = \emptyset$ and, by Corollary 5, is well-positioned. We will thus suppose that $C$ is unbounded.

(ii) implies (i). Suppose $C = \bigcup_{i=1}^n C_i$, where each $C_i$ is well-positioned and let $\|x_n\| \to \infty$ and $x_n/\|x_n\| \to d$. Wlog we can suppose $\{x_n\} \subseteq C_{i_0}$ for some $i_0$. Moreover, in view of Proposition 4-(ii), wlog we can suppose that $C_{i_0}$ allows plastering. Therefore, $\langle x^*, x_n \rangle \geq \|x_n\|$, i.e., $\langle x^*, x_n/\|x_n\| \rangle \geq 1$. This implies $\langle x^*, d \rangle \geq 1$ and so $d \neq 0$.

(i) implies (ii). Suppose first that the dual $V^*$ is separable and that (i) holds. Fix a radius $\rho > 0$ and define the set $\emptyset \neq S_\rho = \{x/\|x\| : x \in C \text{ and } \|x\| \geq \rho\} \subseteq S_V$. We claim that $0 \notin S_{\rho_n}^{\text{seq w}}$ for $\rho$ large enough. Suppose not. Then, there is a sequence $\rho_n \uparrow \infty$ such that $0 \in S_{\rho_n}^{\text{seq w}}$ for all $n$. Taking $n = 1$, there is a sequence $\{u_n\} \subseteq S_{\rho_1}$ for which $u_n \to 0$. On the other hand, $u_n = x_n^1/\|x_n^1\|$ with $\|x_n^1\| \geq \rho_1$. By hypothesis, the sequence $\|x_n^1\|$ is necessarily bounded (otherwise, $x_n^1/\|x_n^1\| \to 0$, thus contradicting (ii)). Therefore, there is some $\rho_{n_2}$ such that $\|x_n^1\| < \rho_{n_2}$ for all $n$. Iterating the same argument for the set $S_{\rho_{n_2}}$, we obtain a new sequence $\{x_n^2\}$ having the properties $\|x_n^2\| \geq \rho_{n_2}$, $x_n^2/\|x_n^2\| \to 0$, and $\|x_n^2\| < \rho_{n_3}$ and so on. Consequently, we get countably many sequences $\{x_n^k\}$ for which $x_n^k/\|x_n^k\| \to 0$ as $n \to \infty$, and $\rho_{n_k} \leq \|x_n^k\| < \rho_{n_{k+1}}$. Since $V^*$ is separable, the unit ball of $V$ is weakly metrizable (see the proof of Lemma 7). Denote by $\delta$ such a metric. For all $k$, there is an element $x_{n_k}^k/\|x_{n_k}^k\|$ of the sequence $\{x_n^k/\|x_n^k\|\}$ such that $\delta (x_{n_k}^k/\|x_{n_k}^k\|, 0) < 1/k$. Therefore, by construction, for the sequence $\{x_{n_k}^k/\|x_{n_k}^k\|\}$, it holds $x_{n_k}^k/\|x_{n_k}^k\| \to 0$ as $k \to \infty$ and $\|x_{n_k}^k\| \to \infty$. This contradicts (i). Hence, $0 \notin S_{\rho_n}^{\text{seq w}}$ for $\rho$ sufficiently large.

By Lemma 7-(i), $S_\rho = \bigcup_{i=1}^n C_i$ where each $C_i$ allows plastering. On the other hand, we have $\{x \in C : \|x\| \geq \rho\} \subseteq \text{cone} S_\rho = \text{cone} \bigcup_{i=1}^n C_i = \bigcup_{i=1}^n \text{cone} C_i$. Clearly each cone $C_i$ allows plastering, and thus $C \subseteq (C \cap \overline{\rho B}) \cup \bigcup_{i=1}^n \text{cone} C_i$, and $C$ is finitely well-positioned.

Suppose now that $V$ is reflexive. The proof proceeds in a similar way until the construction of the sequences $\{x_n^k\}$. Now consider the linear space $W = \text{span} \{x_n^k\}_{n,k}$, which is a

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6See Corollary 21 for a slightly more general result.
separable subspace of $V$. The subspace $W$ is reflexive and its dual $W^*$ is separable. Moreover the $\sigma(W,W^*)$ convergence of sequences in $W$ is equivalent to their $\sigma(V,V^*)$ convergence (see, e.g., [16, Proposition 2.5.22]). Therefore, by using the existing metric on $B_V \cap W$ we can extract a sequence $\{x_n^k\}_k$ such that $x_n^k \to 0$ and $\|x_n^k\| \to \infty$. This leads to a contradiction and the proof proceeds as in the previous case.

We prove that (iii) is equivalent to (i) provided $V$ is reflexive. Assume (i) and let $\{x_n\}$ be an unbounded sequence. Consider the sequence $x_n/\|x_n\|$. As $V$ is reflexive, there is a convergent subsequence $x_{n_k}/\|x_{n_k}\| \to d$ and $d$ does not vanish by (i).

Now assume (iii) and let $\{x_n\} \subseteq C$ with $\|x_n\| \to \infty$ and $x_n/\|x_n\| \to d$. By (iii) there is a subsequence $\{x_{n_k}\}$ a sequence $\{t_k\}$ for which $t_k \to \infty$ and $x_{n_k}/t_k \to d_1 \neq 0$. By (vi) of Proposition 2 there a subsequence $\{x_{n_{k_r}}\}$ for which $x_{n_{k_r}}/\|x_{n_{k_r}}\| \to \lambda d_1 \neq 0$. Hence $x_n/\|x_n\| \to d \neq 0$ and (i) holds.

In Theorem 8 condition (ii) implies (i) without any assumption on $V$. Something more can be actually said.

**Proposition 9** If $C$ is finitely well-positioned, then $0 \notin \overline{B}_C$ and there is no unbounded net $\{x_\alpha\} \subseteq C$ such that $x_\alpha/\|x_\alpha\| \to 0$.

Similarly, it can be proved that $0 \notin \overline{c}_0 B_C$ if $C$ is well-positioned.

**Proof** Let $C$ be finitely well-positioned. As in Theorem 8, $0 \notin B_C$. This is trivially true if $B_C = \emptyset$. Suppose $B_C \neq \emptyset$. In this case cone $B_C = C_\infty$ by Proposition 2. Moreover, $B_C$ is norm closed, and so $0 \notin \overline{B}_C$. If $C$ is finitely well-positioned, $C_\infty$ is finitely well-positioned. Hence, $B_C$ is finitely well-positioned. By Lemma 7, $0 \notin \overline{B}_C$. This proves that $0 \notin \overline{B}_C$.

It remains to prove that there is no unbounded net $\{x_\alpha\} \subseteq C$ such that $x_\alpha/\|x_\alpha\| \to 0$. This property holds for any dual pair $\langle V, W \rangle$. Let $C$ be finitely well-positioned under $\langle V, W \rangle$, then $C \subseteq \bigcup_{i=1}^n K_{w_i}(m_i)$. If $\{x_\alpha\}_{\alpha \in I} \subseteq C$ is a net, consider the sets $I_i = \{\alpha \in I : x_\alpha \in K_{w_i}(m_i)\}$ for $i \in N = \{1, 2, \ldots, n\}$. Clearly $\bigcup_{i=1}^n I_i = I$. We claim that if $\{x_\alpha\}$ is a net, then for some $i \in N$, $I_i$ is a net cofinal into $I$. Thus, $I_i$ is a subnet of $I$. Suppose true this claim. If $\{x_\alpha\} \subseteq C$ for which $x_\alpha/\|x_\alpha\| \to d$ and $\|x_\alpha\| \to \infty$, then this is true for the subnet $\{x_\beta\} \subseteq K_{w_i}(m_i)$ for which $x_\beta/\|x_\beta\| \to d$ and $\|x_\beta\| \to \infty$. Hence, $\langle x_\beta, w_i \rangle \geq \|x_\beta\| - m_i$. Dividing by $\|x_\beta\|$ and taking limit, we get $\langle d, w_i \rangle \geq 1$, that implies $d \neq 0$.

Bounded sets have trivial asymptotic cones and, by Corollary 5, are well-positioned. The next remarkable consequence of Theorem 8 shows that under finite well-positionedness the converse holds. That is, finite well-positionedness turns out to be the property that characterizes bounded sets among the sets that have trivial asymptotic cones.
Corollary 10 A subset \( C \) of a reflexive space \( V \) is bounded if and only if \( C \) is finitely well-positioned and \( C_\infty = \{0\} \).

Proof We prove the “if” part, the converse being trivial. Let \( C \) be finitely well-positioned with \( C_\infty = \{0\} \). Suppose per contra that \( C \) is unbounded. As \( V \) is reflexive, \( B_C \neq \emptyset \). Hence, \( B_C = C_\infty = \{0\} \). By Theorem 8, this is a contradiction.

A consequence of this characterization is that unbounded convex sets that are linearly bounded are not finitely well-positioned.

Example 11 Corollary 10 may fail if \( V \) is not reflexive. The positive cone \( l_1^+ \) of \( l_1 \) allows plastering. For, let \( e = (1,1,\ldots) \in l_\infty \). It holds \( \langle x,e \rangle = \sum_{i=1}^\infty x_i = \|x\| \) for all \( x \in l_1^+ \). Set \( C = \{x \in l_1 : 0 \leq x_i \leq \alpha_i \) for each \( i \}, with \( \sum_{i=1}^\infty \alpha_i = \infty \). The closed and convex set \( C \) is unbounded because \( x^n = (\alpha_1,\ldots,\alpha_n,0,0,\ldots) \in C \) for all \( n \) and \( \|x^n\| = \sum_{i=1}^n \alpha_i \to \infty \). It is also linearly bounded, and so \( C_\infty = \{0\} \), since for each \( 0 \neq x \in l_1^+ \) it holds \( tx \notin C \) for \( t > 0 \)

Example 12 Whether a given set is well-positioned depends on the considered dual pair. The positive cone \( l_1^+ \) of \( l_1 \) is well-positioned with respect to the pair \( \langle l_1, l_\infty \rangle \), but not with respect to \( \langle l_1, c_0 \rangle \). For, if \( \{e_n\} \subseteq l_1^+ \) is the units’ sequence, we have \( e_n \overset{w^*}= 0 \). By Theorem 8 or Proposition 9, \( l_1^+ \) is not finitely well-positioned.

The next result is a further consequence of Lemma 7. Unlike Theorem 8, no assumption is made on \( V \). It is a general version of [3, Lemma 2.1]. Set

\[
S_\rho = \left\{ \frac{x}{\|x\|} : x \in C \text{ and } \|x\| \geq \rho \right\}.
\]

Proposition 13 (i) \( C \) is well-positioned if and only if \( 0 \notin \overline{\partial} S_\rho \) for \( \rho \) large enough.

(ii) \( C \) is finitely well-positioned if and only if \( 0 \notin \overline{S_\rho}^w \) for \( \rho \) large enough.

Proof (i) Suppose \( C \) to be well-positioned. By Proposition 4, \( C \cap \{\|x\| \geq \rho \} \) allows plastering for some \( \rho \). Hence, \( \langle x^*, x \rangle \geq \|x\| \) for all \( x \in C \) and \( \|x\| \geq \rho \). Namely, \( \langle x^*, x/\|x\| \rangle \geq 1 \). We conclude that \( u \in S_\rho \) implies \( \langle x^*, u \rangle \geq 1 \). Clearly \( 0 \notin \overline{\partial} S_\rho \).

Conversely, if \( 0 \notin \overline{\partial} S_\rho \), by Lemma 7 \( S_\rho \) allows plastering. Hence \( \text{cone} (S_\rho) \supseteq C \cap \{\|x\| \geq \rho \} \) allows plastering. By Proposition 4, \( C \) allows plastering.

(ii) Let \( C \) be finitely well-positioned. Then, \( C \cap \{\|x\| \geq \rho \} \) is the finite union of sets that allow plastering, when \( \rho \) is sufficiently large. Therefore, there are nonzero \( \{x_i^*\}_{i=1}^n \subseteq V^* \) such that \( x \in C \cap \{\|x\| \geq \rho \} \) implies \( \langle x_i^*, x \rangle \geq \|x\| \) for some \( i \). Hence, for all \( u \in S_\rho \), \( \langle x_i^*, u \rangle \geq 1 \) for some \( i \). Consequently, if \( v \in V \) is any point such that \( |\langle x_i^*, v \rangle| \leq 1/2 \) for \( i = 1, 2, \ldots, n \),
then \( v \notin S_\rho \). On the other hand, \( |\langle x^*_i, x \rangle| \leq 1/2 \) for all \( i \) is a weak neighborhood of 0. Hence \( 0 \notin \overline{S}_\rho \).

Conversely, suppose \( 0 \notin \overline{S}_\rho \) for some \( \rho \). Lemma 7 implies that \( S_\rho \) is the finite union of sets that allow plastering. So \( C \cap \{ \|x\| \geq \rho \} \subseteq \text{cone } S_\rho \), where \( \text{cone } S_\rho \) is the finite union of sets that allow plastering. This is enough to conclude that \( C \) is finitely well-positioned.

**Corollary 14** Suppose \( V \) is reflexive or has a separable dual. A set \( C \subseteq V \) is finitely well-positioned if and only if \( C \cap W \) is for all closed and separable subspaces \( W \) of \( V \).

**Proof** Suppose that all \( C \cap W \) are finitely well-positioned and that, by contradiction, \( C \) is not. Let \( \rho_m \uparrow \infty \). By Proposition 13, \( 0 \in \overline{S}^w_{\rho_m} \) for all \( m \). By the assumptions on the space \( V \), \( 0 \in \overline{S}^w_{\rho_m} \). Hence there are sequences \( \{x^m_n\} \) such that \( \|x^m_n\| \geq \rho_m \) and \( x^m_n/\|x^m_n\| \rightharpoonup 0 \) as \( n \to \infty \), \( \forall m \). Set \( W = \text{span } \{x^m_n\} \). \( W \) is a closed and separable subspace. By construction, \( C \cap W \) is not finitely well-positioned, which leads to a contradiction.

### 3.1 Polyhedral Cuts

We present a useful criterion for finite well-posedness based on the boundedness of sets’ slices. Specifically, given a set \( C \), a finite set of nonzero functionals \( D = \{x^*_i\}_{i=1}^n \subseteq V^* \) and scalars \( T = \{t_i\}_{i=1}^n \), the polyhedron

\[
C(D,T) = \{ x \in C : \langle x^*_i, x \rangle \leq t_i \text{ for each } i \}
\]

is the slice of \( C \) determined by \( D \) and \( T \).

**Definition 15** A set \( C \) is said to have a (bounded) polyhedral cut if there is a set of nonzero functionals \( D = \{x^*_i\}_{i=1}^n \subseteq V^* \) such that, for each collection \( T = \{t_i\}_{i=1}^n \) of scalars, the slices \( C(D,T) \) are either empty or bounded.

The next result extends the idea behind [3, Lemma 2.2].

**Proposition 16** A convex set \( C \) is finitely well-positioned if and only if it has a polyhedral cut.

The proof of this result rests on couple of lemmas of some independent interest. The first one shows that one direction holds even without convexity.

**Lemma 17** Finite well-positioned sets have a polyhedral cut.

**Proof** Let \( C \) be finitely well-positioned. By Proposition 4 there is \( D = \{x^*_i\}_{i=1}^n \subseteq V^* \setminus \{0\} \) and scalars \( \{m_i\}_{i=1}^n \) such that \( x \in C \) implies \( \langle x^*_i, x \rangle \geq \|x\| - m_i \) for some \( i \). Hence, if
\[ \langle x_i^*, x \rangle \leq t_i \] for all \( i \), it follows \( \|x\| \leq \max_i (m_i + t_i) \) for all \( x \in C \). Consequently, \( C(D, T) \) are bounded or empty for all \( T = \{t_i\}_{i=1}^n \). \hfill \blacksquare

Next we show that the other direction in Proposition 16 holds for a convex set even if there is only a single nonempty and bounded slice, provided it satisfies a Slater-type condition.

**Lemma 18** A convex set \( C \) is well-positioned if there is a nonempty and bounded slice \( C(D, T) \) of \( C \) such that, for some for \( x \in C(D, T) \), it holds \( \langle x_i^*, x \rangle < t_i \) for all \( i \).

**Proof** Suppose \( C \) is convex, that \( C(D, T) \) is nonempty and bounded, and that \( \langle x_i^*, \pi \rangle < t_i \) for all \( i \) and some \( \pi \in C(D; T) \). By translation, wlog we can set \( \pi = 0 \). We can thus assume \( 0 \in C(D, T) \) and \( C(D, T) \) bounded with all \( t_i > 0 \) in \( T \). Namely, there is \( \eta > 0 \) such that \( \|x\| \leq \eta \) for all \( x \in C(D, T) \). Observe that \( x \in C(D, T) \) and \( \langle x_i^*, x \rangle = t_i \) for some \( i \), it follows

\[
\left\langle x_i^*, \frac{x}{\|x\|} \right\rangle = \frac{t_i}{\|x\|} \geq \frac{t_i}{\eta} \tag{2}
\]

Pick now any point \( x \in C \cap \{\|\cdot\| \geq \eta + \varepsilon\} \). Clearly there is some \( x_i^* \in D \) for which \( \langle x_i^*, x \rangle > t_i \). Consider the subset \( \Gamma \subset D \) for which \( \langle x_i^*, x \rangle > t_i \) and choose \( x_j^* \in \Gamma \) such that \( \langle x_j^*, x \rangle / t_j \geq \langle x_i^*, x \rangle / t_i \) for all \( x_i^* \in \Gamma \). As \( C \) is convex and \( 0 \in C \), the points \( \lambda x \in C \) with \( \lambda \in [0, 1] \). Hence there is a scalar \( \lambda_0 \) such that \( \langle x_j^*, \lambda_0 x \rangle = t_j \). Clearly, \( \langle x_j^*, \lambda_0 x \rangle \leq t_i \) for all \( i \neq j \). Hence \( \lambda_0 x \in C(D, T) \) and \( \langle x_j^*, \lambda_0 x \rangle = t_j \). By (2), \( \langle x_j^*, x / \|x\| \rangle \geq t_j / \eta \). To conclude, for all \( x \in C \cap \{\|\cdot\| \geq \eta + \varepsilon\} \) there is \( x_j^* \in D \) such that \( \langle x_j^*, x \rangle \geq (t_j / \eta) \|x\| \). That is, \( C \) is finitely well-positioned. \hfill \blacksquare

**Proof of Proposition 16** In view of the previous two lemmas, it is enough to observe that if a nonempty set \( C(D; T) \) does not satisfy the Slater condition \( \langle x_i^*, x \rangle < t_i \), the set \( C(D; T + \varepsilon) \) does, where \( T + \varepsilon = \{t_i + \varepsilon\}_{i=1}^n \). \hfill \blacksquare

**Corollary 19** A convex set \( C \) is well-positioned if and only if it has a polyhedral cut with \( D \) singleton.

**Example 20** Convexity in Proposition 16 is needed. Let \( f(x) = \sqrt{\|x\|} \) be defined on an infinite dimensional normed space \( V \). Its non-convex epigraph \( \text{epi} f \subset V \times \mathbb{R} \) has a polyhedral cut. For instance, if we consider in \( V \times \mathbb{R} \) the linear functional \((0, 1)\), the set \( \{(x, \lambda) \in \text{epi} f : \lambda \leq t\} \) is bounded in \( V \times \mathbb{R} \) for every \( t \). Nevertheless, \( \text{epi} f \) is not finitely well-positioned, as will be shown later in the paper. \hfill \▲

Observe that Proposition 16 can be formulated in a slightly different equivalent way by saying that a set \( C \) is finitely well-positioned if and only if there is \( 0 \neq x^* \in V^* \) and \( t \in \mathbb{R} \) such that the slice \( \{x \in C : \langle x^*, x \rangle \leq t\} \) is finitely well-positioned and \( \langle x^*, x \rangle < t \) for some \( x \in C \).
After Theorem 8 we observed that finite dimensional subspaces are finitely well-positioned. Something more is proved in the next result, a consequence of Proposition 16.

**Corollary 21** Vector subspaces $H \subseteq V$ are finitely well-positioned if and only if they are finite dimensional. In this case, $\dim H + 1$ is the least number of cones that cover $H$ and allow plastering.

More generally, if $H = \overline{\operatorname{span}} C$, then $C$ is not finitely well-positioned when $\dim H = \infty$ and $\operatorname{ri}_H^w C \neq \emptyset$.

**Proof** (i) By Proposition 6, it is not restrictive to consider $\mathbb{R}^n$, which is clearly finitely well-positioned by Theorem 8. Let $D = \{x_i^*\}_{i=1}^m$ be a set such that, for each $T = \{t_i\}_{i=1}^n$, the slices $C(D, T)$ are either empty or bounded. We will prove that $m \geq n + 1$. Suppose per contra that $m \leq n$. The sets $\langle x_i^*, x \rangle \leq t_i$ are bounded for all $i$. In particular, the cone $K = \{ x \in \mathbb{R}^n : \langle x_i^*, x \rangle \leq 0 \text{ for all } i \}$ is bounded. Being a cone, $K$ is bounded if and only if $K = \{0\}$. Consider the linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ given by $Ax = (\langle x_i^*, x \rangle)_{i=1}^m$. Clearly, $K = A^{-1}(\mathbb{R}^m)$. If $m \leq n$, we have $K \neq \{0\}$. We deduce that $m \geq n + 1$. Take now the $n + 1$ functionals $\langle x_i^*, x \rangle = x_i$ with $i = 1, \ldots, n$ and $\langle y^*, x \rangle = -\sum_{i=1}^n x_i$. The set $\{x_i^*\}_{i=1}^n \cup \{y^*\}$ also determines a polyhedral cut. As proved in Proposition 16 these functionals generate the cones $\langle x_i^*, x \rangle \geq \varepsilon \|x\|$ and $\langle y^*, x \rangle \geq \eta \|x\|$ that cover the space $\mathbb{R}^n$ for sufficiently small $\varepsilon$ and $\eta$.

(ii) That $H$ is never finitely well-positioned when $\dim H = \infty$ has already been observed (see the remark after Lemma 7). Suppose $C$ is finitely well-positioned. Then, $C$ is finitely well-positioned in $H$. If $\overline{x} \in \operatorname{ri}_H^w C$, by the arguments used in the proof of Theorem 25 to show that (i) implies (v), there is weak neighborhood $U \subseteq H$ of $\overline{x}$ that is norm bounded. By translation, we obtain a weak neighborhood $V$ of 0 that is norm bounded. Consequently, there exist functionals $\{x_i^*\}_{i=1}^m$ and $\varepsilon > 0$ for which $|\langle x_i^*, x \rangle| < \varepsilon$ for all $i$ implies $x \in V$. Hence, $\bigcap_{i=1}^m \ker x_i^* \subseteq V$. If $y^*$ is linearly independent of $\{x_i^*\}_{i=1}^m$, there is a point $\overline{x} \in \bigcap_{i=1}^m \ker x_i^* \setminus \ker y^*$. Hence $\overline{x} \neq 0$ and $n\overline{x} \in V$ for all $n$. This contradicts the fact that $V$ is norm bounded. \[\blacksquare\]

**Remark** Corollary 21 implies that the hypotheses on $V$ made in Theorem 8 are needed. For, consider any infinite dimensional subspace $H$ of an infinite dimensional space with the Schur property (e.g., $l_1$). Clearly, $0 \notin B_H$, but $H$ is not finitely well-positioned.\footnote{Note that inter alia we get the well-known result that infinitely dimensional spaces with the Schur property are not reflexive and their duals are not separable.}
4 Asymptotic Compactness

In the literature several notions have been recently introduced to describe the asymptotic behavior of unbounded sets. Here we compare some of them with the notions that we introduced in the paper. Though many of them have been formulated for sets in Hausdorff topological vector spaces, here we consider normed spaces endowed with the weak topology.

We begin with the notion of asymptotic weak compactness due to Dedieu [8] and Zalinescu [18].

**Definition 22** A set $C$ is asymptotically weakly compact if there is $\varepsilon > 0$ and a weak neighborhood $U$ of the origin such that $[0, \varepsilon] C \cap U$ is relatively weakly compact.

The next notion is due to Luc [12].

**Definition 23** A set $C$ has the weak CB property if a bounded set $A$ exists such that $\overline{\text{cone}(C \setminus A)}$ has a weakly compact base.\(^8\)


**Definition 24** A set $C$ is weakly recessively compact if, for any unbounded net $\{x_\alpha\}_\alpha \subseteq C$, there are a subnet $\{x_\beta\}_\beta$ and scalars $t_\beta$ such that $\lim_{t_\beta} t_\beta = \infty$ and $x_\beta / t_\beta \to d \neq 0$.

Finally, recall that a set $C$ is locally weakly compact if each $x \in C$ has a weak neighborhood that is relatively weakly compact.

The following “omnibus” result establishes the equivalences among finite well-positionedness and the asymptotic concepts just introduced. It is important to observe that this equivalence enriches our approach, at least for the weak topology of normed spaces.\(^9\) In fact, it permits to give a geometrical interpretation to the finite well-positionedness property.

**Theorem 25** Let $C$ be a subset of a reflexive space $V$. Consider the following properties:

(i) $C$ is finitely well-positioned;

(ii) $C$ is weakly asymptotic compact;

(iii) $C$ has the weak CB property;

---

\(^8\)That is, $\overline{\text{cone}(C \setminus A)} = \text{cone } B$, where $B$ is a weakly compact set that does not contain the origin.

\(^9\)In other settings the various asymptotic concepts fail to coincide. In particular, recessive compactness seems to be the natural extension of the finite well-positionedness beyond the cases $(V, V^*)$ and $(V^*, V)$. To this end one should use the general version of Definition 24 for Hausdorff topological vector space given by [13].
(iv) $C$ is recessively weakly compact;

(v) $C$ is locally weakly compact.

Then,

$$(i) \iff (ii) \iff (iii) \iff (iv) \implies (v).$$

If, in addition, $C$ is convex, then they are all equivalent.

**Proof**

(i) implies (ii) Let $K_{x^*}(m)$ be with $m \geq 0$. A point $y \in [0, \varepsilon_0] K_{x^*}(m)$ if either $y = 0$ or $y/\varepsilon \in K_{x^*}(m)$ with $\varepsilon \in (0, \varepsilon_0]$. Hence $y$ satisfies $\langle x^*, y \rangle \geq \|y\| - \varepsilon m \geq \|y\| - \varepsilon_0 m$. Consequently, $[0, \varepsilon_0] K_{x^*}(m) \subseteq K_{x^*}(\varepsilon_0 m)$. Consider the weak neighborhood $U = \{y : \langle x^*, y \rangle \leq 1\}$ of 0. Hence,

$$y \in [0, \varepsilon_0] K_{x^*}(m) \cap U \implies \|y\| \leq 1 + \varepsilon_0 m.$$ 

Namely, $K_{x^*}(m)$ is asymptotically weak compact. Now consider any finitely well-positioned set $C \subseteq \bigcup_{i=1}^n K_{x^*}(m_i)$. It follows $[0, \varepsilon_0] C \subseteq \bigcup_{i=1}^n [0, \varepsilon_0] K_{x^*}(m_i)$. Therefore, by using the neighborhood $U = \{x : \|x_i, y\| \leq 1, i = 1, \ldots, n\}$, the set $[0, \varepsilon_0] C \cap U$ is norm bounded.

(ii) implies (iv) and (iii) implies (iv) by ([13, Proposition 2.2]).

(iv) implies (i) Let $C$ be recessively weakly compact. Suppose per contra that $C$ is not finitely well-positioned. By Theorem 8 there is an unbounded sequence $\{x_n\} \subseteq C$ such that $x_n/\|x_n\| \rightharpoonup 0$. As $C$ is recessively weakly compact, there is a subnet $\{x_\beta\} \subseteq \{x_n\}$ and a net $\{t_\beta\}$ such that $\lim_\beta t_\beta = \infty$ and $x_\beta/t_\beta \rightharpoonup d \neq 0$. Hence,

$$\frac{x_\beta}{t_\beta} = \frac{x_\beta}{\|x_\beta\|} \cdot \|x_\beta\| \xrightarrow{t_\beta} d.$$

As a weakly convergent sequence is norm bounded, it follows $0 \leq \|x_\beta\|/t_\beta \leq L$. Passing to a subnet we get $\|x_\gamma\|/t_\gamma \to \lambda$. Moreover, $\lambda \neq 0$, otherwise $x_\beta/t_\beta \rightharpoonup 0$. Consequently,

$$\frac{x_\gamma}{\|x_\gamma\|} = \frac{x_\gamma}{t_\gamma} \cdot \frac{t_\gamma}{\|x_\gamma\|} \xrightarrow{t_\gamma} \frac{1}{\lambda} d \neq 0,$$

a contradiction because $x_\gamma/\|x_\gamma\|$ is a subnet of $x_n/\|x_n\|$ and thus $x_\gamma/\|x_\gamma\| \nrightarrow 0$.

(i) implies (iii) Notice first that if $C$ is a cone that allows plastering, then $C$ satisfies CB with $A = \emptyset$. Actually, if $C \subseteq K_{x^*}$, then $B = \{x : \langle x^*, x \rangle = 1\} \cap C$ is a weak compact base of $C$.

Now, let $C$ be finitely well-positioned. By Proposition 4-(ii), $C \cap \{\|x\| \geq \rho\} = \bigcup_{i=1}^n C_i$, where each $C_i$ allows plastering. Notice that $\|x\| \leq \rho$ is weakly bounded. Therefore, $\text{cone} (C \setminus A) = \text{cone} (\bigcup_{i=1}^n C_i) = \bigcup_{i=1}^n \text{cone} C_i$. We have seen that each $\text{cone} C_i$ has a compact base. Hence $\overline{\text{cone}} C_i = \text{cone} B_i$. It follows that $\overline{\text{cone}} (C \setminus A) = \bigcup_{i=1}^n \text{cone} B_i = \text{cone} (\bigcup_{i=1}^n B_i)$, as desired.
(i) implies (v) Let $C$ be finitely well-positioned. By Proposition 16, there exists $D = \{x_i^*\}_{i=1}^n \subseteq V^* \setminus \{0\}$ that determines a polyhedral cut for $C$. Let $\pi \in C$. The weak neighborhood of $\pi$

$$C \cap \{||x_i^*, x - \pi|| \leq \varepsilon \text{ for each } i\}$$

is norm bounded. Thus, it is relatively weakly compact if $V$ is reflexive.

It remains to prove that (v) implies (i) if $C$ is convex. Pick $\pi \in C$. There is a weak neighborhood of $\pi$ that is relatively weakly compact, and so bounded. That means that there is $\varepsilon > 0$ and $\{x_i^*\}_{i=1}^n$ such that $||x|| \leq \eta$ if $x \in C$ and $|\langle x_i^*, \pi - x \rangle| \leq \varepsilon$ for all $i$. That is, $\{x_i^*\}_{i=1}^n \cup \{-x_i^*\}_{i=1}^n$ determines a polyhedral cut for $C$. By Proposition 16, $C$ is finitely well-positioned.

**Remark** In view of Proposition 8-(iii) and of the equivalence between (i) and (iv), in Definition 24 is enough to consider sequences in place of nets when the space is reflexive.

## 5 Convex Sets

This section is mainly devoted to convex sets. The next lemma provides a few characterizations of convex well-positioned sets. Point (iii), due to [3, Proposition 2.1], says that closed convex sets are well-positioned if and only if they are finitely well-positioned and do not contain any line. For sake of brevity we omit the proof.

**Lemma 26** Let $V$ be a reflexive space. Then:

(i) A closed convex cone $K$ allows plastering if and only if is pointed and $0 \notin B_K$.\(^{10}\)

(ii) A closed convex cone $K$ allows plastering if and only if the set $K \cap \{\langle x^*, x \rangle = 1\}$ is bounded for every strictly positive functional $x^*$ on $K$.\(^{11}\)

(iii) A closed convex set $C$ is well-positioned set if and only if $L_C = \{0\}$ and $0 \notin B_C$.

Next we characterize the convex sets that are finitely well-positioned.

**Proposition 27** Let $C$ be a convex subset of a Banach space $V$. Consider the following conditions:

(i) $C$ is finitely well-positioned;

(ii) there is a projection $P : V \to V$ with finite codimensional range such that its image $P(C)$ is well-positioned;

---

\(^{10}\)A cone $K$ is pointed if $K \cap -K = \{0\}$.

\(^{11}\)That is, $\langle x^*, x \rangle > 0$ for all $0 \neq x \in K$ (see [10, Theorem 2.7] and [11]).
(iii) there is a finite dimensional subspace \( L \) of \( V \) such that \( \pi(C) \) is well-positioned, where \( \pi : V \to V/L \).

Then,

\[
(ii) \iff (iii) \implies (i).
\]

If, in addition, \( C \) is closed and \( V \) is reflexive, then they are all equivalent. In particular, \( C = L_C \oplus C_1 \) with \( C_1 \) well-positioned.

Notice that the direct sum \( C = L_C \oplus C_1 \) means that there exists a closed complementary vector space \( M \) to \( L_C \) that contains \( C_1 \) (i.e., \( V = L_C \oplus M \) and \( C_1 \subseteq M \)).

**Proof** (ii) implies (i). Suppose (ii) holds. \( P_1 = I - P \) is a projection with finite-dimensional range. Set \( L = P_1(V) \). Since \( L \) is finitely well-positioned, by Proposition 16 there exists \( D = \{x_i^*\}_{i=1}^n \subseteq V^* \setminus \{0\} \) that determines a polyhedral cut for \( C \). Likewise, by Corollary 19, since \( P(C) \) is well-positioned there is a functional \( y^* \) such that \( \langle y^*, x \rangle \leq t \) for all \( t \in \mathbb{R} \). The set of functionals \( \{x_i^* \circ P_1\}_{i=1}^n \cup \{y^* \circ P\} \) also determines a polyhedral cut for \( C \). By the decomposition \( x = P_1 x + Px \) for all \( x \in V \), it holds

\[
\{x \in C : \langle x_i^* \circ P_1, x \rangle \leq t_i \ \forall i \text{ and } \langle y^* \circ P, x \rangle \leq \tau\}
\]

\[
= \{x \in P_1(C) : \langle x_i^*, x \rangle \leq t_i\} + \{x \in P(C) : \langle y^*, x \rangle \leq \tau\}
\]

\[
\subseteq \{x \in L : \langle x_i^*, x \rangle \leq t_i\} + \{x \in P(C) : \langle y^*, x \rangle \leq \tau\},
\]

as desired. Hence, \( C \) is finitely well-positioned.

The equivalence between (ii) and (iii) easily follows from the fact that \( M \simeq V/L \) if \( V = M \oplus L \) and \( V \) is complete.

Finally, suppose that \( C \) is finitely well-positioned and \( V \) reflexive. By Corollary 21, \( \dim L_C < \infty \). Hence, it is complemented in \( V \). That is, there is a closed subspace \( M \) of \( V \) for which \( L_C \oplus M = V \). Hence, it holds the decomposition \( C = L_C \oplus (C \cap M) \). Since \( C \cap M \subseteq C \), it is finitely well-positioned. Moreover, the lineality space of \( C \cap M \) is clearly trivial. By Lemma 26-(iii), \( C \cap M \) is well-positioned. Clearly, \( P(C) = C \cap M \), where we denote by \( P \) the projection with range \( M \).

\[\square\]

### 5.1 Dual Properties

Now we study convex sets by using dual properties. To this end, we need some standard notation. The **negative polar cone** \( M^- \) of a set \( M \) is given by

\[M^- = \{x^* : \langle x^*, x \rangle \leq 0 \text{ for each } x \in M\} \]

If \( A \) and \( B \) are subsets of \( V \) and \( V^* \) respectively, we define the **annihilators** by the formulas

\[ A^\perp = \{x^* \in V^* : \langle x^*, x \rangle = 0 \text{ for each } x \in A\} \]

\[ B^\perp = \{x \in V : \langle x^*, x \rangle = 0 \text{ for each } x^* \in B\} \].
The support functional \( \sigma_C \) of a convex set \( C \) is given by \( \sigma_C (x^*) = \sup \{ \langle x^*, x \rangle : x \in C \} \). The domain of \( \sigma_C \) is called the barrier cone \( b(C) \) of \( C \); i.e., \( b(C) = \{ x^* : \sigma_C (x^*) < \infty \} \). Finally, \( C^0 \) denotes the polar of \( C \), i.e., \( C^0 = \{ x^* \in V^* : \sigma_C (x^*) \leq 1 \} \).

As well-known, \( b(C)^{-} = C_\infty \) if \( C \) is closed and convex. By the Bipolar Theorem, \( b(C)^{-} = C_\infty^{-} \). This is equivalent to \( b(C) = C_\infty^\circ \) when \( V \) is reflexive.

In the convex case the next result is due to [3]. Proposition 32 will extend this result to finitely well-positioned sets.

**Proposition 28** Let \( C \) be well-positioned. Then, \( \operatorname{int} b(C) \neq \emptyset \) and the converse holds if \( C \) is convex. If \( V \) is reflexive, then

\begin{enumerate}[(i)]  
  \item \( \operatorname{int} C_\infty^{-} = \operatorname{int} b(C) \);
  \item the functionals \( -x^* + \delta_C \) are coercive for all \( x^* \in \operatorname{int} C_\infty^{-} \);
  \item the functionals \( x^* \in \operatorname{int} C_\infty^{-} \) attain the sup when \( C \) is sw-closed;
  \item \( \overline{\overline{C}}_\infty = [\overline{\overline{C}}]_\infty \).
\end{enumerate}

**Proof** Suppose that \( C \) is well-positioned. Hence, \( C \subseteq K_{x^*} (m) \). Thus \( \sigma_C \leq \sigma_{K_{x^*} (m)} \) and so \( b(C) \supseteq b(K_{x^*} (m)) \). We prove that \( b(K_{x^*} (m)) \supseteq -x^* + B_{V^*} \) and consequently \( b(C) \) has a nonempty interior. Let \( u^* \in B_{V^*} \). Clearly \( \langle u^*, x \rangle \leq \|x\| \). Hence, from \( \langle x^*, x \rangle \geq \|x\| - m \) we get \( \langle x^*, x \rangle \geq \langle u^*, x \rangle - m \). Namely, \( m \geq \langle -x^* + u^*, x \rangle \). Hence \( -x^* + u^* \in b(K_{x^*} (m)) \).

For the converse we need convexity. Observe that \( \sigma_C \geq \sigma_{\overline{\overline{C}}} \) and that \( C \) is well-positioned if and only if \( \overline{\overline{C}} \) is. In particular, \( b(C) = b(\overline{\overline{C}}) \). Hence, \( \operatorname{int} b(\overline{\overline{C}}) \neq \emptyset \) if \( \operatorname{int} b(C) \neq \emptyset \). In this way we can apply the arguments of [3, Theorem 2.1] and infer that \( \overline{\overline{C}} \) is well-positioned. In turn this implies that \( C \) is well-positioned.

Now suppose that \( V \) is reflexive and \( C \) well-positioned. From \( C \subseteq K_{x^*} + x_0 \) it follows \( C_\infty \subseteq K_{x^*} \). Consequently, \( K_{x^*}^{-} \subseteq C_\infty^{-} \). On the other hand, \( K_{x^*}^{-} \supseteq -x^* + B_{V^*} \). Hence, \( \operatorname{int} C_\infty^{-} \neq \emptyset \).

Let \( C \) be well-positioned. We show that \( \operatorname{int} C_\infty^{-} \subseteq b(C) \). Fix \( x^* \in \operatorname{int} C_\infty^{-} \). We can suppose \( C \) to be unbounded, otherwise all claims trivially hold. We claim that there are \( \eta > 0 \) and \( \varepsilon > 0 \) such that \( x \in C \) and \( \|x\| \geq \eta \Rightarrow \langle -x^*, x \rangle \geq \varepsilon \|x\| \). Suppose not. Then, there are two scalar sequences \( \eta_n \uparrow \infty \) and \( \varepsilon_n \downarrow 0 \), as well as a sequence \( \{x_n\} \subseteq C \) for which \( \|x_n\| \geq \eta_n \) and \( \langle -x^*, x_n \rangle < \varepsilon_n \|x_n\| \). Clearly, \( \|x_n\| \to \infty \). As \( V \) is reflexive, passing to a subsequence, we have \( x_n/\|x_n\| \to d \). This implies \( \langle x^*, d \rangle = 0 \). Clearly \( \langle x^*, d \rangle = 0 \) since \( d \in C_\infty \) and \( x^* \in C_\infty^{-} \).

Since \( x^* \in \operatorname{int} C_\infty^{-} \), \( \langle x^* + u^*, d \rangle \leq 0 \) for all \( u^* \in \varepsilon B_{V^*} \). Namely, \( \langle u^*, d \rangle \leq 0 \), which implies \( d = 0 \). But, this is a contradiction because \( C \) is well positioned. Hence, \( \langle -x^*, x \rangle \geq \varepsilon \|x\| \) over \( C \) and \( \|x\| \geq \eta \). Clearly, this means \( \langle -x^*, x \rangle \geq \varepsilon \|x\| - m \) for all \( x \in C \) and for some
m. Consequently, \(\langle x^*, x \rangle \leq -\varepsilon \|x\| + m \leq m\) that implies \(\sup_{x \in C} \langle x^*, x \rangle < \infty\) and \(x^* \in b(C)\) and the claim is proved.

Let us prove (ii). Set \(\lambda_1 < \sup_{x \in C} \langle x^*, x \rangle\) and consider the set of points \(x \in C\) such that \(\langle x^*, x \rangle \geq \lambda_1\). It follows \(\lambda_1 \leq \langle x^*, x \rangle \leq -\varepsilon \|x\| + m\). Namely, \(\|x\| \leq \varepsilon^{-1} (m - \lambda_1)\). The functional \(-x^* + \delta_C\) is thus coercive. Point (iii) easily follows.

We now complete the proof of point (i). We have already proved that \(\text{int } C^-_\infty \subseteq \text{int } b(C)\). Suppose first that \(C\) is closed and convex. Therefore, \(b(C) \subseteq C^-\) holds and so \(\text{int } b(C) \subseteq \text{int } C^-\). It follows \(\text{int } b(C) = \text{int } C^-\).

Suppose now that \(C\) is any well-positioned set and define \(D = \overline{co}(C)\). Clearly \(C^\infty \subseteq D^\infty\) and \(C^-_\infty \supseteq D^-\). For what has been proved, we can write
\[
\text{int } b(C) = \text{int } b(D) = \text{int } D^-_\infty \subseteq \text{int } C^-_\infty \subseteq \text{int } b(C).
\]

Hence point (i) holds. Notice that we get the relation \(\text{int } D^-_\infty = \text{int } C^-_\infty\) that yields
\[
\text{int } D^-_\infty = \text{int } C^-_\infty \Rightarrow D^-_\infty = C^-_\infty.
\]

By the Bipolar Theorem, \(D^-_\infty = \overline{co}(C)\), that is, point (iv).

\[\text{Example 29}\] Proposition 28-(i) may fail when \(V\) is not reflexive. Consider the well-positioned closed convex set \(C = \{x \in l_1 : 0 \leq x_i \leq 1 \text{ for each } i\}\). We have \(C^-_\infty = l_\infty\) since \(C^\infty = \{0\}\). It is not difficult to check that \(b(C) = \{y \in l_\infty : y^+ \in l_1\}\). Clearly, \(\text{int } b(C) \neq \emptyset\), while \(\text{int } b(C) \subseteq b(C) \subseteq l_\infty = \text{int } C^-_\infty\). Observe that even Proposition 28-(ii) fails. Consider for instance the functional \(-e = (-1, -1, ...) \in \text{int } b(C)\). Clearly, \(e + \delta_C\) is not coercive.

We give a few corollaries of the characterization of well-positionedness established in Proposition 28.

\[\text{Corollary 30}\] A cone \(K\) allows plastering if and only if \(\text{int } K^- \neq \emptyset\).

**Proof** Observe that \(\sigma_K = \delta_{K^-}\). Thus, \(b(K) = K^-\). Proposition 28 concludes the proof.

This is the simplest criterion to check whether a cone allows plastering. For instance, as mentioned in the Introduction, the positive cones \(L^p_+\) of \(L^p\) spaces do not allow plastering for \(p > 1\) since \((L^p_+)^- = -L^q_+\) have empty interior, unless they are finite dimensional.

By Proposition 28, \(\text{int } C^-_\infty \neq \emptyset\) when \(C\) is well-positioned. However, in general this is not a characterizing property. For instance, \(C^-_\infty = V\) if \(C\) is linearly bounded, and so \(C^\infty = \{0\}\). Next we show that among finitely well-posed sets, the property \(\text{int } C^-_\infty \neq \emptyset\) indeed characterizes sets that are well-posed.

\[\text{Corollary 31}\] A finitely well-positioned set \(C\) in a reflexive space is well-positioned if and only if \(\text{int } C^-_\infty \neq \emptyset\).
**Proof** If $C$ is well-positioned,

$$
C \subseteq K_{x^*} (m) \Rightarrow C_\infty \subseteq K_{x^*} \Rightarrow C_\infty^- \supseteq K_{x^*}^- \supseteq -x^* + B_{V^*}.
$$

Hence, $\text{int } C_\infty^- \neq \emptyset$. Let us prove the converse implication. Suppose $x^* \in \text{int } C_\infty^-$. We will use an argument similar to that of Proposition 28. We claim that there are $\eta > 0$ and $\varepsilon > 0$ such that $x \in C$ and

$$
\|x\| \geq \eta \Rightarrow \langle -x^*, x \rangle \geq \varepsilon \|x\|.
$$

Suppose not. Then there are two sequences $\eta_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$, as well as a sequence $\{x_n\} \subseteq C$ for which $\|x_n\| \geq \eta_n$ and $\langle -x^*, x_n \rangle < \varepsilon_n \|x_n\|$. Clearly, $\|x_n\| \to \infty$. Hence, $\langle x^*, x_n/\|x_n\| \rangle > -\varepsilon_n$. As the space is reflexive, passing to a subsequence, $x_n/\|x_n\| \rightharpoonup d \in C_\infty$. Consequently, $\langle x^*, d \rangle \geq 0$. This implies $\langle x^*, d \rangle = 0$ since $x^* \in C_\infty^-$. Since $x^* \in \text{int } C_\infty^-$, $\langle x^* + u^*, d \rangle \leq 0$ for all $u^* \in \varepsilon B_{V^*}$. This implies $\langle u^*, d \rangle \leq 0$, which implies $d = 0$. We have proved $x_n/\|x_n\| \rightharpoonup 0$, which contradicts the fact that $C$ is finitely well-positioned. Therefore, the claim is true and $C$ is well-positioned.

We now extend the characterization established in Proposition 28 from well-positioned sets to finitely well-positioned ones.

**Proposition 32** A closed and convex set $C$ of a reflexive space $V$ is finitely well-positioned if and only the following two conditions hold:

1. $Q = \overline{b(C) - b(C)}$ has finite codimension in $V^*$;
2. $\text{ri}_Q b(C) \neq \emptyset$.

In this case, $\text{ri}_Q b(C) = \text{ri}_Q C_\infty^-$ and

$$Q = L_C = b(C) - b(C) = C_\infty^- - C_\infty^-$$

Since $b(C)$ is a cone, $Q$ is the minimal closed affine space containing $b(C)$; that is, $Q = \text{aff } b(C)$.

**Proof** Let $C$ be finitely well-positioned. By Proposition 27, $C = L \oplus C_1$, where $C_1$ is a well-positioned set contained into a closed subspace $M$, $V = L \oplus M$ and $\dim L = n$.

It is well-known that $V = L \oplus M$ implies $V^* = M^\perp \oplus L^\perp$ and $M^\perp \simeq L^*$, $L^\perp \simeq M^*$ (all these properties are true for any Banach space $V$). Notice that $\dim M^\perp = n$ and thus $L^\perp$ has finite codimension. It is easy to see that

$$
\sigma_C (x_1^* \oplus x_2^*) = \sigma_L (x_1^*) + \sigma_{C_1} (x_2^*) = \sigma_{L^*} (x_1^*) + \sigma_{C_1} (x_2^*)
$$

for $x_1^* \oplus x_2^* \in M^\perp \oplus L^\perp$. In the last formula, $\sigma_L$ and $\sigma_{C_1}$ denote the support functionals of $L$ and $C_1$ as subsets of $L$ and $M$, respectively. Clearly, $x_1^*$ and $x_2^*$ in the arguments of $\sigma_L$ and $\sigma_{C_1}$ denote the restrictions of $x_1^*$ to $L$ and $x_2^*$ to $M$, respectively.
On the other hand, $\tilde{\sigma}_L(x_1^*) = \infty$ unless $x_1^* = 0$. Hence, $b(C) \subseteq L^\perp$ and so $b(C) - b(C) \subseteq L^\perp$. Now as $C_1$ is well-positioned in $M$, int $b(C_1) \neq \emptyset$ on the space $M^*$. That is, $\text{ri}_{L^\perp} b(C) \neq \emptyset$. This also implies that $b(C) - b(C) = L^\perp$. To check the other properties, observe that $C_\infty = L \oplus (C_1)_\infty$ and $C_\infty = \{0\} \oplus (C_1)_\infty$. By Proposition 28-(i), $\text{ri}_{L^\perp} b(C) = \text{ri}_{L^\perp} C_\infty$. This concludes the first part of the proof by setting $Q = L^\perp$.

As to the converse, suppose that $C$ satisfies (i) and (ii). Since $Q$ is closed and has finite codimension, $V^*$ admits the decomposition $V^* = N \oplus Q$ where $N \subseteq V^*$ has dimension $n$. By reflexivity, $V = \frac{1}{2}Q \oplus \frac{1}{2}N$ with $\frac{1}{2}Q \simeq N^*$ and $\frac{1}{2}N \simeq Q^*$. Suppose per contra that $C$ is not finitely well-positioned. As $\frac{1}{2}Q$ is finite-dimensional, there are $n + 1$ functionals $\{x_i^* \oplus 0\}_{i=1}^{n+1}$ in $N \oplus Q$ that determine a polyhedral cut for $N$. Pick now any element $0 \oplus y^* \in \text{ri}_Q b(C)$. Then, the collection $\{x_i^* \oplus 0\}_{i=1}^{n+1} \cup \{0 \oplus (-y^*)\}$ would not determine a polyhedral cut. That is, for some scalars $\{t_i\}_{i=1}^{n+1} \cup \{\tau\}$ the slice would be unbounded. On the other hand, the slice is

$$\{x \in P_1(C) : \langle x_i^*, x \rangle \leq t_i \forall i\} + \{x \in P_2(C) : \langle y^*, x \rangle \geq -\tau\}$$

where $P_1 : V \rightarrow \frac{1}{2}Q$ and $P_2 : V \rightarrow \frac{1}{2}N$ are the canonical projections. As $P_1(C) \subseteq \frac{1}{2}Q$, the first set is bounded by construction, so the set $\{x \in P_2(C) : \langle y^*, x \rangle \geq -\tau\}$ would be unbounded. By the Banach-Steinhaus Theorem, there is a sequence $\{x_n\}_n \subseteq P_2(C)$ and a functional $0 \oplus z^*$ such that $\langle z^*, x_n \rangle \geq n$. On the other hand, since $0 \oplus y^* \in \text{ri}_Q b(C)$, $0 \oplus y^* + \lambda (0 \oplus z^*) \in b(C)$ for $\lambda > 0$ small enough. But, $\langle y^* + \lambda z^*, x_n \rangle \rightarrow \infty$, a contradiction. We conclude that $C$ is necessarily finitely well-positioned.

### 6 Functions

A function $f : V \rightarrow \mathbb{R}$ is

(i) well-positioned if its epigraph $\text{epi } f \subseteq V \times \mathbb{R}$ is well-positioned;\(^{12}\)

(ii) finitely well-positioned if its epigraph $\text{epi } f \subseteq V \times \mathbb{R}$ is finitely well-positioned;

(iii) quasi finitely well-positioned if all its nonempty sublevel sets $(f \leq \lambda)$ are finitely well-positioned.

(iv) semi finitely well-positioned if there is a sublevel set $(f \leq \lambda)$, with $\lambda > \inf f$, that is finitely well-positioned.

Clearly, property (i) implies (ii) and (iii) implies (iv). To see that (ii) implies (iii) is enough to consider the equality $\text{epi } f \cap \{\lambda = \lambda\} = (f \leq \lambda) \times \{\lambda\}$.

The next examples show that in general these implications do not have a converse. However, Theorem 39 will show that properties (ii)-(iv) are equivalent for convex functions.

---

\(^{12}\)Well-positioned functions are proper (their epigraphs would otherwise contain a line).
Example 33 The convex function $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi (t) = t$ is finitely well-positioned, but not well-positioned. ▲

Example 34 Let $f (x) = \sqrt{\|x\|}$ be defined over an infinite dimensional reflexive space. It is quasi finitely well-positioned since all nonempty sublevel sets are bounded. However, it is not finitely well-positioned. For, take an unbounded sequence $\{x_n\}$ with $x_n / \|x_n\| \to 0$. Clearly, $\sqrt{\|x\|} / \|x_n\| \to 0$. By Lemma 37 below, $f$ is not finitely well-positioned. The function $f \wedge 1$ is a simple example of a semi finitely well-positioned that is not quasi finitely well-positioned. ▲

The following lemma, whose simple proof is omitted, will be useful in deriving the results of this section. Notice that $V \times \mathbb{R}$ is endowed with the norm $\|(x, \lambda)\| = \|x\| + |\lambda|.$

Lemma 35 Let $\{(x_n, \lambda_n)\}_n \subseteq V \times \mathbb{R}$. Then

$$\frac{(x_n, \lambda_n)}{\|x_n\| + |\lambda_n|} \to 0 \iff \frac{x_n}{\|x_n\|} \to 0 \text{ and } \frac{\lambda_n}{\|x_n\|} \to 0.$$

Next we establish a full characterization of finitely well-positioned functions.

Theorem 36 Let $V$ be reflexive or with separable dual. A function $f : V \to \mathbb{R}$ is finitely well-positioned if and only if there is no unbounded sequence $\{x_n\} \subseteq \text{dom } f$ such that $x_n / \|x_n\| \to 0$ and either $f (x_n) / \|x_n\| \to 0$ or $f (x_n) \downarrow -\infty$.

The proof relies on couple of lemmas.

Lemma 37 Let $V$ be reflexive or with separable dual. A function $f$ bounded from below is finitely well-positioned if and only if there is no unbounded sequence $\{x_n\} \subseteq \text{dom } f$ such that $x_n / \|x_n\| \to 0$ and $f (x_n) / \|x_n\| \to 0$.

Proof By considering $f - \inf f$, wlog we can assume $f \geq 0$. Suppose that the claimed conditions hold and that, per contra, $f$ is not finitely well-positioned. There is a sequence $(x_n, \lambda_n) \in \text{epi } f$ such that $\|x_n\| + |\lambda_n| \to \infty$ and $(x_n, \lambda_n) / (\|x_n\| + |\lambda_n|) \to 0$. By Lemma 35, $x_n / \|x_n\| \to 0$ and $\lambda_n / \|x_n\| \to 0$. The sequence $\|x_n\|$ cannot be bounded. Otherwise, $|\lambda_n|$ would be bounded and thus $\|x_n\| + |\lambda_n|$ cannot go to infinity. Therefore, we can suppose $\|x_n\| \to \infty$. As $0 \leq f (x_n) \leq \lambda_n$, it follows $f (x_n) / \|x_n\| \to 0$, a contradiction.

As to the converse, assume $f$ is finitely well-positioned and that there is a sequence $\{x_n\}$ such that $\|x_n\| \to \infty$, $x_n / \|x_n\| \to 0$ and $f (x_n) / \|x_n\| \to 0$. Consider the points $(x_n, f (x_n)) \in \text{epi } f$. We have $(x_n, f (x_n)) / (\|x_n\| + |f (x_n)|) \to 0$ and $\|x_n\| + |f (x_n)| \to \infty$. This implies that $f$ is not finitely well-positioned. □
Lemma 38 Let V be reflexive or with separable dual. A function \( f : V \to \mathbb{R} \) is finitely well-positioned if and only if it is quasi finitely well-positioned and there is no unbounded sequence \( \{x_n\} \subseteq \text{dom } f \) such that \( x_n / \|x_n\| \to 0 \), \( f(x_n) / \|x_n\| \to 0 \) and \( f(x_n) \uparrow \sup f \).

Proof The conditions are clearly necessary. Let us prove their sufficiency. Let by contradiction \( f \) be not finitely well-positioned under the two claimed conditions. Then, there is a sequence \( (x_n, \lambda_n) \in \text{epi } f \) such that \( x_n / \|x_n\| \to 0 \) and \( \lambda_n / \|x_n\| \to 0 \). Note that necessarily \( \|x_n\| \to \infty \). Examine separately two cases.

(i) \( \limsup_{n \to \infty} f(x_n) < \sup f \). This implies that there is a subsequence \( \{x_{n_k}\} \subseteq (f \leq \lambda) \) for some \( \lambda < \sup f \). Hence \( f \leq \lambda \) would not be finitely well-positioned.

(ii) \( \limsup_{n \to \infty} f(x_n) = \sup f \). In this case a subsequence can be extracted for which \( f(x_n) \uparrow \sup f \). Clearly, \( f(x_n) / \|x_n\| \to 0 \) and this contradicts the hypothesis.

Proof of Theorem 36 The conditions are necessary. For, suppose \( f \) is finitely well-positioned. From \( f \leq |f| \) it follows that \( \text{epi } |f| \subseteq \text{epi } f \). Hence, \( |f| \) is finitely well-positioned. Suppose that a sequence exists as claimed in the statement for which \( f(x_n) / \|x_n\| \to 0 \); i.e., \( |f(x_n)| / \|x_n\| \to 0 \). By Lemma 37, \( |f| \) would not be finitely well-positioned. If \( f(x_n) \downarrow -\infty \), \( \{x_n\} \in (f \leq \lambda) \), and so \( f \) would not be quasi finitely well-positioned. This concludes the proof of necessity.

As to the converse, let us first prove that \( f \) is quasi finitely well-positioned. Suppose not. Then, \( (f \leq \lambda_0) \) is not finitely well-positioned for some \( \lambda_0 \). There is consequently an unbounded sequence \( \{x_n\} \) such that \( x_n / \|x_n\| \to 0 \), and \( f(x_n) \leq \lambda_0 \). Two cases are possible.

(i) \( \liminf_{n \to \infty} f(x_n) = -\infty \). In this case there is a subsequence such that \( f(x_{n_k}) \downarrow -\infty \). But, this leads to a contradiction.

(ii) \( \liminf_{n \to \infty} f(x_n) > -\infty \). In this case the sequence is bounded from below, i.e., \( k \leq f(x_n) \leq \lambda_0 \). This implies \( f(x_n) / \|x_n\| \to 0 \) and we get again a contradiction.

Thus, \( f \) is quasi finitely well-positioned. Suppose that \( f \) is not finitely well-positioned. By Lemma 38, there is an unbounded sequence such that \( x_n / \|x_n\| \to 0 \), \( f(x_n) \uparrow \sup f \), and \( f(x_n) / \|x_n\| \to 0 \), a contradiction. The sufficiency part of the proof is completed.

The next result shows that the three classes of functions introduced at the beginning of the section through finite well-positionedness are equivalent.

Theorem 39 A convex function \( f : V \to \mathbb{R} \) is finitely well-positioned if and only if is semi finitely well-positioned.

The proof of this theorem relies on couple of lemmas.

Lemma 40 Let \( f : V \to \mathbb{R} \) be convex and \( \lambda_1, \lambda_2 > \inf f \). A sublevel set \( (f \leq \lambda_1) \) is unbounded if and only if \( (f \leq \lambda_2) \) is. Moreover, \( B_{(f \leq \lambda_1)} = B_{(f \leq \lambda_2)} \).
**Proof** Set $\lambda_1 > \lambda_2 > \inf f$. Clearly, $(f \leq \lambda_2) \subseteq (f \leq \lambda_1)$ and $B_{(f \leq \lambda_2)} \subseteq B_{(f \leq \lambda_1)}$. Hence $(f \leq \lambda_1)$ is unbounded if $(f \leq \lambda_2)$ is. Suppose $(f \leq \lambda_1)$ is unbounded and let $x_n \in (f \leq \lambda_1)$ with $\|x_n\| \to \infty$. As $\lambda_2 > \inf f$, there is $v \in V$ with $f(v) = \lambda_2 - \varepsilon$. Consider the sequence $(1 - \alpha) v + \alpha x_n$ where $\alpha \in (0, 1)$. By convexity,

$$f((1 - \alpha) v + \alpha x_n) \leq (1 - \alpha) f(v) + \alpha f(x_n) \leq (1 - \alpha) (\lambda_2 - \varepsilon) + \alpha \lambda_1.$$ 

Consequently, we can pick $\alpha$ so that $y_n = (1 - \alpha) v + \alpha x_n \in (f \leq \lambda_2)$. Thus, $(f \leq \lambda_2)$ is unbounded. Let $d \in B_{(f \leq \lambda_1)}$; i.e., $x_n/\|x_n\| \rightharpoonup d$, $f(x_n) \leq \lambda_1$. Consider the above $y_n = (1 - \alpha) v + \alpha x_n \in (f \leq \lambda_2)$. We have

$$\frac{y_n}{\|y_n\|} = \frac{(1 - \alpha) v}{\|y_n\|} + \frac{\alpha \|x_n\|}{\|y_n\|} x_n.$$ 

On the other hand,

$$\frac{\alpha \|x_n\|}{\|y_n\|} = \frac{\alpha \|x_n\|}{\|((1 - \alpha) v + \alpha x_n\|} = \left| \frac{(1 - \alpha) v}{\alpha \|x_n\|} + \frac{x_n}{\|x_n\|} \right|^{-1}$$

that goes to 1 as $n \to \infty$. Hence $y_n/\|y_n\| \rightharpoonup d$, and so $B_{(f \leq \lambda_1)} = B_{(f \leq \lambda_2)}$.  

**Lemma 41** Let $f : V \to \mathbb{R}$ be convex. If $(f \leq \lambda)$, with $\lambda > \inf f$, is finitely well-positioned, then all its nonempty sublevels $(f \leq \lambda)$ are finitely well-positioned. Specifically, if

$$(f \leq \lambda) \subseteq \bigcup_{i=1}^{n} K_{x^*_i}(m_i),$$

then, for each $(f \leq \lambda)$ there are scalars $\varepsilon_i(\lambda) > 0$ and $m_i(\lambda)$ such that

$$x \in (f \leq \lambda) \implies \langle x^*_i, x \rangle \geq \varepsilon_i(\lambda) \|x\| - m_i(\lambda)$$

for some $i = 1, ..., n$.

**Proof** The property is trivially true for the levels $(f \leq \lambda)$ with $\lambda \leq \lambda$. Therefore, it suffices to study the case $\lambda > \lambda$. As $\lambda > \inf f$, there is a point $\bar{x}$ for which $f(\bar{x}) < \lambda$. Pick now scalars $\{t_i\}$ so that $\langle x^*_i, \bar{x} \rangle < t_i$ and consider the convex function $\varphi(x) = \bigvee_{i=1}^{n} \langle x^*_i, x \rangle - t_i \lor (f(x) - \lambda)$. Clearly, $\varphi(\bar{x}) < 0$. Therefore, $(\varphi \leq 0) \neq \emptyset$. Clearly, $(\varphi \leq 0) = \bigcap_i \{ \langle x^*_i, x \rangle \leq t_i \} \cap (f \leq \lambda)$. By (3), $(\varphi \leq 0)$ is bounded. By Lemma 40, all level sets $(\varphi \leq h)$ are bounded for each $h > 0$. On the other hand, the level $(\varphi \leq h)$ is given by the slice $\bigcap_i \{ \langle x^*_i, x \rangle \leq t_i + h \} \cap (f \leq \lambda + h)$, which satisfies the Slater condition in that $\bar{x} \in (f \leq \lambda + h)$ and $\langle x^*_i, \bar{x} \rangle < t_i + h$ for all $i$. By Proposition 16, $(f \leq \lambda + h)$ is finitely well-positioned and (4) holds.  

**Proof of Theorem 39** Assume that $(f \leq \lambda) \subseteq \bigcup_{i=1}^{n} K_{x^*_i}(m_i)$ holds. Consider the set of $n + 2$ functionals $D = \{(x^*_i, 0)\}_{i=1}^{n} \cup \{(0, 1), (0, -1)\}$ on $V \times \mathbb{R}$. By Proposition 41, it is easy
to see that $D$ is such that, for each $T = \{t_i\}_{i=1}^{n+2}$, the slices $\text{epi} f (D, T)$ are either empty or bounded. By Proposition 16, $\text{epi} f$ is finitely well-positioned.

A result similar to Theorem 39 does not hold in general for well-positioned functions, as the example $\varphi (t) = t$ shows. However, if the functions are bounded below the following analogous result holds.

**Proposition 42** Let $V$ be reflexive or with separable dual. A convex and bounded below function $f : V \to \mathbb{R}$ is well-positioned if and only if a sublevel set $(f \leq \lambda)$, with $\lambda > \inf f$, is well-positioned.

**Proof** Suppose first that $f$ is lower semicontinuous. By Theorem 39 $f$ is finitely well-positioned. By Lemma 26 it suffices to show that $\text{epi} f$ does not contain any line. It is easy to see that the unique feasible directions of these lines are $(v, 0)$ otherwise the function would not be bounded from below. This implies that $f (x_0 + tv) \leq \lambda$ for all $t$. Hence $v \in L(f \leq \lambda)$ that contradicts the fact that $(f \leq \lambda)$ is well-positioned.

Clearly, if $f$ is not lower semicontinuous, we can consider its lower semicontinuous hull $\overline{f} \leq f$ and obtaining easily the same result.

The results on support functionals derived in the previous sections can be easily translated into results on Fenchel conjugates through the relation $f^* (x^*) = \sigma_{\text{epi} f} (x^*, -1)$. In this vein, next we characterize well-positioned and finitely well-positioned functions through properties of their Fenchel conjugates. Point (i) is due to [3, Proposition 3.1].

**Proposition 43** Let $f : V \to \mathbb{R}$ be a proper lower semicontinuous and convex function on a reflexive space $V$. Then,

(i) $f$ is well-positioned if and only if $\text{int} \text{ dom} f^* \neq \emptyset$;

(ii) $f$ is finitely well-positioned if and only if $\text{ri dom} f^* \neq \emptyset$ and $\text{aff} (\text{dom} f^*)$ has finite codimension in $V^*$.

**Proof** (i) See [3, Proposition 3.1]. As to (ii), suppose first that $f$ is finitely well-positioned and $f \geq 0$. As $\text{epi} f \subseteq V \times \mathbb{R}_+$, the lineality space of $\text{epi} f$ is contained in $V$. Let $L$ be such a finite-dimensional space and let $V = L \oplus M$ be the usual decomposition. This implies that $f$ is constant over $L$ and by Proposition 27 $f (x_1 \oplus x_2) = \varphi (x_2)$ for $x_1 \oplus x_2 \in L \oplus M$, where $\varphi$ is a well-positioned function over $M$. Calculating the Fenchel conjugate yields

$$f^* (0 \oplus x_2^*) = \varphi^* (x_2^*) \quad \text{and} \quad f^* (x_1^* \oplus x_2^*) = +\infty \text{ if } x_1^* \neq 0,$$

where $x_1^* \oplus x_2^* \in M^\perp \oplus L^\perp$. Hence, $\text{dom} f^* \subseteq L^\perp$. Since $f \geq 0$, $0 \in \text{dom} f^*$ and so $\text{aff} (\text{dom} f^*) \subseteq L^\perp$. Since $\varphi$ is well-positioned, $\text{ri dom} f^* \neq \emptyset$ and $\text{aff} \text{ dom} f^* = L^\perp$. The proof is complete for $f \geq 0$.  

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Let \( f \) be now any finitely well-positioned function. As \( f \) is lower semicontinuous and proper, \( f \geq y^* + \gamma \). Hence \( f_1 = f - y^* - \gamma \geq 0 \) and \( y^* \in \text{dom } f^* \). By applying the previous result, we easily obtain that \( \text{aff } \text{dom } f^* = y^* + L^\perp \) and \( \text{ri } \text{dom } f^* \neq \emptyset \).

As to the converse, suppose that \( f \) satisfies (i) and (ii). Note that (ii) implies \( \text{aff } \text{dom } f^* = \text{aff } (\text{dom } f^*) = Q \). If \( x^* \in \text{ri}_Q \text{dom } f^* \), \((x^*, -1)\) is interior with respect to the affine space \( Q \times \{-1\} \). Hence \((x^*, -1)\) is interior with respect to \( Q \times \mathbb{R} \). So, \((x^*, -1) \in \text{ri } b(\text{epi } f)\) and \( \text{aff } b(\text{epi } f) = Q \times \mathbb{R} \). By Proposition 32, \( \text{epi } f \) is finitely well-positioned.

We close with a noteworthy coercitivity property.

**Proposition 44** Let \( f : V \to \mathbb{R} \) be a proper lower semicontinuous and convex function on a reflexive space \( V \).

(i) If \( f \) is well-positioned, then \( f - x^* \) is coercive for all \( x^* \in \text{int } \text{dom } f^* \);

(ii) If \( f \) is finitely well-positioned, then, \( f - x^* \) is semicoercive for all \( x^* \in \text{ri } \text{dom } f^* \).

**Proof** (i) By Proposition 28 the functional \( -(x^*, -1) + \delta_{\text{epi } f} \) is coercive when \( x^* \in \text{int } (\text{dom } f^*) \). This implies that for a fixed scalar \( \lambda \) the set \( \{(-x^*, x) + \lambda \leq \lambda \} \cap \{f(x) \leq \lambda \} \) is bounded in \( V \times \mathbb{R} \). In particular, it is bounded for \( f(x) = \lambda \). That is, \( \langle -x^*, x \rangle + f(x) \leq \lambda \) is bounded. Namely, \( f - x^* \) is coercive.

(ii) Wlog set \( x^* = 0 \). Thus suppose \( 0 \in \text{ri } \text{dom } f^* \). As \( 0 \in \text{dom } f^* \), \( \inf f > -\infty \). In view of what has been discussed in the proof of Proposition 43, we have that \( f (x_1 + x_2) = \varphi (x_2) \) with \( \varphi \) well-positioned function over \( M \) and \( V = L \oplus M \). As \( 0 \in \text{ri } \varphi^* \). By Proposition 44, \( \varphi \) is coercive over \( M \). Hence \( f \) is semicoercive. If \( \overline{x} \in M \) is a minimum of \( \varphi \), \( \overline{x} + L \) is a set of minimizers of \( f \).

### 7 Some Applications

#### 7.1 Intersections

Given a chain of nonempty sets \( \{C_n\} \), with \( C_{n+1} \subseteq C_n \) for all \( n \), following Bertsekas and Tseng [6] define the asymptotic cones

\[
\{C_n\}_\infty = \left\{ d \in V : \exists t_n \to \infty \text{ and } x_n \in C_n \text{ such that } \frac{x_n}{t_n} \rightharpoonup d \right\}
\]

and

\[
B (\{C_n\}) = \left\{ d \in V : \exists x_n \in C_n \text{ such that } \|x_n\| \to \infty \text{ and } \frac{x_n}{\|x_n\|} \rightharpoonup d \right\}
\]

When \( C_n = C \) for all \( n \), we get back to the usual asymptotic cones, that is, \( \{C_n\}_\infty = C_\infty \) and \( B (\{C_n\}) = B_C \).
Most of the properties in Proposition 2 remain unchanged for this generalization to chains. For instance, cone \( B(\{C_n\}) = \{C_n\}_\infty \). The following relation is key (see, e.g., [15]):

\[
B(\{C_n\}) = \bigcap_{n=1}^\infty BC_n.
\]

It implies, inter alia, that \( B(\{C_n\}) \) is weakly compact and nonempty when \( V \) is reflexive and the sets \( C_n \) are unbounded.

**Definition 45** Given a chain \( \{C_n\} \) and \( d \in \{C_n\}_\infty \), we say that \( \{C_n\} \) retracts along \( d \) if, for any sequence \( x_k \in C_{n_k} \) such that \( x_k/t_k \to d \) with \( t_k \to \infty \), there exists a subsequence \( \{x_{k_r}\} \) and a bounded sequence \( \{\alpha_r\} \subseteq \mathbb{R}_{++} \) such that \( x_{n_r} - \alpha_r d \in C_{n_r} \) for all \( r \).

The chain \( \{C_n\} \) is called retractive if it retracts along all \( d \in \{C_n\}_\infty \). When \( C_n = C \) for all \( n \), we get similar definitions for a fixed set \( C \), with \( \{C_n\}_\infty = C_\infty \).

Any chain trivially retracts along \( 0 \). Hence, \( \{C_n\}_\infty = \{0\} \) is a simple sufficient condition for \( \{C_n\} \) to be retractive.

The next lemma gives an equivalent condition of retractivity in terms of \( B(\{C_n\}) \).

**Lemma 46** A chain \( \{C_n\} \) is retractive if and only if for each \( d \in B(\{C_n\}) \) and for any unbounded sequence \( x_{n_k} \in C_{n_k} \) such that \( x_{n_k}/\|x_{n_k}\| \to d \), there is a subsequence \( \{x_{n_{k_r}}\} \) and a bounded sequence \( \{\alpha_r\} \subseteq \mathbb{R}_{++} \) such that \( x_{n_r} - \alpha_r d \in C_{n_r} \) for all \( r \).

**Proof** We prove the “if,” the converse being trivial. The result is trivially true when \( d = 0 \). Therefore, take a sequence \( x_{n_k} \in C_{n_k} \) with \( x_{n_k}/t_k \to d \neq 0 \) and \( t_k \to \infty \). By the usual argument, there exists a subsequence \( x_{n_{k_r}} \) for which \( \|x_{n_{k_r}}/t_{k_r}\| \to \lambda \neq 0 \). Hence,

\[
\frac{x_{n_{k_r}}}{\|x_{n_{k_r}}\|} = \frac{x_{n_{k_r}}}{t_{k_r}} \cdot \frac{t_{k_r}}{\|x_{n_{k_r}}\|} \to \frac{d}{\lambda},
\]

and so \( \lambda^{-1}d \in B(\{C_n\}) \). Hence, there is a subsequence \( \{x_s\} \subseteq x_{n_{k_r}} \) and a sequence of scalars \( \{\alpha_s\} \) such that \( x_s - \lambda^{-1}\alpha_s d \in C_s \), as desired. \( \blacksquare \)

The next result is an infinite dimensional extension of [6, Proposition 1].

**Theorem 47** A retractive chain of nonempty \( w \)-closed and finitely well-positioned sets of a reflexive space \( V \) has nonempty intersection.

In [15] we derived this result as a consequence of more general results. Here we give a direct proof. We need a geometrical fact. Given a cone \( K = \bigcup_{i=1}^n K_{x_i^*} \), consider the following equivalent norm in \( V \), with \( \alpha > 0 \),

\[
\|x\|_1 = \|x\| + \alpha \sum_{i=1}^n |\langle x_i^*, x \rangle|.
\]

The next property is easy to prove (see [15]).
Lemma 48 If \( x \in K \cap \{\|x\|_1 \leq 1\} \), then \( \|x\| \leq (1 + \alpha)^{-1} \).

Proof of Theorem 47 Let \( C_n \) be a chain of finitely well-positioned sets. We can assume \( C_n \) unbounded otherwise the result is trivial. By Proposition 4 there is some \( \rho > 0 \) such that \( C_n \cap \{\|x\| \geq \rho\} \subseteq K = \bigcup_{i=1}^{n} K_{x_i} \) for all \( n \). We thus renorm the space with the equivalent norm \( (5) \). Set \( x_n \in \arg \min_{x \in C_n} \|x\|_1 \) for all \( n \). Notice that \( \arg \min_{x \in C_n} \|x\|_1 \neq \emptyset \) under our hypotheses. If the sequence \( \{x_n\} \) is bounded, by taking a subsequence, \( x_n \rightharpoonup x_0 \in C_n \) for all \( n \). Therefore, \( x_0 \in \bigcap_{n=1}^{\infty} C_n \) and the theorem is true. It remains to show that the sequence \( \{x_n\} \) is bounded. As \( \|x_n\|_1 \leq \|x_{n+1}\|_1 \), we have \( \|x_n\|_1 \to \infty \) if the sequence is unbounded. Under our assumption, there is a subsequence \( \{x_{n_k}\} \) such that \( x_{n_k}/\|x_{n_k}\|_1 \to d \neq 0 \). By reactivity, passing to a subsequence \( \{x_{n_k}\} \subseteq \{x_{n_r}\} \), \( x_{n_r} - \alpha_r d \in C_{n_r} \) for all \( r \). Hence \( \|x_{n_r} - \alpha_r d\|_1 \geq \|x_{n_r}\|_1 \). That is \( \|x_{n_r}/\|x_{n_r}\|_1 - \beta_r d\|_1 \geq 1 \) with \( \beta_r = \alpha_r/\|x_{n_r}\|_1 \). Observe that

\[
\left\| \frac{x_{n_r}}{\|x_{n_r}\|_1} - \beta_r d \right\|_1 = \left\| (1 - \beta_r) \frac{x_{n_r}}{\|x_{n_r}\|_1} + \beta_r \left( \frac{x_{n_r}}{\|x_{n_r}\|_1} - d \right) \right\|_1 \\
\leq (1 - \beta_r) + \beta_r \left\| \frac{x_{n_r}}{\|x_{n_r}\|_1} - d \right\|_1.
\]

If we prove that \( \|x_{n_r}/\|x_{n_r}\|_1 - d\|_1 < 1 \) for \( r \) large enough, then \( \|x_{n_r}/\|x_{n_r}\|_1 - \beta_r d\|_1 < 1 \), a contradiction. Notice that for \( r \) large enough \( x_{n_r}/\|x_{n_r}\|_1 \in K \cap \{\|x\|_1 \leq 1\} \). Set \( u_r = x_{n_r}/\|x_{n_r}\|_1 \). By Lemma 48,

\[
\|u_r - d\|_1 = \|u_r - d\| + \alpha \sum_{i=1}^{n} |\langle x_i^*, u_r - d \rangle| \leq \|u_r\| + \|d\| + \alpha \sum_{i=1}^{n} |\langle x_i^*, u_r - d \rangle| \\
\leq 2 (1 + \alpha)^{-1} + \alpha \sum_{i=1}^{n} |\langle x_i^*, u_r - d \rangle|.
\]

Since \( \sum_{i=1}^{n} |\langle x_i^*, u_r - d \rangle| \to 0 \), we have \( \|u_r - d\|_1 < 1 \) for \( r \) large enough, provided \( \alpha > 1 \). This completes the proof.

7.2 Closed Images and Algebraic Differences

Next we give a first application of Theorem 47.

Proposition 49 Let \( T : V \to W \) be a continuous linear mapping defined on a reflexive space \( V \), and \( C \subseteq V \) be a finitely well-positioned set.

(i) The image \( T(C) \) is sw-closed if \( C \) is sw-closed and retracts along all directions \( d \in C_\infty \cap \ker T \).

(ii) The image \( T(C) \) is finitely well-positioned if \( W \) is reflexive and if \( C_\infty \cap \ker T \) is a linear space included in \( L_C \); in this case, \( T(C_\infty) = T(C)_\infty \).
Remarks. (i) The hypotheses in point (i) hold when either $C$ is retractive or $C_\infty \cap \ker T = \{0\}$ or $C_\infty \cap \ker T \subseteq L_C$. (ii) Notice that even for non convex sets, $L_C$ is defined as $L_C + C = C$, though in general $L_C$ may not be a vector space.

Proof (i) Suppose first that $W$ is separable and $y_n \in T(C)$ with $y_n \rightharpoonup y$. We have to show that $y \in T(C)$. The sequence is bounded, i.e., $y_n \in \rho W$. As $W$ is separable, it is weakly metrizable over the bounded set $\rho W$. Denote by $\delta$ such a metric. Consider the following sequence of sets in $W$

$$W_n = \{ y \in W : \delta(y, y) \leq \delta(y_n, y) \text{ and } y \in \rho W \},$$

and the associated chain $C_n = C \cap T^{-1}(W_n)$ in $V$. Clearly, the sets $W_n$ are weakly closed and thus the sets $C_n$ are sw-closed. Moreover, $C_n$ are nonempty by construction. If their intersection is nonempty and $\bar{x} \in \bigcap_{n=1}^{\infty} C_n$, then $T(\bar{x}) = \bar{y}$. Notice that it is not restrictive to suppose $\delta(y_n, y) \downarrow 0$ and thus $\{C_n\}$ is a chain.

Let us prove that $\{C_n\}_\infty \subseteq C_\infty \cap \ker T$. Let $d \in \{C_n\}_\infty$. Clearly $d \in C_\infty$. Let $\{x_m\}$ be sequence such that $\{x_m\} \subseteq C_n$, $t_m \to \infty$ and $x_m/t_m \to d$. Consequently, $T(x_m/t_m) \to T(d)$. On the other hand, $T(x_m) \in \rho W$ and so it is bounded. We have $T(d) = 0$ and $d \in \ker T$, and so the inclusion $\{C_n\}_\infty \subseteq C_\infty \cap \ker T$ is proved.

To apply Theorem 47 we must prove that the chain $\{C_n\}$ is retractive. Take any $d \in \{C_n\}_\infty$ and let $\{x_n\}$ be a sequence for which $x_n \in C_n$, $t_n \to \infty$ and $x_n/t_n \to d$. Clearly, $d \in C_\infty \cap \ker T$. By hypothesis, $C$ retracts along the directions in $C_\infty \cap \ker T$. Hence $x_{n_k} - \alpha_k d \in C$ for a subsequence $\{x_{n_k}\}$ and for a scalar sequence $\{\alpha_k\}$. On the other hand, $T(x_{n_k} - \alpha_k d) = T(x_{n_k}) \in T(C_n)$. It follows that $x_{n_k} - \alpha_k d \in C_{n_k}$. We have proved that $C_n$ are retractive and thus $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$, as desired.

Suppose now that $W$ is any space, not necessarily separable, and $y_n \rightharpoonup y$ with $y_n \in T(C)$. It suffices to consider the separable linear space $W_1 = \overline{\text{span}} \{y_n\}_n$ and the linear mapping $\tilde{T} : T^{-1}(W_1) \to W_1$. It is easy to see that all hypotheses hold for the set $C \cap T^{-1}(W_1)$ and so the result follows from the first part of the proof.

(ii) Set $V_0 = \ker T \cap C_\infty$. As $C_\infty$ is finitely well-positioned as well, $V_0$ has finite dimension.

Hence, $V = V_0 \oplus W$. Let $y_n \in T(x_n)$ with $\|y_n\| \to \infty$ and $x_n \in C_n$. The sequence $x_n \in V$ admits the decomposition $x_n = x_n^0 \oplus w_n$. Clearly $T(x_n) = T(w_n) = y_n$. As $T$ is continuous the sequence $w_n$ is unbounded. Note further that $w_n \in C$. Actually, $w_n = x_n - x_n^0 \in C - V = C$. As $C$ is finitely well-positioned, there is a subsequence $w_{n_k}$ and $t_k \to \infty$ such that $w_{n_k}/t_k \to d_1 \neq 0$. Consequently, $y_{n_k}/t_k = T(w_{n_k})/t_k \to T(d_1)$. On the other hand $0 \neq d_1 \in W$. Hence $d_1 \notin V_0 = \ker T \cap C_\infty$. Hence $T(d_1) \neq 0$ and this proves that $T(C)$ is finitely well-positioned.

As regard the last statement, let $y_n/t_n \to d$ with $y_n \in T(x_n)$, $t_n \to \infty$ and $d \neq 0$. Clearly $\|y_n\| \to \infty$ and we know that there exists a subsequence $\{y_{n_k}\}$ such that $y_{n_k}/\|y_{n_k}\| \to \lambda d$
with \( \lambda \neq 0 \). Applying to the sequence \( \{y_{n_k}\} \) the above arguments, we get a subsequence \( \{y_{n_{k_r}}\} \) such that \( y_{n_{k_r}}/\tau_r \to T(d_1) \) where \( 0 \neq d_1 \in C_\infty \). Once again there a subsequence \( \{y_t\} \subseteq \{y_{n_{k_r}}\} \) so that \( y_t/\|y_t\| \to \mu T(d_1) \) with \( \mu \neq 0 \). Hence \( \mu T(d_1) = \lambda d \) and so \( T(\mu \lambda^{-1} d_1) = d \). This shows that \( T(C)_\infty \subseteq T(C_\infty) \). The opposite inclusion \( T(C_\infty) \subseteq T(C)_\infty \) holds in general and thus the claim is proved.

A second noteworthy application of Theorem 47 is in providing very general conditions under which the algebraic difference of two sets sw-closed is sw-closed.

**Proposition 50** Let \( C \) and \( D \) be two sw-closed sets of a reflexive space that retract along the directions in \( C_\infty \cap D_\infty \). If \( C \) finitely well-positioned, then the set \( C - D \) is sw-closed provided either \( D \) is finitely well-positioned or both \( C \) and \( D \) retract completely along the directions in \( C_\infty \cap D_\infty \).

When \( C_\infty \cap D_\infty = \{0\} \), the condition that both \( C \) and \( D \) retract completely along the directions in \( C_\infty \cap D_\infty \) is trivially satisfied. Hence, in this case the set \( C - D \) is sw-closed provided both sets are sw-closed and \( C \) is finitely well-positioned. In view of Theorem 25, Diuedonné [9]'s original result on differences of convex sets is thus a special case of Proposition 50, at least for the weak topology. We refer the reader to Adly et al [2] for more recent results for differences of closed convex sets based on the condition \( C_\infty \cap D_\infty = \{0\} \).

**Proof** If \( D \) is finitely well-positioned it suffices to consider the map \( T : (x,y) \mapsto x - y \) from \( V \times V \) to \( V \) and apply Proposition 49 to the set \( T(C,D) = C - D \). We omit details.

Suppose that \( C \) and \( D \) retract completely along the directions in \( C_\infty \cap D_\infty \). The proof follows closely that of Proposition 49. Suppose first \( V \) separable and \( x_n \in C \), \( y_n \in D \) with \( z_n = x_n - y_n \to \bar{z} \). The sequence \( z_n \) is bounded, i.e., \( z_n \in \rho B_V \). Denote by \( \delta \) the metric on \( \rho B_V \) that generates the weak topology. Consider the sequence of sets \( C_n = C \cap [U_n(\bar{z}) + D] \), where

\[
U_n(\bar{z}) = \{z \in V : \delta(\bar{z},z) \leq \delta(\bar{z},z), z \in \rho B_V\}.
\]

Passing to a subsequence if necessary, we can assume that \( U_{n+1}(\bar{z}) \subseteq U_n(\bar{z}) \). Therefore \( \{C_n\} \) is a chain. Clearly, \( C_n \neq \emptyset \) since \( x_n = z_n + y_n \) and \( x_n \in C \cap [U_n(\bar{z}) + D] \). Moreover, \( [U_n(\bar{z}) + D] \) is sw-closed and so the sets \( C_n \) are sw-closed and finitely well-positioned. Notice further that \( \{U_n(\bar{z}) + D\}_\infty = D_\infty \) and \( \{C_n\}_\infty \subseteq C_\infty \cap D_\infty \). To apply Theorem 47 it remains to show that \( \{C_n\} \) is retractive. Let \( x_n \in C_n \) and \( x_n/t_n \to d \). Clearly \( d \in C_\infty \cap D_\infty \). Under our assumption there is a subsequence \( x_{n_k} \) such that \( x_{n_k} - \alpha d \in C \) for \( \alpha \) small enough. On the other hand, \( x_{n_k} \in [U_{n_k}(\bar{z}) + D] \). Namely, \( x_{n_k} = \xi_{n_k} + y_{n_k} \) with \( \xi_{n_k} \in U_{n_k}(\bar{z}) \) and \( y_{n_k} \in D \). From, \( x_{n_k}/t_{n_k} = \xi_{n_k}/t_{n_k} + y_{n_k}/t_{n_k} \) it follows \( y_{n_k}/t_{n_k} \to d \in C_\infty \cap D_\infty \). By assumptions, passing to a subsequence \( \{y_s\} \subseteq \{y_{n_k}\} \) we have \( y_s - \alpha d \in D \) for small \( \alpha \).

\(^{13}\)That is, in Definition 45 we can take \( \alpha = \alpha_r > 0 \) for all \( r \).
Namely, \( x_s - \alpha d = \xi_s + (y_s - \alpha d) \in U_s(\bar{z}) + D \). Hence for small \( \alpha \) it holds \( x_s - \alpha d \in C_n \), where \( \{x_s\} \) is a subsequence of \( \{x_n\} \). This proves that the sequence \( \{C_n\} \) is retractive. Hence, there is \( \bar{x} \in \bigcap_{n=1}^{\infty} C_n \). This means that for all \( n \) we have \( \bar{x} \in C \) and \( \bar{x} \in U_n(\bar{z}) + D \). That is, \( \bar{x} = \xi_n + y_n \) with \( \xi_n \in U_n(\bar{z}) \) and \( y_n \in D \). Clearly, \( \xi_n \to \bar{z} \). Hence, \( \bar{x} - \xi_n = y_n \to \bar{x} - \bar{z} \). As \( D \) is sw-closed, \( \bar{x} - \bar{z} = \bar{y} \in D \). Therefore, \( \bar{z} = \bar{x} - \bar{y} \).

\[ \square \]

References


