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Using Economic Theory to Guide Numerical Analysis: Solving for Equilibria in Models of Asymmetric First-Price Auctions

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Abstract

In models of first-price auctions, when bidders are *ex ante* heterogeneous, deriving explicit equilibrium bid functions is typically impossible, so numerical methods (such as polynomial approximations) are often employed to find approximate solutions. Recent theoretical research concerning asymmetric auctions has determined conditions under which equilibrium bid functions must cross. When equilibrium bid functions are approximated by low-order polynomials, however, such polynomials may not be flexible enough to satisfy the qualitative predictions of the theory. Plotting the relative expected pay-offs of bidders is a quick, informative way to decide whether the approximate solutions are consistent with theory.

Key words: first-price auctions; asymmetric auctions; numerical methods.

JEL classification: C20, D44, L1.

1. Motivation and Introduction

While first-price auctions have been used extensively, perhaps for centuries, their properties are not yet fully understood by economists. In the most studied model of equilibrium behaviour at first-price auctions, the private valuations of potential buyers of the object for sale are assumed to be independent and identically-distributed draws from a common distribution—the symmetric independent private-values paradigm. In many applications, however, it is natural to assume that bidders are *ex ante* heterogeneous, their valuation draws coming from different distributions, perhaps independently. Under such an assumption, however, formal analysis becomes complicated, even within the otherwise simple independent private-values paradigm. The main problem is technical in nature—a Lipschitz condition at the lower bound is not satisfied by the system of differential equations that characterizes the equilibrium bid functions. Thus, it is typically impossible to derive equilibrium bid functions explicitly.¹ Therefore, beginning with Marshall et al.

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¹For examples in which bid functions can be characterized, see Vickrey [32], Griesmer et al. [13], Plum [28], Cheng [5] as well as Kaplan and Zamir [19].

[25], several researchers have employed different techniques from the numerical analysis literature in an effort to investigate various properties of asymmetric first-price auctions.

In parallel to these efforts, contributors to another literature have pursued theoretical analyses of the problem, basically investigating qualitative features of equilibrium bidder interaction. For example, Lebrun [22] as well as Maskin and Riley [26] have established that a weak bidder bids more aggressively than a strong one: weakness leads to aggression, as Krishna [21] put it. Typically, a bidder is referred to as *strong* if the cumulative distribution function from which his valuation is drawn dominates that of the weak bidder in a particular sense, *first-order stochastic dominance*—the cumulative distribution function of the strong bidder is everywhere to the right of that for the weak bidder. In fact, when deriving their results, Lebrun as well as Maskin and Riley assumed *reversed hazard-rate dominance*, which is stronger than first-order stochastic dominance: under reversed hazard-rate dominance, the ratio of the probability density function to its cumulative distribution function of the strong bidders is point-wise larger than that for the weak bidders.² Recently, Kirkegaard [20] has derived results under much weaker assumptions concerning the primitives of the economic environment. For example, he has shown that when first-order stochastic dominance does not hold, so the cumulative distribution functions of bidders cross, the equilibrium bid functions must cross as well. Indeed, under certain conditions, Kirkegaard has determined the exact number of times equilibrium bid functions will cross.

Although the sharp predictions of the model with strong and weak bidders make it an attractive one, little reason exists to suggest that the model is necessarily an accurate description of real-world bidder asymmetries. For example, in an empirical analysis of winning-bid data from sequential, oral, ascending-price auctions of fish in Denmark, Brendstrup and Paarsch [3] estimated the cumulative distribution functions of valuations for major and minor bidders. They found that the two unconditional cumulative distribution functions crossed twice, while the cumulative distribution functions, conditional on being above the observed reserve price, crossed once.³ The analysis of Kirkegaard [20] allows one to make specific qualitative predictions concerning equilibrium behaviour at first-price auctions in this case. In principle, numerical analysis should permit one to make quantitative predictions as well.

In this paper, we provide a synthesis of these two approaches to analyzing equilibrium behaviour at first-price auctions with *ex ante* bidder asymmetries. Specifically, the qualitative predictions of the theoretical literature are shown to be useful in assessing the performance of numerical approximations to the equilibrium bid functions. At the most general level, the number of times the cumulative distribution functions cross and certain other features of the approximate equilibrium-bid functions (however derived) can be compared to those theoretical predictions. A quick, informative visual “test” is also proposed. We begin, however, by examining one specific approximation strategy in detail—specifically, the third algorithm of Bajari [2], and the modifications made to it by Hubbard and Paarsch [15]. Gayle and Richard [11] have provided the following evaluation of Bajari’s third algorithm:

[it] will often produce accurate numerical approximations [...]. It is found to be fast, at least with good starting values. It remains, however, an approximation whose accuracy needs to be carefully assessed.

²An alternative definition of reversed hazard-rate dominance is that the ratio of the cumulative distribution function of the weak bidder to that of the strong one is monotonically falling.

³These crossings could be an artifact of the interaction of sampling error and the Laguerre polynomials that Brendstrup and Paarsch used to approximate the probability density functions, something that could happen with the kernel-smoothing methods proposed by Guerre et al. [14] as well.

We shall argue, first, that the speed of the algorithm is a distinct advantage; second, however, we shall supply some insights from economic theory that allow a researcher to assess the accuracy of the approximations produced by Bajari’s third algorithm. We have another purpose as well, which is to sound a note of caution: depending on the exact specification, Bajari’s third algorithm may be unable to produce approximations that are consistent with the true qualitative features of the equilibrium bid functions.

Under Bajari’s third algorithm, the equilibrium inverse-bid functions are assumed to be polynomial functions. While the details of this approximation strategy as well as the modifications made to it by Hubbard and Paarsch are described in section 2, here we simply summarize the main idea, which is to estimate the coefficients of the polynomials that come “closest” to satisfying the first-order conditions that characterize the unique Bayes–Nash equilibrium as well as the appropriate boundary conditions. Approximating the equilibrium inverse-bid functions by polynomials requires using a nonlinear least-squares optimization routine. Unlike with the shooting methods of Marshall et al. [25], or the first algorithm of Bajari [2], however, researchers who employ the polynomial approach have much less control because they typically cannot tell the solver to find a better solution. Specifically, with shooting methods, the user can specify a particular error tolerance and take advantage of existing solvers for ordinary differential equations developed by engineers and scientists to treat the two-point, boundary-value problem like an initial-value problem that is solved repeatedly until both boundary conditions are satisfied. (Because these methods begin at the right endpoint, and shoot back toward the left boundary condition, they are sometimes referred to as *backward shooting* methods.) The downside of shooting methods is that, because the system of differential equations is solved iteratively, these methods are much slower to converge than the polynomial approach. While computation time may not be an important consideration when solving one example of an asymmetric first-price auction, many researchers need to solve the equilibrium within the inner loop of an estimation problem that, in turn, involves repeated simulation; see, for example, the Bayesian econometric work in Bajari’s doctoral dissertation [1] concerning asymmetries in procurement auctions, which led to Bajari [2]. In such cases, shooting methods are computationally impractical. Thus, the polynomial approach has some distinct advantages over shooting methods: specifically, it is fast, intuitive, and user-friendly. Moreover, one can also use the polynomial approach to deduce valuations from observed bid data. If researchers need to simulate dynamic games, which require computing the equilibrium inverse-bid functions in each period, then speed is crucial because this may require solving for the equilibrium inverse-bid functions thousands (perhaps millions) of times. For example, Saini [29] solved for a Markov Perfect Bayesian Equilibrium in a dynamic, infinite-horizon, procurement auction in which asymmetries obtain endogenously due to capacity constraints and utilization. Likewise, Saini [30] admitted entry and exit and allowed firms to invest in capacity in a dynamic oligopoly model in which firms compete at auction in each period to investigate the evolution of market structure as well as optimality and efficiency of first- and second-price auctions in dynamic settings. To do this, he needed to solve for the equilibrium inverse-bid functions for each firm, at each state (which was determined by each firm’s capacity), at each iteration, when computing the Markov Perfect Bayesian Equilibrium.

The polynomial approach also avoids the inherent instability of backward shooting algorithms, which was shown recently by Fibich and Gavish [6]. Fibich and Gavish proved analytically that, regardless of the numerical method used to integrate the system backwards, the solutions are unstable. Moreover, this instability increases as the number of players increases (or the number of different classes of players increases in an asymmetric setting). Because our constrained-polynomial approximations avoid the use of numerical ordinary differential equation

solvers, we avoid the instability property that characterizes backwards shooting methods used to solve asymmetric first-price auctions. Fibich and Gavish proposed a fixed-point iteration method using a transformation of the standard system of differential equations (which requires solving for the inverse-bid functions) that involves one player's valuation as the independent variable.⁴ In contrast, we consider the standard system of differential equations, but complement it with theoretical constraints and employ an alternative method used to solve boundary value problems which is related to collocation methods.⁵ Nonetheless, our theory carries over to the approach of Fibich and Gavish—or any other approach, as the tests we propose can be used to evaluate any candidate solution.

Clearly, the polynomial approach is attractive. The question is: Can researchers trust the solutions it returns? Here, theory can play an important and useful role. In our research, we have identified some qualitative predictions that equilibrium inverse-bid functions must satisfy. While this sort of analysis is no substitute for a convergence criterion, our predictions at least provide some minimal standards that should be met. Because the polynomial approach is now commonly used by researchers, both theoretical and empirical, our predictions will allow these researchers to perform both qualitative and quantitative checks that will help them defend the validity of the approximate equilibrium inverse-bid functions they have calculated.

Perhaps the most obvious consideration that a researcher faces when applying the polynomial approach involves the fact that the assumed order of the polynomial will affect both the speed and the performance of the algorithm. While Bajari used polynomials of order five, and argued that polynomials of even lower order may be good enough, the examples he provided are fairly simple. In particular, the cumulative distribution functions he considered can be ordered according to reversed hazard-rate dominance. Thus, the equilibrium bid functions can never cross, either in theory or under his approximations. Following Bajari's lead, Brendstrup and Paarsch [3] used fourth-order polynomials to approximate the equilibrium inverse-bid functions in their application. As noted above, their conditional cumulative distribution functions crossed once, so the equilibrium bid functions are theoretically known to cross once. Below, we illustrate the importance of choosing the order of the approximating polynomial.

Specifically, we demonstrate that low-order polynomials are not flexible enough to generate approximate equilibrium-bid functions that can cross several times, even when it is theoretically known that those bid functions must cross several times. Thus, in practice, it is important to specify a polynomial of sufficiently high order. While the visual test that we propose can be used to decide whether proposed approximations to the equilibrium bid functions are consistent with the theory, the test does not allow one to evaluate numerical precision. So we counsel a researcher to err on the side of caution and to specify high-order polynomials—for example, say, twenty-five or thirty. We have found very little difference between approximations that are of, say, order twenty-five and those of order thirty: approximations of these orders are the same to any reasonable tolerance. The reader may then think that this will increase computing time, so that the polynomial approach is then no longer a competitive alternative to shooting methods. Our experience with the modern computer software that we have used to solve such optimization problems suggests that a polynomial of order twenty-five is not appreciably more time-consuming to com-

⁴Fibich and Gavish noted that this choice is *ad hoc* and that the wrong choice can lead to a divergent solution—see footnote 7 of their paper. In their concluding discussion, they suggest this choice might be worth pursuing in future research.

⁵We have a two-point boundary-value problem over a domain which is unknown *a priori*. Thus, our problem can be considered a free boundary-value problem. Our reliance on economic theory is what helps us to pin this boundary down.

pute than one of order five. (We used the modelling language AMPL, which is described in detail in Fourer et al. [9].) In fact, the more flexible is the polynomial (the higher the order) the easier it is to fit the Bayes–Nash equilibrium, a result that some economists have found paradoxical, but which several applied mathematicians have found obvious. Basically, even though an higher-order polynomial has more parameters than a lower-order one, the higher-order one can require fewer iterations (and, thus, less time) to find the best fit.

In solved examples and through the application of simulation methods, we have found several interesting facts of both a quantitative and a qualitative nature. First, the order of the approximating polynomial can affect expected-revenue rankings, substantially. For instance, we investigate an example in which one distribution involves a mean-preserving spread over another. In this example, when the order is five, the first-price auction is calculated to dominate the second-price auction; however, when the order is twenty-five, that ranking is reversed. This qualitative result is important because, in the theoretical literature concerning revenue rankings, researchers have focused largely on environments in which first-order stochastic dominance exists. One exception is the work of Vickrey [32] who presented an analytical example in which, for a particular parameterization, one distribution is a mean-preserving spread over another. In Vickrey’s example, the first-price auction dominates the second-price one. Thus, our example establishes that Vickrey’s result is not robust and suggests that additional research in this area may be worthwhile. In other words, accurate numerical approximations can be used to guide future directions for theoretical research as well.

Second, the order of the approximating polynomial can affect the incidence of inefficiencies at first-price auctions. In one example, when the order is five, the proportion of auctions at which the bidder with the highest value does not win the first-price auction is around 0.056; in the same example, when the order is twenty-five, it is around 0.073, around twenty-five percent higher. We have also found that different pricing rules may favour one class of bidder over another. A corollary to this is that, in some political environments, the choice of auction format and pricing rule can actually favour one class of bidder over another.

Our paper is organized as follows: in the next section, we outline the third algorithm of Bajari [2] as well as the modifications made to it by Hubbard and Paarsch [15], while in section 3, we describe how the results of Kirkegaard [20] can be used to evaluate the consistency of numerical approximations, however derived, when the cumulative distribution functions cross. We provide several numerical examples to illustrate the main points. In section 4, we provide some simulation results based on the solved examples of section 3. Finally, we summarize and conclude our research in section 5. To reduce clutter, we present the proof of a proposition in an appendix at the end of the paper.

2. Theoretical Model and Numerical Methods

Consider a set $\mathcal{N} = \{1, 2, \dots, N\}$ containing N risk-neutral bidders who are vying to win an object sold at a first-price auction—to wit, either a first-price, sealed-bid auction or an oral, descending-price (Dutch) auction. A particular bidder, indexed by n , is assumed to draw a valuation V_n independently from a continuously differentiable cumulative distribution function F_n that has a finite and strictly positive probability density function f_n on the common support $[\underline{v}, \bar{v}]$ where $\bar{v} > \underline{v} \geq 0$, $n = 1, 2, \dots, N$.⁶ Here, V_n represents bidder n ’s monetary value of the object

⁶As in Bajari [2], the support is assumed to be the same for all bidders. While Bajari studied procurement auctions, here, as in Paarsch and Hong [27], we have translated his approach to standard auctions.

for sale.

2.1. Some Properties of Equilibrium Bid Functions

Let $\sigma_n(v)$ denote bidder n 's equilibrium bid function. Thus, when bidder n has value v_n , he tenders s_n which equals $\sigma_n(v_n)$. Denote by $\varphi_n(s)$ bidder n 's equilibrium inverse-bid function, so $\varphi_n(\cdot)$ is $\sigma_n^{-1}(\cdot)$. Under relatively weak assumptions, Lebrun [23] has demonstrated that an equilibrium (in strictly increasing strategies) exists, and that equilibrium is unique.

Because the supports of the distributions of valuations are the same for all bidders, it can also be shown that all bidders submit bids in the same interval $[\underline{v}, \bar{s}]$ where $\bar{s} \in (\underline{v}, \bar{v})$. To wit, the following conditions must be satisfied:

- 1a. $\varphi_n(\underline{v}) = \underline{v}$ for all $n = 1, 2, \dots, N$;
- 1b. $\varphi_n(\bar{s}) = \bar{v}$ for all $n = 1, 2, \dots, N$.

Conditions 1a and 1b mean that $\sigma_n(\underline{v})$ equals $\underline{v}$ and $\sigma_n(\bar{v})$ equals $\bar{s}$, the highest observable equilibrium bid, for all $n = 1, 2, \dots, N$. Below, some attention will be devoted to the number of times the equilibrium bid functions cross in the interior of $(\underline{v}, \bar{v})$, or the number of times the equilibrium inverse-bid functions cross in the interior of $(\underline{v}, \bar{s})$.

Lebrun proved that $\varphi_n(s)$ is differentiable on $(\underline{v}, \bar{s}]$. When bidder n has value v_n , his decision problem is to choose a bid s to maximize expected profit, or

$$\max_{s < v_n} (v_n - s) \prod_{m \neq n} F_m[\varphi_m(s)].$$

In equilibrium, bidder n tenders s if his value v_n equals $\varphi_n(s)$. Substituting into the first-order condition and re-arranging yields the following:

$$1 - \sum_{m \neq n} \frac{[\varphi_n(s) - s] f_m[\varphi_m(s)] \varphi'_m(s)}{F_m[\varphi_m(s)]} = 0 \quad n = 1, 2, \dots, N \quad (1)$$

as formulated by Bajari [2]. Thus, the equilibrium inverse-bid functions are the solution to a system of differential equations, characterized by (1), with the boundary conditions mentioned above. Although it is typically impossible to solve this system of differential equations in closed-form, some properties can be deduced by studying the system at the endpoints, as s approaches $\underline{v}$ or $\bar{s}$. For example, Fibich et al. [7] proved the following properties, the first of which follows directly from (1):

- 2a. $\sum_{m \neq n} (\bar{v} - \bar{s}) f_m(\bar{v}) \varphi'_m(\bar{s}) = 1$ for all $n = 1, 2, \dots, N$.
- 2b. If $f_n(\underline{v}) \in \mathbb{R}_{++}$ and $\varphi_n(s)$ is differentiable at $s = \underline{v}$ for all $n = 1, 2, \dots, N$, then $\varphi'_n(\underline{v}) = [N/(N-1)]$.

However, as mentioned above, Lebrun proved only that $\varphi_n(s)$ is differentiable on $(\underline{v}, \bar{s}]$. It is unknown whether this property holds at $\underline{v}$, as assumed in Property 2b; see Lebrun [23], footnote 8, for a nice discussion of this point. This issue is discussed further below. The assumption that $f_n(\underline{v}) \in \mathbb{R}_{++}$ is also important since Property 2b is demonstrably false otherwise. For example, Cheng [5] considers a setting where two bidders draw valuations from different power distributions with $\underline{v}$ equal zero (but where bidders' supports have different upper end-points). At least one of the power distributions is non-linear and, therefore, does not satisfy $f_n(\underline{v}) \in \mathbb{R}_{++}$. Cheng identified situations in which the two bidders use different linear bidding strategies. Thus, bidding strategies, or their inverse, cannot be tangent to one another at s equal $\underline{v}$ (or anywhere else for that matter), contrary to Property 2b.

2.2. Approximating Bid Functions by Polynomials

Since it is typically impossible to solve the system of differential equations characterized by (1) in closed-form, Bajari [2] proposed approximating $\varphi_n(s)$ by an ordinary polynomial. That is, he assumed that $\varphi_n(s)$ can be approximated by

$$\hat{\varphi}_n(s) = \bar{s} + \sum_{k=0}^K \alpha_{n,k}(s - \bar{s})^k, \quad s \in [\underline{v}, \bar{s}], \quad n = 1, 2, \dots, N. \quad (2)$$

Without loss of generality, assume that the polynomials are of the same order for all bidders. We assume K is three, or greater. The problem is to estimate $\bar{s}$ as well as the $\alpha_{n,k}$ s for all $n = 1, 2, \dots, N$ and $k = 0, 1, \dots, K$. To accomplish this, Bajari proposed selecting a large number T of grid points uniformly from the interval $[\underline{v}, \bar{s}]$. Under the functional form in (2), the left-hand side of (1) can then be calculated at any grid point. Bajari proposed choosing the parameters that minimize

$$H(\bar{s}, \alpha) = \sum_{n=1}^N \sum_{t=1}^T [\text{left-hand side of (1) for bidder } n \text{ at grid point } t]^2 + \sum_{n=1}^N [\underline{v} - \hat{\varphi}_n(\underline{v})]^2 + \sum_{n=1}^N [\bar{v} - \hat{\varphi}_n(\bar{s})]^2 \quad (3)$$

where α denotes a vector that collects the $N \times (K + 1)$ coefficients of the polynomials. If all the first-order conditions as well as the boundary conditions are satisfied, then $H(\bar{s}, \alpha)$ will equal zero.

Note that, under Bajari's approach, Conditions 1a and 1b (viz., that $\varphi_n(\underline{v})$ equal $\underline{v}$ and $\varphi_n(\bar{s})$ equal $\bar{v}$) are not imposed. Rather, they are only approximately satisfied at a minimum of H . Because these endpoints are easy to identify visually, particularly in a graph, researchers might choose instead to minimize

$$H_A(\bar{s}, \alpha) = \omega_{\text{FOC}} \sum_{n=1}^N \sum_{t=1}^T [\text{left-hand side of (1) for bidder } n \text{ at grid point } t]^2 + \omega_{\underline{v}} \sum_{n=1}^N [\underline{v} - \hat{\varphi}_n(\underline{v})]^2 + \omega_{\bar{v}} \sum_{n=1}^N [\bar{v} - \hat{\varphi}_n(\bar{s})]^2 \quad (4)$$

where ω_{FOC} , $\omega_{\underline{v}}$, and $\omega_{\bar{v}}$ are positive weights chosen by the researcher. Then, relatively large values for $\omega_{\underline{v}}$ and $\omega_{\bar{v}}$ can be chosen to ensure that the approximated endpoints are close to the theoretical predictions.⁷ Likewise, Property 2a is not imposed. Once again, however, because the property derives from the first-order conditions, it should also be satisfied when the fit of the approximation is good. As for Property 2b, the functional form assumed by Bajari is clearly differentiable. Hence, a good fit would also produce the prediction in Property 2b. Indeed, the illustrations of the approximate equilibrium-bid functions presented in Bajari [2] as well as Brendstrup and Paarsch [3] depict equilibrium bid functions that are tangent at the left endpoints (right in Bajari's case because he investigated procurement auctions).

⁷For example, Bajari [2] set ω_{FOC} equal to one and $\omega_{\underline{v}}$ as well as $\omega_{\bar{v}}$ equal to T .

If Conditions 1a and 1b as well as Properties 2a and 2b are not satisfied, then the approximate equilibrium-bid functions are flawed, perhaps in important ways. Since it is difficult to know *a priori* how important such flaws may be, one strategy would be to consider only those approximations that “come close” to satisfying the conditions. The advantage of Conditions 1a and 1b as well as Properties 2a and 2b is one can verify that they are satisfied in a straightforward way: no prior knowledge of the equilibrium bid functions in the interior is required because Conditions 1a and 1b as well as Properties 2a and 2b concern the boundaries. Thus, these corners offer a quick first “test” of the plausibility of the approximate equilibrium-bid functions. Alternatively, only candidate solutions that satisfy these conditions should be entertained: to wit, they should be introduced as constraints on candidate solutions. Note, too, that by assuming that the equilibrium inverse-bid functions can be represented by polynomials, one is assuming that the functions are continuously differentiable everywhere on the support. As such, the implicit assumption made by Fibich et al. [7] (that $\varphi_n(s)$ be differentiable at s equal $\underline{v}$) is satisfied.

Hubbard and Paarsch [15] modified Bajari’s third algorithm in three ways: first, instead of regular polynomials, they employed Chebyshev polynomials, which are orthogonal polynomials and thus more stable numerically;⁸ second, they cast the problem within the Mathematical Programming with Equilibrium Constraints (MPEC) approach advocated by Su and Judd [31]. (For more on the MPEC approach, see Luo et al. [24].) Hubbard and Paarsch used the MPEC approach to discipline the estimated Chebyshev coefficients in the approximations so that the first-order conditions defining the equilibrium inverse-bid functions are approximately satisfied, subject to constraints that the boundary conditions defining the equilibrium strategies are satisfied. Finally, they imposed monotonicity on candidate solutions.

Under their assumptions, the approximate equilibrium inverse-bid function for bidder n can be expressed as

$$\hat{\varphi}_n(s; \alpha_n, \bar{s}) = \sum_{k=0}^K \alpha_{n,k} \mathbb{T}_k[x(s; \bar{s})] \quad n = 1, 2, \dots, N \quad (5)$$

where $x(\cdot)$ lies in the interval $[-1, 1]$ and where, for completeness, we have explicitly defined it as a transformation of the bid s under consideration. Here, $\mathbb{T}_k(\cdot)$ denotes the k^{th} Chebyshev polynomial of the first kind and the vector α_n collects the polynomial coefficients for bidder n . Thus,

$$\begin{aligned} \mathbb{T}_0(x) &= 1 \\ \mathbb{T}_1(x) &= x \\ \mathbb{T}_{k+1}(x) &= 2x\mathbb{T}_k(x) - \mathbb{T}_{k-1}(x) \quad k = 1, 2, \dots, K-1. \end{aligned}$$

Instead of using equi-spaced points like Bajari, Hubbard and Paarsch used the Chebyshev nodes on the interval $[-1, 1]$, which are

$$x_t = \cos\left(\frac{2t-1}{2T}\pi\right), \quad t = 1, \dots, T.$$

The points $\{s_t\}_{t=1}^T$ are found using the following transformation:

$$x_t \equiv x(s_t; \bar{s}) = \frac{2s_t - \underline{v} - \bar{s}}{\bar{s} - \underline{v}},$$

⁸Judd [18] has advocated using Chebyshev polynomials, which are orthogonal with respect to the L_2 norm, the basis of the nonlinear least squares objective.

or

$$s_t = \frac{\bar{s} + \underline{v} + (\bar{s} - \underline{v})x_t}{2},$$

which maps the s_t s from the Chebyshev nodes. The Chebyshev nodes have the property of minimizing the maximum interpolation error when approximating a function. As such, they are often considered the best choice for the approximating grid.

In summary, Hubbard and Paarsch approximated the equilibrium inverse-bid functions by Chebyshev polynomials and chose the parameters α and $\bar{s}$ to minimize

$$H_I(\bar{s}, \alpha) = \sum_{n=1}^N \sum_{t=1}^T [\text{left-hand side of (1) for bidder } n \text{ at grid point } t]^2 \quad (6)$$

subject to constraints that Conditions 1a and 1b are satisfied and that the approximated solutions are all monotonic for each bidder.

In going forward, we maintain the spirit of the approach taken by Hubbard and Paarsch, but incorporate Properties 2a and 2b as well. In doing so, for each bidder, four constraints are imposed on the equilibrium inverse-bid functions, which are approximated by Chebyshev polynomials of degree K . The unknown parameters are chosen to minimize (6) subject to the constraints described by Conditions 1a and 1b as well as Properties 2a and 2b. We employ this approach in deriving the observations that follow.

As mentioned above, four conditions apply to each bidder. Thus, there are $4N$ conditions or constraints in total and TN points that enter the objective function. By comparison, there are $N(K + 1) + 1$ parameters to be estimated—the parameters in α plus $\bar{s}$. For the number of conditions (boundary and first-order together) to equal the number of unknowns

$$N(T + 4) = N(K + 1) + 1$$

or

$$(T + 4) = (K + 1) + \frac{1}{N}. \quad (7)$$

Since at auctions, N weakly exceeds two, and T and K are integers, this equality cannot hold for any (T, K) choice.

We believe our approach is related to the spectral methods used to solve partial differential equations; for more on these methods, see Gottlieb and Orszag [12]. Related to the spectral family of methods is a family referred to as *collocation* methods. Under collocation methods, it is assumed that the solution can be represented by a candidate approximation, typically a polynomial; a solution is selected that solves the system exactly at a set of points over the interval of interest; such points are referred to as the *collocation points*. Because equality (7) cannot hold, collocation is infeasible in this case, but the MPEC-based approach can be thought of as an hybrid between collocation and least-squares as some constraints are explicitly imposed, leading to a constrained nonlinear optimization problem. It will be impossible to make all residual terms equal to zero. In other words, the fit is necessarily imperfect in a quantitative sense. The main point in this section is that there may be qualitative inadequacies as well. When comparing the $N(K + 1) + 1$ parameters with the $4N$ conditions, note that, if K equals three and all the conditions are satisfied, then only one degree of freedom remains.

In order to compare the equilibrium inverse-bid functions of bidders m and n , define

$$D_{n,m}(s) = \hat{\varphi}_n(s) - \hat{\varphi}_m(s)$$

and

$$D_{n,m}^r(s) = \frac{\hat{\varphi}_n(s) - \hat{\varphi}_m(s)}{(s - \underline{v})^2}.$$

Note that $D_{n,m}^r$ is a measure of the difference between equilibrium inverse-bid functions in relation to the size of the bids.

EXAMPLE 1: Assume that N is two, that $[\underline{v}, \bar{v}]$ are $[0, 1]$ and that $f_1(\bar{v})$ exceeds $f_2(\bar{v})$. The ordering of the probability density functions at $\bar{v}$ implies that bidder 1 is “strong at the top” when compared to bidder 2. That is, F_1 dominates F_2 near $\bar{v}$ in the sense of both first-order stochastic dominance and reversed hazard-rate dominance. Assume that a third-order polynomial is used to approximate each equilibrium inverse-bid function, so K is three. Solving the $4N$ or eight conditions for the eight coefficients in α , while allowing $\bar{s}$ to vary, yields the following conclusion:

$$D_{1,2}(s) = \hat{\varphi}_1(s) - \hat{\varphi}_2(s) = \left[\frac{f_1(1) - f_2(1)}{f_1(1)f_2(1)} \right] \left[\frac{s^2(\bar{s} - s)}{\bar{s}^2(1 - \bar{s})} \right].$$

Recall $\bar{v}$ exceeds $\bar{s}$, so regardless of the exact value of $\bar{s}$, $[\hat{\varphi}_1(s) - \hat{\varphi}_2(s)]$ is proportional to $s^2(\bar{s} - s)$ which, in turn, is strictly positive for all $s \in (0, \bar{s})$. Thus, $\hat{\varphi}_1(s)$ must exceed $\hat{\varphi}_2(s)$ for all $s \in (0, \bar{s})$, or $\sigma_1(v)$ is strictly less than $\sigma_2(v)$ for all $v \in (0, 1)$. This conclusion rests *only* on the assumption that $f_1(1)$ exceeds $f_2(1)$. In this example,

$$D_{1,2}^r(s) = \left[\frac{f_1(1) - f_2(1)}{f_1(1)f_2(1)} \right] \left[\frac{\bar{s} - s}{\bar{s}^2(1 - \bar{s})} \right].$$

Kirkegaard [20] showed that if $F_1(v)$ crosses $F_2(v)$, then theory predicts that the equilibrium bid functions must cross as well. Yet, when K is three, such a crossing cannot obtain for these approximate equilibrium-bid functions. $\square$

When K is four or more, then, for a given $\bar{s}$, there are more parameters than we have conditions, so it is impossible to get “closed-form” expressions for these absolute and relative differences, respectively: too many degrees of freedom exist. While the analytic differences cannot be derived explicitly for higher order polynomials, EXAMPLE 1 illustrates a general phenomenon, described in the following Proposition which describes some qualitative features of $D_{n,m}(s)$ for higher K .

Proposition 1. Assume (i) $f_n(\underline{v}) \in \mathbb{R}_{++}$ and (ii) $\varphi_n(s)$ is a polynomial of order K , $K \geq 3$, with real coefficients that satisfy Conditions 1a and 1b as well as Properties 2a and 2b, for all $n = 1, 2, \dots, N$. If $f_n(\bar{v}) \neq f_m(\bar{v})$, then $\varphi_n(s)$ and $\varphi_m(s)$ cross at most $(K - 3)$ times on $(\underline{v}, \bar{s})$, $m, n = 1, 2, \dots, N$.

Proof. See the Appendix. $\blacksquare$

To be clear, Proposition 1 does not state that the third algorithm of Bajari [2] will produce at most $(K - 3)$ interior crossings of the approximate equilibrium-bid functions: Bajari does not impose Conditions 1a and 1b as well as Properties 2a and 2b. However, as mentioned, if the estimates are “good,” then they should “come close” to satisfying the conditions. Proposition 1 does, however, imply that a limited number of ways exist in which equilibrium inverse-bid functions approximated by polynomials can interact under Conditions 1a and 1b as well as Properties 2a and 2b. Proposition 2, below, contains a complementary result.

Figure 1: $F_1(v)$ and $F_2(v)$ for EXAMPLE 2

Proposition 2. *Under Condition 1a and Property 2b, $D_{n,m}^r$ is a polynomial of order $(K - 2)$.*⁹

Proof. From the proof of Proposition 1, Condition 1a and Property 2b imply that $D_{n,m}$ can be written

$$D_{n,m}(s) = \sum_{k=2}^K (\gamma_{n,k} - \gamma_{m,k})(s - \underline{v})^k,$$

which implies Proposition 2. ■

When K is three, then Proposition 2 implies that $D_{n,m}^r$ is linear. If K is four, then $D_{n,m}^r$ is a quadratic. Adding Condition 1b implies that $D_{n,m}^r(\bar{s})$ equals zero, while adding Property 2a determines $dD_{n,m}^r(\bar{s})/ds$. Thus, when K is small, if Conditions 1a and 1b as well as Properties 2a and 2b are imposed, then the relationships among the approximate equilibrium inverse-bid functions are almost predetermined. Above, in EXAMPLE 1, $D_{n,m}^r$ is completely determined by $\bar{s}$ when K is three.

We now introduce a specific pair of cumulative distribution functions $F_1(\cdot)$ and $F_2(\cdot)$ which are plotted in figure 1, and pursue a more quantitative analysis than in EXAMPLE 1.

EXAMPLE 2: Assume that N is two and that $[\underline{v}, \bar{v}]$ are $[0, 1]$. Here, $F_1(\cdot)$ is a standard uniform distribution, while $F_2(\cdot)$ is a cumulative distribution function constructed using piecewise polynomials so that the function is monotonic, has $F_2(0)$ equal zero, and has $F_2(1)$ equal one. Furthermore, $d^2F_2(v)/dv^2$ is continuous, which means that $df_2(v)/dv$ exists. This latter property is unnecessary under this approach; we require only that the probability density functions be continuous,

⁹If only Condition 1a is imposed, then the function $[D_{n,m}(s)/(s - \underline{v})]$ is a polynomial of order $(K - 1)$.

Figure 2: $\hat{D}_{1,2}(s)$ for $K = 3, 5, 25$ for EXAMPLE 2

strictly positive, and finite everywhere. Finally, F_2 has been constructed so that F_1 is a mean-preserving spread over F_2 ; the common mean is 0.5. In figure 2, we plot the absolute differences in the approximate equilibrium inverse-bid functions $\hat{D}_{1,2}(s)$ when the order of the approximating polynomials K increases from three to five to twenty-five.¹⁰

We denote by $\bar{s}_K$ the approximated high bid in the case when an order K polynomial is used to approximate the equilibrium inverse-bid functions. When K equals three, the absolute difference in the approximate equilibrium inverse-bid functions and the theoretical absolute difference coincide exactly. We do not plot the theoretical difference, but the two are indistinguishable. Note that, when K equals three, $D_{1,2}(s)$ is maximized at $\frac{2}{3}\bar{s}$, which we identify in figure 2. Note, too, that the difference does not cross zero in the interior of $(0, \bar{s}_3)$, which implies that the approximate equilibrium-bid functions do not cross. As the order of the approximating polynomials

¹⁰As mentioned in the introduction, we used the modelling language AMPL (which is described in detail in Fourer et al. [9]) to calculate the coefficients of our approximating polynomials. For most of our work, we used the solver SNOPT 7.2-8, but we also checked to see whether the results were sensitive to the solver used; for example, we used the solver KNITRO 6.0.0 as well. While there are always some differences in solutions across solvers, we did not find these to be important. To get an initial guess for the high-bid parameter $\bar{s}_K$, we used the third-order approximation first because, in that case, $\bar{s}_3$ is the only free parameter. For the parameters of subsequent, higher-order polynomials, we used the previous convergent estimates as starting values, and set the highest-order coefficient to zero; this is reasonable because the Chebyshev polynomials are orthogonal polynomials. The important point to note is that, in practice, the absolute value (or square) of the approximated coefficients decays quickly to zero as K increases; for example, often, if the estimated coefficient on the first-order polynomial is 0.5, then the next will be around 0.05, the next 0.005, and so forth. The approximations for a given degree K take less than one second to solve and the objective (6) is decreasing in K . In practice, a researcher can select a K which satisfies the theoretical predictions we describe in the next section and meets some error tolerance criterion—for example, the objective (6) for our degree 25 approximation is below 10^{-6} . While all of our work was done in the L^2 norm, one could carry out the work in the L^1 norm as well. This is discussed in some detail in Hubbard and Paarsch [16] as well as in Hubbard et al. [17].

increases to five, a crossing obtains: the approximate equilibrium-bid functions cross in the interior of $(0, \bar{s}_5)$ which is shown by the absolute difference crossing the zero line once. Finally, in the twenty-fifth order case, the absolute difference in the approximate equilibrium inverse-bid functions equals zero twice in the interior of $(0, \bar{s}_{25})$, meaning that the approximate equilibrium-bid functions cross one another twice. Note, too, that the value of the high bid $\bar{s}_K$ changes as the order of the approximations change. For low values of K , $\bar{s}_K$ exceeds the common mean of the two distributions (which is 0.5). However, the opposite holds for large values of K . Using a different approximation technique, Fibich and Gavish [6] provided an example in which it is also the case that $\bar{s}$ falls below the common mean. This observation is also consistent with Gavious and Minchuk [10] who noted that if both bidders' distributions are "almost" uniform, small asymmetries between them will tend to suppress $\bar{s}$. In comparison, if the two bidders are symmetric, then it is well known that $\bar{s}$ will equal the mean of the distribution. $\square$

Note that the difference $D_{1,2}(s)$, which we were able to characterize explicitly in the case of K equal three, is most definitely *not* the true difference between the equilibrium bid functions. It is derived from the constraints and requires knowledge only of $f_1(\bar{v})$ and $f_2(\bar{v})$, so it cannot possibly describe the true differences in the equilibrium bid functions. This is precisely our point: with too few degrees of freedom, the approximations are essentially predetermined. Thus, the absolute difference in the approximations when K equals three will look like the analytic expression we derived for this example. One criticism of the polynomial approach is that it works well, if the practitioner has a good initial guess.¹¹ When K equals three, the researcher obtains an initial guess that already satisfies some theoretical properties at essentially no cost because there is only one free parameter, $\bar{s}$.

While our results thus far illustrate why polynomials of too low an order may result in poor approximations, we have yet to show how to evaluate whether an approximation is sufficiently "good"—i.e., consistent with theory. In the next section, we discuss the theoretical predictions that can be made when the cumulative distribution functions of valuations cross.¹² Armed with such predictions, it is possible to assess better the accuracy of approximate equilibrium-bid functions.

3. Theoretical Predictions and a Test

To begin, let

$$P_{n,m}(v) = \frac{F_m(v)}{F_n(v)}, \quad v \in (\underline{v}, \bar{v}]$$

measure bidder n 's strength (power) relative to bidder m at a given value v . The larger is $P_{n,m}$, the stronger bidder n when compared to bidder m at that value. For example, if $P_{n,m}(v)$ exceeds one,

¹¹The most common alternative solution technique involves using the shooting methods described above. Shooting methods work well, but they too require a good initial guess (for $\bar{s}$) if they are to be at all competitive (in terms of computation speed) with the polynomial approach. Nonetheless, the initial guess is relatively more important under the polynomial approach which involves more parameters. For a comparison of computational strategies used to solve for equilibria in models of asymmetric first-price auctions, see Hubbard and Paarsch [16].

¹²If they do not cross, then one distribution dominates the other in the first-order stochastic sense, and the results described in the introduction may provide some guidance.

then $F_n(v)$ is less than $F_m(v)$. Similarly, define $U_n(v)$ as bidder n 's equilibrium expected pay-off (profit) at an auction if his value is v , and let

$$R_{n,m}(v) = \frac{U_n(v)}{U_m(v)}, \quad v \in [\underline{v}, \bar{v}]$$

denote bidder n 's equilibrium pay-off relative to bidder m 's equilibrium pay-off at a given value. Note that $P_{n,m}(v)$ is exogenous, while $R_{n,m}(v)$ is endogenous.

Kirkegaard [20] demonstrated that the two ratios can be used to make predictions concerning the properties of $\sigma_n(v)$ and $\sigma_m(v)$ or, equivalently, $\varphi_n(s)$ and $\varphi_m(s)$. At v equal $\bar{v}$, the two bids coincide and so too do the two ratios, or $\sigma_n(\bar{v})$ equals $\sigma_m(\bar{v})$ and $R_{n,m}(\bar{v})$ equals $P_{n,m}(\bar{v})$, which is one. In fact, comparing the two ratios at any $v \in (\underline{v}, \bar{v}]$ is equivalent to comparing the equilibrium bids at v , or

$$R_{n,m}(v) \gtrless P_{n,m}(v) \iff \sigma_n(v) \gtrless \sigma_m(v), \quad \text{for } v \in (\underline{v}, \bar{v}]. \quad (8)$$

Moreover, it turns out that the motion of $R_{n,m}$, which is endogenous, is determined by how it compares to $P_{n,m}$, which is exogenous. Specifically,

$$R'_{n,m}(v) \gtrless 0 \iff R_{n,m}(v) \gtrless P_{n,m}(v), \quad \text{for } v \in (\underline{v}, \bar{v}]. \quad (9)$$

Coupled with the condition that

$$R_{n,m}(\bar{v}) = P_{n,m}(\bar{v}) = 1, \quad (10)$$

(9) allows one to make a number of predictions.

In figures 3.a and 3.b, we depict (8) and (9). In this example, F_n and F_m cross twice in the interior, so $P_{n,m}$ equals one twice in the interior. The boundary condition in (10) and the relationship in (9) imply that $R_{n,m}$ and $P_{n,m}$ must cross exactly twice in the interior. Therefore, by (8), the equilibrium bid functions must cross *exactly* twice in the interior. Note, too, that if the assumptions in Property 2b are satisfied, then by l'Hôpital's rule

$$\lim_{v \rightarrow \underline{v}} R_{n,m}(v) = \frac{f_m(\underline{v})}{f_n(\underline{v})} = \lim_{v \rightarrow \underline{v}} P_{n,m}(v). \quad (11)$$

Thus, simply by plotting $P_{n,m}$, it is often possible to predict qualitative behaviour with great accuracy. The example in figures 3.a and 3.b satisfies the *diminishing wave property*. What does this mean? Well, first, the extrema of $P_{n,m}(v)$ —including $\lim_{v \rightarrow \underline{v}} P_{n,m}(v)$ —alternate between being above and below one; second, the extrema that are above one get closer to one as v increases; and, third, the extrema that are below one also get closer to one as v increases; see Kirkegaard [20] for a formal definition. In this case, the exact number of times equilibrium bid functions cross can be identified, as in figure 3.a. The number of crossings between $R_{n,m}$ and $P_{n,m}$ in the interior is identical to the number of interior stationary points of $P_{n,m}$. When the diminishing wave property is not satisfied, the number of interior stationary points, instead, provides an upper bound on the number of times equilibrium bid functions can cross in the interior.

If approximate equilibrium-bid functions do not cross the theoretically-predicted number of times, then these approximations are clearly suspect.¹³ However, even when the approximate

¹³Note that these methods can be applied to approximations derived using shooting methods, the fixed-point or Newton iterations of Fibich and Gavish [6], or any other method of approximating the equilibrium in a model of an asymmetric first-price auction.

3.a: Comparing $R_{n,m}(v)$ and $P_{n,m}(v)$

3.b: A possible path for $R_{n,m}(v)$

Figure 3: Comparing $R_{n,m}(v)$ and $P_{n,m}(v)$ and a path consistent with (8)–(11).

equilibrium-bid functions are consistent with theoretical predictions, the theoretical relationship between $R_{n,m}$ and $P_{n,m}$ suggests a visual test of the accuracy of approximate equilibrium-bid functions. Based on the approximate equilibrium-bid functions, denoted $\hat{\sigma}_n$, the ratio of expected pay-offs can be computed. Denote the estimated ratio by $\hat{R}_{n,m}$. If $P_{n,m}$ and $\hat{R}_{n,m}$ are plotted in the same figure, then they should interact in a manner consistent with (9), as illustrated in figure 3.b. The steepness of $\hat{R}_{n,m}$ at a point of intersection with $P_{n,m}$ and the location of the intersections can be used to eliminate inaccurate solutions.

1. **Slope:** At any point where $P_{n,m}$ and $\hat{R}_{n,m}$ intersect (i.e., where $\hat{\sigma}_n$ equals $\hat{\sigma}_m$), the latter should be *flat*, have a derivative that equals zero. If $\hat{R}_{n,m}$ is steep at such a point, then this is an indication that the approximate equilibrium-bid function is inaccurate as the first-order conditions are not close to being satisfied. Note, too, that this is true any time bids coincide (for any $v > \underline{v}$, including $\bar{v}$).
2. **Location:** The location of the intersections of $P_{n,m}$ and $\hat{R}_{n,m}$ must also be consistent with theory. In particular, $P_{n,m}$ and $R_{n,m}$ can cross *at most* once between any two peaks of $P_{n,m}$; with diminishing waves, they *must* cross between any two peaks (not counting v equals $\bar{v}$). In figure 3.b, for example, $P_{n,m}$ and $R_{n,m}$ must cross once to the left of the point where $P_{n,m}$ is minimized, and once between the two interior stationary points.

EXAMPLE 2 (CONTINUED): Above, we demonstrated that the third-order polynomial approximations were poor by appealing to the absolute difference in the approximate equilibrium-bid functions. Now consider plotting the exogenous ratio $P_{2,1}$ and the approximate endogenous ratio $\hat{R}_{2,1}$ for these distributions. In figure 4.a, we depict the two ratios when polynomials of order five are used to approximate the equilibrium inverse-bid functions. First, note that this example does not satisfy the diminishing wave property as both $\lim_{v \rightarrow \bar{v}} P_{2,1}(v)$ and the first interior stationary point of $P_{2,1}$ are greater than one. Because the diminishing wave property is not satisfied, we can only use theory to bound the number of times the equilibrium bid functions should cross: if $P_{2,1}$ does not get closer to one as v increases, then the equilibrium bid functions can cross at most twice. (Remember that they have to cross at least once because the cumulative distributions cross once.) The fifth-order approximations are an improvement over the third-order approximations in the sense that the approximate equilibrium-bid functions at least cross; the relationship in (8) implies this since $\hat{R}_{2,1}$ equals $P_{2,1}$ once over $v \in (\underline{v}, \bar{v})$. However, we can use the slope property formalized above to demonstrate why this approximation is insufficient. Specifically, note that $\hat{R}_{2,1}$ is not at a stationary point when it intersects $P_{2,1}$. This observation is a corollary of the more general property that $\hat{R}_{2,1}$ should be decreasing (increasing) when it is below (above) $P_{2,1}$. It is clear that this is violated for the fifth-order approximations as $\hat{R}_{2,1}$ peaks around v equal to 0.2 in figure 4.a.

In contrast, we depict in figure 4.b the exogenous and endogenous ratios for the twenty-fifth order approximations, which are consistent with all of the theoretical properties. $\hat{R}_{2,1}$ intersects $P_{2,1}$ twice in the interior and, at each crossing, the slope of the endogenous ratio is zero. Furthermore, the locations of the crossings are appropriate: $\hat{R}_{2,1}$ intersects once between $\underline{v}$ and the first interior stationary point and once between the two interior extrema. Equation (10) specifies that both ratios take a value of one at $\bar{v}$. Thus, because $\hat{R}_{2,1}$ must be increasing when it exceeds $P_{2,1}$, as v approaches the high valuation $\bar{v}$, $\hat{R}_{2,1}$ should be bounded between $P_{2,1}$ and one, as it is in figure 4.b. In short, $\hat{R}_{2,1}$ is horizontal at $\bar{v}$. While this convergence obtains in figure 4.a as well,

4.a: $K = 5$

4.b: $K = 25$

Figure 4: $P_{2,1}(v)$ and $\hat{R}_{2,1}(v)$ for Example 2

$\hat{R}_{2,1}$ approaches at a much steeper angle and we know from our slope property that $\hat{R}_{2,1}$ should be flat when it intersects $P_{n,m}$, as in figure 4.b. $\square$

While EXAMPLE 2 has allowed us to investigate polynomial approximations of the equilibrium inverse-bid functions when the diminishing wave property is not satisfied, theory provides precise predictions concerning the number of crossings when the diminishing wave property is satisfied. In the next example, we consider a situation in which the diminishing wave property is satisfied and the cumulative distribution functions cross twice.

EXAMPLE 3: Assume that N is two and that $[\underline{v}, \bar{v}]$ are $[0, 1]$. Here, $F_1(\cdot)$ is again a standard uniform distribution, while $F_2(\cdot)$ is a cumulative distribution function (different from that used in EXAMPLE 2) constructed using piecewise polynomials such that the function is monotonic, has $F_2(0)$ equal zero, and has $F_2(1)$ equal one. Furthermore, $d^2 F_2(v)/dv^2$ is continuous, which means that $df_2(v)/dv$ exists. As mentioned above, this is not required for our approach—we require only that the probability density functions be continuous, strictly positive, and finite everywhere, which they are. We plot the cumulative distribution functions in figure 5.a. In figure 5.b, we depict $P_{2,1}$, the ratio of the cumulative distribution functions; note that the diminishing wave property is satisfied in this example.

When the equilibrium inverse-bid functions are approximated by third-order or fourth-order polynomials, the approximate equilibrium-bid functions do not cross, suggesting that serious inadequacies exist with the approximations: for these cases, $\hat{R}_{2,1}$ is everywhere below $P_{2,1}$, but not monotonic. However, the fifth-order approximations cross twice, the exact number of times predicted by theory. In figure 6.a, we depict the exogenous and endogenous ratios in the case when the equilibrium inverse-bid functions are approximated by polynomials of order five. We restrict the figure to the interval $v \in [0.2, 1]$ so that the crossings can be seen clearly.¹⁴ While the number of crossings is correct, the slope property described above is again violated because $\hat{R}_{2,1}$ is not flat when it intersects $P_{2,1}$; the location property is nearly violated as well. Note, too, that both intersections are close to lying between the interior stationary points of $P_{2,1}$. In figure 6.b, we depict the ratios for the case when the equilibrium inverse-bid functions are approximated by polynomials of order twenty-five. These approximations are consistent with all of our theoretical checks. By comparing figure 6.a with figure 6.b, we note that the location of the crossings are clearly different—the twenty-fifth order approximations cross at lower values of v relative to the fifth-order approximations. $\square$

It is also interesting to note that if the diminishing wave property were satisfied, then the number of times $P_{n,m}$ and $R_{n,m}$ (or σ_m and σ_n) cross would be *independent* of either the number of rival bidders they face or which cumulative distribution functions characterize these opponents, provided their supports are the same. That is, theory tells us that only pairwise comparisons between bidders are required. Thus, if bidders were to draw valuations from three (or more) different distributions (so F_1 , F_2 , and F_3), then our insights could be used to compare pairs of bidders—class 1 versus class 2, class 1 versus class 3, and class 2 versus class 3. To wit, all that matters is the relationship between pairs of distribution functions.

¹⁴Over the interval $v \in [0, 0.2]$ both $\hat{R}_{2,1}$ and $P_{2,1}$ increase as v decreases. In fact, $\hat{R}_{2,1}$ converges to $P_{2,1}$ as $v \rightarrow \underline{v}$ which is expected. Given that, by assumption, the equilibrium inverse-bid functions can be represented by a series of polynomials, the conditions for Property 2b hold and the relationship in (11) should obtain. In this example, $\hat{R}_{2,1}$ converges to $P_{2,1}$ from below as the first interior stationary point obtains below one, whereas in EXAMPLE 2, $\hat{R}_{2,1}$ converged to $P_{2,1}$ from above as the first interior stationary point obtained above one.

5.a: $K = 5$

5.b: $K = 25$

Figure 5: $F_1(v)$, $F_2(v)$, and $P_{2,1}$ for Example 3

6.a: $K = 5$

6.b: $K = 25$

Figure 6: $P_{2,1}(v)$ and $\hat{R}_{2,1}(v)$ for Example 3

4. Some Simulation Results

While much of our research has provided qualitative predictions, we also undertook a simulation study to investigate the quantitative importance of using good approximations. Several of the results we have found are interesting in their own right. For instance, we have found that an existing result dating back to Vickrey [32] is not robust. Thus, numerical methods may prove a useful starting point in providing directions of future theoretical research.

In our simulation experiments, we generated information concerning one trillion auctions. Specifically, we generated pairs of independent uniform random draws which represented the cumulative distribution functions of the two players. We then inverted the respective cumulative distribution functions introduced above in EXAMPLE 2 and EXAMPLE 3 at these uniform draws to obtain the respective valuation draws. We conducted the auctions, first assuming a first-price rule, where we varied the order of the polynomials used to approximate the equilibrium inverse-bid functions, and then mapped the random valuations into random bids, and also assuming a second-price rule. (Within the independent private-values paradigm, at second-price auctions, each player has a dominant strategy to bid his valuation.)

In table 1, we present some statistics of interest which summarize the simulation study. Specifically, we computed expected revenues under the different pricing rules as well as with different orders of approximating polynomials, the number of auctions that involved inefficient allocations, the number of times each bidder won the auction, as well as a measure of the *ex ante* equilibrium expected pay-off for bidder n for a given approximation. We computed the *ex ante* equilibrium expected pay-off using the following formula:

$$\mathbb{E}[U_n^{\text{FP}}(V)] = \int_v^{\bar{v}} [v - \hat{\varphi}_n^{-1}(v)] F_m(\hat{\varphi}_m[\hat{\varphi}_n^{-1}(v)]) f_n(v) dv$$

where $\hat{\varphi}_n(\cdot)$ corresponds to player n 's approximated equilibrium inverse-bid function.

We divide our discussion of the simulation results into three parts; expected revenues gained by the seller, inefficient allocations among buyers, and the *ex ante* equilibrium expected pay-offs of bidders.

Consider first expected revenues to the seller in EXAMPLE 2. It is interesting to compare expected revenues across pricing rules. A second-price auction yields an expected revenue of 0.3445. Note, too, from the entries in table 1 that, when the polynomial is of a low degree, the first-price auction appears more profitable, on average. On the other hand, when polynomials of higher order are used, the results are exactly the opposite.¹⁵ In summary, while expected revenues change only a bit quantitatively as K increases, in terms of expected-revenue ranking, the qualitative predictions are sensitive to changes in K .

Now, Vickrey [32] was able, analytically, to derive equilibrium strategies and expected revenues when F_1 is the uniform distribution and F_2 is degenerate. In this case, a first-price auction is more profitable, on average, than a second-price one, provided F_2 has all of its mass located at a point greater than 0.43. In particular, F_1 can be viewed as a mean-preserving spread over F_2 where the latter has all of its mass concentrated at 0.5. In table 2, we summarize some properties of this model (based on Vickrey's analysis, not simulations), as a counterpart to Table 1. Recall that, in our EXAMPLE 2, F_1 is also uniform and a mean-preserving spread over F_2 . However, in

¹⁵Incidentally, the same pattern emerges in EXAMPLE 3, where expected revenue is 0.3399 at second-price auctions.

Table 1: Results of Simulation Study with One Trillion Auctions

	Order of $\hat{\varphi}_n$	Expected Revenue	Proportion of Inefficiencies	Prop. Wins Player 1	Prop. Wins Player 2	$\mathbb{E}(U_1)$	$\mathbb{E}(U_2)$
EXAMPLE 2	$K = 3$	0.3563	0.0193	0.4807	0.5193	0.1553	0.1436
	$K = 4$	0.3473	0.0223	0.5209	0.4791	0.1645	0.1434
	$K = 5$	0.3458	0.0227	0.5089	0.4911	0.1634	0.1459
	$K = 10$	0.3431	0.0337	0.5028	0.4972	0.1635	0.1481
	$K = 15$	0.3432	0.0338	0.5027	0.4973	0.1634	0.1481
	$K = 20$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K = 25$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K = 30$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	SPA	0.3445	0.0000	0.5000	0.5000	0.1555	0.1555
EXAMPLE 3	$K = 3$	0.3475	0.0721	0.5491	0.4509	0.1568	0.1568
	$K = 4$	0.3452	0.0670	0.5287	0.4713	0.1584	0.1613
	$K = 5$	0.3364	0.0557	0.5041	0.4959	0.1605	0.1678
	$K = 10$	0.3334	0.0704	0.4964	0.5036	0.1632	0.1699
	$K = 15$	0.3330	0.0719	0.4947	0.5053	0.1634	0.1702
	$K = 20$	0.3328	0.0725	0.4944	0.5056	0.1635	0.1703
	$K = 25$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	$K = 30$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	SPA	0.3399	0.0000	0.4867	0.5133	0.1601	0.1733

Table 2: Vickrey's Example with F_1 a Mean-Preserving Spread over F_2

Pricing Rule	Expected Revenue	Probability Player 1 Wins	Probability Player 2 Wins	$\mathbb{E}(U_1)$	$\mathbb{E}(U_2)$
First-Price	0.3833	0.5625	0.4375	0.1677	0.0625
Second-Price	0.3750	0.5000	0.5000	0.1250	0.1250

EXAMPLE 2, for large K , we found that the second-price auction garners more revenue, on average, for the seller than the first-price auction. In this sense, Vickrey's result is not robust. While most expected-revenue comparisons in the literature concern distributions that can be ordered according to first-order stochastic dominance, EXAMPLE 2 illustrates that relaxing this assumption may prove interesting.

Fibich et al. [8] and Hubbard et al. [17] used perturbation analysis to examine auctions with "small" asymmetries and concluded that small departures from symmetry have the same first-order effect on revenue in both the first-price auction and the second-price auction. Gavious and Minchuk [10] then considered second-order effects. In their model, bidders' distributions were derived by slightly perturbing a uniform distribution. Although their examples all involve first-order stochastic dominance, the theory they developed does not require this. Consider, for example, an environment where

$$F_1(v) = v + \varepsilon v(1 - v) \left(\frac{1}{2} - v \right)$$

and

$$F_2(v) = v - \varepsilon v(1 - v) \left(\frac{1}{2} - v \right),$$

where the weakly-positive ε is “small.” In this case, $F_1(v)$ is a mean-preserving spread over $F_2(v)$ whenever ε is strictly positive and bidders are symmetric if ε equals zero—in which case the Revenue Equivalence Proposition applies. Theorem 2 in Gaviols and Minchuk [10], however, can be applied to show that a small increase in ε , starting at ε equal to zero, leads the second-price auction to become more profitable than the first-price auction. Thus, this theoretical example is consistent with the findings for EXAMPLE 2 of our simulation study.

Now consider inefficiencies. In both EXAMPLE 2 and EXAMPLE 3, the larger is K the larger is the quantitative effect on the efficiency properties of the first-price auction. Starting from a K of four (so dismissing a K of three as pathological, based on the analysis presented in EXAMPLE 1), the number of inefficiencies increases by fifty percent in EXAMPLE 2 as K grows to thirty. Note, too, that the number of inefficiencies is non-monotonic in K in EXAMPLE 3. Likewise, in EXAMPLE 3, small K also leads to incorrect predictions regarding which bidder is more likely to win the first-price auction.

Our examples also reveal some new insights into the *ex ante* equilibrium expected pay-offs of bidders and, hence, the preferences of bidders over the two pricing rules. Denote by μ_n the mean of F_n , $n = 1, 2$, and let $m \neq n$ denote bidder n ’s opponent. Bidder n ’s *ex ante* expected pay-off at a second-price auction is

$$\mathbb{E}[U_n^{\text{SP}}(V)] = \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^v (v - u) f_m(u) \, du \, f_n(v) \, dv$$

Now, by Leibniz’s rule,

$$\frac{d}{dv} \int_{\underline{v}}^v (v - u) f_m(u) \, du = (v - v) f_m(v) + \int_{\underline{v}}^v f_m(u) \, du = F_m(v).$$

Also, using integration-by-parts, we have

$$\begin{aligned} \mathbb{E}[U_n^{\text{SP}}(V)] &= \int_{\underline{v}}^{\bar{v}} \left[\int_{\underline{v}}^v (v - u) f_m(u) \, du \right] f_n(v) \, dv \\ &= \left[F_n(v) \int_{\underline{v}}^v (v - u) f_m(u) \, du \right]_{\underline{v}}^{\bar{v}} - \int_{\underline{v}}^{\bar{v}} F_n(v) F_m(v) \, dv \\ &= F_n(\bar{v}) \left[\int_{\underline{v}}^{\bar{v}} (\bar{v} - u) f_m(u) \, du \right] - \int_{\underline{v}}^{\bar{v}} F_n(v) F_m(v) \, dv \\ &= \bar{v} - \mu_m - \int_{\underline{v}}^{\bar{v}} F_n(v) F_m(v) \, dv. \end{aligned}$$

Hence,

$$\mathbb{E}(U_n^{\text{SP}}) - \mathbb{E}(U_m^{\text{SP}}) = \mu_m - \mu_n. \quad (12)$$

Thus, in EXAMPLE 2, at a second-price auction, in expectation, both bidders are equally well-off. From table 1, it is interesting to note that $\mathbb{E}[U_1^{\text{FP}}]$ exceeds $\mathbb{E}[U_2^{\text{FP}}]$. EXAMPLE 2 can easily be perturbed, by slightly increasing μ_2 , to generate a new example where $\mathbb{E}[U_1^{\text{SP}}]$ is less than $\mathbb{E}[U_2^{\text{SP}}]$, but where $\mathbb{E}[U_1^{\text{FP}}]$ still exceeds $\mathbb{E}[U_2^{\text{FP}}]$. Of course, the same argument also applies to Vickrey’s example of table 2. Nevertheless, we believe we are the first to point out that there

are environments in which the two standard auctions favour different bidders, in the sense that different bidders win.¹⁶

Continuing with expected utility, note that, in EXAMPLE 2, $\mathbb{E}(U_1^{\text{FP}}) + \mathbb{E}(U_2^{\text{FP}})$ exceeds $\mathbb{E}(U_1^{\text{SP}}) + \mathbb{E}(U_2^{\text{SP}})$. That is, collectively, the bidders prefer the first-price auction, while the seller prefers the second-price auction. Since the latter pricing rule is efficient, the “total pie” is larger, but this benefits the seller, not the bidders. In contrast, in Vickrey’s example of table 2, bidders collectively prefer the second-price auction to the first-price auction, while the opposite holds for the seller.

A final observation pertains to the probabilities of winning for each bidder. When F_1 is the uniform distribution, bidder 2’s *ex ante* probability of winning a second price auction is

$$\int_0^1 F_1(v)f_2(v) dv = \int_0^1 v f_2(v) dv = \mu_2.$$

In EXAMPLE 2, both μ_1 and μ_2 equal one-half, so the two bidders are *ex ante* equally likely to win the auction.¹⁷ Changing the pricing rule barely changes the probabilities of winning. However, although bidder 1’s probability of winning is only about one percent greater than bidder 2’s winning probability, $\mathbb{E}(U_1^{\text{FP}})$ is about ten percent greater than $\mathbb{E}(U_2^{\text{FP}})$. Thus, the *ex ante* probabilities of winning may not be a particularly good indication of the distribution of pay-offs among bidders. Indeed, it is tempting to speculate that in some environments the bidder who wins more often may be worse off, at least *ex ante*.

5. Summary and Conclusions

In a model of a first-price auction, if the draws of potential bidders are independent, but from different distributions, then a Lipschitz condition at the lower bound $\underline{v}$ is not satisfied by the system of differential equations that characterizes the equilibrium bid functions. In order to appeal to classical results concerning existence and uniqueness of a solution, the Lipschitz condition must hold; in order to apply many standard methods to solve the system of equations, it is convenient for the Lipschitz condition to hold. When the Lipschitz condition does not hold, solutions can typically only be calculated numerically. Thus, analyzing asymmetric first-price auctions is challenging—for theorists as well as for those who employ numerical methods to gain insight. Perhaps this is why numerical analysis can suggest directions for theoretical work; for example, Cantillon [4] “rationalized” many of the results discovered by Marshall et al. [25]. Theoretical work is also useful in informing numerical analysis; for example, our work uses many of the results of Kirkegaard [20]. Certainly, this interdependence will continue as researchers push the boundaries of understanding behaviour at asymmetric first-price auctions forward in both strands of the literature.

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¹⁶In Maskin and Riley [26], the weak bidder prefers the first-price auction to the second-price auction, while the strong bidder has the opposite preference. However, *ex ante*, the strong bidder is better off than the weak bidder under *both* pricing rules.

¹⁷The example in table 2 has the same property. Note, too, that in that example that $\mathbb{E}(U_1^{\text{SP}})$ equals $\mathbb{E}(U_2^{\text{SP}})$, which is consistent with equation (12).

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A. Appendix

Proof of Proposition 1. It is convenient to rewrite (2) as

$$\hat{\varphi}_n(s) = \sum_{k=0}^K \gamma_{n,k}(s - \underline{v})^k, \quad s \in [\underline{v}, \bar{s}], \quad (13)$$

which can be obtained by expanding the terms under the summation in (2), collecting terms with like exponents, and defining the appropriate coefficients. Note that $\gamma_{n,k}$ implicitly depends on $\bar{s}$.

Now, compare the equilibrium bid functions of two bidders n and m , $n \neq m$,

$$D_{n,m}(s) \equiv \hat{\varphi}_n(s) - \hat{\varphi}_m(s) = \sum_{k=0}^K (\gamma_{n,k} - \gamma_{m,k})(s - \underline{v})^k. \quad (14)$$

Condition 1a implies $D_{n,m}(\underline{v})$ is zero, so $\gamma_{n,0}$ must equal $\gamma_{m,0}$. Likewise, Property 2b implies $D'_{n,m}(\underline{v})$ is also zero, so $\gamma_{n,1}$ must also equal $\gamma_{m,1}$.¹⁸ Hence,

$$\begin{aligned} D'_{n,m}(s) &= \sum_{k=2}^K (\gamma_{n,k} - \gamma_{m,k})k(s - \underline{v})^{k-1} \\ &= (s - \underline{v}) \times \sum_{k=2}^K (\gamma_{n,k} - \gamma_{m,k})k(s - \underline{v})^{k-2}. \end{aligned}$$

The second term forms a polynomial of degree $(K - 2)$, which has $(K - 1)$ terms and whose coefficients can change signs at most $(K - 2)$ times. Therefore, by Descartes' rule of signs, the polynomial has at most $(K - 2)$ roots for (strictly) positive values of $(s - \underline{v})$. In other words, $D_{n,m}$ has at most $(K - 2)$ stationary points on $(\underline{v}, \bar{s}]$. By assumption (Property 2b) it also has a stationary point at $\underline{v}$. Recall, too, that $D_{n,m}(\underline{v})$ equals zero as does $D_{n,m}(\bar{s})$. Armed with these observations, the Proposition can now be proven.

The assumption that $f_m(\bar{v})$ does not equal $f_m(\bar{v})$ in combination with Property 2a imply that the two bidders use different strategies near the top. Thus, it can be ruled out that equilibrium bid functions coincide everywhere: $D_{n,m}(s)$ cannot equal zero everywhere. Therefore, by Descartes' rule of signs, $D_{n,m}$ has a finite number of roots; there can be no non-degenerate interval on which $D_{n,m}$ is zero. Let Q denote the total number of roots on $(\underline{v}, \bar{s})$, and let $s_{(q)}$ denote the q^{th} root, with $\underline{v} < s_{(1)} < s_{(2)} < \dots < s_{(Q)} < \bar{s}$ if Q exceeds zero. Since $D_{n,m}(\underline{v})$ equals zero as does $D'_{n,m}(\underline{v})$, $D_{n,m}(s)$ cannot equal zero on the interval $(\underline{v}, \underline{v} + \varepsilon)$. In order for $D_{n,m}(s_{(1)})$ to equal zero, $D'_{n,m}(s)$ equals zero for some $s \in (\underline{v}, s_{(1)})$. That is, one of the at most $(K - 2)$ interior stationary points is between $\underline{v}$ and $s_{(1)}$. Likewise, there must be at least one stationary point between $s_{(1)}$ and $s_{(2)}$,

¹⁸Thus, the formulation in (13) lends itself to easy tests of approximated equilibrium-bid functions as well.

and so on. Since there are at most $(K - 2)$ interior stationary points, there can be at most $(K - 2)$ roots on $(\bar{v}, \bar{s}]$. However, since there is a root at $\bar{s}$, because $D_{n,m}(\bar{s})$ equals zero, there can be at most $(K - 3)$ roots on $(\bar{v}, \bar{s})$. ■

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