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Dynamically Stable Preferences[☆]

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Abstract

In the framework of dynamic choice under uncertainty, we define dynamic stability as a combination of two assumptions prevalent in the literature: dynamic consistency and the requirement that updated preferences belong to the same class as ex ante ones. Maxmin preferences are shown to be not dynamically stable, and any dynamically stable subset in that class can contain only expected utility preferences. Dynamic stability also turns out to be a defining characteristic of the multiplier preferences of Hansen and Sargent (2001) within the scope of variational preferences. Restrictions imposed by dynamic stability are shown to be related to invariance of preferences.

Keywords: dynamic consistency, dynamic stability, ambiguity, invariance, consequentialism, Sure Thing Principle, multiplier preferences

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1. Introduction

The literature on choice under ambiguity, originating from the seminal work of Ellsberg (1961), aims to formalize the idea that a single prior probability distribution may not be sufficient to represent the attitudes of an agent towards uncertainty. In the last two decades, this literature became a mature branch of decision theory, to say the least. However, extending the theory of choice under ambiguity in a way that incorporates the gradual resolution of uncertainty in a dynamic setting is found rather challenging. Sensitivity to ambiguity appears to combine poorly with familiar dynamic choice features — in particular, dynamic consistency — that are appealing in the standard realm of expected utility. Consider, for example, a decision maker whose ex ante *and* conditional preferences belong to the class of maxmin preferences of Gilboa and Schmeidler (1989). As extensively argued in the literature, these preferences can be dynamically consistent only under very special circumstances (see Example 1 below). This has caused some authors to resort to weaker (or different) consistency conditions, while others have chosen to reject ambiguity averse preferences entirely.

In this work, we perform a systematic analysis of the difficulties surrounding ambiguity sensitive preferences in the dynamic framework, and point to the fact that they stem from *combining* two distinct assumptions. The first is dynamic consistency, which is salient in the literature on dynamic decision making, both in and out of the realm of ambiguity. This assumption, however, merely provides a link between ex ante and conditional preferences and is unrestrictive for ex ante preferences. Therefore, it alone does not create any tension with ambiguity. The second assumption, prevalent but rarely emphasized,¹ postulates that the conditional (updated) preferences preserve the structure of the ex ante ones. (To wit, if an ex ante preference relation belongs, for example, to the class of maxmin preferences, it is assumed that the conditional preferences are also maxmin.) Loosely speaking, we refer to this assumption as “dynamic stability;” that is, we say that a class of preferences is *dynamically stable* if dynamically consistent updating of any preference relation in this class with respect to any event yields a preference relation that also belongs to this class.

Unsurprisingly, dynamic stability is not an innocuous assumption. For instance, some popular classes of preferences (such as maxmin preferences) are *not* dynamically stable. At the same time, we find substantial variation in the degree of dynamic stability across various classes of preferences studied in the literature.

The lack of dynamic stability of maxmin preferences can be expected in light of the existing literature. We, however, can make a stronger claim: *Any dy-*

¹The notable exceptions are Gilboa and Schmeidler (1993), Eichberger and Kelsey (1996), Sarin and Wakker (1998), Hanany and Klibanoff (2009). We discuss the related literature in Section 6.

namically stable subclass of maxmin preferences can contain only expected utility preferences as the ex ante ones. This result provides motivation for seeking alternatives to dynamic consistency when maxmin preferences are placed in the dynamic context. Examples of other classes that are not dynamically stable include variational, vector expected utility, and confidence preferences. Nevertheless, not all classes of ambiguity sensitive preferences are alike, and there are well known classes of preferences that are dynamically stable, such as Hansen and Sargent’s (2001) multiplier preferences and uncertainty averse preferences of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011a).

Although the class of variational preferences is not dynamically stable, its dynamic properties differ from those of the class of maxmin preferences: It contains dynamically stable subclasses that extend beyond the standard expected utility preferences. The class of multiplier preferences turns out to be the largest dynamically stable subclass of variational preferences. More precisely: *If an ex ante preference relation belongs to the variational class, and all conditional preferences obtained from it in a dynamically consistent manner are also variational, then this ex ante preference relation must have the multiplier form.* Among other implications, this characterization provides a new behavioral axiomatization of the multiplier preferences.

What determines dynamic properties of preferences? We find that the attitude towards ambiguity, represented by uncertainty aversion or uncertainty loving, does not affect dynamic stability: If two classes of preferences differ only in whether or not a particular attitude towards ambiguity is assumed, they have the same structure of dynamically stable subclasses. At the same time, the classes of preferences listed earlier perform differently in the dynamic setting in spite of the fact that they are all uncertainty averse. The main difference among those classes of preferences is the type of independence axiom on which they are based, and, as we show, it is this difference that is responsible for them having different dynamic stability properties. We further link dynamic properties of those classes to invariance — a static property of preferences that we define later in the paper. Two special cases of invariance — constant absolute ambiguity aversion (invariance with respect to changes in payoff levels) and constant relative ambiguity aversion (changes in scale)² — are known well in the literature and represent attitudes towards ambiguity that are conceptually similar to the familiar CARA and CRRA attitudes towards risk. Among the classes of preferences listed earlier, variational preferences have a constant absolute ambiguity aversion property, confidence preferences have a constant relative ambiguity aversion property, and maxmin preferences have both of these properties. Going beyond these popular

²See, e.g., Grant and Polak (2011), Klibanoff, Marinacci and Mukerji (2005, Def. 6), and Strzalecki (2010, §2) for the definition of these concepts.

classes of preferences, we show that the dynamic stability assumption has very specific and strong implications for classes of preferences that are invariant. More precisely: *A class of preferences that is invariant to any sufficiently rich class of transformations can be dynamically stable only if it satisfies a quite restrictive postulate in the static setting — Savage’s Sure Thing Principle.* Moreover, preferences in such dynamically stable classes necessarily admit a certain additive representation.

The rest of the paper is organized as follows. Section 2 discusses our assumptions, formally defines the notion of dynamic stability, and presents two illustrative examples in Subsection 2.4. Section 3 derives dynamic stability properties of preferences that satisfy the Weak Certainty Independence Axiom, such as variational and vector expected utility preferences. The section also elaborates on dynamic stability as the defining property of multiplier preferences. Section 4 considers preferences that are based on other versions of the independence condition, such as maxmin preferences. Section 5 states our most general result connecting dynamic stability to invariance of preferences, consequentialism, and the Sure Thing Principle. Finally, Section 6 surveys the related literature, elaborates on the relationship between ambiguity and dynamic consistency, and concludes with a discussion about tradeoffs in modeling ambiguity in the dynamic setting and extensions of our results to other frameworks.

2. Setup

2.1. Framework

We adopt here a slight extension of the classical Anscombe-Aumann framework. In what follows, we let Z stand for the set of outcomes, a nonempty closed and convex subset of a topological vector space,³ and set Ω to be a finite state space with $|\Omega| \geq 3$. From the decision maker’s point of view, the members of Z^Ω represent state-contingent payoff vectors, and we will refer to them simply as “acts” in what follows.

The dynamics in this setting are represented by gradual resolution of uncertainty in the form of the information that an event E from $\mathcal{E}_0 := 2^\Omega \setminus \{\emptyset, \Omega\}$ occurs. There is no explicit time in the model, and the elements of Z represent the final (and only) outcomes for the decision maker. As is fairly common in the literature on preference updating, we assume that the state space is finite to avoid the complications that would result from conditioning on null events.⁴ Our

³In the classical Anscombe-Aumann framework, Z would be the collection of all Borel probability measures on a Polish space of (riskless) outcomes. Thus, one can view members of Z here as generalizations of roulette wheel lotteries.

⁴In the standard theory of Bayesian updating, this corresponds to restricting attention to discrete probability distributions and leaving out the Bayes rule for densities.

objects of study are preference relations (complete preorders) on Z^Ω and Z^E for each $E \in \mathcal{E}_0$. Let $\mathcal{E} := \mathcal{E}_0 \cup \{\Omega\}$. A preference relation $\succeq$ on Z^E , where $E \in \mathcal{E}$, is called **degenerate** if $f \sim g$ for all $f, g \in Z^E$,⁵ and nondegenerate otherwise.

As usual, for $E \in \mathcal{E}_0$, $f \in Z^E$, and $h \in Z^{\Omega \setminus E}$, we denote by $f E h$ the act in Z^Ω such that $(f E h)(\omega) = f(\omega)$ if $\omega \in E$ and $(f E h)(\omega) = h(\omega)$ if $\omega \notin E$. We also define null events in the standard way: For any given preference relation $\succeq$ on Z^Ω , an event $E \in \mathcal{E}_0$ is called **$\succeq$ -null** if $f E h \sim g E h$ for all $f, g \in Z^\Omega$ and $h \in Z^{\Omega \setminus E}$.

Finally, for $E \in \mathcal{E}$, we let $\Delta(E)$ denote the set of all probability distributions on E .

2.2. Dynamic Stability

Our notion of dynamic stability builds upon two main ingredients: dynamic consistency and preservation of the structure of preferences.

Dynamic consistency is broadly understood as the willingness of the decision maker to carry out plans that were chosen ex ante, and it is by far the most frequently discussed concept in the literature on dynamic choice. In the expected utility realm, dynamic consistency may be thought of as a manifestation of rationality, in contrast to behavioral traits such as temptation. In the context of ambiguity, it is not viewed unequivocally (see, in particular, Siniscalchi, 2009a,b). Nevertheless, given the central role of dynamic consistency in the literature and a number of normative and practical arguments in its favor, it is surely a concept worth investigating, at least to the extent of what can be achieved under ambiguity.

As a matter of fact, without additional qualifications, dynamic consistency does not restrict the ex ante preferences: There always exists a system of conditional preferences that is dynamically consistent with a given ex ante preference relation for all possible events. Indeed, as we formalize in Subsection 2.3, conditional preferences can be *defined* through dynamic consistency: Given that a certain event occurs, one may require a decision maker to compare two acts by looking at what he would have chosen ex ante. Consequently, dynamic consistency alone does not bring much discipline to the picture. What is often desirable besides dynamic consistency is retaining certain structural properties for the conditional preferences. In fact, most of the literature on ambiguity in the dynamic setting makes strong assumptions in this respect, presuming that the conditional preferences have the *same* structure as the ex ante ones. So, loosely put, our main goal here is to study the extent to which preference relations from well known classes satisfy dynamic consistency in addition to maintaining their ex ante structure under conditioning. We refer to such preferences as *dynamically stable*.

⁵Here, as usual, $\sim$ stands for the symmetric (indifference) part of $\succeq$.

We now formalize this concept.

Definition 1. A **class of preferences** $\mathcal{P}$ is a set of the form $\bigcup_{E \in \mathcal{E}} \mathcal{P}^E$, where each $\mathcal{P}^E$ is a nonempty collection of preferences on Z^E .

In words, a class of preferences is a collection of preferences, where each of the preference relations compares either acts defined on the grand state space Ω or acts defined on one of its nonempty subsets. A preference relation over acts defined on a subset E of the state space can be thought of as representing the decision maker's preferences when he knows that the true state of nature belongs to E . We will refer to preferences over acts defined on the entire state space Ω as the **ex ante** preferences in a given class of preferences.

As mentioned earlier, we impose dynamic consistency by means of *defining* conditional preferences through this property, as in Machina and Schmeidler (1992).

Definition 2. Let $\succsim$ be a preference relation on Z^Ω for some nonempty finite set Ω , let E be a nonempty proper subset of Ω , and $h \in Z^{\Omega \setminus E}$ (counterfactual payoffs). A **conditional** of $\succsim$ on E given h (denoted by $\succsim_{E,h}$) is a preference relation over Z^E defined as

$$f \succsim_{E,h} g \iff f E h \succsim g E h \quad (1)$$

for all $f, g \in Z^E$.

Since we do not consider violations of dynamic consistency in this paper, we use the term conditional preferences (conditionals) to refer exclusively to preferences obtained from the ex ante ones through dynamic consistency. Relegating the discussion of this definition to Subsection 2.3, we next define the dynamically stable classes of preferences.

Definition 3. A class of preferences $\mathcal{P}$ is said to be **dynamically stable** if the conditional of any ex ante preference $\succsim$ in this class on E given h belongs to $\mathcal{P}$ for all $E \in \mathcal{E}_0$ and $h \in Z^{\Omega \setminus E}$.

To reiterate, a dynamically stable class of preferences is, per force, dynamically consistent and, in addition, it is “closed under conditioning.”⁶ Depending on the choice of the class of preferences, the latter property captures the preservation of the “structure” of the ex ante preferences in that class by the conditional

⁶Although our definition applies conditioning only to preferences from $\mathcal{P}^\Omega$, it is easy to see that, given any $\succsim \in \mathcal{P}^\Omega$, the preference relation $(\succsim_{E,h})_{E',h'}$ — a further update of a conditional preference relation $\succsim_{E,h}$ — is identical to $\succsim_{E',(h' E h)}$ and will belong to $\mathcal{P}$ for any nonempty $E' \subsetneq E \subsetneq \Omega$, $h \in Z^{\Omega \setminus E}$, and $h' \in Z^{E \setminus E'}$ if $\mathcal{P}$ is dynamically stable.

Figure 1: Decision tree conditional on learning event $\{\omega_1, \dots, \omega_k\}$
 Square nodes labeled “D” are the decision points; circular nodes represent resolution of uncertainty.

preferences. For instance, if a class of ex ante preferences is defined through a particular functional form and is dynamically stable, then the functional form remains the same for the updated preferences. This is surely convenient for applications, because the related decision problems become stationary and well suited for analysis by dynamic programming. Dynamic stability is also a reasonable assumption from the axiomatic point of view. If one believes in a certain set of axioms and assumes that the ex ante preferences satisfy them, then it is natural to maintain the same set of axioms for the updated preferences. [Epstein and Le Breton \(1993, p. 10\)](#) argue for this principle in application to one particular axiom, Savage’s P4:

It makes little sense to impose P4 on $\succeq$ but not on its updates; indeed, $\succeq$ itself is presumably an updated version of some “earlier” preference ordering.

Clearly, this argument is equally appealing for any structural property for preference relations. The explicit requirement that conditional preferences have the same properties as ex ante ones is also key in [Gilboa and Schmeidler \(1993\)](#), [Eichberger and Kelsey \(1996\)](#), [Sarin and Wakker \(1998\)](#), [Hanany and Klibanoff \(2009\)](#), as discussed in Section 6. Equipped with the notion of dynamic stability, our goal in this paper is twofold: to study the restrictions that the requirement of dynamic stability imposes on ex ante preferences; and to find which classes of preferences are admissible from the point of view of this requirement.

2.3. Dynamic Consistency and Consequentialism

As we stated earlier, our definition of “conditionals” is based on the principle of dynamic consistency. We postulate that, conditional on learning that the true state of the world belongs to E , the comparison of two alternatives f and g agrees with the ex ante ranking of acts $f E h$ and $g E h$ for a suitable h . As illustrated in Figure 1, this postulate reflects the idea that choices in the lower part of the tree should follow ex ante plans, since the ex ante view of the payoff vectors offered by the left and the right branches of the tree are exactly $f E h$ and $g E h$.

Relationship (1) in Definition 2 represents one possible formalization of dynamic consistency and can be traced back to the work of Savage. Machina and Schmeidler (1992) and Epstein and Le Breton (1993) also employ the idea of using dynamic consistency as a definition of conditional preferences, rather than a link between two exogenously given relations. A formalization close to (1) is used by Ghirardato (2002) and Ghirardato, Maccheroni and Marinacci (2008). Gilboa and Schmeidler (1993) refer to rule (1) as the h -Bayesian update rule.

Definition 2 has a noteworthy feature: The conditional preferences have subscripts corresponding to payoffs in states that are known to be impossible. This means that we allow the decision maker to violate Hammond’s (1988) principle of consequentialism. Hammond argued that “regrets, sunk costs, even the structure of the decision tree itself” should be irrelevant for choices (or, if they are, they should be accounted for in the outcomes). Consequentialism can be imposed on preferences by requiring that all conditionals $\succsim_{E,h}$ be, in fact, independent of h .⁷ In our setting, consequentialism is equivalent to a static property of preferences — Savage’s Sure Thing Principle.

Although consequentialism is a postulate that is frequently assumed together with dynamic consistency, we do not want to restrict attention only to settings where it holds. One reason for looking beyond consequentialism is that it loses part of its normative appeal outside of the realm of expected utility preferences, as argued extensively by, for example, Machina (1989) and Hanany and Klibanoff (2007). Nevertheless, even though we do not impose consequentialism at the outset of our analysis, it may arise in some cases as an *implication* of dynamic stability. This endogenous appearance of consequentialism, discussed in Section 6, provides some new insight into its relationship with dynamic consistency, and it emphasizes the strength of dynamic stability as a behavioral assumption.

Taking stock, we keep our focus on two postulates in tandem — dynamic consistency and the assumption that the ex ante and conditional preferences belong to the same class — without imposing any additional assumptions. These two requirements alone have strong implications, and our primary goal is to understand what kind of structure they impose on preferences.

2.4. Two Examples

To fix ideas, we next provide two examples of classes of preferences, one being dynamically stable and the other not.

⁷Considering the system of conditional preferences as indexed by a pair of an event and a vector of counterfactual payoffs, (E, h) , permits a wider range of possible violations of consequentialism in comparison, for example, with the setup of Ghirardato et al. (2008). There, conditional preferences are defined over Z^Ω , indexed by an event only, and $\Omega \setminus E$ can, in principle, be non-null for the preference relation $\succsim_E$. Our setting is slightly more general because we do not require the decision maker to have well-defined and transitive preferences over acts such as $f E h'$ and $g E h''$ ($h' \neq h''$) conditional on an event E .

Example 1. For any $E \in \mathcal{E}$, a preference relation $\succsim$ on 2^E is said to have a **maxmin representation** (or that it is a maxmin preference) if there exist a nonconstant and affine function $u : Z \rightarrow \mathbb{R}$ and a nonempty, closed and convex subset M of $\Delta(\Omega)$ such that

$$f \succsim g \quad \Leftrightarrow \quad \min_{\mu \in M} \int_{\Omega} (u \circ f) d\mu \geq \min_{\mu \in M} \int_{\Omega} (u \circ g) d\mu.$$

The other way of saying this is that $\succsim$ can be represented by a mapping

$$f \mapsto \min_{\mu \in M} \int_{\Omega} (u \circ f) d\mu.$$

Let $\mathcal{P}^E$ stand for the set of all preference relations on Z^E that admit a maxmin representation. Then, the collection $\mathcal{P} = \bigcup_{E \in \mathcal{E}} \mathcal{P}^E$ is called the **class of maxmin preferences**. Now we want to illustrate (and prove later in Proposition 4) that this class is not dynamically stable.

To see this, consider the case where $Z = \mathbb{R}$ and $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Consider a preference relation on Z^{Ω} that admits a maxmin representation with a linear utility function u and the set of priors $M = \{(1/3, 1/3 - \alpha, 1/3 + \alpha) : -\alpha^* \leq \alpha \leq \alpha^*\}$. Set $E = \{\omega_1, \omega_2\}$, let h_x stand for the map $\{\omega_3\} \mapsto x$ for any real number x , and observe that the conditional $\succsim_{\{\omega_1, \omega_2\}, h_x}$ on Z^E is represented by the functional

$$\tilde{V}(f) = \frac{1}{3}f(\omega_1) + \left(\frac{1}{3} + \alpha^*\right)\min(f(\omega_2), x) + \left(\frac{1}{3} - \alpha^*\right)\max(f(\omega_2), x).$$

Clearly, for any fixed $x \in \mathbb{R}$, preference relation $\succsim_{\{\omega_1, \omega_2\}, h_x}$ does not admit a maxmin representation because it is not homothetic and, hence, does not satisfy the Certainty Independence Axiom.

It is worth emphasizing that in our definition of dynamic stability, we require *all* conditionals to belong to the same class. The above example demonstrates that, *in general*, this requirement is violated for maxmin preferences. Nevertheless, it is possible that certain maxmin preference relations have maxmin conditionals relative to *some* events or counterfactual payoffs.⁸

Contrary to the class of maxmin preferences, the class of expected utility preferences is obviously dynamically stable. There also exist classes of preferences that are strictly ambiguity averse and dynamically stable at the same time. One such class is the class of multiplier preferences, introduced by Hansen and Sargent (2001).

⁸The former case is studied in Sarin and Wakker (1998) and Epstein and Schneider (2003), and the latter in Gilboa and Schmeidler (1993). We compare our results to theirs in Subsection 4.1 and Section 6.

Recall that, for any $E \in \mathcal{E}$, a preference relation $\succeq$ on Z^E is said to be a **multiplier preference relation** if it is represented by a mapping

$$f \mapsto \min_{\mu \in \Delta(E)} \left(\int_E (u \circ f) d\mu + \theta R(\mu \parallel q) \right), \quad (2)$$

where $u : Z \rightarrow \mathbb{R}$ is nonconstant and affine, $q \in \Delta(E)$ has full support,⁹ $\theta \in (0, \infty]$, and R (relative entropy or Kullback-Leibler distance) is defined as

$$R(\mu \parallel q) = \int_E \ln \left(\frac{d\mu}{dq} \right) d\mu,$$

if $\mu, q \in \Delta(E)$ are such that μ is absolutely continuous with respect to q , and $+\infty$ otherwise. The probability distribution q in this representation is called the **reference prior**. When $\theta = +\infty$, the multiplier preference representation reduces to

$$f \mapsto \int_E (u \circ f) q(d\omega),$$

which corresponds to the case of expected utility preferences.

Example 2. For any $E \in \mathcal{E}$, let $\mathcal{P}^E$ stand for the set of all multiplier preference relations on Z^E . We refer to the collection $\mathcal{P} = \bigcup_{E \in \mathcal{E}} \mathcal{P}^E$ as the **class of multiplier preferences**. We claim that this class *is* dynamically stable.

To see this, let $\succeq$ be a preference relation on Z^Ω that admits a multiplier preference representation with the utility index v , parameter $\vartheta \in (0, \infty]$, and reference prior $p \in \Delta(\Omega)$. To check whether its conditionals are multiplier preferences, let the real map $V_E(\cdot; u, \theta, q)$ be the multiplier preference representation on Z^E , where $E \in \mathcal{E}$, with the parameters u , θ , and q , that is,

$$V_E(f; u, \theta, q) := \min_{\mu \in \Delta(E)} \left(\int_E (u \circ f) d\mu + \theta R(\mu \parallel q) \right).$$

Take any $E \in \mathcal{E}_0$, $f \in Z^E$, $h \in Z^{\Omega \setminus E}$, and observe that $V_\Omega(f E h; v, \vartheta, p)$ can be rearranged to obtain

$$\begin{aligned} V_\Omega(f E h; v, \vartheta, p) = \min_{\pi \in (0,1)} & \left(V_E(f; v, \vartheta, p \mid E) \pi + \right. \\ & V_{\Omega \setminus E}(h; v, \vartheta, p \mid (\Omega \setminus E)) (1 - \pi) + \\ & \left. \vartheta \ln \frac{\pi}{p(E)} + \vartheta \ln \frac{1 - \pi}{1 - p(E)} (1 - \pi) \right). \end{aligned}$$

By the Envelope Theorem, this expression is strictly increasing in $V_E(f; v, \vartheta, p \mid E)$. Therefore, we have $V_\Omega(f E h; v, \vartheta, p) \geq V_\Omega(g E h; v, \vartheta, p)$ if and only if $V_E(f; v, \vartheta, p \mid E) \geq V_E(g; v, \vartheta, p \mid E)$ for every $f, g \in Z^E$. Consequently, $\succeq_{E,h}$ is represented by $V_E(\cdot; v, \vartheta, p \mid E)$, and, thus, it belongs to the class of multiplier preferences.

⁹That is, $q(\{\omega\}) > 0$ for all $\omega \in \Omega$.

Motivated by these examples, the rest of the paper investigates the notion of dynamic stability in a more systematic way.

3. Preferences Satisfying Weak Certainty Independence

3.1. Dynamically Stable Subclasses

We now turn to presenting our results about dynamic stability of specific classes of preferences, starting with preferences satisfying the Weak Certainty Independence Axiom of [Maccheroni, Marinacci and Rustichini \(2006a\)](#). Although this weakening of the Independence Axiom is not the first one studied in the literature on ambiguity averse preferences, this class of preferences gives nontrivial results that are best at illustrating the present approach.

To be precise, we consider preference relations on Z^E , where $E \in \mathcal{E}$, that satisfy the following axioms.¹⁰

Axiom (Nondegeneracy). There exist $f, g \in Z^E$ such that $f \succ g$.

Axiom (Mixture Continuity). The sets $\{\alpha \in [0, 1] : \alpha f + (1 - \alpha)g \succeq h\}$ and $\{\alpha \in [0, 1] : h \succeq \alpha f + (1 - \alpha)g\}$ are closed for any $f, g, h \in Z^E$.

Axiom (Monotonicity). For all $f, g \in Z^E$, if $f(\omega) \succeq g(\omega)$ for all $\omega \in E$, then $f \succeq g$.

Axiom (Weak Certainty Independence). For all $f, g \in Z^E$, $x, y \in Z$, and $\alpha \in (0, 1]$,

$$\alpha f + (1 - \alpha)x \succeq \alpha g + (1 - \alpha)x \quad \Rightarrow \quad \alpha f + (1 - \alpha)y \succeq \alpha g + (1 - \alpha)y.$$

Axiom (Uncertainty Aversion). For all $f, g \in Z^E$ and $\alpha \in (0, 1)$, $f \sim g$ implies $\alpha f + (1 - \alpha)g \succeq f$.

A preference relation that satisfies these five axioms is said to be **variational**. As [Maccheroni et al. \(2006a\)](#) showed, a preference relation satisfies these axioms if and only if it can be represented by a mapping

$$f \mapsto \min_{\mu \in \Delta(\Omega)} \left(\int_{\Omega} (u \circ f) d\mu + c(\mu) \right),$$

where $u : Z \rightarrow \mathbb{R}$ is a nonconstant and affine function representing preferences over pure risk, and $c : \Delta(\Omega) \rightarrow [0, \infty]$ is a convex and lower semicontinuous function with $\inf c(\Delta(\Omega)) = 0$. Clearly, this representation generalizes that of maxmin preferences and multiplier preferences. For subsequent discussion, we define a **class of variational preferences** as the collection $\mathcal{P} = \bigcup_{E \in \mathcal{E}} \mathcal{P}^E$, where each

¹⁰As is common in the literature, we abuse notation by identifying $x \in Z$ with a constant act from Z^E taking the value x in all states of the world.

Figure 2: Ex ante decision tree

$\mathcal{P}^E$ is the collection of all preferences on Z^E that satisfy the five axioms stated above.

The following proposition identifies the implications of imposing this set of axioms jointly on the ex ante preference $\succsim$ and its conditionals.

Proposition 1. *Let $\succsim$ be a variational preference relation on Z^Ω . If $\succsim_{E,h}$ is also a variational preference relation for each $E \in \mathcal{E}_0$ and $h \in Z^{\Omega \setminus E}$, then $\succsim$ must be a multiplier preference relation.*

An immediate implication of this proposition is that the class of variational preferences is not dynamically stable. Moreover, preferences from the variational class, whose conditionals are variational as well, necessarily belong to the relatively small class of multiplier preferences.

One way to explain the gist of Proposition 1 is by using the language of decision trees.¹¹ We postulate that:

1. A variational preference relation governs the decision maker's choices in all trees that involve decisions at the ex ante stage only (as illustrated in Figure 2);
2. he is dynamically consistent in all *simple* (one-stage) conditional choice trees (see Figure 1, Section 2.3); and
3. his conditional preferences in the simple trees are also variational.

Then, Proposition 1 says:

1. The decision maker is dynamically consistent in arbitrary trees (in the sense that his ex ante choice coincides with the outcome of the backward induction process using conditional preferences at each node);
2. he is a consequentialist; and
3. his preferences actually have a very special form: They are multiplier preferences.

¹¹See, e.g., Siniscalchi (2009a) for a formal introduction of trees for the analysis of decisions in the presence of ambiguity.

Proposition 1 complements Example 2, which has demonstrated that conditionals of any preference relation from the class of multiplier preferences retain their functional form, thereby remaining within the same class. The proposition above establishes the converse: There are no other preferences with this property within the class of variational preferences. Informally, the two points — dynamic stability of the class of multiplier preferences and the relationship between the classes of multiplier and variational preferences — can be summarized as follows: *The class of multiplier preferences is the largest dynamically stable subclass of the class of variational preferences.*

3.2. An Axiomatization of Multiplier Preferences

The results of the previous subsection can be viewed from two perspectives — as describing the properties of either the variational preferences or the multiplier preferences. In this subsection, we elaborate on the second perspective.

Until the recent axiomatization by Strzalecki (2011), it was not entirely clear whether multiplier preferences have any behavioral grounds and/or normative appeal beyond technical convenience. Our results show that, in fact, these preferences occupy a distinguished position from the dynamic perspective: They correspond to those variational preferences whose conditionals are dynamically consistent and continue to be variational.

In addition to demonstrating the salience of the multiplier preferences, Proposition 1 provides a foundation for a new axiomatic characterization of these preferences in a static setting. Indeed, the requirement that all conditional preferences satisfy the Weak Certainty Independence Axiom can be condensed to the following condition.

Axiom (Conditional Weak Certainty Independence). For any $E \in \mathcal{E}$, $f, g \in Z^E$, $h \in Z^{\Omega \setminus E}$, $x, y \in Z$, and $\alpha \in (0, 1)$,

$$(\alpha f + (1 - \alpha)x) E h \succsim (\alpha g + (1 - \alpha)x) E h \implies (\alpha f + (1 - \alpha)y) E h \succsim (\alpha g + (1 - \alpha)y) E h.$$

A reformulation of Proposition 1, thus, reads as follows:

Corollary 2. *A preference relation $\succsim$ on Z^Ω satisfies the Nondegeneracy, Mixture Continuity, Monotonicity, Conditional Weak Certainty Independence, and Uncertainty Aversion Axioms if and only if it is a multiplier preference relation.*

The above result complements Strzalecki's (2011) axiomatic foundation of multiplier preferences. In his characterization, Strzalecki combines Savage's Sure Thing Principle (Axiom P2) with the axioms characterizing variational preferences. By contrast, we do not explicitly impose P2 or any other new axiom.

Instead, we modify one of the axioms (namely, Weak Certainty Independence) to ensure that it holds on all subsets of the state space in addition to the entire space, thereby guaranteeing dynamic stability. Proposition 1, then, yields the sought multiplier preference representation.

3.3. Vector Expected Utility Preferences

Among preferences satisfying the Weak Certainty Independence Axiom, another interesting type of preferences, called vector expected utility (VEU) preferences, was introduced by Siniscalchi (2009c). The collection of such preferences is neither a subset nor a superset of variational preferences. The main axiom characterizing the vector expected utility preferences is the Complementary Independence Axiom, which imposes the independence condition on the so-called complementary acts. For a preference relation $\succsim$ on Z^E , where $E \in \mathcal{E}$, such acts are defined as follows.

Definition 4. Two acts $f, \bar{f} \in Z^E$ are *complementary* if and only if, for any two states $\omega, \omega' \in E$,

$$\frac{1}{2}f(\omega) + \frac{1}{2}\bar{f}(\omega) \sim \frac{1}{2}f(\omega') + \frac{1}{2}\bar{f}(\omega').$$

If two acts $f, \bar{f} \in Z^E$ are complementary, then $(f, \bar{f})$ is referred to as a *complementary pair*.

The exact statement of Siniscalchi's axiom is as follows.

Axiom (Complementary Independence). For any two complementary pairs $(f, \bar{f})$ and $(g, \bar{g})$ in Z^E , and $\alpha \in [0, 1]$, $f \succsim \bar{f}$ and $g \succsim \bar{g}$ imply $\alpha f + (1-\alpha)g \succsim \alpha \bar{f} + (1-\alpha)\bar{g}$.

In comparison with the postulates of Proposition 1, the Complementary Independence Axiom adds a further restriction on preferences and delivers the following result.

Proposition 3. Let $\succsim$ be a preference relation on Z^Ω that satisfies the Nondegeneracy, Mixture Continuity, Monotonicity, Weak Certainty Independence, and Complementary Independence Axioms. If $\succsim_{E,h}$ also satisfies these axioms for each $E \in \mathcal{E}$ and $h \in Z^{\Omega \setminus E}$, then $\succsim$ is an expected utility preference relation.

The Complementary Independence Axiom, therefore, substantially changes the extent to which preferences are dynamically stable. As the characterization of the VEU preferences includes the axioms listed in Proposition 3, we conclude that the largest dynamically stable subclass within the class of VEU preferences is just the class of expected utility preferences.

4. Preferences Satisfying Other Versions of the Independence Axiom

4.1. Strengthening of the Axiom

In this section, we study the dynamic stability of some preference classes that are based on other types of weakenings of the classical Independence Axiom. The natural starting point in this inquiry is the Certainty Independence Axiom introduced by Gilboa and Schmeidler (1989).

Axiom (Certainty Independence). For any $f, g \in Z^E$, $x \in Z$, and $\alpha \in (0, 1)$,

$$f \succeq g \quad \Rightarrow \quad \alpha f + (1 - \alpha)x \succeq \alpha g + (1 - \alpha)x.$$

As Gilboa and Schmeidler (1989) show, the class of maxmin preferences, defined in Example 1 through its representation, can equivalently be defined as the collection of preferences $\mathcal{P} = \bigcup_{E \in \mathcal{E}} \mathcal{P}^E$, where each $\mathcal{P}^E$ consists of preferences on Z^E that satisfy the Nondegeneracy, Mixture Continuity, Monotonicity, Certainty Independence, and Uncertainty Aversion Axioms.

Our next result shows that *any* class of preferences satisfying the Certainty Independence Axiom, with or without the Uncertainty Aversion Axiom,¹² can be dynamically stable only if all ex ante preferences in this class have the expected utility form.

Proposition 4. *Let $\succeq$ be a preference relation that satisfies the Nondegeneracy, Mixture Continuity, Monotonicity, and Certainty Independence Axioms. If $\succeq_{E,h}$ satisfy the same axioms for each $E \in \mathcal{E}_0$ and $h \in Z^{\Omega \setminus E}$, then $\succeq$ is an expected utility preference relation.*

As an immediate corollary, we observe that the class of maxmin preferences is not dynamically stable, and that its largest dynamically stable subset is the class of expected utility preferences. Thus, our Example 1, illustrating the failure of conditional preferences to have maxmin form, is not special in any respect. For any maxmin preference relation with a non-neutral attitude towards ambiguity, there exists a simple decision tree such that the conditional preference relation does not belong to the class of maxmin preferences.¹³

Proposition 4 provides an additional insight into the rectangularity condition of Epstein and Schneider (2003). This condition restricts jointly a maxmin preference relation and a partition of the state space, and guarantees the existence

¹²Preferences satisfying the Certainty Independence Axiom but not necessarily the Uncertainty Aversion Axiom are studied in Ghirardato, Maccheroni and Marinacci (2004a) and Amara (2009).

¹³This property of maxmin preferences was noted in Eichberger and Kelsey (1996), who proved a similar result for Choquet expected utility preferences.

of conditionals with respect to the cells of this partition that are dynamically consistent and maxmin. More precisely, let $\succsim$ be a maxmin preference relation on Z^Ω , $M_\succsim \subseteq \Delta(\Omega)$ be the set of priors in the multiprior representation of $\succsim$, and $\Pi = \{E_1, \dots, E_k\}$ be a partition of Ω . The set of priors is said to be **rectangular** with respect to Π if

$$M_\succsim = \left\{ \sum_{i=1}^k \mu(E_i) \mu_i(\cdot \mid E_i) \mid \mu, \mu_1, \dots, \mu_k \in M_\succsim \right\}.$$

Theorem 3.2 in Epstein and Schneider (2003) implies that (i) the preference relation $\succsim$ whose set of priors $M_\succsim$ is rectangular has conditionals $\succsim_{E_i, h}$ for all $i = 1, \dots, k$ that are maxmin and independent of h ; and (ii) the sets of priors representing these conditionals can be computed as $M_{\succsim_{E_i, h}} = \{\mu(\cdot \mid E_i) \mid \mu \in M_\succsim\}$.¹⁴ The following simple corollary of our Proposition 4 shows that when one starts with a maxmin preference relation that has a rectangular set of priors with respect to some partition Π and that does not have the expected utility form, then rectangularity *necessarily* fails with respect to some unions of the cells of Π , or, put differently, with respect to some coarsenings of Π .

Corollary 5. *Let $\Pi = \{E_1, \dots, E_k\}$ be a partition of Ω with $k \geq 3$, and $\succsim$ be a preference relation that admits a maxmin representation via a utility function $f \mapsto \min_{\mu \in M} \int_\Omega (u \circ f) d\mu$ with a nonconstant and affine function $u : Z \rightarrow \mathbb{R}$ and a set of beliefs $M \subseteq \Delta(\Omega)$ that satisfies the rectangularity condition with respect to Π , and such that every measure in M has full support. Then, either*

- (i) *the restriction of $\succsim$ to acts measurable with respect to Π is an expected utility preference relation, or*
- (ii) *there exists a coarsening Π' of Π such that M does not satisfy the rectangularity condition with respect to Π' .*

Proof. Let $\succsim'$ be a preference relation defined on $Z^{\{1, \dots, k\}}$ by

$$f' \succsim' g' \Leftrightarrow \sum_{i=1}^k f'(i) \mathbb{1}_{E_i} \succsim \sum_{i=1}^k g'(i) \mathbb{1}_{E_i} \quad \text{for all } f', g' \in Z^{\{1, \dots, k\}}.$$

Clearly, $\succsim'$ admits a maxmin representation with the utility function u and the set of priors $M' := \{\sum_{i=1}^k \mu(E_i) \delta_i \mid \mu \in M\}$.¹⁵ If M satisfies the rectangularity

¹⁴Note, however, that $\succsim_{E, h}$ need not be maxmin for $E \subsetneq \Omega$ if $E \notin \Pi$. Moreover, fixing $E \in \Pi$ and $h \in Z^{\Omega \setminus E}$, it is quite possible that the priors $M_{\succsim_{E, h}}$ from the representations of the conditional $\succsim_{E, h}$, in turn, do not satisfy the rectangularity condition with respect to any partition of E .

¹⁵As usual, δ_i denotes a probability measure assigning weight one to state i .

condition with respect to all coarsenings of Π , then, for every nonempty $S \subset \{1, \dots, k\}$, $\succeq'$ has a conditional on S that is dynamically consistent, independent of the counterfactual payoffs, and admits a maxmin representation with the utility function u and the set of priors $\left\{ \sum_{i \in S} \frac{\mu(E_i)}{\sum_{j \in S} \mu(E_j)} \delta_i \mid \mu \in M \right\}$, as follows from [Epstein and Schneider \(2003, Theorem 3.2\)](#). Since all conditionals of $\succeq'$ turn out to be maxmin, [Proposition 4](#) implies that $\succeq'$ must be an expected utility preference relation. $\square$

4.2. Incomparable Versions of the Independence Axiom

[Chateauneuf and Faro \(2009\)](#) introduce another class of preferences satisfying the Uncertainty Aversion Axiom that they call confidence preferences. Dynamic properties of these preferences turn out to be similar to the ones of variational preferences: Although the class of confidence preferences is not dynamically stable, it contains a nontrivial dynamically stable subclass that is precisely the intersection of this class with the class of Second-Order Expected Utility (SOEU) preferences. In order to formally present this result, we first introduce two axioms defining this class.

Axiom (Worst Independence). There exists $x_* \in Z$ is such that $f \succeq x_*$ for all $f \in Z^E$, and, for all $f, g \in Z^E$ and $\alpha \in (0, 1)$,

$$f \sim g \quad \Rightarrow \quad \alpha f + (1 - \alpha)x_* \sim \alpha g + (1 - \alpha)x_*.$$

Axiom (Bounded Attraction for Certainty). There exists $\delta \geq 1$ such that for all $f \in Z^\Omega$ and $x, y \in Z$,

$$x \sim f \quad \Rightarrow \quad \frac{1}{2}x + \frac{1}{2}y \succeq \frac{1}{2}f + \frac{1}{2}\left(\frac{1}{\delta}y + \left(1 - \frac{1}{\delta}\right)x_*\right).$$

The **class of confidence preferences** is defined as the collection of preferences $\mathcal{P} = \bigcup_{E \in \mathcal{E}} \mathcal{P}^E$, where each $\mathcal{P}^E$ is the collection of all preferences on Z^E that satisfy the Nondegeneracy, Mixture Continuity, Monotonicity, Uncertainty Aversion, Risk Independence, Worst Independence and Bounded Attraction for Certainty Axioms. These preferences can be represented by the mapping

$$f \mapsto \min_{\substack{\mu \in \Delta(\Omega): \\ \phi(\mu) \geq \alpha_0}} \frac{1}{\phi(\mu)} \int_{\Omega} (u \circ f) d\mu,$$

where $u : Z \rightarrow \mathbb{R}_+$ is a nonconstant and affine function such that $u(x_*) = 0$, $\alpha_0 \in (0, 1]$ is the minimal “confidence level,” $\phi : \Delta(\Omega) \rightarrow [0, 1]$ is a confidence function that satisfies certain technical conditions (see [Chateauneuf and Faro \(2009\)](#) for details).

We are now ready to state our result about dynamic stability of this class.

Proposition 6. *Let $\succeq$ be a confidence preference relation on Z^Ω . If $\succeq_{E,h}$ is also a confidence preference relation for each $E \in \mathcal{E}$ and $h \in Z^{\Omega \setminus E}$, then there exist a nonconstant and affine function $u : Z \rightarrow \mathbb{R}_+$, $\mu \in \Delta(\Omega)$ with full support, and $\gamma \in (0, 1]$ such that $\succeq$ has a utility representation via the mapping*

$$f \mapsto \int_{\Omega} (u \circ f)^\gamma d\mu. \quad (3)$$

Preferences admitting representation (3) constitute a subset of Second-Order Expected Utility (SOEU) preferences — i.e., preferences having representation

$$f \mapsto \int_{\Omega} \varphi(u \circ f) d\mu$$

for some nonconstant and affine function $u : Z \rightarrow \mathbb{R}$, strictly increasing function $\varphi : u(Z) \rightarrow \mathbb{R}$, and probability measure $\mu \in \Delta(\Omega)$. Moreover, as Grant, Polak and Strzalecki (2009) observe, the set of preferences having representation (3) is exactly the intersection of the classes of confidence and SOEU preferences. Hence, the class of confidence preferences has the property that its largest dynamically stable subclass is exactly the intersection of this class with the SOEU preferences, and this property is common between confidence and variational preferences (see Proposition 13 and the proof of Proposition 1 in the Appendix).

4.3. Weakening of the Independence Axiom

It is also interesting to inquire how Proposition 1 would change if we used, instead, a condition less restrictive than the Weak Certainty Independence Axiom. The weakest such condition proposed in the literature is the Risk Independence Axiom, which is essentially the standard Independence Axiom imposed only on constant acts:

Axiom (Risk Independence). For any $x, y, z \in Z$ and $\alpha \in (0, 1)$

$$x \sim y \quad \Rightarrow \quad \alpha x + (1 - \alpha)z \sim \alpha y + (1 - \alpha)z.$$

The class of ambiguity averse preferences corresponding to this form of independence is studied in Cerreia-Vioglio et al. (2011a). To present our result about dynamic stability properties of this class, we also need to introduce the following strengthening of the standard monotonicity axiom.

Axiom (Strict Monotonicity). For any $f, g \in Z^E$, if $f(\omega) \succeq g(\omega)$ for all $\omega \in E$, then $f \succeq g$; if, in addition, $f(\omega) > g(\omega)$ for some $\omega \in E$, then $f > g$.

Proposition 7. *For any $E \in \mathcal{E}$, let $\mathcal{P}^E$ stand for the set of all preference relations on Z^E that satisfy the Mixture Continuity, Strict Monotonicity, Risk Independence, and Uncertainty Aversion Axioms. Then, the class of preferences $\mathcal{P} = \bigcup_{E \in \mathcal{E}} \mathcal{P}^E$ is dynamically stable.*

Dynamic stability of broad classes of preferences such as the uncertainty averse preferences of Cerreia-Vioglio et al. (2011a) should not be regarded as a surprise — after all, the class of all transitive binary relations is obviously dynamically stable as well. From this perspective, Proposition 7 establishes, in the first place, that the Risk Independence axiom is already permissive enough to achieve dynamic stability, and that other axioms listed in the proposition do not interfere with it.¹⁶

Summary

The results in Sections 3–4 about the dynamic stability of known classes of preferences can be summarized using the following diagram, which depicts mappings from classes of preferences to their largest dynamically stable subclasses.¹⁷

The above diagram illustrates our claim that the classes of ambiguity averse preferences suggested in the literature have quite different dynamic properties. Preferences in the top row of the diagram — expected utility, maxmin, variational, and uncertainty averse — are nested, and their axiomatic foundations differ from one another by progressive weakening of the independence condition.

Our results also demonstrate that the attitude towards ambiguity in the form of uncertainty aversion does not interfere with dynamic consistency and does not play a role in dynamic properties of preferences. Proposition 7 shows that even a stronger property — dynamic stability — is compatible with uncertainty aversion plus a few standard postulates. In fact, if one considers a class of preferences characterized solely by the Uncertainty Aversion Axiom, it can easily be shown that such a class of preferences is also dynamically stable. At the same time, the differences between dynamic properties of preferences appear to be completely unrelated to uncertainty aversion. Moreover, dropping the uncertainty aversion

¹⁶The fact that the Uncertainty Axiom alone does not interfere with dynamic stability is observed in Eichberger and Kelsey (1996).

¹⁷To keep this diagram simple, we assume that the definitions of all classes of preferences depicted include the Strict Monotonicity Axiom. More precisely, we illustrate the relationships between strictly monotone maxmin preferences, strictly monotone variational preferences, and so on. Adding the strict monotonicity assumption does not change the relationships between classes of preferences established in Propositions 1 and 4.

postulate from the list of axioms characterizing preferences discussed above will keep the relationships between classes of preferences intact, as illustrated by the diagram below (and proven by Proposition 4 in Subsection 4.1, as well as by Proposition 13 and Lemma 12 in the Appendix).¹⁸

This picture of classes of preferences and their relationships is by no means exhaustive. The classes of preferences that we selected for the analysis share a lot in terms of their axiomatic foundations, and, as the next section makes clear, we have a common tool to study models that have that or similar structure. Nevertheless, our analysis does not include such important classes as the class of smooth ambiguity preferences of [Klibanoff et al. \(2005\)](#). It would be interesting and useful to know their dynamic stability properties as well; this, however, has to be left outside of the scope of this paper.

5. Dynamic Stability and Invariance

What is the exact role of the type of the independence condition in the dynamic stability of a class of preferences? The weakest form of independence condition known in the literature — Risk Independence — does not have implications beyond the decision maker's choices in situations of pure risk. However, other independence conditions — Worst Independence, Weak Certainty Independence, and Certainty Independence — imply *invariance* of preferences with respect to certain transformations of acts. One example of this kind of invariance is constant absolute ambiguity aversion of preferences, postulating that the attitude of the decision maker towards ambiguity does not change when the state-contingent payoffs (or, more precisely, utility levels) are raised uniformly across all the states. Although invariance is a property of preferences that has a purely static nature, we argue that it is, in fact, the main driving force behind our results for variational and maxmin preferences. It is this property that makes the inheritance of the structure of preferences such a restrictive assumption.

¹⁸Classes of preferences listed in the top row of this diagram are studied in [Ghirardato, Maccheroni and Marinacci \(2004b\)](#), [Grant and Polak \(2011\)](#), [Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci and Siniscalchi \(2011b\)](#).

Informally, the invariance of a preference relation with respect to a certain transformation of payoffs can be understood as follows. Consider two acts f and g , and assume that the state-contingent payoffs of those acts are represented by their respective utility levels rather than by physical prizes. Now, transform the payoffs in each of the states using a real-valued map. Then, the preference relation is said to be invariant with respect to the given transformation if the decision maker prefers the transformed vector f' over the transformed vector g' whenever he prefers f over g , and this holds for all possible pairs of f and g . Maxmin, variational, and confidence preferences are all invariant to certain transformations, and the distinction among these classes of preferences lies in the functional form of these transformations.

Formally, we consider preference relations on Z^E , where $E \in \mathcal{E}$, that satisfy the basic assumptions of Nondegeneracy, Mixture Continuity, Monotonicity, and Risk Independence. Note that this system of axioms is very weak and holds for all preferences studied in the literature on ambiguity, including the class of uncertainty averse preferences of Cerreia-Vioglio et al. (2011a). Its main implication is the existence of a von Neumann-Morgenstern utility function that represents the restriction of preferences to constant acts: There exists a nonconstant and affine function $u_{\succsim} : Z \rightarrow \mathbb{R}$ such that for any $x, y \in Z$, $x \succsim y \Leftrightarrow u_{\succsim}(x) \geq u_{\succsim}(y)$. In the standard interpretation of the Anscombe-Aumann framework, $u_{\succsim}$ embodies the decision maker's attitude towards pure risk.

Now, let $\mathcal{A}$ be a group of real line automorphisms; that is, $\mathcal{A}$ is a nonempty collection of bijections from $\mathbb{R}$ to $\mathbb{R}$ such that

- (i) if $A, B \in \mathcal{A}$, then $A \circ B \in \mathcal{A}$, and
- (ii) if $A \in \mathcal{A}$, then $A^{-1} \in \mathcal{A}$.

For any $x \in \mathbb{R}$ and $A \in \mathcal{A}$, we will frequently use the parenthesis-free notation Ax instead of $A(x)$.

Definition 5. Given $E \in \mathcal{E}$, a preference relation $\succsim$ on Z^E with an affine utility representation u over constant acts, is said to be **invariant** with respect to a group $\mathcal{A}$ of real line automorphisms if

$$f \succsim g \quad \Leftrightarrow \quad u^{-1} \circ A \circ u \circ f \succsim u^{-1} \circ A \circ u \circ g \quad (4)$$

for all $f, g \in Z^E$ and $A \in \mathcal{A}$ such that $(A \circ u \circ f)(\omega), (A \circ u \circ g)(\omega) \in u(Z)$ for all $\omega \in E$.¹⁹

¹⁹First, note that the utility function $u : Z \rightarrow \mathbb{R}$ in Definition 5 is generally not an injection. However, if Condition (4) holds for some choice of preimages, it will hold for any such

Put differently, take a preference relation $\succeq$ on Z^E with a utility representation u over constant acts, and consider the induced preference relation $\succeq^*$ defined on $u(Z)^E$ through an equivalence $f \succeq g \Leftrightarrow (u \circ f) \succeq^* (u \circ g)$. We say that $\succeq$ is $\mathcal{A}$ -invariant if $\tilde{f} \succeq^* \tilde{g}$ implies $A\tilde{f} \succeq^* A\tilde{g}$, and this implication holds for all $\tilde{f}, \tilde{g} \in u(Z)^E$ and $A \in \mathcal{A}$.

Now, the Weak Certainty Independence Axiom of [Maccheroni et al. \(2006a\)](#) can be translated into the language of invariance of preferences by saying that the preference relation is invariant with respect to the collection of mappings $\mathcal{A}_v := \{x \mapsto x + b \mid b \in \mathbb{R}\}$.²⁰ Similarly, the Worst Independence Axiom of [Chateauneuf and Faro \(2009\)](#) corresponds to positive homogeneity of preferences — i.e., invariance with respect to $\mathcal{A}_h := \{x \mapsto kx \mid k > 0\}$ — and the Certainty Independence Axiom of [Gilboa and Schmeidler \(1989\)](#) implies both types of invariance simultaneously, yielding invariance of preferences with respect to $\mathcal{A}_{hv} := \{x \mapsto kx + b \mid k > 0, b \in \mathbb{R}\}$. Note that the three examples of collections of mappings, $\mathcal{A}_v$, $\mathcal{A}_h$, and $\mathcal{A}_{hv}$, clearly have the structure of a group, satisfying the two conditions stated in the definition of a group of automorphisms.

Our next result brings together dynamic stability and invariance with respect to abstract groups of automorphisms. First, though, we need to introduce one technical condition to ensure that the groups we consider are sufficiently rich.

Let $\mathcal{A}$ be a group of real line automorphisms, $K \subset \mathbb{R}$ a nonempty compact set, and $\varepsilon > 0$. Then, let

$$\mathcal{A}_{K,\varepsilon} := \{A \in \mathcal{A} : \sup_{x \in K} |Ax - x| < \varepsilon\}.$$

In other words, $\mathcal{A}_{K,\varepsilon}$ denotes an ε -ball in $\mathcal{A}$ around the identity function, where the size of the ball is computed in the sup-norm on K . Finally, for any $x \in \mathbb{R}$, let $\mathcal{A}_{K,\varepsilon}x := \{Ax \mid A \in \mathcal{A}_{K,\varepsilon}\}$.

Definition 6. A group of real line automorphisms $\mathcal{A}$ is said to **cover** $S \subseteq \mathbb{R}$ if for any nonempty compact $K \subset \text{int } S$, any $\varepsilon > 0$, and any $x \in \text{int } S$, there exists an open interval $N_x \ni x$ such that $\mathcal{A}_{K,\varepsilon}x \supseteq N_x$.

choice: Indeed, the basic axioms that we imposed on preferences guarantee that, given any acts $h_1, h'_1, h_2, h'_2 \in Z^E$ for some $E \in \mathcal{E}$, if $(u \circ h_1)(\omega) = (u \circ h'_1)(\omega)$ and $(u \circ h_2)(\omega) = (u \circ h'_2)(\omega)$ for all $\omega \in E$, then $h_1 \succeq h_2 \Leftrightarrow h'_1 \succeq h'_2$.

Second, an affine utility function representing the restriction of $\succeq$ to constant acts is unique only up to a positive affine transformation. To claim that $\succeq$ is invariant according to our definition, it is sufficient to find one utility function u and one group $\mathcal{A}$ such that Condition (4) holds. Note, however, that if one takes an $\mathcal{A}$ -invariant preference relation having the utility representation u over constant acts, but then chooses another utility representation over constant acts, $\tilde{u} = L \circ u$ where $L : \mathbb{R} \rightarrow \mathbb{R}$ is positive affine, then $\succeq$ will clearly be invariant to $\tilde{\mathcal{A}} := \{L \circ A \circ L^{-1} \mid A \in \mathcal{A}\}$.

²⁰[Maccheroni et al. \(2006a\)](#) used the term “vertical invariance” for the corresponding invariance property of functionals representing the preferences.

As an illustration, consider the collection of mappings $\mathcal{A}_{hv} = \{x \mapsto kx + b \mid k > 0, b \in \mathbb{R}\}$. It is easy to verify that this group covers any interval, including the entire real line: Take an arbitrary $\varepsilon > 0$, $x \in \text{int } S$, and let $N_x := \{y \in \mathbb{R} : |x - y| < \varepsilon\}$. Then, for an arbitrary $y \in N_x$, the automorphism $z \mapsto z + y - x$ from $\mathcal{A}$ lies within the ε distance from the identity function on any compact $K \subset \mathbb{R}$, and does the job of mapping x to y , proving that $\mathcal{A}_{K,\varepsilon}x \supseteq N_x$.

We are now ready to present our main theorem, which establishes necessary and sufficient conditions for inheritance of properties of an ex ante preference relation by its conditionals, assuming that the preferences are invariant to some transformations.

Theorem 8. *Suppose that $\succeq$ is a preference relation that satisfies the Nondegeneracy, Mixture Continuity, Monotonicity, and Risk Independence Axioms, and let $u : Z \rightarrow \mathbb{R}$ be its utility representation over constant acts. Suppose that $\mathcal{A}$ is a group of real line automorphisms that covers $u(Z)$, and suppose that $\succeq$ is $\mathcal{A}$ -invariant. Then, the following conditions are equivalent.*

- (i) *Conditional preferences $\succeq_{E,h}$ satisfy the Nondegeneracy, Mixture Continuity, Monotonicity, and Risk Independence Axioms and are $\mathcal{A}$ -invariant for all $E \in \mathcal{E}_0$ and $h \in Z^{\Omega \setminus E}$.*
- (ii) *$\succeq$ is consequentialist and satisfies the Strict Monotonicity Axiom.*
- (iii) *There exists a function $V : u(Z) \times \Omega \rightarrow \mathbb{R}$, strictly increasing and continuous in the first argument, such that the mapping $Z^\Omega \rightarrow \mathbb{R}$ defined as*

$$f \mapsto \sum_{\omega \in \Omega} V((u \circ f)(\omega), \omega) \quad (5)$$

is a utility representation for $\succeq$.

- (iv) *There exist a function $V : u(Z) \times \Omega \rightarrow \mathbb{R}$, strictly increasing and continuous in the first argument, and functions $\alpha : \mathcal{A} \rightarrow \mathbb{R}_{++}$ and $\beta : \mathcal{A} \times \Omega \rightarrow \mathbb{R}$ such that mapping (5) is a utility representation for $\succeq$, and*

$$V(Ax, \omega) = \alpha(A)V(x, \omega) + \beta(A, \omega)$$

for all $\omega \in \Omega$, $A \in \mathcal{A}$, and all $x \in u(Z)$ such that $Ax \in u(Z)$.

Statement (i) of the theorem postulates that the conditional preferences have the same properties as the ex ante ones — namely, invariance with respect to the same set of transformations (as well as the basic axioms of the Anscombe-Aumann framework). Therefore, this condition reflects the idea of the inheritance of properties by conditional preferences, which is the essence of dynamic stability. Of course, there may be other properties of ex ante preferences besides invariance

that a modeler might find desirable for the conditionals to inherit. Nevertheless, we restrict our attention solely to invariance because this is the property that drives our earlier results characterizing the largest dynamically stable subclasses within the classes of maxmin, variational, and confidence preferences.

Statement (ii) of the theorem consists of two parts — the essential one (Consequentialism), and a technical one (Strict Monotonicity). We defined consequentialism earlier as the property that $\succsim_{E,h} = \succsim_{E,h'}$ for all $E \in \mathcal{E}_0$, and $h, h' \in Z^{\Omega \setminus E}$. In the current setting, it is equivalent to Savage's Sure Thing Principle (Axiom P2), which requires that

$$f E h \succsim g E h \quad \Rightarrow \quad f E h' \succsim g E h'$$

for any $E \in \mathcal{E}_0$, $f, g \in Z^E$, and $h, h' \in Z^{\Omega \setminus E}$. The fact that Statement (i) implies (ii) is by far the most important part of Theorem 8. In particular, it implies that any dynamically stable class that consists of invariant preferences can contain only preferences that satisfy the Sure Thing Principle. The Sure Thing Principle, in turn, is a substantial restriction on admissible preferences, and imposes a lot of structure even when the attention is restricted to static choices. In other words, the Sure Thing Principle limits the types of behavior that can be captured by the preferences and, essentially, restricts the modeler's choice since only a few classes of preference studied in the ambiguity literature satisfy this principle.

The technical part of Statement (ii) of the theorem — the Strict Monotonicity — is not a substantial condition. Its role in the theorem is limited to making Statements (i) and (ii) exactly equivalent, since strict monotonicity follows from nondegeneracy of conditional preferences.

Statements (iii) and (iv) of the theorem are relatively simple observations: The implication from (ii) to (iii) has been known since [Debreu \(1960\)](#), and the equivalence of Statements (iii) and (iv) is intuitive, given our background assumption of invariance of ex ante preferences. Nevertheless, we state these conditions here for the purpose of giving a more complete picture of the implications of dynamic stability. They establish that preferences in dynamically stable and invariant classes have a certain structure in terms of their representation, as well: Such preferences have to admit an additive representation with state-dependent utilities. Moreover, these state-dependent utilities have to satisfy certain functional equations derived from the group of automorphisms in question. These functional equations are able to pin down the precise representation of preferences for the classes of preferences studied earlier in the paper. If a preference relation is invariant to the group of affine transformations (as a result of imposing the Certainty Independence Axiom), the only solutions to the corresponding functional equation are the linear functions, and representation (5) turns into the expected utility representation. Invariance to $\mathcal{A}_v$ (resulting from The Weak Certainty Independence Axiom) gives either linear or exponential functions as the

solutions, and the corresponding dynamically stable and invariant class becomes a subclass of the Second-Order Expected Utility preferences that either admit an expected utility representation or use an exponential function as the transformation applied to utility-acts — i.e., are represented by a mapping $f \mapsto \int_{\Omega} e^{\gamma(u \circ f)} d\mu$ for some nonconstant and affine $u : Z \rightarrow \mathbb{R}$, $\gamma \neq 0$, and $\mu \in \Delta(\Omega)$.²¹ Restricting attention further to uncertainty averse preferences in that subclass, the theorem implies that any dynamically stable subclasses of the class of variational preferences can contain only multiplier preferences as the ex ante ones.²²

Statements (ii)–(iv) of our theorem can also be viewed as sufficient conditions for dynamic stability. As such, they make it clear, for instance, that the class of multiplier preferences not only contains dynamically stable subclasses, but is itself dynamically stable (cf. Example 2). The sufficient conditions can also be stated in a slightly different and more general manner that does not require invariance as a background assumption: Suppose that $u : Z \rightarrow \mathbb{R}$ is nonconstant and affine and that $V : u(Z) \times \Omega \rightarrow \mathbb{R}$ is strictly increasing and continuous in its first argument. Then, the collection of preferences on Z^E that admit a utility representation by the mapping

$$f \mapsto \sum_{\omega \in E} V((u \circ f)(\omega), \omega)$$

for all $E \in \mathcal{E}$, is dynamically stable.

Theorem 8 derives relatively strong implications (such as the Sure Thing Principle) out of two nontechnical assumptions: dynamic stability and the invariance of preferences. To see the intuition behind this result, the key is to observe that these assumptions imply the invariance of preferences when transformations are applied only to *some* components of the vectors of state-contingent payoffs. First, invariance of conditional preferences implies that the comparison of two acts of the form $f E h$ and $g E h$ remains unchanged if the f and g parts of these acts are transformed using an automorphism from group $\mathcal{A}$, but the h part is left intact. In other words, if $f E h \succeq g E h$, then $(Af) E h \succeq (Ag) E h$ for all $A \in \mathcal{A}$. Second, since the entire acts $f E h$ and $g E h$ can also be transformed preserving the comparison, and since each automorphism in $\mathcal{A}$ has an inverse, this implies that the comparison of acts $f E h$ and $g E h$ is preserved if the h part is transformed

²¹Grant et al. (2009) provide the most relevant analysis of the Second-Order Expected Utility preferences for this discussion.

²²This result can also be derived from Statement (ii) of our theorem (the Sure Thing Principle) using the axiomatization of the multiplier preferences of Strzalecki (2011). However, the characterization of the largest dynamically stable subclasses cannot be obtained by using Statement (ii) and the Strzalecki's (2011) argument for classes of preferences for which the Uncertainty Aversion Axiom does not hold. Nevertheless, Statement (iv) of the theorem can still be used in those cases (see Proposition 13 in the Appendix).

instead, and the f and g parts are left intact. In other words, if $f E h \succeq g E h$, then $f E (Ah) \succeq g E (Ah)$ for all $A \in \mathcal{A}$. Finally, when the group $\mathcal{A}$ is sufficiently rich, this implies that the h part in the pair of acts $f E h$ and $g E h$ can simply be replaced with an arbitrary $h' \in Z^{\Omega \setminus E}$ without changing how the acts rank, which is exactly what the Sure Thing Principle postulates.

6. Discussion

Related Literature

Recent literature on ambiguity in the dynamic setting, such as Epstein and Schneider (2003), Hanany and Klibanoff (2007), Eichberger, Grant and Kelsey (2007), Ghirardato et al. (2008), Al-Najjar and Weinstein (2009), leaves one with the impression that there is an intrinsic tension between ambiguity aversion and dynamic consistency. For instance, dynamic consistency of a system of maxmin preferences can be guaranteed only in a few special cases. The rectangularity condition of Epstein and Schneider (2003), for example, describes one such case in terms of a joint restriction on a set of priors of a maxmin preference relation and a partition of the state space. Under the additional assumption of consequentialism, this condition is both necessary and sufficient for maxmin conditional preferences that are defined on the cells of the fixed partition to be dynamically consistent with an ex ante maxmin preference relation.²³ This condition is quite useful for creating examples of systems of preferences that are dynamically consistent in the case of a fixed information structure. Nevertheless, only a very limited subset of all maxmin preferences satisfies this condition with respect to nontrivial partitions. In particular, the rectangularity condition does not hold in the standard formalization of the Ellsberg three-color choice problem. The results in Subsection 4.1 show when rectangularity cannot be achieved: For instance, we argue that for any ex ante preference relation and any partition of the state space with at least three cells, there exists a coarsening of that partition with respect to which rectangularity fails.

A condition similar to the rectangularity condition of Epstein and Schneider (2003) is described in an earlier paper of Sarin and Wakker (1998). They recognize that desirability of using the same functional form of the utility function at different stages constitutes an independent postulate, and introduce the notion of sequential consistency: “Sequential consistency requires that if a decision maker has committed to a family of models (e.g., the rank-dependent family), then he should use that family of models throughout.” This postulate is then applied to the maxmin model in the context of uncertainty and a number of other models in

²³The approach of Epstein and Schneider (2003) has also been applied to other classes of preferences (see, e.g., Maccheroni, Marinacci and Rustichini, 2006b, and Siniscalchi, 2010).

the context of risk. Overall, the approach of [Sarin and Wakker \(1998\)](#) is similar to the one of [Epstein and Schneider \(2003\)](#): they study a decision maker within a fixed information structure, whereas the object of analysis in this paper is a class of preferences. Their most relevant result says that both dynamic consistency and sequential consistency of a maxmin decision maker can be achieved with a rectangular set of priors, which gives a reduced version of one of the directions of [Epstein and Schneider \(2003, Theorem 3.2\)](#).

[Gilboa and Schmeidler \(1993\)](#) present a different special case of dynamic consistency, and, using our terminology, their Theorem 3.3 can be re-stated as follows. Suppose that preferences $\succeq$ are maxmin, and the set of outcomes has a $\succeq$ -maximal element $\bar{z} \in Z$. Then, preferences $\succeq_{E,h}$ for $h(\omega) \equiv \bar{z}$ on $\Omega \setminus E$ are maxmin for all $E \in \mathcal{E}_0$. Closest to our results are the ones of [Eichberger and Kelsey \(1996\)](#), who study the Choquet expected utility model and prove that dynamically consistent updating of such preferences does not lead to preferences satisfying the same axioms (unless the preferences are expected utility ones).

Given the sparseness of positive results, the literature seems to have turned to studying various weakenings of the dynamic consistency requirement. [Hanany and Klibanoff \(2007\)](#) simultaneously relax dynamic consistency and consequentialism to allow updating rules to depend on the feasible sets of acts. Their conditional preferences preserve ex ante maximal elements in closed convex sets of acts, and are not required to respect ex ante ordering otherwise. More precisely, they consider arbitrary collections of maxmin preferences $\succeq_{E,g,B}$, where E is an event, g is an act, and B is a closed and convex set of acts. These preferences are deemed to be consistent with an ex ante preference relation $\succeq$ if for any act g that is a $\succeq$ -maximal act in B , it is $\succeq_{E,g,B}$ -maximal in B , as well. Indexing conditional preferences by (E, g, B) and their definition of consistency also permits a wider range of departures from consequentialism than our setting. They further develop these ideas in [Hanany and Klibanoff \(2009\)](#), in which they explicitly seek the “closure” of a model of preferences in updating. With that aim, they suggest functional forms of updating rules for a number of classes of preferences that jointly achieve the closure and their weaker version of the consistency requirement. [Eichberger et al. \(2007\)](#) postulate Conditional Certainty Equivalent Consistency, which is essentially a restriction of the dynamic consistency requirement to the case when the counterfactual payoff vector is a constant equal to the certainty equivalent of the act, and find the functional form of the corresponding updating rule for Choquet Expected Utility preferences. [Ghirardato et al. \(2008\)](#) argue that “dynamic consistency is a compelling property only for comparisons of acts that are not affected by the possible presence of ambiguity.” Restricting attention to preferences satisfying the Certainty Independence Axiom, they derive auxiliary “unambiguously preferred” preference relations, and impose dynamic consistency only on the latter.

In a different branch of the literature, Siniscalchi (2009a) opts to reject the dynamic consistency assumption in the context of uncertainty/ambiguity. He instead formalizes dynamic choice over “decision trees” that is based on backward induction reasoning. The resulting theory enables an interesting discussion about the value of information and commitment. Al-Najjar and Weinstein (2009) argue that the entire concept of ambiguity should be normatively rejected because of the difficulties in combining the Ellsberg-type behavior with dynamic consistency and consequentialism at the same time.

In this paper, we investigate when it is possible to achieve dynamic consistency in its classical form and without imposing consequentialism at the outset. We start with observing that ambiguity does not create a tension with dynamic consistency and then analyze what other common assumptions may impede or promote dynamic consistency when imposed alongside this property. By contrast to many of the cited papers, we do not restrict our attention to one particular class of preferences. Rather, we view our strengthening of dynamic consistency — dynamic stability — as a tool to discriminate among various classes of preferences. On that front, we find which known classes of preferences are dynamically stable, and we show what can be achieved in terms of dynamic stability within classes of preferences that are not dynamically stable in their entirety.

Dynamic Stability, Consequentialism, and Invariance

The most intriguing finding of our paper is the fact that consequentialism may sometimes arise as an implication of combining dynamic consistency with the assumption that updated preferences have the same form as ex ante ones. Whether this implication holds or not depends on how broadly the notion of the same form is interpreted. More precisely, this implication holds if the definition of the form of preferences includes invariance, such as constant absolute ambiguity aversion. On the other hand, if the notion of the form of preferences is interpreted very broadly, and includes, for example, only the Uncertainty Aversion Axiom, then the assumptions of dynamic consistency and inheritance of the form do not impose any restrictions on the ex ante preferences.

The interpretation of our result depends on whether consequentialism is viewed as a desirable or an undesirable property. On one hand, it may be viewed as desirable because it holds for conventional expected utility preferences and makes calculations in applied work much easier. However, it may be undesirable because, for example, it makes the accommodation of dynamically consistent choices in the three-color Ellsberg Paradox quite difficult (if not impossible).²⁴ In the first case, our result implies that the postulate of consequentialism does not

²⁴Subsection 2.3 provides references to other arguments against adopting consequentialism in the context of ambiguity.

necessarily require special attention, since it follows from other assumptions. In the second case, undesirability of consequentialism turns invariance of preferences into a problematic property.

On a more general level, we consider four properties of preferences that turn out to be relevant in the dynamic setting:

- dynamic consistency;
- consequentialism (or, equivalently, Savage’s Sure Thing Principle);
- inheritance of the structure of preferences by conditionals;
- invariance (implied by assuming one of the well known classes of preferences).

Our Theorem 8 shows that not all of the 2^4 ways to combine these properties can be realized. Many of our earlier propositions can be thought of as manifestations of impossibilities implied by that theorem. For instance, the lack of dynamic stability of maxmin preferences reflects the impossibility of combining the two dynamic properties — dynamic consistency and inheritance of the structure of preferences by conditionals — with the intrinsic static properties of maxmin preferences: These preferences are invariant and, generally, violate the Sure Thing Principle.

Our results clarify the modeler’s choices in bringing ambiguity into the dynamic context: One option is to use preference classes that are found to be dynamically stable and consequentialist at the same time, such as the class of multiplier preferences. The other option, as indicated by Theorem 8, is to drop either the dynamic consistency assumption or the assumption that conditional preferences have the same form as ex ante ones. Finally, the third option is to switch from the popular classes of preferences, such as maxmin or variational, to preferences for which invariance does not hold.

Extensions

We conclude the paper with a discussion about the dependence of our results on the framework assumptions and their possible extensions.

The Anscombe-Aumann framework that we use makes specific assumptions about the structure of the space of outcomes (“roulette wheel lotteries”), as well as the decision maker’s evaluation of outcomes (the Risk Independence axiom). Since the main goal of these postulates is to simplify the analysis rather than to improve the descriptive power of the theory, one may wonder whether the analysis of this paper can be adapted to the Savagean framework. For instance, consider the class of preferences over acts X^E , where $E \in \mathcal{E}$ and X is an arbitrary metric space, that admit a utility representation via the mapping $f \mapsto \min_{\mu \in M} \int_E (u \circ$

$f)d\mu$, where $u : X \rightarrow \mathbb{R}$ is a nonconstant and continuous utility index, and $M \subseteq \Delta(\Omega)$ is a nonempty, closed, and convex set of beliefs. Now one may ask whether ex ante preferences and their dynamically consistent conditionals can simultaneously have this form. The answer to this question can be obtained in the same way as in the Anscombe-Aumann framework using our Theorem 8 under one additional assumption — the assumption that the ex ante and conditional preferences share the same utility index u , which is consistent with the idea that information should affect beliefs but not tastes.

Completeness of preferences is another background assumption that we made without any discussion. However, our approach is applicable to incomplete preferences as well. For instance, it is easy to see that the class of preferences described by the Bewley's (1986) model is dynamically stable. The main general result of our paper — the observation that dynamic stability and invariance imply consequentialism and the Sure Thing Principle — also holds for incomplete preferences that are complete in ranking constant acts and closed continuous over utility acts.²⁵

Finally, the ideas of this paper can be re-cast in a very different framework of preferences over consumption bundles rather than uncertainty.²⁶ There, in parallel to the relationship between ex ante and conditional preferences, one might be interested in the relationship between preferences over all possible products (total consumption of a household) and preferences over a small subset of those products (i.e., selection in one grocery store). “Dynamic stability” in that context would mean that choices (preferences) in a subset of products are consistent with the preferences over all consumption bundles, and that preferences in the smaller domain have the same structure as preferences in the larger one. A re-cast of our Theorem 8 would describe preferences satisfying these two properties if their structure includes the property of invariance (e.g., homotheticity).

Appendix A. Proofs

Appendix A.1. Proof of Theorem 8

In our proofs, we will operate with the spaces of so-called utility acts, $\mathbb{R}^\Omega$ or $\mathbb{R}^E$ (for $E \in \mathcal{E}_0$), instead of spaces of Anscombe-Aumann acts Z^Ω or Z^E , respectively. This step is formalized by the following lemma.

Lemma 9. *Take $E \in \mathcal{E}$, and suppose that $\succsim$ is a preference relation on Z^E that satisfies the Nondegeneracy, Mixture Continuity, Monotonicity, and Risk*

²⁵That is, it holds whenever the consequent of our Lemma 9 holds with a possibly incomplete preference relation $\succeq$.

²⁶We are thankful to Ariel Rubinstein for suggesting this reinterpretation.

Independence Axioms, and $u : Z \rightarrow \mathbb{R}$ is its utility representation over constant acts with range $\mathcal{U}$. Then, there exists a unique nondegenerate preference relation $\succeq$ on $\mathcal{U}^E$ such that

$$f \succeq g \iff (u \circ f) \succeq (u \circ g) \quad \forall_{f,g \in Z^E}. \quad (6)$$

Preference relation $\succeq$ satisfies (i) closed continuity: sets $\{\varphi \in \mathcal{U}^\Omega : \varphi \succeq \psi\}$ and $\{\varphi \in \mathcal{U}^\Omega : \psi \succeq \varphi\}$ are closed for all $\psi \in \mathcal{U}^E$; and (ii) monotonicity: for any $\varphi, \psi \in \mathcal{U}^E$, $(\forall_\omega \varphi(\omega) \geq \psi(\omega)) \Rightarrow \varphi \succeq \psi$.

Proof. Preference relation $\succeq$ defined through (6) is well-defined: By the Monotonicity Axiom, if $f', g' \in Z^E$ are such that $(u \circ f)(\omega) = (u \circ f')(\omega)$ and $(u \circ g)(\omega) = (u \circ g')(\omega)$ for all $\omega \in E$, then $f \succeq g$ if and only if $f' \succeq g'$. Clearly, $\succeq$ is reflexive, transitive, and complete. The monotonicity of $\succeq$ also immediately follows from the monotonicity of $\succeq$. The continuity of $\succeq$ follows from Lemma 67 of [Cerreia-Vioglio et al. \(2011a\)](#), which shows the existence of a continuous functional $I : \mathcal{U}^E \rightarrow \mathbb{R}$ that represents $\succeq$. $\square$

The following lemmas preceding the proof of the main theorem prove statements connecting Strict Monotonicity of ex ante preferences to Nondegeneracy and Risk Independence of conditionals.

Lemma 10. *Let $\succeq$ be a nondegenerate preference relation on Z^Ω satisfying the Mixture Continuity, Monotonicity, and Risk Independence Axioms, and fix any $E \in \mathcal{E}_0$, $h \in Z^{\Omega \setminus E}$. If $\succeq_{E,h}$ also satisfies the Risk Independence Axiom, then either*

(i) $x \succeq y \iff x \succeq_{E,h} y$ for all $x, y \in Z$, or

(ii) $x \sim_{E,h} y$ for all $x, y \in Z$ (i.e., $\succeq_{E,h}$ is degenerate).

Proof. By the result of [Herstein and Milnor \(1953\)](#), there exist affine functions $u, u' : Z \rightarrow \mathbb{R}$ representing the restriction of preferences $\succeq$ and $\succeq_{E,h}$, respectively, to constant acts. Let $Y := \langle Z \rangle$, where $\langle \cdot \rangle$ denotes the linear span. Functions u and u' can be extended uniquely to affine functions $Y \rightarrow \mathbb{R}$. Assume without loss of generality that $u(0) = u'(0) = 0$. By the rank-nullity theorem, there exists $y_0 \in Y$ such that $Y = \ker u \oplus \langle y_0 \rangle$ and $u(y_0) = 1$. The key observation is that conditioning preserves indifferences: If $z_1 \sim z_2$ for some $z_1, z_2 \in Z$, then the monotonicity of $\succeq$ implies that $z_1 E h \sim z_2 E h$, which means that $z_1 \sim_{E,h} z_2$. Consequently, $u'(y) = 0$ for all $y \in \ker u$, so $u'(y) = u'(y_0)u(y)$ for all $y \in Y$. If $u'(y_0) = 0$, then $\succeq_{E,h}$ is degenerate. Otherwise, pick any $z_1, z_2 \in Z$ such that $z_1 > z_2$, and, hence, $u(z_1) > u(z_2)$. If $u'(y_0) < 0$, then $u'(z_1) < u'(z_2)$, meaning that $z_2 >_{E,h} z_1$, so, in turn, $z_2 E h > z_1 E h$, which contradicts the monotonicity of $\succeq$. Conclude that $u'(y_0) > 0$, which means that u' is a positive affine transformation of u , and the restrictions of $\succeq$ and $\succeq_{E,h}$ to constant acts coincide. $\square$

Corollary 11. *Let $\succeq$ be a nondegenerate preference relation on Z^Ω satisfying the Mixture Continuity, Monotonicity, and Risk Independence Axioms. If $\succeq_{E,h}$ satisfy the Nondegeneracy and Risk Independence Axioms for all $E \in \mathcal{E}_0$ and $h \in Z^{\Omega \setminus E}$, then $\succeq$ satisfies the Strict Monotonicity Axiom.*

Proof. Suppose that $f, g \in Z^\Omega$ are such that $f(\omega) \succeq g(\omega)$ for all $\omega \in \Omega$, and $f(\omega_0) \succ g(\omega_0)$ for some $\omega_0 \in \Omega$. Let $E := \{\omega_0\}$ and $h := f|_{\Omega \setminus E}$. Since $\succeq_{E,h}$ is nondegenerate, $f(\omega_0) \succ_{E,h} g(\omega_0)$ by Lemma 10, and, therefore, $f = f E h \succ g E h \succeq g$. $\square$

Lemma 12. *Let $\succeq$ be a preference relation on Z^Ω satisfying the Nondegeneracy, Mixture Continuity, Strict Monotonicity, and Risk Independence Axioms. Then, $\succeq_{E,h}$ also satisfies the Nondegeneracy, Mixture Continuity, Strict Monotonicity, and Risk Independence Axioms for all $E \in \mathcal{E}_0$, $h \in Z^{\Omega \setminus E}$.*

Proof. Mixture Continuity and Strict Monotonicity of $\succeq_{E,h}$ immediately follow from the corresponding properties of $\succeq$. Nondegeneracy of $\succeq_{E,h}$ follows from the Strict Monotonicity of $\succeq$. It remains to verify that $\succeq_{E,h}$ satisfies the Risk Independence Axiom. Take $y_1, y_2, z \in Z$ and $\lambda \in (0, 1)$ and observe

$$\begin{aligned} y_1 \succeq_{E,h} y_2 &\Leftrightarrow y_1 E h \succeq y_2 E h \stackrel{sMON}{\Leftrightarrow} y_1 \succeq y_2 \stackrel{R-Ind}{\Leftrightarrow} \\ &\lambda y_1 + (1 - \lambda) z \succeq \lambda y_2 + (1 - \lambda) z \stackrel{sMON}{\Leftrightarrow} \lambda y_1 + (1 - \lambda) z \succeq_{E,h} \lambda y_2 + (1 - \lambda) z, \end{aligned}$$

which establishes the claim. $\square$

Proof of Theorem 8. (i) $\Rightarrow$ (ii). As follows from Lemma 10 and nondegeneracy of conditional preferences, there is a single utility function $u : Z \rightarrow \mathbb{R}$ that represents the restrictions of $\succeq$ and all its conditionals to constant acts. Using the result of Lemma 9, we switch from the original preferences $\succeq$ and $\succeq_{E,h}$ ($E \in \mathcal{E}_0$, $h \in Z^{\Omega \setminus E}$) to preferences $\succeq$ and $\succeq_{E,h}$ over utility acts from $\mathcal{U}^\Omega$ and $\mathcal{U}^E$, respectively. $\mathcal{A}$ -invariance of $\succeq$ can now be stated as $f \succeq g \Leftrightarrow A \circ f \succeq A \circ g$ for any $A \in \mathcal{A}$ and $f, g \in \mathcal{U}^\Omega$ such that $A \circ f, A \circ g \in \mathcal{U}^\Omega$ as well, and analogously for $\succeq_{E,h}$ ($E \in \mathcal{E}_0$, $h \in Z^{\Omega \setminus E}$).

Observe that preference relation $\succeq$ satisfies the Coordinate Independence condition: For all $f, g \in \mathcal{U}^\Omega$, $\omega_0 \in \Omega$, $v, w \in \mathcal{U}$,

$$f E v \succeq g E v \quad \Rightarrow \quad f E w \succeq g E w, \quad (7)$$

where $E = \Omega \setminus \{\omega_0\}$. This Coordinate Independence condition applied in a state-by-state manner immediately implies Savage's Sure Thing Principle, and the Strict Monotonicity follows from Corollary 11.

To prove the Coordinate Independence, take first any $f, g \in (\text{int}\mathcal{U})^\Omega$, $v \in \text{int}\mathcal{U}$, and assume that $f E v \succeq g E v$. Let $S := \{t \in \text{int}\mathcal{U} : f E t \succeq g E t\}$, and note that $S \neq \emptyset$ since $v \in S$. S is closed in $\text{int}\mathcal{U}$ by the continuity of $\succeq$. We

claim that S is also open in $\text{int}\mathcal{U}$. Let $m := \min_{\omega \in E} \min\{f(\omega), g(\omega)\}$ and $M := \max_{\omega \in E} \max\{f(\omega), g(\omega)\}$. Take any $t \in S$, note that $\mathcal{U}$ is a convex set, and pick $\varepsilon > 0$ such that $\min\{m, t\} - \varepsilon \in \text{int}\mathcal{U}$ and $\max\{M, t\} + \varepsilon \in \text{int}\mathcal{U}$. Since $\mathcal{A}$ covers $\mathcal{U}$, there exists an open interval $N_t \ni t$ such that $\mathcal{A}_{[m, M], \varepsilon} t \supseteq N_t$. Take any $y \in N_t$, find $A \in \mathcal{A}_{[m, M], \varepsilon}$ such that $At = y$, and observe that

$$\begin{aligned} fEt \succeq gEt &\Rightarrow AfEy \succeq AgEy \Rightarrow \\ &Af \succeq_{E, y} Ag \Rightarrow f \succeq_{E, y} g \Rightarrow fEy \succeq gEy. \end{aligned}$$

Therefore, $N_t \subseteq S$ and S is open. Since $\text{int}\mathcal{U}$ is connected, $S = \text{int}\mathcal{U}$, which proves property (7) when all parts of the expression are in $\text{int}\mathcal{U}$. By the continuity of $\succeq$, it extends to the boundaries.

(ii) $\Rightarrow$ (iii). Similar to the first part of the proof, we employ Lemma 9 and switch to preferences $\succeq$ over $\mathcal{U}^\Omega$, where $\mathcal{U}$ is the range of the utility function representing the restriction of $\succeq$ to constant acts. By Theorem III.4.1 of Wakker (1989), the Coordinate Independence condition (implied by the Sure Thing Principle) and the assumption that $|\Omega| \geq 3$ guarantee the existence of a collection of continuous functions $V(\cdot, \omega) : \mathcal{U} \rightarrow \mathbb{R}$ indexed by $\omega \in \Omega$, such that $\succeq$ has an additive utility representation by a mapping

$$f \mapsto \sum_{\omega \in \Omega} V(f(\omega), \omega).$$

The Strict Monotonicity condition ensures that $V(\cdot, \omega)$ are strictly increasing for all $\omega \in \Omega$.

(iii) $\Rightarrow$ (iv). The uniqueness part of the same Theorem III.4.1 states that the collection of functions $(V(\cdot, \omega))_{\omega \in \Omega}$ is jointly cardinal. That is, if $\succeq$ has an additive representation via the mapping $f \mapsto \sum_{\omega \in \Omega} V(f(\omega), \omega)$, then the mapping $f \mapsto \sum_{\omega \in \Omega} W(f(\omega), \omega)$ is another utility representation of $\succeq$ if and only if there exist constants $\alpha > 0$ and $(\beta_\omega)_{\omega \in \Omega}$ such that $W(\cdot, \omega) \equiv \alpha V(\cdot, \omega) + \beta(\omega)$ for all $\omega \in \Omega$. Since $\succeq$ is $\mathcal{A}$ -invariant, then for any $A \in \mathcal{A}$, the mapping $f \mapsto \sum_{\omega \in \Omega} V(Af(\omega), \omega)$ is a utility representation of $\succeq$. Therefore, each $A \in \mathcal{A}$ defines terms $\alpha(A)$ and $(\beta(A, \omega))_{\omega \in \Omega}$ such that $V(Ax, \omega) = \alpha(A)V(x, \omega) + \beta(A, \omega)$ for all $x \in \mathcal{U}$.

(iv) $\Rightarrow$ (i). Fix any $E \in \mathcal{E}_0$, $h \in Z^{\Omega \setminus E}$. The strict monotonicity of V implies that all the required properties of $\succeq_{E, h}$ except the invariance hold by Lemma 12. The mapping

$$f \mapsto V_E(f) := \sum_{\omega \in E} V((u \circ f)(\omega), \omega)$$

is a utility representation of $\succeq_{E, h}$ by the definition of conditional preferences. The functional equation for V implies that V_E satisfies

$$V_E(u^{-1} \circ A \circ u \circ f) = \alpha(A)V_E(f) + \sum_{\omega \in \Omega} \beta(A, \omega)$$

for all $f \in Z^E$ and $A \in \mathcal{A}$ such that $(A \circ u \circ f)(\omega) \in u(Z)$ for all $\omega \in E$. Consequently,

$$V_E(u^{-1} \circ A \circ u \circ f) \geq V_E(u^{-1} \circ A \circ u \circ g) \iff V_E(f) \geq V_E(g)$$

for all $f, g \in Z^E$ and $A \in \mathcal{A}$ such that $(A \circ u \circ f)(\omega), (A \circ u \circ g)(\omega) \in u(Z)$ for all $\omega \in E$, which proves the invariance of $\succeq_{E,h}$ with respect to $\mathcal{A}$. $\square$

Appendix A.2. Proofs of the Remaining Results

Proposition 13. *Let $\succeq$ be a preference relation on Z^Ω , and suppose that $\succeq$ and $\succeq_{E,h}$ for all $E \in \mathcal{E}_0$, $h \in Z^{\Omega \setminus E}$ satisfy the Nondegeneracy, Continuity, Monotonicity, and Weak Certainty Independence Axioms. Then, either*

- (i) *there exist nonconstant and affine $u : Z \rightarrow \mathbb{R}$ and $q \in \Delta(\Omega)$ with full support such that $\succeq$ admits an expected utility representation via the mapping $f \mapsto \int_\Omega (u \circ f) dq$, or*
- (ii) *there exist nonconstant and affine $u : Z \rightarrow \mathbb{R}$, $q \in \Delta(\Omega)$ with full support, and a nonzero (possibly negative) real number γ such that $\succeq$ admits a utility representation via the mapping*

$$f \mapsto \int_\Omega \frac{1}{\gamma} e^{\gamma(u \circ f)} dq. \quad (8)$$

Proof. First, observe that the Weak Certainty Independence Axiom implies Risk Independence, so there exists nonconstant and affine $u : Z \rightarrow \mathbb{R}$ that represents $\succeq$ over constant acts. Axioms assumed for $\succeq$ and its conditionals imply that preferences are invariant to the group of automorphisms $\mathcal{A} := \{z \mapsto z + t \mid t \in \mathbb{R}\}$, as essentially proven by Lemma 31 of [Maccheroni, Marinacci and Rustichini \(2004\)](#). Then, by Theorem 8, there exists an additive representation for $\succeq$ on Z^Ω . Function $V : u(Z) \times \Omega \rightarrow \mathbb{R}$ given by that representation solves a system of functional equations

$$V(x + t, \omega) = \alpha(t)V(x, \omega) + \beta(t, \omega),$$

where $\alpha : \mathbb{R} \rightarrow \mathbb{R}_{++}$ and $\beta : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ are some functions. Since V is strictly increasing in its first argument, the corollary of Theorem 2 of [Aczél \(2005\)](#) establishes that there exist $\gamma \neq 0$, $\delta : \Omega \rightarrow \mathbb{R} \setminus \{0\}$, $B : \Omega \rightarrow \mathbb{R}$ such that either $V(x, \omega) = \delta(\omega)x + B(\omega)$ for all $x \in u(Z)$ and $\omega \in \Omega$, or $V(x, \omega) = \delta(\omega)e^{\gamma x} + B(\omega)$ for all $x \in u(Z)$ and $\omega \in \Omega$. For the purpose of representing $\succeq$, intercept B is immaterial; the monotonicity of the preference relation ensures that $\delta(\omega) > 0$ for all $\omega \in \Omega$ in the first case and $\gamma\delta(\omega) > 0$ for all $\omega \in \Omega$ in the second one. Normalization of the coefficients $\delta(\omega)$ then yields a probability measure on Ω with full support, and gives the sought representations. $\square$

Proof of Proposition 1. Any variational preference relation $\succeq$ satisfies the conditions of Proposition 13, so the conclusion of that proposition applies.

Then, if $\succeq$ is not an expected utility preference relation, it admits representation (8), and the Uncertainty Aversion implies that $\gamma < 0$. Defining $\theta := -\frac{1}{\gamma}$ and observing that

$$-\theta \ln \int_{\Omega} e^{-f(\omega)/\theta} q(d\omega) = \min_{\mu \in \Delta(\Omega)} \left(\int_{\Omega} f(\omega) \mu(d\omega) + \theta R(\mu \parallel q) \right)$$

for all $\theta > 0$, $q \in \Delta(\Omega)$, $f \in \mathbb{R}^{\Omega}$ (Proposition 1.4.2 of Dupuis and Ellis, 1997), representation (8) reduces to the multiplier preference representation. $\square$

Proof of Proposition 3. As the proof of Proposition 1 shows, a preference relation $\succeq$ that satisfies the conditions of this proposition is either an expected utility preference relation, or has the exponential representation (8). Suppose that it does not have the expected utility form.

Let $\mathcal{U} := u(Z)$, and assume without loss of generality that $0 \in \text{int}\mathcal{U}$. Take any three states $\omega_1, \omega_2, \omega_3 \in \Omega$, some vector $h \in Z^{\Omega \setminus \{\omega_1, \omega_2, \omega_3\}}$, and consider $\succeq_{\{\omega_1, \omega_2, \omega_3\}, h}$. This preference relation is represented by a mapping

$$f \mapsto a_1 \frac{1}{\gamma} e^{\gamma u(f(\omega_1))} + a_2 \frac{1}{\gamma} e^{\gamma u(f(\omega_2))} + a_3 \frac{1}{\gamma} e^{\gamma u(f(\omega_3))},$$

where $a_1, a_2, a_3 > 0$ and $\gamma \neq 0$.

Similar to Step 1 of the proof of Proposition 1, we switch to working with the preferences over utility acts from $\mathcal{U}^{\Omega}$. Now, we can finish the proof using only elementary tools. Define $z : \mathcal{U}^2 \rightarrow \mathbb{R}$ through equality

$$a_1 e^{\gamma x} + a_2 e^{\gamma y} + a_3 e^{\gamma z(x, y)} = a_1 e^{-\gamma x} + a_2 e^{-\gamma y} + a_3 e^{-\gamma z(x, y)}. \quad (9)$$

Note that the solution of this equation is unique, so the function z is well-defined. It is also continuous and satisfies $z(0, 0) = 0$. Let $D \subseteq \mathcal{U}$ be a sufficiently small open interval such that $0 \in D$ and $z(D, D) \subseteq \mathcal{U}$.

Observe that utility acts $f = (x, y, z(x, y))$ and $\bar{f} = (-x, -y, -z(x, y))$ are complementary and satisfy $f \sim \bar{f}$ for all $x, y \in D$. Then, the Complementary Independence Axiom applied to two pairs of acts having such structure yields

$$\begin{aligned} (\lambda x + (1 - \lambda)x', \lambda y + (1 - \lambda)y', \lambda z(x, y) + (1 - \lambda)z(x', y')) \sim \\ (-\lambda x - (1 - \lambda)x', -\lambda y - (1 - \lambda)y', -\lambda z(x, y) - (1 - \lambda)z(x', y')) \end{aligned}$$

for all $x, y, x', y' \in D$ and $\lambda \in [0, 1]$, which means that

$$z(\lambda x + (1 - \lambda)x', \lambda y + (1 - \lambda)y') = \lambda z(x, y) + (1 - \lambda)z(x', y')$$

by the definition of the function z . The latter, in conjunction with $z(0,0) = 0$, establishes that z is a linear function on D^2 : There exist $k_1, k_2 \in \mathbb{R}$ such that $z(x, y) = k_1x + k_2y$ for all $x, y \in D$.

Taking $y = 0$, identity (9) can be rewritten as

$$\frac{a_1}{a_3} = -\frac{e^{\gamma k_1 x} - e^{-\gamma k_1 x}}{e^{\gamma x} - e^{-\gamma x}},$$

which can hold for all $x \in D \setminus \{0\}$ only if $a_1 = a_3$ and $k_1 = -1$. Similarly, taking $x = 0$ yields $a_2 = a_3$ and $k_2 = -1$. Finally, taking $y = x$, identity (9) can be rewritten as

$$2 = \frac{e^{2\gamma x} - e^{-2\gamma x}}{e^{\gamma x} - e^{-\gamma x}},$$

which cannot hold for all $x \in D \setminus \{0\}$, a contradiction. $\square$

Proof of Proposition 4. Let $\succsim$ be a preference relation on Z^Ω that satisfies the Mixture Continuity, Monotonicity, and Certainty Independence Axioms, and such that its conditionals satisfy these axioms, as well. As can be seen from the proof of Lemma 3.3 of Gilboa and Schmeidler (1989), $\succsim$ and its conditionals are invariant to the group of automorphisms $\mathcal{A} := \{z \mapsto kz + t \mid k > 0, t \in \mathbb{R}\}$. By Theorem 8, $\succsim$ admits an additive utility representation by the mapping $f \mapsto \sum_{\omega \in \Omega} V((u \circ f)(\omega), \omega)$, where $u : Z \rightarrow \mathbb{R}$ is the utility representation of $\succsim$ over constant acts. Since the collection of automorphisms $\{z \mapsto z + t \mid t \in \mathbb{R}\}$ is a subgroup of $\mathcal{A}$, functions V have to be either linear, $V(x, \omega) = \delta(\omega)x + B(\omega)$ for all $x \in u(Z)$ and $\omega \in \Omega$, or exponentials, $V(x, \omega) = \delta(\omega)e^{\gamma x} + B(\omega)$ for all $x \in u(Z)$ and $\omega \in \Omega$, for some $\gamma \neq 0$, $\delta : \Omega \rightarrow \mathbb{R}_{++}$, and $B : \Omega \rightarrow \mathbb{R}$. The exponentials do not satisfy the functional equation $V(kx, \omega) = \tilde{\alpha}(k)V(x, \omega) + \tilde{\beta}(k, \omega)$, which has to hold for some $\tilde{\alpha} : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$, $\tilde{\beta} : \mathbb{R}_{++} \times \Omega \rightarrow \mathbb{R}$ and all $x \in u(Z)$ and $k > 0$. Therefore, $V(\cdot, \omega)$ are linear, and the expected utility representation obtains. $\square$

Proof of Proposition 7. Take any preference relation $\succsim$ on Z^Ω that satisfies Mixture Continuity, Strict Monotonicity, Risk Independence, and Uncertainty Aversion Axioms, any $E \in \mathcal{E}_0$ and $h \in Z^{\Omega \setminus E}$. Uncertainty Aversion of $\succsim_{E,h}$ immediately follows from the Uncertainty Aversion of $\succsim$, and the remaining axioms follow from Lemma 12. $\square$

Proof of Proposition 6. Let a preference relation $\succsim$ on Z^Ω belong to the class of confidence preferences, and suppose that all its conditionals also belong to this class. As follows from Lemmas 19 and 21 of Chateauneuf and Faro (2009), $\succsim$ and its conditionals are invariant to the group of automorphisms $\mathcal{A} := \{z \mapsto kz \mid k > 0\}$ if the utility function $u : Z \rightarrow \mathbb{R}$ representing $\succsim$ over constant acts is normalized such that $\inf u(Z) = u(x_*) = 0$. By Theorem 8, $\succsim$ admits an additive utility

representation via the mapping $f \mapsto \sum_{\omega \in \Omega} V((u \circ f)(\omega), \omega)$. Each of the functions $V(\cdot, \omega)$ has to solve the following functional equation:

$$V(kx, \omega) = \alpha(k)V(x, \omega) + \beta(k, \omega) \quad \forall_{x \in u(Z), \omega \in \Omega, k > 0}. \quad (10)$$

Substituting $x = 0$ and subtracting the resulting equation from (10) yields a new equation

$$\tilde{V}(kx, \omega) = \alpha(k)\tilde{V}(x, \omega) \quad \forall_{x \in u(Z), \omega \in \Omega, k > 0}, \quad (11)$$

where $\tilde{V} : u(Z) \times \Omega$ is defined as $\tilde{V}(x, \omega) := V(x, \omega) - V(0, \omega)$. Clearly, α is a multiplicative function, $\alpha(k_1 k_2) = \alpha(k_1)\alpha(k_2)$ for any $k_1, k_2 > 0$, and function $\tilde{V}$ can be extended to $\mathbb{R}_+ \times \Omega$ preserving equation (11) even if $u(Z)$ is bounded. Theorem 4 in Aczél (1966, §3.1) establishes that there exist $\gamma \neq 0$ and $\delta : \Omega \rightarrow \mathbb{R} \setminus \{0\}$ such that $\tilde{V}(x, \omega) = \delta(\omega)x^\gamma$ for all $x \in \mathbb{R}_+$ and $\omega \in \Omega$. Monotonicity and Uncertainty Aversion ensure that $\delta(\omega) > 0$ for all $\omega \in \Omega$ and $\gamma \in (0, 1]$. Then, the desired representation can be obtained by normalization. $\square$

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