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Income drawdown option with minimum guarantee

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Abstract

This paper deals with a constrained investment problem for a defined contribution (DC) pension fund where retirees are allowed to defer the purchase of the annuity at some future time after retirement.

This problem has already been treated in the unconstrained case in a number of papers. The aim of this work is to deal with the more realistic case when constraints on the investment strategies and on the state variable are present. Due to the difficulty of the task, we consider, as a first step, the basic model of [Gerrard, Haberman & Vigna, 2004], where interim consumption and annuitization time are fixed. We extend their model by adding a no short-selling constraint on the control variable and a final capital requirement constraint on the state variable. This implies, in particular, no ruin.

The mathematical problem is naturally formulated as a stochastic control problem with constraints on the control and the state variable, and is approached by the dynamic programming method. We write the non-linear Hamilton-Jacobi-Bellman equation for the problem and transform it into a dual one that is semi-linear, following a well-established duality procedure. In the special relevant case without running cost, we explicitly compute the value function for the problem and give the optimal strategy in feedback form. A numerical application ends the paper and shows the extent of applicability of the model to a DC pension fund in the decumulation phase.

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Keywords: pension fund, decumulation phase, constrained portfolio, stochastic optimal control, dynamic programming, Hamilton-Jacobi-Bellman equation.

1 Introduction

In countries where immediate annuitization is the only option available in defined contribution (DC) pension schemes, members who retire at a time of low bond yield rates have to accept a pension

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lower than the one available with higher bond yields (so-called annuity risk). In many countries, including Argentina, Australia, Brazil, Canada, Chile, Denmark, El Salvador, Japan, Peru, UK, US, the retiree is allowed to defer annuitization at some time after retirement, withdraw periodic income from the fund, and invest the rest of it in the period between retirement and annuitization. This allows the retiree to postpone the decision to purchase an annuity until a more propitious time. This flexibility is usually referred to as "income drawdown option" or "programmed withdrawal (option)".[1] For a detailed survey on the several forms of benefits provided by the programmed withdrawals option, we refer the interested reader to [Antolin, Pugh & Stewart, 2008]. There are often limits imposed on both the consumption and on how long the annuity purchase can be deferred. On the other hand, there is virtually unlimited freedom to invest the fund in a broad range of assets. While this option allows the retiree to aim for a final annuity higher than that purchasable at retirement, the evident drawback consists in the possibility of ruin, i.e. exhausting the fund while still alive. The three degrees of freedom of the retiree (amount of consumption, investment allocation, and time of annuitization), together with the important issue of ruin possibility, have been investigated in the actuarial and financial literature in many papers. Among others we recall [Albrecht & Maurer, 2002], [Blake, Cairns & Dowd, 2003], [Di Giacinto & Vigna, 2012], [Gerrard, Haberman & Vigna, 2006], [Gerrard, Højgaard & Vigna, 2011] [Milevsky, 2001], [Milevsky, Moore & Young, 2006], [Milevsky & Young, 2007].

While the issue of ruin has been tackled in many papers, the problem of providing a minimum guarantee to the pensioner who takes the income drawdown option has not been considered in the literature, up to our knowledge. Nevertheless, the guarantee of a minimum level of ultimate annuity should be strong reason for taking programmed withdrawals. Moreover, the introduction of restrictions – which is more in line with financial regulatory environment – makes possible a more accurate judgement on the effective tradeoff between the risks and the benefits provided by the income drawdown option. Motivated by these considerations, in this paper we fill in this gap in the literature, by defining and solving an optimal investment problem for the decumulation phase of a DC plan, where a minimum guarantee is provided and short-selling is forbidden.

The natural way of dealing with this problem is to formulate it as a stochastic optimal control problem with an appropriate choice of the state and control variables, of the constraints that they must satisfy, and of the optimizing criterion (utility or loss function). We choose the framework of [Gerrard, Haberman & Vigna, 2004], [Gerrard, Haberman & Vigna, 2006], and [Gerrard, Højgaard & Vigna, 2011] taking a quadratic target-based loss function. Indeed our approach could be used also to treat different objective functions. The three mentioned works consider similar models with an increasing number of degrees of freedom (i.e. control variables) but they do not solve the problems when constraints on the wealth and on the investment strategies are present.[2] Since the introduction of these constraints makes the problem very hard to attack and non treatable in the general case with the results of the known literature, we consider the simplest model [Gerrard, Haberman & Vigna, 2004], where the retiree is given only one degree of freedom, namely the investment allocation. The income withdrawn from the fund in the unit time is assumed to be fixed and the retiree is obliged to annuitize at a fixed future time T. In view of this fact, this paper must be considered as a first step towards a satisfactory treatment of the problem.

[1] Other equivalent expressions are: phased withdrawal, scheduled withdrawal, allocated annuities, allocated pensions, allocated income streams.

[2] In [Gerrard, Haberman & Vigna, 2004], the only control variable is the investment strategy, in [Gerrard, Haberman & Vigna, 2006] the control variables are the investment and the consumption policies, while in [Gerrard, Højgaard & Vigna, 2011] the retiree is allowed to choose the annuitization time, together with the investment-consumption policies.

From the methodological point of view, following the approach of the papers mentioned above, we tackle the problem by the dynamic programming approach studying the associated Hamilton-Jacobi-Bellman (HJB) equation.

Notice that other methods can be used to deal with the same problem. In particular, in the case of optimal portfolio problems with capital constraints, both probabilistic duality methods and methods based on backward stochastic differential equations (BSDE) have been successfully employed in the literature. In this respect we observe that, among others, [Basak & Shapiro, 2001], [El Karoui, Jeanblanc & Lacoste, 2005], [Korn, 1997], [Tepla, 2001], and [Bielecki, Jin, Pliska & Zhou, 2005] are concerned with this kind of problems. More precisely, [Basak & Shapiro, 2001], [Korn, 1997] and [Tepla, 2001] deal with direct duality methods on the control problem - while we use duality at an analytic stage applying it to the HJB equation. However, differently from our case, in [Korn, 1997] the constraint is on the final average of the wealth, while in [Basak & Shapiro, 2001] there is a VaR-type constraint. The paper closest to ours seems to be [Tepla, 2001], where, as in our case, an almost sure constraint on the terminal wealth is imposed. What we get is from a purely analytic point of view the same result as in [Tepla, 2001].

Moreover, we should mention also the link of our problem with the rich class of mean-variance (MV) optimization problems in continuous-time. The well-known equivalence between MV-problems and expected utility function in the single-period framework can be extended to the continuous-time case (see for instance [Korn, 1997], [Zhou & Li, 2000], [Bielecki, Jin, Pliska & Zhou, 2005], [Vigna, 2012]). In the rich stream of literature on MV-optimization originating by the seminal paper [Zhou & Li, 2000], the work by [Bielecki, Jin, Pliska & Zhou, 2005] solve a problem similar to ours, in a more general setting regarding the financial market. Their methodology is an extension of the risk neutral approach introduced by [Pliska, 1986] and boils down in presenting the optimal portfolio as the solution of a linear backward stochastic differential equation (BSDE).

However, if one chooses to deal with a dynamic programming approach - as we do - it turns out that the presence of a state constraint leads to suitable boundary conditions for the HJB equation bringing a loss of the possibility of finding, unlike the papers mentioned above, simple explicit solutions to such an equation. Also a straight theoretical approach to the HJB equation – dealing, for instance, with a characterization of the value function as the unique viscosity solution and then with the proof of the regularity of viscosity solutions – is very problematic, since the HJB equation is a fully nonlinear, degenerate, non autonomous parabolic PDE. Therefore, we use a known procedure applied in this kind of equations arising in portfolio optimization problems, which allows to transform the equation into a dual one looking nicer. This procedure has been used, e.g., in [Elie & Touzi, 2008], [Gao, 2008], [Gerrard, Højgaard & Vigna, 2011], [Milevsky, Moore & Young, 2006], [Milevsky & Young, 2007] and [Xiao , Zhai & Qin, 2007]. In all such papers the dual equation is always linear.[3]

In our case the dual equation is in general semilinear and becomes linear when the current cost is zero. The general semilinear case is studied in the extended version of this paper [Di Giacinto, Federico, Gozzi, & Vigna, 2010] (to which the interested reader is referred) proving regularity of solutions to the dual equation and, consequently, to the original one. In this paper, for brevity, we set up the general model and then focus on the special and still significant case with no running cost, which allows, with a procedure similar to the above quoted papers, to find an explicit solution to the dual problem and come back with an explicit solution to the original one.

[3]To this regard, it is worth to stress that in [Schwartz & Tebaldi, 2006] this procedure is applied in an incomplete market (due to the presence of an uninsurable income) giving rise to a dual equation which is still fully nonlinear. The authors approach this dual equation by means of a series expansion.

So we can characterize the optimal strategy and wealth and perform a numerical simulation which allows to get some insights on the effects of the constraints on the optimal paths.

The availability of closed-form solution to this problem is particularly important in the context of DC pension schemes. Indeed, using this model the retiree who takes the income drawdown option can decide about both the level of the minimum guarantee and that of a desired target. These levels are driven by the retiree's risk profile, the determination of which is typically an issue. Application of closed-form optimal policies coupled with numerical simulations for the risky asset provide the distribution of the annuity received upon ultimate annuitization, that in turn helps the retiree to determine her own risk profile. All these features make this model – or possible evolutions of it – applicable by pension fund advisors in the decision-making process of retirees entering the decumulation phase of a DC scheme.

The remainder of the paper is organized as follows. In Section 2, we introduce the model and define the problem to be solved. Section 3 represents the theoretical core of the paper. Therein, we consider the problem, pass to the dual formulation, and show equivalence between the dual and the original problem. We then find closed-form solutions for the special case with no running cost. In Section 4, we show a numerical application that highlights the potential applicability to a DC pension plan. Section 5 concludes and outlines further research.

2 The model

In this section, we outline the model and describe the problem faced by a representative member of a pension scheme.

We consider the position of an individual who chooses the income drawdown option at retirement. We assume that final annuitization is compulsory at a certain age. Thus, the individual withdraws a certain fixed income until she achieves the age when the purchase of the annuity is compulsory. Without loss of generality, we assume that the individual retires at time $s = 0$ and that compulsory annuitization occurs at time $s = T$. In a real context the pensioner could annuitize at any time between $s = 0$ and $s = T$. Here, for mathematical convenience and as in [Gerrard, Haberman & Vigna, 2004], we assume that the pensioner annuitizes exactly at time T. Introducing the possibility of choosing the annuitization time would increase remarkably the difficulty of the problem. The problem of finding the optimal investment-consumption couple as well as optimal annuitization time has been treated rigorously as a combined stochastic control and optimal stopping problem in [Milevsky, Moore & Young, 2006], [Milevsky & Young, 2007] and [Gerrard, Højgaard & Vigna, 2011]. However, none of these papers treats the case with constraints on the investment strategy and the state variable.

The fund is invested in two assets: a riskless asset with constant instantaneous rate of return $r \geq 0$, and a risky asset whose price follows a geometric Brownian motion with constant volatility $\sigma > 0$ and drift $\mu := r + \sigma\beta$, where $\beta > 0$ is the so-called Sharpe ratio or risk premium. The random variable T_D representing the remaining lifetime of the pensioner is assumed to be exponentially distributed with parameter $\delta > 0$, and is assumed to be independent of the Brownian motion. The pensioner withdraws a fixed amount $b_0 > 0$ in the unit of time. Bequests motives are absent and the only reason for taking programmed withdrawals is the hope of being better off than immediate annuitization when deferred annuitization takes place. Indeed, the basic idea driving this option (see also [Milevsky, 2001]) is that, since the annuity price is calculated with the riskfree rate, in the years after retirement the equity risk premium pays more than the mortality credits (due to annuitants who die earlier than average). To support this intuition, [Gerrard, Haberman & Vigna, 2006] have also found that in the presence of sufficiently good risky assets the income drawdown option should be preferred to immediate annuitization. The choice of a fixed consumption rate throughout the decumulation phase, made mainly for mathematical convenience, is consistent with the hope of being better off with deferred annuitization. In fact, by selecting b_0 equal to the pension rate produced by immediate annuitization, the comparison between immediate annuitization and programmed withdrawals is then straightforward, the former producing a fixed income of b_0 until death, the latter producing an income of b_0 until time T and a different (hopefully higher) pension rate from T until death.

According to [Merton, 1969] the state equation that describes the dynamics of the fund wealth $X(\cdot)$ is the following

$$\begin{cases} dX(s) = [rX(s) + (\mu - r)\pi(s) - b_0]\,ds + \sigma\pi(s)dB(s), \quad s \in [0, T], \\ X(0) = x_0, \end{cases} \tag{1}$$

where $x_0 > 0$ is the fund wealth at the retirement date $s = 0$, $B(\cdot)$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t^B)_{t\geq 0}, P)$ and $\pi(\cdot)$ is the investment strategy representing the amount of portfolio invested in the risky asset.[4]

We consider the problem where short-selling is not allowed and the final fund cannot be lower than a certain pre-determined level $S \geq 0$. Therefore, S is the minimum guarantee and the set of admissible strategies is

$$\{\pi(\cdot) \geq 0 \;:\; X(T) \geq S \text{ a.s.}\}\,. \tag{2}$$

As we will show in Proposition 3.1, the final capital requirement $X(T) \geq S \geq 0$ almost surely implies in particular no ruin, i.e. $X(t) \geq 0$ almost surely for every $t \in [0, T]$.

Almost all works of the literature on optimization problems in DC pension schemes have been solved without constraints on the control variables and the state variable. It is worth noticing that to the best of our knowledge only [Di Giacinto, Federico & Gozzi, 2011] solve a constrained portfolio selection problem in a DC pension scheme, but adopting the point of view of the fund manager. This is due to the mathematical difficulty of the problem with constraints and justifies the choice of simplifying assumptions in this setup, such as the fixed consumption rate and the fixed annuitization time. We intend to relax these assumptions in future work. Given the hard mathematical tractability of this kind of problem, here we do not tackle the no-borrowing constraint, that is also left to future research.

The preferences of the pensioner are described by the loss function

$$L(s, x) := (F(s) - x)^2\,, \tag{3}$$

where the target function $F(\cdot)$, i.e. the target that the agent wishes to track at any time $s \in [0, T]$, is given by

$$F(s) := \frac{b_0}{r} + \left(F - \frac{b_0}{r}\right) e^{-r(T-s)}.$$

The quantity $F \in (0, b_0/r)$ is the target fund desired at terminal time T and can be chosen arbitrarily. Then, the interpretation of $F(s)$ is pretty clear. Should the fund hit $F(s)$ at time $s \leq T$, the pensioner would be able to consume b_0 from s to T by investing the whole portfolio in the riskless asset, and achieve the desired target F at time T of compulsory annuitization. Clearly,

[4] $\mathbb{F}$ is the Brownian filtration augmented with the P-null sets, so it satisfies the usual conditions.

in this case the loss function computed on the state trajectory corresponding to this riskless strategy would be 0 at any time $s \leq t \leq T$. As a matter of fact, it will be shown that if the fund is equal to the target, the optimal strategy is the null one. Typically, the final target F as well as the fixed consumption rate b_0 will depend on the initial wealth x_0 or on the replacement ratio achievable with it. The quantity F is also associated to the risk profile of the member: a high F is associated to a less risk averse retiree, and vice versa.

It is important to underline at this stage the fundamental difference between S and F. Clearly, $S < F$. While the latter is the target desired by the pensioner, the former is the minimum wealth guaranteed by the pension fund at time T. In other words, S is guaranteed, F is not. In real applications, the selection of both S and F should be made by the retiree and should reflect her risk attitude. Intuitively, the lower the gap between them, the higher the risk aversion, and vice versa.

Notice that minimizing the loss function $L(s,x)$ is equivalent to maximizing the utility function $U(s,x) = -(F(s)-x)^2$. Thus, this optimization problem is apparently based on the assumption that there exists a wealth level that maximizes the utility. This drawback is only apparent, since we prove that the wealth level $F(s)$ cannot be reached by the fund under optimal control by construction of the model. Another important implication of this result is the fact that this utility function in the region of interest is increasing and concave, that are the standard requirements for a well-behaved utility function.

The choice of this quadratic loss function is also motivated by the fact that, expectedly, it has been shown to produce an optimal portfolio that is mean-variance efficient (see [Vigna, 2012]). Indeed, there is no other portfolio that provides a (strictly) higher expected value with the same variance, and no other portfolio that provides a (strictly) lower variance with the same mean. Although mean-variance is not the only criterion available for decision-making in pension funds, we regard it as appropriate for a variety of reasons, including the fact that it is still one of the most common criteria used to assess and compare funds performances (see [Chiu & Zhou, 2011]).

Furthermore, this choice of the target function has several advantages.

Firstly, as noted above, the interpretation of $F(s)$ is pretty clear.

Secondly, as mentioned above, the possibility that deviations above the target can produce unreasonable positive loss is prevented. [Gerrard, Haberman & Vigna, 2004] show that the optimal fund never reaches the target, provided that at initial time $s=0$ the fund x_0 is lower than the target $F(0)$; here we extend the result by proving that the fund under optimal control cannot reach $F(s)$.

Thirdly, it can be shown that, by choosing a loss function that does not penalize deviations above the target, such as

$$\tilde{L}(s,x) := \begin{cases} (F(s)-x)^2, & \text{if } x \leq F(s), \\ 0, & \text{if } x > F(s), \end{cases} \tag{4}$$

the optimal policy in the region of interest (i.e. below the target level) is equal to that found with the loss function (3). This feature is desirable, as stressed in Remark 3.4.

The general optimization problem consists in minimizing over the set of admissible strategies (2) the functional

$$\mathbb{E}\left[\int_0^{T_D} \kappa e^{-\theta s} L(s,X(s)) \mathbf{1}_{\{s \leq T\}} ds + e^{-\theta T} L(T,X(T)) \mathbf{1}_{\{T < T_D\}}\right] \tag{5}$$

where θ is the subjective discount factor, $\kappa \geq 0$ is a weighting constant which measures the importance of the running cost in the period before annuitization relative to the final cost at time

T, and T_D is the random variable representing the remaining lifetime. When $\kappa > 0$, the presence of a running cost keeps the trajectory of the wealth closer to the target at any time $t \leq T$, thus limiting the occurrence of undesirable events – as getting close to the terminal minimum capital requirement S.

Due to the assumed exponential distribution of T_D and its independence from the Brownian motion, it is easy to show that minimizing the functional (5) is equivalent to minimizing

$$\mathbb{E}\left[\int_0^T \kappa e^{-\rho s} L(s, X(s)) ds + e^{-\rho T} L(T, X(T))\right], \tag{6}$$

where $\rho = \theta + \delta$, i.e. ρ is the sum of the subjective discount factor and the force of mortality.

3 Solution by Dynamic Programming

We apply the Dynamic Programming technique to approach the problem of minimizing (6). The first step consists in writing the problem for generic initial data (t, x). So, given the probability space $(\Omega, \mathcal{F}, P)$ and the Brownian motion of the previous section, let us define for $t \in [0, T]$ the filtration $\mathbb{F}^t := (\mathcal{F}_s^t)_{s \in [t,T]}$, where $\mathcal{F}_s^t$ is the σ-algebra generated by $(B(u) - B(t))_{u \in [t,s]}$ and completed by the P-null sets. Consider the equation

$$\begin{cases} dX(s) = [rX(s) + (\mu - r)\pi(s) - b_0]\, ds + \sigma \pi(s) dB(s), & s \in [t, T], \\ X(t) = x, \end{cases}$$

where $x \in \mathbb{R}$ and $\pi(\cdot) \in L^2(\Omega \times [t, T]; \mathbb{R})$ is progressively measurable with respect to $\mathbb{F}^t$. This equation admits a unique strong solution on $(\Omega, \mathcal{F}, P)$ (see [Karatzas & Shreve, 1998], Section 5.6.C) that we denote by $X(\cdot; s, x, \pi(\cdot))$. Let us define the set of the admissible strategies depending on the initial (t, x) by

$$\Pi_{ad}^0(t, x) := \left\{\pi(\cdot) \in L^2(\Omega \times [t, T]; \mathbb{R}) \mid \pi(\cdot) \text{ prog. meas. w.r.t. } \mathbb{F}^t,\ \pi(\cdot) \geq 0,\ X(T; t, x, \pi(\cdot)) \geq S\right\}.$$

We are interested in the following optimization problem: for given $(t, x) \in [0, T] \times \mathbb{R}^+$,

$$\text{minimize} \quad J(t, x; \pi(\cdot)) := \mathbb{E}\left[\int_t^T \kappa e^{-\rho s} \left(F(s) - X(s)\right)^2 ds + e^{-\rho T} \left(F(T) - X(T)\right)^2\right] \tag{7}$$

over the set of admissible strategies $\Pi_{ad}^0(x, t)$, where we have set $X(s) := X(s; t, x, \pi(\cdot))$ in (7).

We denote the value function associated to this optimization problem by V, i.e.

$$V(t, x) := \inf_{\pi(\cdot) \in \Pi_{ad}^0(t,x)} J(t, x; \pi(\cdot)), \quad t \in [0, T],\ x \in \mathbb{R},$$

with the agreement that $\inf \emptyset = +\infty$. Clearly we have $V \geq 0$.

3.1 The set of admissible strategies

Let us set

$$S(t) := \frac{b_0}{r} - \left(\frac{b_0}{r} - S\right) e^{-r(T-t)}, \quad t \in [0, T]. \tag{8}$$

The function $S(\cdot)$ represents a sort of safety level for the wealth. Should the fund hit this barrier at time t, the null strategy (i.e. $\pi(\cdot) \equiv 0$) from t onwards would guarantee the fulfillment of the capital requirement. Moreover, it will be shown that the null strategy is indeed the only admissible one for $x = S(t)$, and therefore the optimal one. In the next proposition we formalize this intuition.

Proposition 3.1. *Let $t \in [0,T]$, $x \in \mathbb{R}$. Then*

(1) $\Pi^0_{ad}(t,x) \neq \emptyset$ *if and only if* $0 \in \Pi^0_{ad}(t,x)$. *This happens if and only if* $x \geq S(t)$.

(2) *If* $x = S(t)$, *then* $\Pi^0_{ad}(t,x) = \{0\}$ *and* $X(s;t,x,0) = S(s)$ *on* $[t,T]$.

(3) *Let* $x \geq S(t)$. *Then* $\pi(\cdot) \in \Pi^0_{ad}(t,x)$ *if and only if*

$$\pi(s) = \pi(s)\mathbf{1}_{\{t \leq s < \tau\}},$$

where

$$\tau := \inf\{s \in [t,T] \mid X(s;t,x,\pi(\cdot)) = S(s)\}$$

with the convention that $\inf \emptyset = T$.

(4) *If* $x > S(t)$, *then* $\Pi^0_{ad}(s,x) \supsetneq \{0\}$.

(5) *The state constraint* $X(T) \geq S$ *is equivalent to*

$$X(t) \geq S(t), \quad P\text{-a.s. } \forall t \in [0,T].$$

Proof. The proof is in the Appendix. □

Proposition 3.1 states some intuitive properties, in particular the fact that the "European guarantee" $X(T) \geq S$ is equivalent to the "American guarantee" $X(t) \geq S(t)$ for all $t \in [0,T]$ (see Remark 2.1 in [El Karoui, Jeanblanc & Lacoste, 2005], or also Proposition 2.1 in [Bielecki, Jin, Pliska & Zhou, 2005]). Hence, the problem makes sense only when the fund lies above the barrier given by $S(t)$, $t \in [0,T]$. The safety level is an absorbing barrier. As a relevant consequence, occurrence of ruin – which is one of the major risks implied by programmed withdrawals – is prevented a priori. Finally, another consequence in the application of the model is that the subjective choice of the guaranteed final fund S cannot be too high. In fact, due to (8) and to $x \geq S(t)$, it must be

$$S \leq xe^{r(T-t)} - \frac{b_0}{r}\left(e^{r(T-t)} - 1\right). \tag{9}$$

This restriction plays a role in the numerical application (see Remark 4.1).

3.2 Reduction of the problem on a bounded domain

Due to Proposition 3.1, the value function V is finite (and non-negative) on the set

$$\mathcal{D} = \{(t,x) \in [0,T] \times \mathbb{R} \mid x \geq S(t)\}.$$

Due to the specific structure of the target level $F(\cdot)$ and since we are interested to start from initial $x_0 < F(0)$, we can restrict the problem to a bounded domain. This is due basically to the following lemma, which shows what happens to the value function and to the optimal strategy when the fund reaches the fund $F(t)$.

Lemma 3.2. *Let $t \in [0,T]$ and $x = F(t)$. Then $X(s;t,x,0) = F(s)$ for all $s \in [t,T]$. Moreover, $V(t,x) = 0$ and the optimal strategy is $\pi(\cdot) \equiv 0$.*

Proof. Let $t \in [0,T]$, $x \in \mathbb{R}$, and set $X(\cdot) := X(\cdot;t,x,0)$. The dynamics of $X(\cdot)$ are given by

$$\begin{cases} dX(s) = (rX(s) - b_0)\, ds, & s \in [t,T], \\ X(t) = x. \end{cases}$$

The dynamics of the target $F(\cdot)$ after t is given by

$$\begin{cases} dF(s) = (rF(s) - b_0)\, ds, & s \in [t,T], \\ F(t) = F(t). \end{cases}$$

Therefore $X(\cdot)$ and $F(\cdot)$ solve the same initial value problem, so they coincide.

Moreover, since we have $J(t,x;0) = 0$ and $V(\cdot,\cdot) \geq 0$, we get that $\pi(\cdot) \equiv 0$ is optimal for the initial (t,x) and $V(t,x) = 0$. □

Lemma 3.2 suggests that the graph of $F(\cdot)$ works as a barrier for the problem, so that we are led to consider the region

$$\mathcal{C} = \{(t,x) \mid t \in [0,T], \quad S(t) \leq x \leq F(t)\} \subset \mathcal{D}. \tag{10}$$

As a matter of fact we are interested to start from an initial $(t,x) \in \mathcal{C}$. In this case, the optimal strategy keeps the state always in $\mathcal{C}$. This is a consequence of the following proposition.

Proposition 3.3. *Let $(t,x) \in \mathcal{C}$, $\pi(\cdot) \in \Pi^0_{ad}(t,x)$. Set $X(\cdot) := X(\cdot;t,x,\pi(\cdot))$ and define the stopping time*

$$\tau := \inf\{s \geq t \mid X(s) = F(s)\},$$

with the convention $\inf \emptyset = T$. Define the strategy

$$\pi^\tau(s) := \begin{cases} \pi(s), & \text{if } s < \tau, \\ 0, & \text{if } s \geq \tau. \end{cases}$$

Then $J(t,x;\pi^\tau(\cdot)) \leq J(t,x;\pi(\cdot))$. As a consequence, on $\mathcal{C}$ the value function admits the representation

$$V(t,x) = \inf_{\pi(\cdot) \in \Pi_{ad}(t,x)} J(t,x;\pi(\cdot)),$$

where

$$\Pi_{ad}(t,x) = \left\{\pi(\cdot) \in \Pi^0_{ad}(t,x) \mid S(s) \leq X(s;t,x,\pi(\cdot)) \leq F(s), \ s \in [t,T]\right\} \subset \Pi^0_{ad}(t,x).$$

Proof. It follows straightly from Lemma 3.2. □

Remark 3.4. *A relevant consequence of the argument leading to Proposition 3.3 is that if we replace the loss function (3) with (4), i.e. with*

$$\tilde{L}(s,x) = \begin{cases} (F(s) - x)^2, & \text{if } x \leq F(s), \\ 0, & \text{if } x > F(s), \end{cases} \tag{11}$$

and call $\tilde{V}$ the value function associated to such loss function, we have $\tilde{V} = V$ on $\mathcal{C}$ and, starting from $(t,x) \in \mathcal{C}$, we have the same optimal feedback strategy. This different formulation of the problem might be more appealing to financial advisors of pension funds. In fact, a model based on a loss function such as (11) can be immediately understood and accepted by any pensioner, without entering the mathematical technicalities of the model.

Proposition 3.3 says that on the set $\mathcal{C}$ the original problem is equivalent to the problem with state constraint

$$S(s) \le X(s) \le F(s), \qquad s \in [t,T].$$

The analogue of Proposition 3.1 is the following.

Proposition 3.5. *Let $(t,x) \in \mathcal{C}$. Then*

(1) $0 \in \Pi_{ad}(t,x)$.

(2) *If $x = S(t)$ (respectively, $x = F(t)$), then $\Pi_{ad}(t,x) = \{0\}$ and $X(s;t,x,0) = S(s)$ (respectively, $X(s;t,x,0) = F(s)$) on $[t,T]$.*

(3) *$\pi(\cdot) \in \Pi_{ad}(t,x)$ if and only if*

$$\pi(s) = \pi(s)\mathbf{1}_{\{t \le s < \tau\}},$$

where

$$\tau := \inf\{s \in [t,T] \mid X(s;t,x,\pi(\cdot)) \in \{S(s), F(s)\}\}$$

with the convention that $\inf \emptyset = T$.

(4) *If $S(t) < x < F(t)$, then $\Pi_{ad}(s,x) \supsetneq \{0\}$.*

Proof. The claims can be obtained exactly as in the proof of Proposition 3.1. $\square$

Notice that, rephrasing the problem in these new terms, both the lateral boundaries

$$\partial_F^* \mathcal{C} := \{(t,x) \in [0,T] \times \mathbb{R} \mid x = F(t)\}, \quad \partial_S^* \mathcal{C} := \{(t,x) \in [0,T] \times \mathbb{R} \mid x = S(t)\}$$

are absorbing for the problem, in the sense that if $x = S(t)$ (respectively, $x = F(t)$), then the only admissible strategy is $\pi(\cdot) \equiv 0$ and $X(s;t,x,0) = S(s)$ for $s \in [t,T]$ (respectively, $X(s;t,x,0) = F(s)$ for $s \in [t,T]$).

3.3 Reducing the problem to a rectangle

Here we perform a change of variable in order to work with a simpler stochastic control problem. The domain $\mathcal{C}$ will be transformed into a rectangle and our value function V will be related to the value function H of this new control problem.

Let us consider the diffeomorphism $\mathcal{L} : [0,T] \times [S,F] \to \mathcal{C}$,

$$(t,z) \longmapsto (t,x) = \mathcal{L}(t,z) = (t, \mathcal{L}_1(t,z)) := \left(t, ze^{-r(T-t)} + \frac{b_0}{r}\left(1 - e^{-r(T-t)}\right)\right).$$

Remark 3.6. *The relationship between x and z given by $x = \mathcal{L}_1(t,z)$ is clear: z is the fund that one would have at time T with the riskless strategy from t onwards. In other words, $z = X(T;t,x,0)$. In particular, $F(t) = \mathcal{L}_1(t,F)$ and $S(t) = \mathcal{L}_1(t,S)$.*

Let $(t,x) \in \mathcal{C}$ and $\pi(\cdot) \in \Pi_{ad}(t,x)$. By application of Ito's formula to the process

$$Z(s) = [\mathcal{L}_1(s,\cdot)]^{-1}\left(X(s;t,x,\pi(\cdot))\right), \quad s \in [t,T], \tag{12}$$

we see that Z solves

$$\begin{cases} dZ(s) = \sigma\tilde{\pi}(s)\left[\beta dt + dB(s)\right], & s \in [t,T], \\ Z(t) = z := [\mathcal{L}_1(t,\cdot)]^{-1}(x), \end{cases} \tag{13}$$

where $\tilde{\pi}(s) = e^{r(T-s)}\pi(s)$. For $(t,z) \in [0,T] \times [S,F]$ define

$$\tilde{\Pi}_{ad}(t,z) := \left\{ \tilde{\pi}(\cdot) \in L^2(\Omega \times [t,T];\mathbb{R}) \mid \tilde{\pi}(\cdot) \text{ is prog. meas. w.r.t. } \mathbb{F}^t,\ S \le Z(s;t,z,\tilde{\pi}(\cdot)) \le F,\ s \in [t,T] \right\}.$$

Due to (12), we have $\tilde{\Pi}_{ad}(t,z) = \Pi_{ad}(t,\mathcal{L}_1(t,z))$.

Consider the objective functional

$$\tilde{J}(t,z;\tilde{\pi}(\cdot)) := \mathbb{E}\left[\int_t^T \kappa\,\eta(s)(F - Z(s))^2 ds + \eta(T)(F - Z(T))^2\right], \tag{14}$$

where

$$\eta(s) := e^{-\rho s - 2r(T-s)}$$

and $Z(\cdot) := Z(\cdot;t,z,\tilde{\pi}(\cdot))$ follows the dynamics (13). Then consider the associated optimization problem: for given $(t,z) \in [0,T] \times [S,F]$,

$$\text{minimize} \quad \tilde{J}(t,z;\tilde{\pi}(\cdot)) \quad \text{over} \quad \tilde{\pi}(\cdot) \in \tilde{\Pi}_{ad}(t,z). \tag{15}$$

As usual, define the value function for this problem as

$$H(t,z) := \inf_{\tilde{\pi}(\cdot) \in \tilde{\Pi}_{ad}(t,z)} \tilde{J}(t,z;\tilde{\pi}(\cdot)), \quad (t,z) \in [0,T] \times [S,F]. \tag{16}$$

We can easily see that

$$H(t,z) = V(t,\mathcal{L}_1(t,z)). \tag{17}$$

It follows that all the analysis which will be done for the problem (15) and for its associated value function H can be suitably rephrased for the problem (7) and for its associated value function V. Therefore, from now on within this section, we will study the problem (15) and the associated value function H, which is simpler. The advantage is that the lateral boundaries are now

$$[0,T) \times \{S\}, \qquad [0,T) \times \{F\}.$$

They are absorbing for this new problem, in the sense that if $z = S$ (respectively, $z = F$), then the only admissible strategy is $\tilde{\pi}(\cdot) \equiv 0$ and $Z(s;t,S,0) = S$ for all $s \in [t,T]$ (respectively, $Z(s;t,F,0) = F$ for all $s \in [t,T]$).

It can be shown that the value function H is convex, nonincreasing and continuous. In particular, it has minimum at F. The proofs are standard and can be found in the extended paper [Di Giacinto, Federico, Gozzi, & Vigna, 2010].

Concerning the value of H at the lateral boundaries, as immediate consequence of the absorbing property of the lateral boundaries we have

$$H(t,F) = 0,\ \forall t \in [0,T), \tag{18}$$

$$H(t,S) = \psi(t) + \eta(T)(F - S)^2,\ \forall t \in [0,T), \tag{19}$$

where

$$\psi(t) = \int_t^T \kappa\,\eta(s)(F - S)^2 ds. \tag{20}$$

3.4 The HJB equation

In this section we write down the HJB equation for the value function H. Despite the change of variable of the previous subsection, the difference between the differential problem we are going to set defining the HJB equation and the one studied in [Gerrard, Haberman & Vigna, 2004] is that the domain is smaller in space and suitable boundary conditions must be imposed. The presence of boundary conditions is due to the state constraint that we are imposing here. As usual, the presence of boundary conditions makes the HJB equation much more difficult to treat.

The current value Hamiltonian is

$$\begin{array}{rccc} \mathcal{H}_{cv}: & \mathbb{R}^2 \times [0,+\infty) & \longrightarrow & \mathbb{R}, \\ & (p,P;\tilde{\pi}) & \longmapsto & \frac{1}{2}\sigma^2 P \tilde{\pi}^2 + \sigma\beta p \tilde{\pi} \end{array}$$

and the Hamiltonian is

$$\begin{array}{rccc} \mathcal{H}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \cup \{-\infty\}, \\ & (p,P) & \longmapsto & \inf_{\tilde{\pi} \geq 0} \mathcal{H}_{cv}(p,P;\tilde{\pi}). \end{array}$$

Given $(p,P) \in [0,T) \times \mathbb{R} \times (0,+\infty)$, the function $\pi \mapsto \mathcal{H}_{cv}(p,P;\tilde{\pi})$ has a unique minimum point on $[0,+\infty)$ given by

$$\tilde{\pi}^*(t,p,P) = \left(-\frac{\beta\, p}{\sigma P}\right) \vee 0, \tag{21}$$

hence in this case the Hamiltonian can be written as

$$\mathcal{H}(p,P) = \begin{cases} -\dfrac{\beta^2 p^2}{2P}, & \text{if } p < 0, \\ 0, & \text{if } p \geq 0. \end{cases} \tag{22}$$

If $P \leq 0$ the Hamiltonian is

$$\mathcal{H}(p,P) = \begin{cases} -\infty, & \text{if } p < 0, \\ 0, & \text{if } p \geq 0. \end{cases}$$

We recall that in the Hamiltonian, p is the formal argument where to insert H_z, and P is the formal argument where to insert H_{zz} (if these derivatives exist). However, as mentioned, it can be proved in standard way that the value function H is convex and nonincreasing. Therefore, only negative values of p and positive values of P are consistent with H_z, H_{zz}. Nevertheless, we allow in principle the formal arguments p, P of $\mathcal{H}$ to range in the whole $\mathbb{R}^2$.

The HJB equation (see e.g. [Yong & Zhou, 1999, Ch. 4]) reads as

$$h_t(t,z) + \kappa\eta(t)(F-z)^2 + \mathcal{H}(h_z(t,z), h_{zz}(t,z)) = 0, \quad (t,z) \in [0,T) \times (S,F). \tag{23}$$

3.4.1 Passage to a dual equation

Now we associate a semilinear PDE to the fully nonlinear PDE (23), by means of a dual transformation of the state variable. This technique has been already used in the case of HJB equations coming from optimal portfolio allocation problems (for which the nonlinearity in the second order term takes the form v_x^2/v_{xx}). We refer, e.g., to [Elie & Touzi, 2008] and [Schwartz & Tebaldi, 2006] in a lifetime consumption and investment problem, to [Gao, 2008] and [Xiao , Zhai & Qin, 2007] in the accumulation phase of a pension fund, to [Milevsky, Moore & Young, 2006], [Milevsky & Young, 2007] and [Gerrard, Højgaard & Vigna, 2011] in the decumulation phase of a pension fund.

When $\kappa = 0$, like in all the above quoted papers[5] the resulting dual PDE is linear and we can find an explicit solution that will be presented in Subsection 3.5. In the general case $\kappa \neq 0$ the resulting dual PDE is just semilinear and we are not able to find explicit solutions, so it becomes necessary to study the regularity of its solutions. We perform this analysis in the extended paper [Di Giacinto, Federico, Gozzi, & Vigna, 2010], while in the remainder of this section we limit ourselves to show, in the general case $\kappa \geq 0$, the formal connection between the original and the dual problem.

Consider the following formal argument. For every $(t, y) \in [0, T) \times (0, +\infty)$, take the function

$$[S, F] \to \mathbb{R}^+, \quad z \mapsto H(t, z) + zy.$$

Assume that for each $(t, y) \in [0, T) \times (0, +\infty)$ such function has a unique minimizer $g(t, y) \in (S, F)$. Then, if H is C^1 in the space variable, such minimizer is characterized by the relation

$$H_z(t, g(t, y)) = -y. \tag{24}$$

We want to write an equation for g. To do that suppose that the value function H belongs to $C^{1,3}([0, T) \times (S, F); \mathbb{R})$. Then, by standard arguments (see [Yong & Zhou, 1999]), H is a solution of the HJB equation (23). Differentiating (23) we get

$$H_{tz}(t, z) = 2\kappa\, \eta(t)(F - z) + \frac{\beta^2}{2} \frac{2H_z(t, z)H_{zz}(t, z)^2 - H_z^2 H_{zzz}(t, z)}{H_{zz}(t, z)^2}, \quad (t, z) \in [0, T) \times (S, F). \tag{25}$$

Moreover, if $g \in C^{1,2}([0, T) \times (0, +\infty); \mathbb{R})$, differentiating (24) with respect to t, with respect to y, and twice with respect to y, we obtain

$$H_{tz}(t, g(t, y)) + H_{zz}(t, g(t, y))g_t(t, y) = 0, \tag{26}$$

$$H_{zz}(t, g(t, y))g_y(t, y) = -1, \tag{27}$$

$$H_{zzz}(t, g(t, y))g_y^2(t, y) + H_{zz}(t, g(t, y))g_{yy}(t, y) = 0. \tag{28}$$

Plugging (26), (27), and (28) into (25) we get the following semilinear equation for g:

$$g_t(t, y) + \beta^2 y g_y(t, y) + \frac{\beta^2}{2} y^2 g_{yy}(t, y) - 2\kappa\eta(t)(F - g(t, y))g_y(t, y) = 0, \text{ on } [0, T) \times (0, +\infty). \tag{29}$$

A natural set of boundary conditions for g is to take Dirichlet conditions at the space state boundary and a terminal condition at time T. Concerning the terminal boundary condition, it is easy to find it using the terminal boundary condition for H. Indeed by (24) it has to be

$$g(T, y) = \left(F - \frac{y}{2\eta(T)} \right) \vee S.$$

Concerning the Dirichlet boundary conditions, it follows from (24) that they have to be related to Dirichlet boundary conditions on H_z. Such conditions are the following:[6]

$$\text{(i) } H_z(t, F) = 0, \quad \text{(ii) } \lim_{z \downarrow S} H_z(t, z) = -\infty, \quad \forall t \in [0, T). \tag{30}$$

[5] Apart from [Schwartz & Tebaldi, 2006] where the dual equation is still fully nonlinear.

[6] Concerning these conditions we notice that the first one can be proved directly (see [Di Giacinto, Federico, Gozzi, & Vigna, 2010]), while the second one is more difficult to prove, but it has the following intuition behind. The marginal loss when the fund approaches the safety level is huge. This is clear if one thinks to the main idea of this paper. The retiree takes the income drawdown option in order to be better off than immediate annuitization, and therefore she aims at reaching the target. Her worst scenario is (and has to be) falling into the safety level.

If

$$H_{zz}(t,\cdot) > 0, \tag{31}$$

then $H_z(t,\cdot)$ is invertible and the latter boundary conditions on H_z imply the following ones for g:

$$\text{(i) } g(t,F) = 0, \quad \text{(ii) } \lim_{y\to+\infty} g(t,y) = S. \tag{32}$$

All these arguments can be made rigorous ending up with the following proposition.

Proposition 3.7. *Suppose that the value function H belongs to the class $C^{1,3}([0,T)\times(S,F);\mathbb{R})$, that it satisfies (30)-(31). Let g be defined by (24). Then $g \in C^{1,2}([0,T)\times(0,+\infty);\mathbb{R})$ and it is a solution of (29)-(32).*

The converse passage, i.e. the passage from a solution g of (29)-(32) to a solution h of (23)-(30) is much more technical and is proven in the general case $\kappa \geq 0$ in [Di Giacinto, Federico, Gozzi, & Vigna, 2010]. However, when explicit solutions are available for (29)-(32), the proof becomes easier. This happens in the special case $\kappa = 0$, as it is shown in the next subsection.

3.5 Solution in the case with no running cost

It is very hard – maybe impossible – to find explicit solutions to (29)-(32) when $\kappa \neq 0$. In this paper, we treat only the case $\kappa = 0$, while the general case $\kappa \neq 0$ – that needs the viscosity approach and a remarkably more technical analysis as well as non-standard results – is treated in the extended paper [Di Giacinto, Federico, Gozzi, & Vigna, 2010]. Notice that, in this context, the case $\kappa \neq 0$ is mainly interesting from the mathematical point of view. Indeed, we observe that the main assumption of this paper is that the retiree enters retirement at time $t = 0$ and takes the income drawdown option until compulsory annuitization at time $t = T$, with no action in the intervening period. Therefore, the desire of closeness to a target fund over time, although perfectly reasonable, does not seem to be strictly necessary and can be dropped without rendering the problem unrealistic or less interesting.

On the other hand, we notice that finding optimal strategies that avoid ruin a priori is of great interest in itself, given that a substantial stream of literature addresses the relevant issue of avoiding ruin or minimizing its probability in income drawdown problems. See, among others, [Albrecht & Maurer, 2002], [Gerrard, Højgaard & Vigna, 2011], [Milevsky, Moore & Young, 2006], and [Milevsky & Robinson, 2000]. In these papers the flexibility of choosing a guaranteed final fund $S > 0$ and the guarantee of a positive income b_0 from retirement to final annuitization – that is the motivation of this paper – are missing. Indeed, up to our knowledge, this is the first model in the literature on income drawdown option that allows the pensioner to choose a minimum guaranteed level of wealth at the time T of ultimate annuitization, while guaranteeing a fixed consumption rate up to time T.

So, let $\kappa = 0$. In this case (29) becomes

$$g_t(t,y) + \beta^2 y g_y(t,y) + \frac{\beta^2}{2} y^2 g_{yy}(t,y) = 0, \quad \text{on } [0,T)\times(0,+\infty) \tag{33}$$

with boundary conditions (32). This is a linear PDE of Black-Scholes type and, as known, its (classical) solution admits the Kolmogorov probabilistic representation

$$g(t,y) = \mathbb{E}\left[g(T, Y(T;t,y))\right], \quad (t,y) \in [0,T]\times[0,+\infty),$$

where $Y(\cdot;t,y)$ is the solution of

$$\begin{cases} dY(s) = \beta^2 Y(s)ds + \beta Y(s)dB(s), \quad s \in [t,T], \\ Y(t) = y, \end{cases} \tag{34}$$

Since the law of $Y(T;t,y)$ is known (it is the log-normal law), we can explicitly compute g. Indeed, the unique classical solution of (33) with boundary conditions (32) is the function

$$\begin{cases} g(t,y) = (F-S)\Phi(k(t,y)) - \dfrac{y}{2\eta(T)} e^{\beta^2(T-t)}\Phi(k(t,y)-\beta\sqrt{T-t}) + S, \ (t,y) \in [0,T)\times[0,+\infty), \\ g(T,y) = \left(F - \dfrac{y}{2\eta(T)}\right) \vee S, \end{cases} \tag{35}$$

where

$$k(t,y) = \frac{-\log\left(\dfrac{y}{2\eta(T)(F-S)}\right) - \dfrac{\beta^2}{2}(T-t)}{\beta\sqrt{T-t}}$$

and where Φ is the cumulative distribution function of a standard Gaussian random variable, i.e. $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi$. Having at disposal the closed-form solution for $g(t,y)$, we can now complete Proposition 3.7 and prove the second half of the equivalence between the original problem and the dual one.

Proposition 3.8. *Let g be defined by* (35)*. Then*

$$[g(t,\cdot)]^{-1} \textit{ is integrable at } S^+, \ \forall t \in [0,T). \tag{36}$$

Moreover let

$$\begin{cases} h(t,z) = \eta(T)(F-S)^2 - \displaystyle\int_S^z [g(t,\cdot)]^{-1}(\xi)d\xi, \quad (t,z) \in [0,T)\times[S,F], \\ h(T,z) = \eta(T)(F-z)^2, \quad z \in [S,F]. \end{cases} \tag{37}$$

Then h is a classical solution of (23)-(30) *with* $\kappa = 0$.

Proof. It follows by direct computations. It can be seen also as a particular case of Proposition 4.18 of [Di Giacinto, Federico, Gozzi, & Vigna, 2010]. □

Proposition 3.8 provides a function h which is a good candidate to be the value function H. We are going to show that it is actually the value function through a verification theorem and use it to construct optimal feedback strategies for the problem. First we associate to the function h provided by Proposition 3.8 a feedback map which formally describes the optimal strategy in feedback form. Due to (21) this map is

$$G(t,z) := \begin{cases} -\dfrac{\beta}{\sigma}\dfrac{h_z(t,z)}{h_{zz}(t,z)}, & (t,z) \in [0,T)\times(S,F), \\ 0, & (t,z) \in [0,T)\times\{S,F\}, \end{cases} \tag{38}$$

where h is defined in (37). It is more convenient to rewrite it in terms of the solution g provided by (35). Using (26), (27), and (28), it reads as

$$G(t,z) = -\frac{\beta}{\sigma}\ [g(t,\cdot)]^{-1}(z)\ g_y\left(t, [g(t,\cdot)]^{-1}(z)\right), \quad \text{on } [0,T)\times(S,F).$$

Then, taking into account the explicit expression (35), we see that G is continuous and bounded on $[0,T)\times[S,F]$. It can also be shown (see [Di Giacinto, Federico, Gozzi, & Vigna, 2010]) that the map G defined in (38) is not Lipschitz continuous with respect to z, but is α-Hölder continuous with respect to z uniformly in $t \in [0,t_0]$ for every $t_0 \in [0,T)$ and $\alpha \in (0,1)$. This result would be suitable to study directly the closed loop equation, proving existence and uniqueness of a strong solution.[7] However, here we follow a different approach, and prove existence and uniqueness of a strong solution by means of the process $Y^{*,t,y^{-1}} = (Y^{t,y})^{-1}$; this process will play a role on the interpretation of the optimal wealth in Remark 3.12. Let $y := [[g(t,\cdot)]^{-1}(z)]^{-1}$, i.e. $y^{-1} := [g(t,\cdot)]^{-1}(z)$. Then $Y^{*,t,y^{-1}}$ solves

$$\begin{cases} dY^{*,t,y^{-1}}(s) = -\beta Y^{*,t,y^{-1}}(s)dB(s), \quad s\in[t,T], \\ Y^{*,t,y^{-1}}(t) = [g(t,\cdot)]^{-1}(z). \end{cases} \tag{39}$$

Consider the process

$$Z^*(s;t,z) := g\big(s, Y^{*,t,y^{-1}}(s)\big), \qquad y^{-1} := [g(t,\cdot)]^{-1}(z). \tag{40}$$

We notice that by definition of $Z^*(\cdot;t,z)$, since $Y^{*,t,z}(s) > 0$ and $g(t,\cdot) \in (S,F)$ on $(0,+\infty)$ for every $t \in [0,T)$, we have

$$Z^*(s;t,z) \in (S,F), \ \forall s \in [t,T]. \tag{41}$$

Ito's formula and the fact that g solves (33) yield that $Z^*(\cdot;t,z)$ solves the closed loop equation associated with the map G, i.e.

$$\begin{cases} dZ(s) = \sigma G(s,Z(s))\,[\beta ds + dB(s)]\,, \quad s\in[t,T], \\ Z(t) = z \in (S,F), \end{cases} \tag{42}$$

on the interval $[t,T]$. Furthermore, equation (42) admits the solution $Z(\cdot) \equiv F$ (respectively, $Z(\cdot)\equiv S$) if $z=F$ (respectively, if $z=S$). So we also set $Z^*(\cdot;t,S)\equiv S$ and $Z^*(\cdot;t,F)\equiv F$. Then define the feedback strategy

$$\pi^*_{t,z}(s) = \begin{cases} G(s,Z^*(s;t,z)), & s\in[t,T). \\ 0, & s=T, \end{cases} \tag{43}$$

Of course $\tilde\pi^*_{t,z}(s) \in L^2(\Omega\times[t,T])$ as G is bounded. Moreover

$$Z^*(s) = Z(s;t,z,\tilde\pi^*_{t,z}(\cdot)). \tag{44}$$

Indeed, both $Z(\cdot;t,z,\tilde\pi^*_{t,z}(\cdot))$ and $Z^*(\cdot;t,z)$ solve the state equation under the control $\tilde\pi^*_{t,z}(\cdot)$. Therefore we conclude that $\tilde\pi^*_{t,z}(\cdot)$ is admissible. We now show that it is indeed the unique optimal strategy starting from (t,z). To prove the uniqueness, first we need to prove that the functional (14) is strictly convex on $\tilde\Pi_{ad}(t,z)$. This is the result of the following proposition.

Proposition 3.9. *Let $(t,z)\in[0,T)\times[S,F]$. Then the functional $\tilde\Pi_{ad}(t,z)\to\mathbb{R}$, $\tilde\pi(\cdot)\mapsto\tilde J(t,z;\tilde\pi(\cdot))$ is strictly convex.*

Proof. Let $(t,z)\in[0,T)\times[S,F]$ and take $\tilde\pi_1(\cdot),\tilde\pi_2(\cdot)\in\tilde\Pi_{ad}(t,z)$. Further, for $\lambda\in(0,1)$ let $\tilde\pi_\lambda(\cdot) := \lambda\tilde\pi_1(\cdot)+(1-\lambda)\tilde\pi_2(\cdot)$. Defining $Z_1(\cdot) := Z(\cdot;t,z,\tilde\pi_1(\cdot))$, $Z_2(\cdot) := Z(\cdot;t,z,\tilde\pi_2(\cdot))$, and

[7] In this case, we would use the theory treated in [Yamada & Watanabe, 1971] to prove pathwise uniqueness and then existence of strong solutions. For a similar approach see, e.g., [Di Giacinto, Federico & Gozzi, 2011].

$Z_\lambda(\cdot) = \lambda Z_1(\cdot) + (1-\lambda) Z_2(\cdot)$ we see that $Z_\lambda(\cdot) := Z(\cdot; t, z, \tilde{\pi}_\lambda(\cdot))$. Therefore, due to the convexity of $z \mapsto (F-z)^2$ we have

$$\begin{aligned} \tilde{J}(t, z; \tilde{\pi}_\lambda(\cdot)) &= \mathbb{E}\left[(F - Z_\lambda(T))^2\right] \leq \lambda \mathbb{E}\left[(F - Z_1(T))^2\right] + (1-\lambda)\mathbb{E}\left[(F - Z_2(T))^2\right] \\ &= \lambda \tilde{J}(t, z; \tilde{\pi}_1(\cdot)) + (1-\lambda)\tilde{J}(t, z; \tilde{\pi}_2(\cdot)). \end{aligned}$$

Moreover, by linearity of the state equation we have $Z_1 \neq Z_2$ when $\tilde{\pi}_1 \neq \tilde{\pi}_2$ and by the strict convexity of $z \mapsto (F-z)^2$ the above inequality is strict; so the claim is proved. □

Theorem 3.10. *Let h be the function defined in* (37)*. Then $h = H$ and for each $(t, z) \in [0, T) \times [S, F]$ the strategy $\tilde{\pi}^*_{t,z}(\cdot)$ defined by* (43) *is the unique optimal strategy for the problem* (15) *with $\kappa = 0$.*

Proof. We distinguish the two cases

(i) $(t, z) \in [0, T) \times (S, F)$;

(ii) $(t, z) \in [0, T) \times \{S, F\}$.

In case (ii) the only admissible strategy is the null one and the claim is trivial.

Let us consider case (i). Given an admissible strategy $\tilde{\pi}(\cdot) \in \tilde{\Pi}_{ad}(t, z)$ and using the fact that h solves (33) we can show by standard verification arguments that

$$\tilde{J}\left(t, z; \pi(\cdot)\right) \geq h(t, z).$$

On the other hand consider $\tilde{\pi}^*_{t,z}(\cdot)$. As we have observed $\tilde{\pi}^*_{t,z}(\cdot)$ is admissible. Moreover (41) ensures that $Z(\cdot; t, z, \tilde{\pi}^*_{t,z}(\cdot)) \in (S, F)$. Then the fact that h is a classical solution of the HJB equation (23) on $[0, T) \times (S, F)$ yields (again by standard verification arguments)

$$\tilde{J}(t, z; \tilde{\pi}(\cdot)) = h(t, z).$$

Therefore as a byproduct we can conclude that $h(t, z) = H(t, z)$ and that $\tilde{\pi}^*_{s,z}(\cdot)$ is optimal.

The uniqueness of this optimal strategy follows from the strict convexity of $\tilde{J}(t, z; \cdot)$ proved in Proposition 3.9. □

Remark 3.11. *The uniqueness of the optimal strategy yields uniqueness of solution for the closed loop equation* (42)*. Indeed, suppose to have another solution $\tilde{Z}$ of the closed loop equation. Applying the Dynamic Programming Principle with the stopping time*

$$\tau := \inf\left\{ s \in [t, T] \mid \tilde{Z}(s) \in \{S, F\} \right\}$$

(with the agreement that $\inf \emptyset = T$), and using the fact that H is a smooth solution of the HJB equation (23) *in $[0, T) \times (S, F)$, we can see that the strategy*

$$\tilde{\pi}_{t,z}(s) = \begin{cases} G(s, \tilde{Z}(s)), & s \in [t, \tau), \\ 0, & s \geq \tau, \end{cases}$$

is optimal for the problem. By uniqueness of optimal strategies (Proposition 3.9) it must be $\tilde{\pi}_{t,z} = \pi^*_{t,z}$. Moreover we can see that G is monotone in z. Hence we have also $\tilde{Z} = Z^*$, where Z^* is defined in (40)*.*

Remark 3.12. *By a different approach, we have obtained the same result as in [Tepla, 2001]. Indeed, let us look at the optimization problem we have defined in Subsection 3.3.*[8] *Consider the definition of the optimal wealth Z^* we have given in* (40) *for initial data* $(0, z_0)$ *and the risk neutral probability measure Q, the process $B^Q(t) = \beta t + B(t)$ is a Browinan motion, therefore, due to* (42)*, the process Z^* is a Q-martingale. So, setting $Y^* = Y^{*,0,y^{-1}}$, with $y^{-1} = [g(0,\cdot)]^{-1}(z_0)$, we have*

$$Z^*(t) = \mathbb{E}_Q\left[Z^*(T) \mid \mathcal{F}_t\right] = \mathbb{E}_Q\left[g(T, Y^*(T)) \mid \mathcal{F}_t\right] = S + \mathbb{E}_Q\left[g(T, Y^*(T)) - S \mid \mathcal{F}_t\right].$$

Hence, the value of the optimal wealth at time t is equal to the minimum guarantee S plus the value of an option with underlying Y^* and with payoff function $g(T, y) - S$. This is indeed the same result obtained in [Tepla, 2001]. We observe also that there is also an insurance interpretation to the solution found, as mentioned in [Tepla, 2001] and widely described in [El Karoui, Jeanblanc & Lacoste, 2005]: it is the so called Option Based Portfolio Insurance method introduced by [Leland & Rubinstein, 1976].

4 Numerical application

In this section we show a numerical application of the model presented so far using a MATLAB code. We consider the position of a male retiree aged 60 with initial wealth $x_0 = 100$. Consistently with [Gerrard, Haberman & Vigna, 2004], we set $T = 15$. The market parameters are $r = 0.03, \mu = 0.08, \sigma = 0.15$, implying a Sharpe ratio equal to $\beta = 0.33$. The amount withdrawn in the unit time, b_0, is set equal to the pension rate purchasable at retirement, using Italian projected mortality tables (RG48). Thus, we set $b_0 = 6.22$. This choice is consistent with previous literature on the topic.

The choice of the final target F and the final guarantee S are evidently subjective and depend on the member's risk aversion. High risk aversion will lead to a high guarantee and a low level of the target, while a high target and a low guarantee will be driven by low risk aversion. We have tested three levels of risk aversion. Thus, high risk aversion is associated to terminal safety level $S = \frac{2}{3}b_0 a_{75}$ and final target equal to $F = 1.5 b_0 a_{75}$, where a_{75} is the actuarial value of a unitary lifetime annuity issued to an individual aged 75; medium risk aversion is associated to terminal safety level $S = \frac{1}{2}b_0 a_{75}$ and final target equal to $F = 1.75 b_0 a_{75}$; low risk aversion is associated to terminal safety level equal to $S = 0$ and final target equal to $F = 2 b_0 a_{75}$. These values are reported in Table 1 below.

	S	F
High risk aversion	$\frac{2}{3}b_0 a_{75}$	$1.5 b_0 a_{75}$
Medium risk aversion	$\frac{1}{2}b_0 a_{75}$	$1.75 b_0 a_{75}$
Low risk aversion	0	$2 b_0 a_{75}$

Table 1: Terminal safety level S and final target F for different risk profiles.

The interpretation of these choices is immediate. With high and medium risk aversion, the minimum pension rate guaranteed is, respectively, two third and half of the annuity rate that was possible

[8] In this case, the interest rate has been removed - as well as the benefit streaming $b_0 dt$ - so we are in presence of a market composed by a riskless asset with vanishing return rate and a risky asset with volatility σ and risk premium β. We consider this problem in place of the original one just for convenience.

to have on immediate annuitization at retirement, $b_0 a_{75}$; the targeted wealth is sufficient to fund a final pension that amounts to, respectively, 1.5 and 1.75 of b_0. With low risk aversion, ruin is always avoided but in the worst scenario no money is left for annuitization at age 75; on the other hand, the targeted pension pursued is twice b_0.

Remark 4.1. *It is worth mentioning that even the most risk averse individual has some restrictions in choosing the minimum income guaranteed. Indeed, from the formulation of the problem (see restriction* (9) *on* S*) we have that the value of* S *has to satisfy* $S \leq z_0 = x_0 e^{rT} - \frac{b_0}{r}(e^{rT} - 1)$*. The most risk averse choice would be* $S = z_0$*, but in this case the only admissible strategy would be* $\pi(\cdot) \equiv 0$*, i.e. the whole fund wealth must be invested in the riskless asset (see Proposition (3.1)), and one would end up after 15 years with an annuity lower than that purchasable at retirement.*[9] *This choice makes little sense in a realistic framework, given that here the bequest motive is disregarded and the individual takes the income drawdown option only in the hope of being able to buy a better annuity than* b_0*. For this reason, we here consider only cases where* $S < z_0$*, which in this particular example translates into* $S < \frac{2}{3} b_0 \, a_{75}$*.*

We have carried out 1000 Monte Carlo simulations for the behaviour of the risky asset, with discretization step equal to one week. In order to do so, we have simulated the process Y^* given by equation (39) with starting point

$$Y^*(0) = [g(0, \cdot)]^{-1}(z_0)$$

and inserting the corresponding values of S, F and z_0 as above. With each risk aversion we have generated the same 1000 scenarios, by applying in each case the same stream of pseudo random numbers.

For each risk aversion choice, we report the following results:

- Evolution of the fund under optimal control during the 15 years time, by showing a graph with mean and standard deviation and a graph with some percentiles.
- Behaviour of the optimal investment strategy over the 15 years time, by showing a graph with some percentiles. Notice that we report the optimal share of portfolio invested in the risky asset $\pi^*(\cdot)/X^*(\cdot)$, rather than the optimal amount $\pi^*(\cdot)$. This is standard, and is done in order to facilitate comparisons between different situations.
- Distribution of the final annuity that can be bought with the final fund at age 75 and comparison with the annuity purchasable at retirement. The conversion of the final fund into annuity has been done with the same basis as above.

Figures 1–4 report results for high risk aversion, Figures 5–8 those for medium risk aversion, Figures 9–12 those for low risk aversion. In particular, Figures 1, 5, and 9 report, over 15 years time, the mean and dispersion of the fund trajectories, while Figures 2, 6, and 10 report their percentiles. Figures 3, 7, and 11 report some percentiles of the distribution of the optimal investment allocation $\pi^*(\cdot)/X^*(\cdot)$ over 15 years. Finally, Figures 4, 8, and 12 report the distribution of the final annuity upon annuitization at time T.

[9] This is clear, considering that the investment in an insurance product benefits from mortality credits, that enhance the riskless rate.

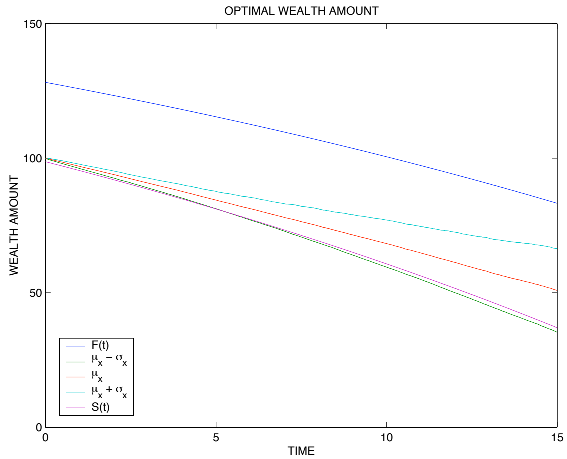

Figure 1: High risk aversion.

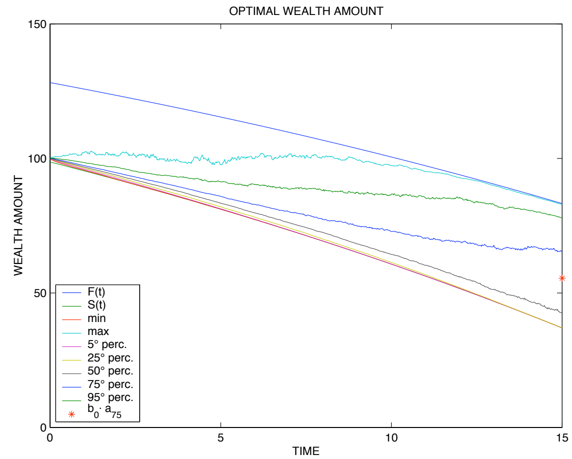

Figure 2: High risk aversion.

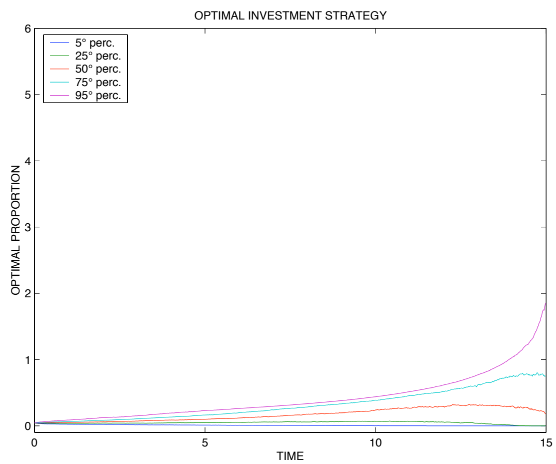

Figure 3: High risk aversion.

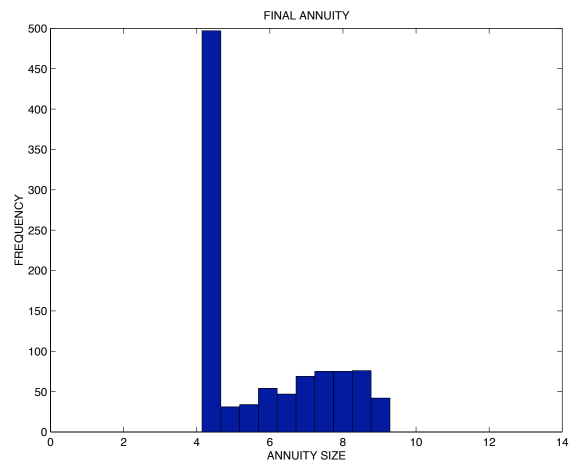

Figure 4: High risk aversion.

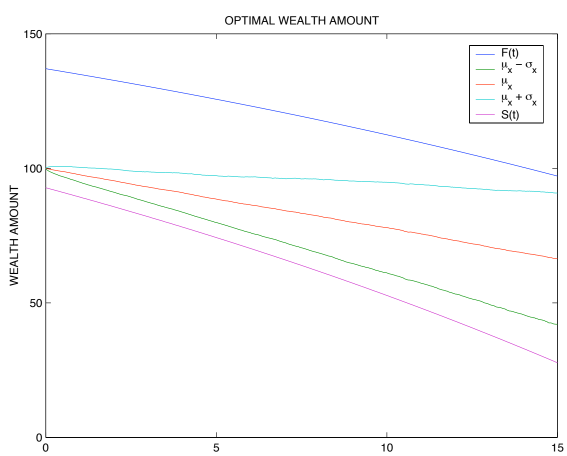

Figure 5: Medium risk aversion.

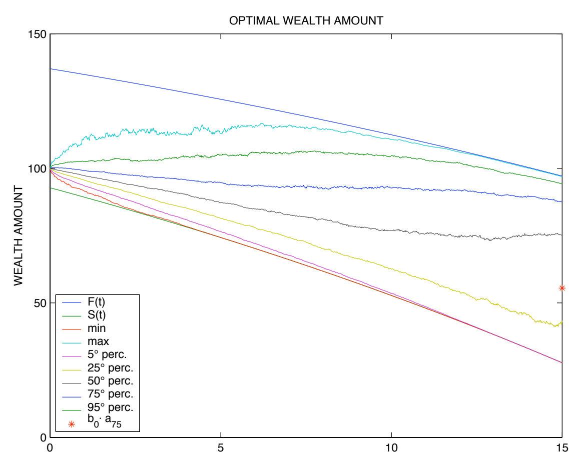

Figure 6: Medium risk aversion.

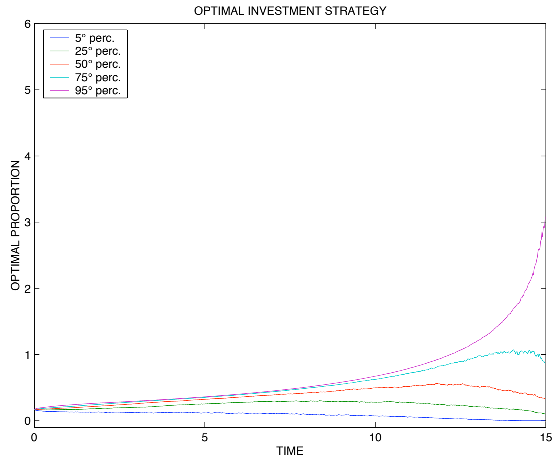

Figure 7: Medium risk aversion.

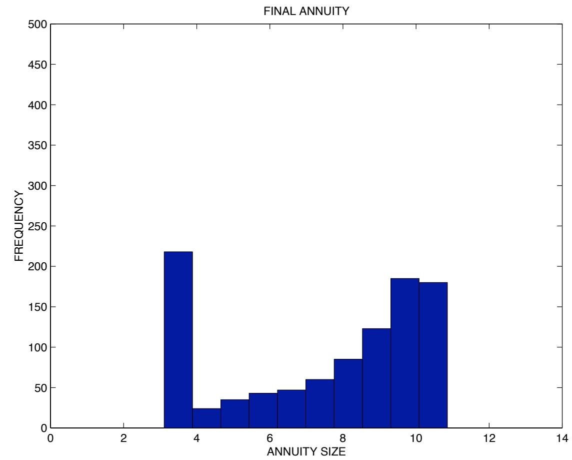

Figure 8: Medium risk aversion.

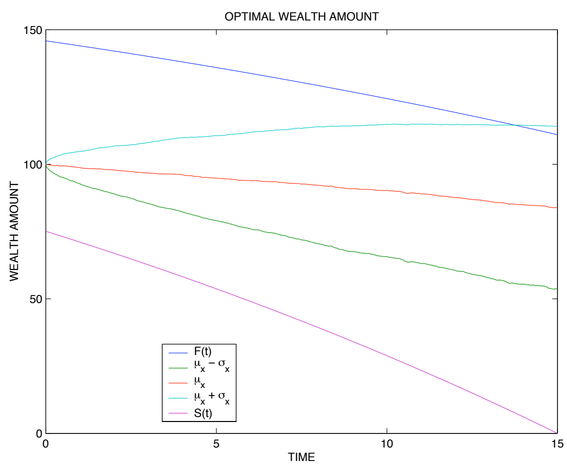

Figure 9: Low risk aversion.

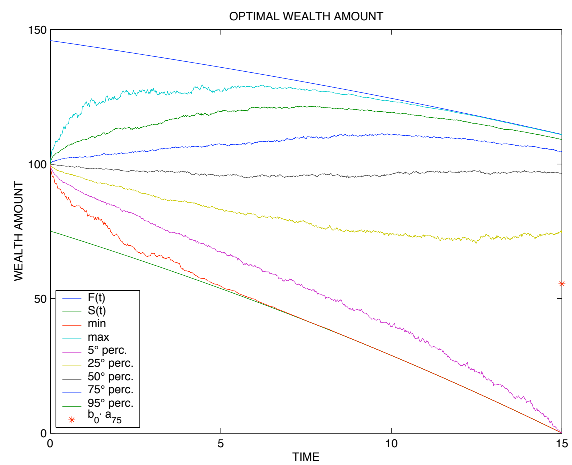

Figure 10: Low risk aversion.

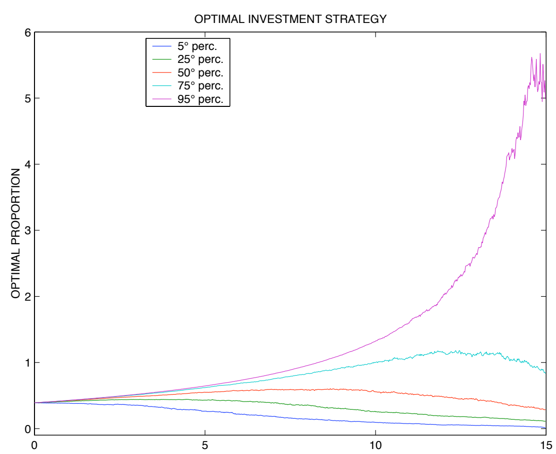

Figure 11: Low risk aversion.

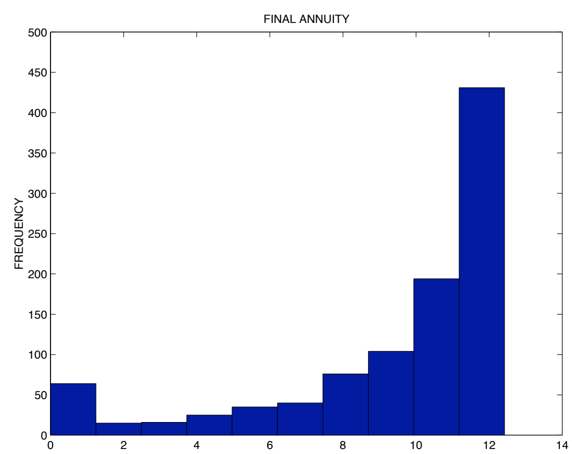

Figure 12: Low risk aversion.

From the graphs we can make the following comments:

- The wealth trajectories lie strictly between the two barriers $S(t)$ and $F(t)$ for $t < T$. In fact, due to (40) and (41), the two bottom and upper absorbing barriers cannot be reached before time T. [10]

- Due to our choice of S and F, when the risk aversion decreases, the boundaries for the wealth process become larger. Also the simulated wealth process results to be more spread out around the mean. This is due to the intuitive fact that the optimal strategies are more aggressive (see next item) and the range of final outcomes increases, both in the positive and in the negative direction.

- Inspection of Figures 3, 7, and 11 shows that when the risk aversion decreases, the optimal strategies become riskier. In fact, with high risk aversion the 95^{th} percentile of the optimal investment allocation $\pi^*(\cdot)/X^*(\cdot)$ stays below 2 even immediately prior to time T, whereas with low risk aversion it lies between 5 and 6 close to T. On the other hand, clearly, all strategies are bounded away from 0.

- Comparing Figures 4, 8, and 12 it is immediate to see that the distribution of the final annuity becomes more and more spread when the risk aversion decreases. Moreover, with high risk aversion one can observe a considerable concentration around the guaranteed income $\frac{2}{3}b_0 = 4.15$. In fact, in almost 50% of the cases, the fund approaches $S(t)$ and stays close to it until T (this can be noticed also by thorough inspection of Figure 2). On the contrary, the distribution of final annuity looks very favourable in the case of low risk aversion, where in most of the cases the annuity lies between 9 and 12, and unfavourable scenarios leading to final income equal to 0 happen in about 5% of the cases.

- We observe that the standard deviation of the investment allocation increases over time, especially towards time T. This can be observed in Figures 3, 7, and 11. The optimal share of portfolio becomes very variable in the 2-3 years before time T. This feature makes this case substantially different from the (state) unconstrained one, where the higher variability of the investment strategy is experienced in the first years after retirement. This interesting difference is evidently due to the inclusion of the absorbing lower barrier $S(t)$. The explanation can be the following: when time T approaches the risk of collapsing onto the safety level reduces remarkably, and many pensioners may be willing to take more risk than in previous years when the risk of locking their position into the safety level is more important.

One should not forget that the real goal of the pensioner who opts for phased withdrawals is to be better off than immediate annuitization when final annuitization takes place. Thus, it is of greatest interest to provide her with detailed information regarding the distribution of the final annuity achieved. To some extent, this has been already shown in Figures 4, 8, and 12. However, the histograms cannot report relevant information that are of immediate use for the member who has to choose a risk profile. In particular, for the member's decision making the comparison between the final annuity achievable by taking income drawdown option and b_0 (the pension rate purchasable at retirement) is relevant. Table 2 reports useful statistics of the distribution of the final annuity achieved at age 75, for each risk aversion. The first nine lines report mean, standard deviation, min, max and some percentiles of the distribution of the final annuity. Lines 10 and 11 report,

[10] Looking at the graphs reporting the percentiles of the trajectories, however, it seems that in some cases the fund touches the bottom target $S(t)$. This is due to the approximation error made by the machine, that is unavoidable. In fact, for not too low values of x, $\Phi(x)$ is so close to 0 that it cannot be distinguished from it. The result is that in the practical applications for not too high values of y^* one has $\Phi\left(k(t, y^*)\right) = \Phi\left(k\left(t, y^*\right) - \beta\sqrt{T-t}\right) = 0$. This implies $g(t, y^*) = S$, meaning that the fund is on the safety level $S(t)$, which is theoretically impossible.

respectively, the guaranteed income S/a_{75} and the targeted income F/a_{75} (as chosen in Table 1), while the last line reports the probability (i.e. the frequency over 1000 scenarios) that the final annuity is higher than b_0.

	HIGH RISK AVERSION	MEDIUM RISK AVERSION	LOW RISK AVERSION
mean	5.70	7.44	9.40
st.dev.	1.74	2.73	3.38
min	4.15	3.11	0.00
5th perc.	4.15	3.11	0.00
25th perc.	4.15	4.87	8.45
50th perc.	4.75	8.41	10.80
75th perc.	7.33	9.80	11.72
95th perc.	8.71	10.55	12.21
max	9.29	10.86	12.42
guaranteed income S/a_{75}	4.15	3.11	0
targeted income F/a_{75}	9.33	10.89	12.44
prob(final annuity $> b_0$)	39.20%	68.80%	84.10%

Table 2: Distribution of final annuity at age 75 when the annuity on immediate annuitization is $b_0 = 6.22$.

The following comments can be made:[11]

- The mean of the final annuity is 5.70, 7.44, 9.40 with high, medium and low risk aversion, respectively. The probability of being able to afford a final annuity higher than $b_0 = 6.22$ is 39.20%, 68.80% and 84.10% with high, medium and low risk aversion, respectively.
- This shows that if the risk aversion is too high,[12] the price for having a high guarantee on the final income is that the chances of reaching the desired annuity reduce dramatically. In fact, in 60% of the cases the individual ends up with a final annuity lower than b_0 and, even worse, in almost 50% of the cases the individual receives exactly the guaranteed income, that is only two third of b_0. This is likely to be an undesirable result for the pensioner and it seems to indicate that if the member's risk aversion is too high, it is not convenient to take the income drawdown option. This feature was already observed by [Gerrard, Haberman & Vigna, 2006].
- On the other hand, with medium and low risk aversion the chances of being better off with annuitization at time T are almost 70% and 85%, respectively. This is an encouraging result,

[11] Notice that, due to the approximation error made by the machine (see previous footnote) the values indicated by the minimum and by the first low percentiles coincide with the guaranteed income.

[12] Observe, in fact, that the value of $S = 0.67\, b_0\, a_{75}$ is chosen to be very close to the upper boundary $z_0 = 0.70\, b_0\, a_{75}$.

given that from retirement to T the pensioner has withdrawn the prescribed rate of b_0 and that she was also guaranteed with a minimum lifetime income at retirement, or at worst against ruin.

- The low risk aversion profile could turn out to be particularly attractive to a member whose global post-retirement income is not heavily affected by the second pillar provision. In fact:
  - the chances of exceeding the immediate annuitization income b_0 are extremely high (84%);
  - in 750 cases out of 1000 the member ends up with an annuity higher than 8.45, that is well above $b_0 = 6.22$;
  - in about 50 cases out of 1000 the final annuity is null (see also Figure 12);
  - ruin never occurs.
- Clearly, the price to pay for having a favourable distribution of final income is to take more risk, which translates into more aggressive investment policies. This is highlighted by Fig.11, that reports the optimal investment strategies for low risk aversion. In more than 25% of the cases, the optimal strategy consists in borrowing considerable amounts of money to be invested in the risky asset. This kind of strategy is evidently not feasible in the presence of real world constraints. Hence, the importance and the need of approaching the problem with a no-borrowing constraint in future research.

5 Conclusions and further research

In this paper we have considered the investment allocation problem for a member of a DC pension scheme in the decumulation phase. We have extended the basic unconstrained model of [Gerrard, Haberman & Vigna, 2004] by introducing a no short-selling constraint and a final capital requirement. It turns out that the wealth process must lie between two barriers: the bottom one representing a natural safety level for the fund, and the upper one representing a sort of target to be pursued. In particular, the presence of the bottom safety level implies that the undesirable event of ruin is avoided. The problem has been formulated in general trough the dynamic programming approach and the associated HJB equation. We have transformed the original problem into a dual one and have shown equivalence between the two problems. In the special but relevant case without running cost, we have solved the dual problem, finding closed-form solution for the value function and the optimal strategy. A numerical application shows the impact of the model on retiree's choices.

Some remarks on the practical relevance of our analysis are the following. To the best of our knowledge this is the first model in the literature on this topic that allows the pensioner to choose a minimum guaranteed level of wealth at the time of ultimate annuitization. The introduction of a minimum guaranteed wealth in the income drawdown scheme should further encourage the selection of this option by retirees. The model is quite flexible for it allows for subjective choices regarding both the safety and the target levels. These choices are typically driven by the needs and the risk profile of the pensioner. In particular, the less risk-averse pensioner can aim to a high target such as double the annuity, while still keeping the guarantee of avoiding ruin; the most risk-averse individual can aim to a lower target, while still guaranteeing a minimum income level upon final annuitization. Moreover, the availability of closed-form expressions for the optimal policy makes the model quite useful for practical purposes. Indeed, supported by encouraging results of the numerical application performed, we think that this model can provide a number of information useful for the setup of a decision-making tool in the decumulation phase of a DC pension plan.

Due to the difficulty of the task, we have not analyzed the more important problem of short-selling and borrowing constraints plus final capital requirement and the case when the investor can choose the consumption rate and the annuitization date. This problem could be tackled at theoretical level with the viscosity approach coupled with the dual transformation – similarly to what done in the extended paper [Di Giacinto, Federico, Gozzi, & Vigna, 2010]. The search for explicit solutions, at least in some special case, seems to be very challenging and is in the agenda for future research.

Appendix

Proof of Proposition 3.1

Proof. (1) Clearly, if $0 \in \Pi^0_{ad}(t,x)$, then $\Pi^0_{ad}(t,x) \neq \emptyset$. Conversely, suppose that $\Pi^0_{ad}(t,x) \neq \emptyset$ and let $\pi(\cdot) \in \Pi^0_{ad}(t,x)$. This means that $X(T;t,x,\pi(\cdot)) \geq S$ almost surely; therefore $\tilde{\mathbb{E}}[X(T;t,x,\pi(\cdot))] \geq S$, where $\tilde{\mathbb{E}}$ denotes the expectation under the probability $\tilde{P} = e^{-\beta B(T)-\frac{\beta^2}{2}T} \cdot P$ given by the Girsanov transformation. Writing the dynamics of $X(\cdot;t,x,\pi(\cdot))$ under $\tilde{P}$ and taking the expectations under $\tilde{\mathbb{E}}$, we see that

$$X(T;t,x,0) = \tilde{\mathbb{E}}[X(T;t,x,\pi(\cdot)] \geq S,$$

hence $0 \in \Pi^0_{ad}(t,x)$. This proves the first part of the claim.

For the second part, notice that the state equation yields

$$X(s;t,x,0) = \frac{b_0}{r} - \left(\frac{b_0}{r} - x\right) e^{r(s-t)},$$

so that from the expression of $S(\cdot)$ in (8) we obtain the claim.

(2) If $x = S(t)$, by the state equation and (8)) we have $X(s;t,x,0) = S(s)$ on $[t,T]$; therefore $0 \in \Pi^0_{ad}(t,x)$. On the other hand, take a strategy $\pi(\cdot) \in \Pi^0_{ad}(t,x)$ and let $X(\cdot) := X(\cdot;t,x,\pi(\cdot))$. Arguing as before, one can see that in this case

$$S = X(T;t,x,0) = \tilde{\mathbb{E}}[X(T)] \geq S.$$

Since $\pi(\cdot)$ is admissible, we have $X(T) \geq S$. Thus, it must be $X(T) \equiv S$ under P. Therefore

$$\mathrm{Var}[X(T)] = 0, \quad \text{under } P \sim \tilde{P}, \tag{45}$$

where $\tilde{P}$ is the probability measure given by the Girsanov transformation described in part (1). We claim that this happens if and only if $\pi(\cdot) \equiv 0$. Let us consider the system under $\tilde{P}$. As shown we have

$$\tilde{\mathbb{E}}[X(s)] = X(s;t,x,0) = \frac{b_0}{r} - \left(\frac{b_0}{r} - x\right) e^{r(s-t)}. \tag{46}$$

Applying Ito's formula to $X(\cdot)$ with the square function under $\tilde{P}$ we get

$$d(X(s))^2 = 2X(s)\left[[rX(s) - b_0]ds + \sigma\pi(s)d\tilde{B}(s)\right] + \sigma^2\pi(s)^2 ds. \tag{47}$$

We want to take the expectations in (47), getting rid of the stochastic integral. This can be done with a localization argument. So we get, passing (47) to the expectations,

$$\tilde{\mathbb{E}}\left[X(T)^2\right] = x^2 e^{2r(T-t)} - \int_t^T e^{2r(T-u)} 2b_0 \tilde{\mathbb{E}}[X(u)]du + \int_t^T e^{2r(T-u)} \sigma^2 \tilde{\mathbb{E}}\left[\pi^2(u)\right] du. \tag{48}$$

Using (45), (46), and (48), straightforward computations yield

$$\int_t^T e^{2r(T-u)} \sigma^2 \tilde{\mathbb{E}}\left[\pi^2(u)\right] du = 0,$$

which proves what we have claimed.

(3) This is obvious, given the previous item.

(4) Let $x > S(t)$. Define the strategy

$$\pi^\tau(s) := \begin{cases} 1, & \text{if } s \in [t, \tau], \\ 0, & \text{if } s \in [\tau, T], \end{cases}$$

where

$$\tau := \inf\left\{s \in [t, T] \mid X(s; t, x, 1) = S(s)\right\}.$$

Then, by the previous item, $\pi^\tau(\cdot) \in \Pi^0_{ad}(t, x)$. Moreover, since $x > S(t)$, we have $\tau > t$. Therefore $\pi(\cdot)$ is not identically null, so the claim.

(5) The claim reduces to show that, for every $\pi(\cdot) \in \Pi^0_{ad}(t, x)$, we have $X(s) \geq S(s)$ almost surely for any $s \in [t, T]$. This follows by items (2) and (3). $\square$

References

[Albrecht & Maurer, 2002] Albrecht P., and Maurer R., *Self-Annuitization, Consumption Shortfall in Retirement and Asset Allocation: The Annuity Benchmark*, Journal of Pension Economics and Finance, **1**, 269–288, 2002.

[Antolin, Pugh & Stewart, 2008] Antolin P., Pugh C. and Stewart F., *Forms of Benefit Payment at Retirement*, OECD Working Paper on Insurance and Private Pensions No.26, OECD publishing, 2008. DOI:10.1787/238013082545.

[Basak & Shapiro, 2001] Basak S, Shapiro A., *Value-at-Risk-Based Risk Management: Optimal Policies and Asset prices*, **14**, 371–405, 2001.

[Bielecki, Jin, Pliska & Zhou, 2005] Bielecki, T.R., Jin, H., Pliska, S.R. and Zhou, X.Y., *Continuous-time mean-variance portfolio selection with bankruptcy prohibition*, Mathematical Finance, **15**, 213–244, 2005.

[Blake, Cairns & Dowd, 2003] Blake D., Cairns A.J.G., and Dowd K., *Pensionmetrics 2: Stochastic pension plan design during the distribution phase*, Insurance: Mathematics and Economics, **33**, 29–47, 2003.

[Chiu & Zhou, 2011] Chiu C.H. and Zhou X.Y., *The premium of dynamic trading*, Quantitative Finance, **11**, 115–123, 2011.

[Di Giacinto, Federico & Gozzi, 2011] Di Giacinto M., Federico S., and Gozzi F., *Pension funds with a minimum guarantee: a stochastic control approach*, Finance and Stochastics, **15**, 297–342, 2011.

[Di Giacinto, Federico, Gozzi, & Vigna, 2010] Di Giacinto M., Federico S., Gozzi F., and Vigna E., *Constrained portfolio choices in the decumulation phase of a pension plan*, Carlo Alberto Notebooks No. 155, 2010. Available at SSRN: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1600130.

[Di Giacinto & Vigna, 2012] Di Giacinto M. and Vigna E., *On the sub-optimality cost of immediate annuitization in DC pension funds*, Central European Journal of Operation Research, forthcoming.

[Elie & Touzi, 2008] Elie R., and Touzi N., *Optimal lifetime consumption and investment under drawdown constraint*, Finance and Stochastics, **12**, 299–330, 2008.

[El Karoui, Jeanblanc & Lacoste, 2005] El Karoui N., Jeanblanc M., and Lacoste V., *Optimal portfolio management with American capital guarantee*, Journal of Economics, Dynamics and Control, **29**, 449–468, 2005.

[Gao, 2008] J. Gao, *Stochastic optimal control of DC pension funds*, Insurance: Mathematics and Economics,**42**, 1159–1164, 2008.

[Gerrard, Haberman & Vigna, 2004] Gerrard R., Haberman S., and Vigna E., *Optimal investment choices post retirement in a defined contribution pension scheme*, Insurance: Mathematics and Economics, **35**, 321–342, 2004.

[Gerrard, Haberman & Vigna, 2006] Gerrard R., Haberman S., and Vigna E., *The Management of Decumulation Risks in a Defined Contribution Environment*, North American Actuarial Journal, **10**, 84–110, 2006.

[Gerrard, Højgaard & Vigna, 2011] Gerrard R., Højgaard B., and Vigna E., *Choosing the optimal annuitization time post retirement*, Quantitative Finance, **12**, 1143–1159, 2012.

[Karatzas & Shreve, 1998] Karatzas I., and Shreve S.E., *Methods of mathematical Finance*, Springer-Verlag, New York, 1998.

[Korn, 1997] Korn, R. *Optimal portfolios*, World Scientific, Singapore, 1997.

[Leland & Rubinstein, 1976] Leland H.E., Rubinstein M., *The evolution of portfolio insurance*. In: Luskin, D.L. (Ed.), Portfolio Insurance: A Guide to Dynamic Hedging. Wiley, New York, 1976.

[Merton, 1969] Merton R.C., *Lifetime Portfolio Selection under Uncertainty: the Continuous-Time Case*, Review of Economics and Statistics, **51**, 247–257, 1969.

[Milevsky, 2001] Milevsky M.A., *Optimal Annuitization Policies: Analysis of the Options*, North American Actuarial Journal, **5**, 57–69, 2001.

[Milevsky, Moore & Young, 2006] Milevsky M. A., Moore K. S. and Young V. R., *Optimal Asset Allocation and Ruin-Minimization Annuitization Strategies*, Mathematical Finance, **16**, 647–671, 2006.

[Milevsky & Robinson, 2000] Milevsky M. A., and Robinson C., *Self- Annuitization and Ruin in Retirement*, North American Actuarial Journal, **4**, 112–129, 2000.

[Milevsky & Young, 2007] Milevsky M.A., and Young V. R., *Annuitization and Asset Allocation*, Journal of Economic Dynamics and Control, **31**, 3138–3177, 2007.

[Pliska, 1986] Pliska S.R., *A stochastic calculus model of continuous trading: optimal portfolios*, Mathematics of Operations Research, **11**, 371–384, 1986.

[Schwartz & Tebaldi, 2006] Schwartz E.S., and Tebaldi C., *Illiquid assets and optimal portfolio choice*, NBER Working Paper, No. W12663, 2006.

[Tepla, 2001] Tepla L., *Optimal investment with minimum performance constraints*, Journal of Economics, Dynamics and Control, **25**, 1629–1645, 2001.

[Vigna, 2012] Vigna E., *On efficiency of mean-variance based portfolio selection in DC pension schemes*, Quantitative Finance, forthcoming.

[Xiao , Zhai & Qin, 2007] J. Xiao, Zhai H., and Qin C., *The constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts*, Insurance: Mathematics and Economics, **40**, 302–310, 2007.

[Yamada & Watanabe, 1971] Yamada T., and Watanabe S., *On the uniqueness of solutions of stochastic differential equations*, Journal of Mathematics of Kyoto University, **11**, 155–167, 1971.

[Yong & Zhou, 1999] Yong J., and Zhou X.Y., *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer Verlag, New York, 1999.

[Zhou & Li, 2000] Zhou X.Y., and Li, D., *Continuous-Time Mean-Variance Portfolio Selection: A Stochastic LQ Framework*, Applied Mathematics and Optimization, **42**, 19–33, 2000.