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Penny Auctions are Unpredictable*

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Abstract

I study an auction format called penny auctions. In these auctions, every bid increases the price by a small amount, but it is costly to place a bid. The auction ends if more than some predetermined amount of time has passed since the last bid. Outcomes of real penny auctions are surprising: even selling cash can give the seller an order of magnitude higher or lower revenue than the nominal value. Sometimes the winner of the auction pays very little compared to many of the losers at the same auction. The unexpected outcomes have led to the accusations that the penny auction sites are either scams or gambling or both. I propose a tractable model of penny auctions and show that the high variance of outcomes is a property of the auction format. Even absent of any randomization, the equilibria in penny auctions are close to lotteries from the buyers' perspective.

JEL: D11, D44, C73

Keywords: penny auction, Internet auctions, bid fees, gambling

1 Introduction

A typical *penny auction*[1] sells a new brand-name gadget, at a starting price of \$0 and a timer at 1 minute. When the auction starts, the timer starts to tick down and players may submit bids. Each bid costs \$1 to the bidder, increases price by \$0.01, and resets the timer to 1 minute. Once the timer reaches 0, the last bidder is declared the winner and can purchase the object at the final price. The structure of penny auctions is similar to dynamic English auctions, but with one significant difference: in penny auctions bidders have to pay bid fees for each bid they make.

Penny auctions have surprising properties in practice[2]. First and foremost, the relation between the final price and the value of the object is stochastic, has a high mass on very low values and a long tail on high values. Second, the winner of the auction pays very often, but not always, less than the value of the object. However, since the losers collectively pay large amounts, the revenue to the seller is often higher than the value of the object. In fact, sometimes the losers pay more than the winner.

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[1] There are two kinds of auctions where the name penny auctions has been used previously. First type was observed during the Great Depression, foreclosed farms were sold at the auctions. In these auctions sometimes the farmers colluded to keep the farm in the community at marginal prices. These low sales prices motivated the name penny auctions. Second use of the term comes from the Internet age, where in the auction sites auctions are sometimes started at very low starting prices to generate interest in the auctions. Both uses of the term are unrelated to the auctions analyzed in this paper.

[2] For detailed analysis and data description, see Appendix A.

We are interested in penny auctions for several reasons. First, since the penny auction is not a special case of any well-known auction format, it is interesting to know the properties of the game and its equilibria. As we will document in this paper, even under the standard assumptions—fully rational risk-neutral bidders, common value of the prize, common knowledge on the number of players—all symmetric equilibria will necessarily have the flavor described above: highly uncertain outcomes, with positive mass both on the very low values and very high values.

Second, the popularity of penny auctions and other pay-to-bid auction formats has posed a new policy question: how should a policy-maker think of these auction formats? In various countries penny auctions have been accused of being scams or gambling sites. In fact, Better Business Bureau listed penny auctions as one of the top ten scams in 2011, by saying[3]

> Sales scams are as old as humanity, but the Internet has introduced a whole new way to rip people off. Penny auctions are very popular because it seems like you can get something useful—cameras, computers, etc.—for way below retail. But you pay a small fee for each bid (usually 50¢ to $1.00) and if you aren't the winner, you lose that bid money. Winners often are not even the top bidder, just the last bidder when time runs out. Although not all penny auction sites are scams, some are being investigated as online gambling. BBB recommends you treat them the same way you would legal gambling in a casino—know exactly how the bidding works, set a limit for yourself, and be prepared to walk away before you go over that limit.

In this paper we show that although penny auctions do not use any randomization devices, the equilibrium outcomes are still highly uncertain. Therefore for the individual bidder's perspective, the auction format is similar to a lottery, which means that perhaps the definition of gambling must be extended to include auction formats like penny auctions.

Third, penny auctions provide an interesting case study for behavioral economics. In this paper we are assuming fully rational agents and argue that although the structure of equilibria generated by the model is similar to what we observe in practice, the high revenue cannot be explained by a fully rational behavior. Therefore penny auctions provide a good platform for empirical behavioral economics. After all, if the critics are right, the auction format may have been invented to exploit bidders' behavioral biases.

This paper along with Augenblick (2012) and Platt, Price, and Tappen (2013) was the first to study penny auctions[4]. The main focus of the other two papers is on the empirical analysis of penny auctions. Both are able to match bidding behavior relatively well, but to be able to use the model on the data both papers make assumptions that are in some sense strategically less flexible than the model in this paper. The theoretical model introduced by Platt, Price, and Tappen (2013) assumes that the bidders never make simultaneous decisions, which gives a simple unique characterization of equilibria. The theoretical model in Augenblick (2012) is much closer to mine, but with one significant difference in the bid costs that will be pointed out when we introduce the model. This gives Augenblick a simple equilibrium characterization. Kakhbod (2013) extended the analysis for risk-loving bidders and Caldara (2012) showed that Theorem 4.6 in this paper can be extended for asymmetric equilibria.

Several papers have tried to find an empirical explanation for the unusually high revenues. Augenblick (2012) found evidence of sunk-cost bias, whereas Platt, Price, and Tappen (2013) considered risk-loving preferences as the best explanation. More recent papers Wang and Xu (2012) and Caldara (2012) found that unusually high revenue comes mainly from the agents who are not strategically sophisticated, i.e. new players who still learn the game.

[3] `http://www.bbb.org/us/article/bbb-names-top-ten-scams-of-2011-31711`
[4] All three papers were written independently in 2009.

In addition to penny auctions, there are a few other forms of *pay-to-bid auctions* that have been used in practice and studied by economists. Similarly to penny auctions, the main revenue in these auctions does not come from the winning price, but rather from bid fees. *Price reveal auction* is a descending-price auction where the current price is hidden and bidders can privately observe the price for a fee. Gallice (2012) showed that under the standard assumptions those auctions would end very quickly and would be unprofitable. Di Gaetano (2011) extended the model for endogenous entry and showed that in this case, an equilibrium with high number of bids is possible. In *unique price auctions*, the bidders submit positive integers as bids, and the winner is the one who submitted the lowest[5] or the highest unique number. Similarly to penny and price reveal auctions, the main revenue does not come from the sales price, but rather from the bid fees. Studies by Rapoport, Otsubo, Kim, and Stein (2007); Raviv and Virag (2009); Östling, Wang, Chou, and Camerer (2011); Eichberger and Vinogradov (2008) have found that in these auctions there is a surprising degree of convergence towards the equilibrium and have offered various explanations for the deviations.

Both the name and the general idea of penny auctions is similar to the *dollar auction* introduced by Shubik (1971). In this auction, cash is sold to the highest bidder, but the two highest bidders will pay their bids. Shubik used it to illustrate potential weaknesses of traditional solution concepts and described this auction as an extremely simple, highly amusing, and usually highly profitable for the seller. Dollar auction is a version of *all-pay auctions*, that has been used to describe rent-seeking, R&D races, political contests, and job-promotions. Full characterization of equilibria under full information in one-shot (first-price) all-pay auctions is given by Baye, Kovenock, and de Vries (1996). A second-price all-pay auction, also called *war of attrition*, was introduced by Smith (1974) and has been used to study evolutionary stability of conflicts, price wars, bargaining, and patent competition. Full characterization of equilibria under full information is given by Hendricks, Weiss, and Wilson (1988). Although penny auction is an all-pay auction, it is not a special case of previously documented auction formats, because in contrast to standard all-pay auctions, the winner in penny auction may pay less than the losers.

The paper is organized as follows. Section 2 introduces a theoretical model and discusses its assumptions. The analysis is divided into two parts. Section 3 analyzes the case when the price increment—"the penny" in the auction name—is zero, which means that the auction game will be infinite. Section 4 discusses the case, where the price increment is strictly positive. Section 5 gives some concluding remarks and suggests extensions for the future research.

2 The Model

The auctioneer sells an object with market price of V dollars. We assume that this is fixed and common value to all the participants. There are $N+1 \geq 2$ players (bidders) participating in the auction, denoted by $i \in \{0, 1, \dots, N\}$. We assume that all bidders are risk-neutral and at each point of time maximize the expected continuation value of the game (in dollars).

The auction is dynamic, bids are submitted in discrete time points $t \in \{0, 1, \dots\}$. Auction starts at initial price P_0. At each period $t > 0$ exactly one of the players is the current leader and other N players are non-leaders. At time $t = 0$ all $N+1$ bidders are non-leaders.

At each period t, the non-leaders simultaneously choose whether to submit a bid or pass. Each submission of a bid costs C dollars and increases price by price increment ε. If $K > 0$ non-leaders submit a bid, each of them will be the leader in the next period with equal probability, $\frac{1}{K}$. So, if $K > 0$ bidders

[5] The lowest unique price auction is also called *lowest unique positive integer game.*

submit a bid at t, then $P_{t+1} = P_t + K\varepsilon$, and each of these K players pays C dollars to the seller[6]. The other non-leaders and the current leader will not pay anything at this round and will be non-leaders with certainty. The current leader cannot anything[7]. If all non-leaders pass at time t, the auction ends. If the auction ends at $t = 0$, then the seller keeps the object and if it ends at $t > 0$, then the object is sold to the current leader at price P_t. Finally, if the game never ends, all bidders get payoffs $-\infty$ and the seller keeps the object. All the parameters of the game are commonly known and the players know the current leader and observe all the previous bids by all the players.

We will use the following normalizations. In case $\varepsilon > 0$, we normalize $v = \frac{V-P_0}{\varepsilon}$, $c = \frac{C}{\varepsilon}$, and $p_t = \frac{P_t-P_0}{\varepsilon}$. In games where $\varepsilon = 0$ we use $v = V - P_0$, $c = C$, and $p_t = P_t - P_0$. Therefore both in all cases $p_0 = 0$. Given the assumption and normalizations, a penny auction is fully characterized by (N, v, c, ε), where ε is only used to distinguish between infinite games that we will discuss in Section 3 and finite games in Section 4.

Assumption 2.1. *We assume*[8] $v - c > \lfloor v - c \rfloor$ *and* $v > c + 1$.

The first assumption says that $v - c$ is not a natural number. It is just a technical assumption to avoid considering some tie-breaking cases, where the players are indifferent between submitting one last bid and not. The second assumption just ignores irrelevant cases, since $c + 1$ is the absolute minimum amount of money a player must spend to win the object. So, if the assumption does not hold, the game never starts.

To discuss the outcomes of the auction, we will use the following notation. Given a particular equilibrium, the probability that the game ends without any bids (with the seller keeping the object) is denoted by Q_0. Conditional on the object being sold, the probability that there was exactly p bids is denoted by $Q(p)$. The unconditional probability of having p bids is denoted by $\overline{Q}(p)$, so that $\overline{Q}(0) = Q_0$ and $\overline{Q}(p) = (1 - Q_0)Q(p)$ for all $p > 0$. The normalized revenue to the seller is denoted by R and the expected revenue, conditional on object being sold, by $\overline{R}$.

As the solution concept we are considering Symmetric Stationary Subgame Perfect Nash Equilibrium (SSSPNE). We will discuss the formal details of this equilibrium in Appendix B and show that in the cases we consider SSSPNE are Subgame Perfect Nash Equilibria that satisfy two requirements. First property is Symmetry, which means that the players' identity does not play any role (so it could also be called Anonymity). The second property is Stationarity, which means that instead of conditioning their behavior on the whole histories of bids and identities of leaders, players only condition their behavior on the current price and number of active bidders.

In case $\varepsilon > 0$ this restriction means that we can use the current price p (independent on time or history how we arrived to it) as the current state variable and solve for a symmetric Nash equilibrium in this state, given the continuation values at states that follow each profile of actions. So the equilibrium is fully characterized by a $q : \{0, 1, \dots\} \to [0, 1]$, where $q(p)$ is the probability of submitting a bid that each non-leader independently uses at price p.

In case $\varepsilon = 0$ the equilibrium characterization is even simpler, since there are only two states. In the beginning of the game there is $N + 1$ non-leaders, and in any of the following histories the number of non-leaders is N. So, the equilibrium is characterized by $(\hat{q}_0, \hat{q})$ where $\hat{q}_0$ is the the probability that a

[6]This is the assumption where our model differs from Augenblick (2012), which assumes that only the submitting bidder who was chosen to be the next leader has to incur the bid cost. This simplifies the game, since whenever there is at least C dollars of surplus available, all non-leaders would want to submit bids. In our model, since even unsuccessful attempts to become the leader are costly, the behavior of opponents is much more relevant.

[7]This is a simplifying assumption. However, thinking about the practical auctions, it seems to be a plausible assumption to make. We will assume that the practical design of the auction is constructed so that whenever a current leader submitted a bid, the auctioneer or system assumes that it was just a mistake and ignores the bid.

[8]$\lfloor \cdot \rfloor$ is the floor function $\lfloor x \rfloor = \min\{k \in \mathbb{Z} : k \leq x\}$.

player submits a bid at round 0 and $\hat{q}$ is the probability that a non-leader submits a bid at any of the following rounds. The SSSPNE can be found simply by solving for Nash equilibria at both states, taking into account the continuation values.

Lemmas B.1, B.2, and B.3 in Appendix B show that any equilibria found in this way are SPNE satisfying Symmetry and Stationarity, and vice versa, any SSSPNE can be found using the described methods. It must be noted that restricting the attention to this particular subset of Subgame Perfect Nash equilibrium, is restrictive and simplifies the analysis. As we will argue later, in general there are many other Subgame Perfect Nash equilibria in these auctions. The restrictions correspond to a situation where the players are only shown the current price. In practice players have more information, but in the case when they for one reason or another do not want to put in enough effort to keep track on all the bids (or believe that most of the opponents will not do it), the situation is similar. As an approximation this assumption should be quite plausible.

3 Auction with zero price increment

We will first look at a case where the price increment $\varepsilon = 0$.[9] One could also argue that this could be a reasonable approximation of a penny auction where ε is positive, but very small, so that the bidders perceive it as 0.

In this case the auction very close to an infinitely repeated game, since there is nothing that would bound the game at any round[10]. After each round of bids, bid costs are already sunk and the payoffs for winning are the same.

This is a well-defined game and we can look for SSSPNE in this game. As argued above and proved in the Appendix B, the SSSPNE is fully characterized by a pair $(\hat{q}_0, \hat{q})$, where $\hat{q}_0$ is the probability that a non-leader will submit at round 0 and $\hat{q}$ the probability that a non-leader submits a bid at any round after 0. Let $\hat{v}^*, \hat{v}$ be the leader's and non-leaders' continuation values (after period 0). The following theorem shows that the SSSPNE is unique and gives full characterization for this equilibrium.

Theorem 3.1. *In the case $\varepsilon = 0$, there is a unique SSSPNE $(\hat{q}_0, \hat{q})$, such that*

(i) $\hat{q} \in (0, 1)$ is uniquely determined by equality $(1 - \hat{q})^N \Psi_N(\hat{q}) = \frac{c}{v}$,

(ii) for $N + 1 = 2$, then $\hat{q}_0 = 0$; otherwise $\hat{q}_0 \in (0, 1)$ is uniquely determined by $(1 - \hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{v}$,

where $\Psi_N(q) = \frac{1-(1-q)^N}{qN}$.

Proof First notice that there is no (symmetric) pure strategy equilibria in this game, since if $\hat{q} = 1$, then the game never ends and all players get $-\infty$, which cannot be an equilibrium. Also, if $\hat{q} = 0$, then $\hat{v}^* = v$ and $\hat{v} = 0$. This cannot be an equilibrium, since a non-leader would want to deviate and submit a bid to get $\hat{v}^* - c$, which is higher than $\hat{v}$, since $v > c + 1 > c$ by assumption. Therefore, in any equilibrium $\hat{q} \in (0, 1)$.

We will start with the case when $N + 1 = 2$. Since the equilibrium is in mixed strategies, non-leader's value must be equal when submitting a bid or not. If she submits a bid, she will be the next leader with certainty and the value of not submitting a bid is 0, since the game ends with certainty. Thus $\hat{v} = \hat{v}^* - c = 0$, and so $\hat{v}^* = c$. Being the leader, there is $(1 - \hat{q})$ probability of getting the object and $\hat{q}$

[9] This specification was called "Free auction" (if $P_0 = 0$) or "Fixed-price auction" (if $P_0 > 0$) by Swoopo.

[10] This is in contrast to $\varepsilon > 0$ case, where the game always ends in finite time. We will establish this in Lemma 4.1 in Section 4.

probability of getting $\hat{v} = 0$ in the next round, so $\hat{v}^* = (1-\hat{q})v = c$ and therefore $\hat{q} = 1 - \frac{c}{v}$. At $t = 0$, if $\hat{q}_0 > 0$ then expected value from bid is strictly negative[11], therefore the only possible equilibrium is such that $\hat{q}_0 = 0$, i.e. with no sale. This is indeed an equilibrium, since by submitting a bid alone gives $\hat{v}^* = c$ with certainty and costs c with certainty, so it is not profitable to deviate. Note that when $N + 1 = 2$, then $\Psi_1(\hat{q}) = 1$, so $(1-\hat{q})^N \Psi_1(\hat{q}) = 1 - \hat{q}$ and $(1-\hat{q})v = \frac{c}{v}v = c = \hat{v}^*$, so the results are a special case of the claim from the theorem.

Suppose now that $N + 1 \geq 3$. Look at any round after 0. Again, this is a mixed strategy equilibrium, where $\hat{q} \in (0, 1)$, so non-leader's value is equal to the expected value from not submitting a bid. The other $N - 1$ non-leaders submit a bid each with probability $\hat{q}$, which means that the game ends with probability $(1-\hat{q})^{N-1}$ and continues from the same point with probability $1 - (1-\hat{q})^{N-1}$. Therefore

$$\hat{v} = [1 - (1-\hat{q})^{N-1}]\hat{v} + (1-\hat{q})^{N-1}0 \iff \hat{v} = 0,$$

since $0 < \hat{q} < 1$. This gives the leader $(1-\hat{q})^N$ chance to win the object and with the rest of the probability to become a non-leader who gets 0, so

$$\hat{v}^* = (1-\hat{q})^N v + [1 - (1-\hat{q})^N]\hat{v} = (1-\hat{q})^N v.$$

The value of $\hat{q}$ is pinned down by the mixing condition of a non-leader

$$\hat{v} = 0 = \sum_{K=0}^{N-1} \binom{N-1}{K} \hat{q}^K (1-\hat{q})^{N-1-K} \left[\frac{1}{K+1}\hat{v}^* + \frac{K}{K+1}\hat{v}\right] - c \iff \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}),$$

where function $\Psi_N(q)$ is the player i's probability of becoming the new leader in after submitting a bid when $N - 1$ other non-leaders submit their bids independently, each with probability q and since

$$\Psi_N(q) = \sum_{K=0}^{N-1} \binom{N-1}{K} q^K (1-q)^{N-(K+1)} \frac{1}{K+1} = \frac{1-(1-q)^N}{qN},$$

it is straightforward to verify that it is strictly decreasingcontinuous function with limits 1 and $\frac{1}{N}$ as $\hat{q} \to 0$ and $\hat{q} \to 1$ correspondingly. As $\hat{q}$ changes in $(0, 1)$, it takes all values in the interval $\left(\frac{1}{N}, 1\right)$, each value exactly once. Now, $(1-\hat{q})^N$ is also strictly decreasing continuous function with limits 1 and 0, so the function $(1-\hat{q})^N \Psi_N(q)$ is a strictly decreasing continuous function in $\hat{q}$ and takes all values in the interval $(0, 1)$. Since $0 < \frac{c}{v} < 1$ and there exists unique $\hat{q} \in (0, 1)$ that solves the equation $(1-\hat{q})^N \Psi_N(\hat{q}) = \frac{c}{v}$.

Let us now consider period 0 to find the equilibrium strategy at $\hat{q}_0$. Denote the expected value that a player gets from playing the game by $\hat{v}_0$. We claim that $\hat{q}_0 \in (0, 1)$. To see this, suppose first that $\hat{q}_0 = 0$, which means that the game ends instantly and all bidders get 0. By submitting a bid, a player could ensure becoming the leader with certainty in the next round and therefore getting value $\hat{v}^* - c = (1-\hat{q})^N v - c$. Equilibrium condition says that this must be less than equilibrium payoff 0, but then

$$(1-\hat{q})^N v - c \leq 0 \iff (1-\hat{q})^N \leq \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}),$$

so $\Psi_N(\hat{q}) \geq 1$. This is contradiction, since $\Psi_N(\hat{q}) < 1$ for all $\hat{q} > 0$.

Suppose now that $\hat{q}_0 = 1$ is an equilibrium, so that each bidder must weakly prefer bidding to not

[11]The cost is certainly c, but expected benefit is weighted average c and 0 both with strictly positive probability.

bidding and getting continuation value of a non-leader, $\hat{v} = 0$. This gives equilibrium condition

$$\frac{1}{N+1}\hat{v}^* - c = \frac{1}{N+1}(1-\hat{q})^N v - c \geq 0 \iff \frac{(1-\hat{q})^N}{N+1} \geq \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}),$$

so $\Psi_N(\hat{q}) \leq \frac{1}{N+1} < \frac{1}{N}$, which is a contradiction.

Thus, in equilibrium $0 < \hat{q}_0 < 1$ is defined by

$$0 = \sum_{K=0}^{N} \binom{N}{K} \hat{q}_0^K (1-\hat{q}_0)^{N-K} \frac{1}{K+1}\hat{v}^* - c \quad \iff \quad (1-\hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{v}.$$

To show that this equation defines $\hat{q}_0$ uniquely (for a fixed $\hat{q} \in (0,1)$), we can rewrite it as follows.

$$(1-\hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}) \quad \iff \quad \Psi_{N+1}(\hat{q}_0) = \Psi_N(\hat{q}).$$

Now, we already showed that $\Psi_N(\hat{q}) \in \left(\frac{1}{N}, 1\right)$. As argued above (continuous, strictly decreasing) $\Psi_{N+1}(\hat{q}_0)$ takes values in the interval $\left(\frac{1}{N+1}, 1\right) \supset \left(\frac{1}{N}, 1\right)$, so the equation must have unique solution $\hat{q}_0$. $\square$

Corollary 3.2. *From Theorem 3.1 we get the following properties of the auctions with $\varepsilon = 0$:*

(i) $\hat{q}_0 < \hat{q}$.

(ii) If $N+1 > 2$, then the probability of selling the object is $1 - (1-\hat{q}_0)^{N+1} > 0$. If $N+1 = 2$, the seller keeps the object.

(iii) Expected ex-ante value to the players is 0.

(iv) Expected revenue to the seller, conditional on sale, is v,

Proof We will prove each part and also give some intuition where applicable.

(i) It is straightforward to verify that $\Psi_N(q)$ is strictly increasing function of $N \in \mathbb{N}$. Since $\Psi_{N+1}(\hat{q}_0) = \Psi_N(\hat{q})$ and $\Psi_K(q)$ is strictly decreasing function of q, we have $\hat{q}_0 < \hat{q}$.

This is intuitive, since from the perspective of a non-leader, the two situations are identical in terms of continuation values, but at $t = 0$ there is one more opponent trying to become the leader.

(ii) This is just reading from the theorem. By the rules of the game, the seller sells the object whenever there was at least one bid, so the object is not sold only in the case when all bidders choose not to submit a bid at round 0. Therefore, the object is sold with probability $P(p > 0) = 1 - (1-\hat{q}_0)^{N+1}$.

If $N+1 = 2$, then $\hat{q}_0 = 0$, so $P(p > 0) = 0$. If $N+2 > 2$, then $\hat{q}_0 > 0$, so $P(p > 0) > 0$.

(iii) Let $\hat{v}_0$ be the expected ex ante value to the players. If $N+1 = 2$, then $\hat{v}_0 = 0$, since players pass with certainty. If $N+1 > 2$, then each bidder is at round 0 indifferent between bidding and not bidding, and not bidding gives 0 if none of the other players bid and $\hat{v} = 0$ of some bid. Therefore $\hat{v}_0 = 0$.

(iv) There is another way how the ex-ante value to the players, $\hat{v}_0$, can be computed. Let the actual number of bids the players submitted in a particular realization of uncertainty be B. Conditioning on sale means that $B > 0$.

Since the value to the winner is v, and collectively all the players paid Bc in bid costs, the aggregate value to the players is $v - Bc$. By symmetry and risk-neutrality, ex-ante this value is divided equally among all players, so

$$0 = (N+1)v(0) = \sum_{B=1}^{\infty} [v - Bc] E(B|B > 0) = v - c\overline{R}.$$

Expected revenue to the seller, given that the object is sold, is Bc from all the bids. So

$$\overline{R} = E(Bc|B > 0) = cE(B|B > 0) = v.$$

□

The following observations illustrate, that although in expected terms all the payoffs are precisely determined, in actual realizations almost anything can happen with positive probability.

Observation 3.3.

(i) With probability $(N+1)(1-\hat{q}_0)^N \hat{q}_0 (1-\hat{q})^N > 0$ the seller sells the object after just one bid and gets $R = c$. The winner gets $v - c$ and the losers pay nothing.

(ii) When we fix arbitrarily high number M, then there is positive probability that revenue $R > M$. This is true since there is positive probability of sale and at each round there is positive probability that all non-leaders submit bids.

(iii) With positive probability we can even get a case where revenue is bigger than M, but the winner paid just c.

Observation 3.4. None of the qualitative results in this case were dependent on the parameter values, so changes in parameters only affect the numerical outcomes.

(i) In particular, given that Assumption 2.1 is satisfied, the expected revenue and the total payoff to the bidders does not depend on the parameter values other than the fact that $\overline{R} = v$.

(ii) Equilibrium conditions were $(1-\hat{q})^N \Psi_N(\hat{q}) = \frac{c}{v}$ and $\Psi_{N+1}(\hat{q}_0) = \Psi_N(\hat{q})$ and functions $(1-\hat{q})^N \Psi_N(q), \Psi_N(q)$, and $\Psi_{N+1}(q)$ are strictly decreasing. Therefore, as $\frac{c}{v}$ increases, both $\hat{q}$ and $\hat{q}_0$ will decrease.

This means that for a fixed v, as c decreases, the probability of sale decreases. Note that in the limit as $c \to 0$, we get an auction that can be approximately interpreted as dynamic English auction. The puzzling fact is that in this auction the object is never sold.

(iii) As N increases, since $\Psi_N(q)$ is decreasing in N, both $\hat{q}$ and $\hat{q}_0$ decrease.

Remark The discussion above was about SSSPNE. If we do not require stationarity and symmetry, then almost anything is possible in terms of equilibrium strategies, expected revenue to the seller, and the payoffs to the bidders. It is easy to see this from the following argument

(i) Fix $i \in \{1, \ldots, N+1\}$. One possible SPNE is such that player i always bids and all the other players always pass. This is clearly an equilibrium since given i's strategy, any $j \neq i$ can never get the object and can never get more than 0 utility. Also, given that none of the opponents bid, i wants to bid, since $v - c > 0$. This equilibrium gives $v - c$ to i and 0 to all the other bidders.

(ii) Using this continuation strategy profile as a "punishment" we can construct other equilibria, including one where no-one bids (if i bids at the first round then some $j \neq i$ will punish him by always bidding in the next rounds that, so that the deviator i pays c and gets nothing, whereas punisher j will get $v - c > 0$).

(iii) Or we can construct an equilibrium where all the players bid $\lfloor v/c \rfloor$ times and then quit. If the bidding rule is constructed so that all bidders get non-negative expected value and are punished as described above, this is indeed a possible equilibrium. This will be the highest possible revenue from a pure strategy equilibrium with symmetry on the path of play.

(iv) With suitable randomizations it is possible to construct equilibria that extract any revenue from c to v.

4 Auction with positive price increment

In this section we analyze the auctions with positive price increment. Since the equilibria are different with two players (where after the first period, there are no simultaneous moves), we are focusing on $N + 1 \geq 3$ case in this section and the equilibria for $N + 1 = 2$ is provided in Appendix C. As argued above and proved in the Appendix B, we can characterize any SSSPNE by a vector $q = (q(0), q(1), \dots)$, where $q(p)$ is the non-leaders' probability to bid at price p. We showed that it is both necessary and sufficient to check for stage-game Nash equilibria, given the continuation payoffs induced by the chosen actions. In a given equilibrium, we will denote leader's continuation value at price by $v^*(p)$ and non-leaders' continuation value by $v(p)$.

Define $\tilde{p} = \lfloor v - c \rfloor$ and $\gamma = (v - c) - \lfloor v - c \rfloor \in [0, 1)$, so that $v = c + \tilde{p} + \gamma$. Note that by Assumption 2.1, $\gamma > 0$ and $\tilde{p} > 0$.

If price increment is positive and game goes on, the price rises. This means if the game does not end earlier, then sooner or later the price rises to a level where none of the bidders would want to bid. The following Lemma establishes this obvious fact formally and gives upper bound to the prices where bidders are still active.

Lemma 4.1. *Fix any equilibrium. None of the players will place bids at prices $p_t \geq \tilde{p}$. That is, $q(p) = 0$ for all $p \geq \tilde{p}$.*

Proof First note that if $p > v$, then the upper bound of the winner's payoff in this game is $v - p < 0$ and therefore any continuation of this game is worse to all the players than end at this price. So, we know that the prices where $q(p) > 0$ are bounded by v.

Let $\hat{p}$ be the highest price where $q(\hat{p}) > 0$. Suppose by contradiction that $\hat{p} \geq \tilde{p} = \lfloor v - c \rfloor$. Since $q(\hat{p} + K) = 0$ for all $K \in \mathbb{N}$, the game ends instantly if arriving to these prices. Therefore $v(\hat{p} + K) = 0 < c$, and so

$$v^*(\hat{p} + K) = v - \hat{p} - K = (c + \tilde{p} + \gamma) - \hat{p} - K = \underbrace{\tilde{p} - \hat{p}}_{\leq 0} + \underbrace{\gamma - K}_{<0} + c < c,$$

So, if $K - 1 \in \{0, \dots, N - 1\}$ opponents bid, by submitting a bid the agent gets strictly negative expected value. By not submitting a bid, any non-leader can ensure getting 0. Thus each non-leader has strictly dominating strategy not to bid at $\hat{p}$, which is a contradiction. Therefore $q(p) = 0$ for all $p \geq \tilde{p}$. $\square$

Finally, to get cleaner results the technical Assumption 2.1 is not enough in some cases. In these cases we will use the following Assumption 4.2, which is slightly stronger.

Assumption 4.2. *$v > c + 2$ and $v - c < \lfloor v - c \rfloor + (N - 1)c$.*

The first assumption says that $v - c > 2$ which is same as saying $\tilde{p} > 1$ (instead of $\tilde{p} > 0$). The second assumption says that $\gamma < (N - 1)c$, i.e. neither c and N are not too small. Both assumptions are mild and easily satisfied in practical applications, where $v \gg c > 1$, so $\gamma < 1 < (N - 1)c$ whenever $N > 1$.

Corollary 4.3. *With $\varepsilon > 0$, in any equilibrium:*

(i) Price level $\max\{\tilde{p} - 1 + N, N + 1\}$ is an upper bound of the support of realized prices. Under Assumption 4.2, the upper bound is just $\tilde{p} - 1 + N$.

If $\tilde{p} > 1$, then the last price where bidders could make bids with positive probability is $\tilde{p} - 1$ and if all N non-leaders make bids, we will reach the price $\tilde{p} + N - 1$. If $\tilde{p} = 1$, then the bidders only make bids at 0 and there are $N + 1$ non-leaders at this stage, so the upper bound is $N + 1$.

Combination of these two cases gives us the upper bound. Assumption 4.2 and specifically the assumption that $v - c > 2$ ensures that $\tilde{p} > 1$ and therefore we do not have to us the max *operator.*

(ii) The game is finite and there exists a a point of time $\tau \leq \tilde{p} + N$, where game has ended with certainty at any equilibrium. This is true since at each period when the game does not end, the price has to increase at least by 1.

(iii) All non-leaders have strictly dominating strategy not to bid at prices $p_t \geq \tilde{p}$ and at $t + 1$ the game has ended with certainty. This means that we can use backwards induction to find any SPNE.

Before we characterize the equilibrium, let's consider an typical equilibria, illustrated by Example 4.4.

Example 4.4. Let $N + 1 = 3, v = 4.1, c = 2$, and $\varepsilon > 0$. The unique SSSPNE for this game is given in the Table 1. Since $q(0) \in (0, 1)$, the expected utility for all players is $v(0) = 0$ and expected revenue for

p	$q(p)$	$v^*(p)$	$v(p)$	$\overline{Q}(p)$	$Q(p)$
0	0.2299		0	0.4567	
1	0.0645	2.7129	0	0.358	0.6588
2	0	2.1	0	0.1715	0.3157
3	0	1.1	0	0.0139	0.0255
4	0	0.1	0	0	0

Table 1: Example 4.4, solution

the seller $E(R|p > 0) = v = 4.1$.

Note that ex-ante expectation of the sales price is going to be non-trivial. In fact, with 2.5% probability we observe price 3, which implies revenue $3(2 + 1) = 9$, which is significantly higher than 4.1. From this, $c + 3 = 5 > 4.1 = v$ is paid by the winner and both losers will pay 2.

By Lemma 4.1 in any game $q(p) = 0, \forall p \geq \tilde{p}$. When we take $p = \tilde{p} + K$ for $K = 0, 1, \ldots,$ then $q(\tilde{p} + K) = 0$ and

$$v^*(\tilde{p} + K) = v - (\tilde{p} + K) = v - c - \tilde{p} + c - K = c + \gamma - K, \quad v(\tilde{p} + K) = 0.$$

So, we can consider the rest of the game to be finite and solve it using backwards induction. Take $p \in \{0, \ldots, \tilde{p} - 1\}$. If $p > 0$, there are N non-leaders and if $p = 0$, there are $N + 1$. Denote the number of non-leaders by $\bar{N}$. Then one of the following three situations characterizes $q(p), v^*(p)$, and $v(p)$.

First, a stage-game equilibrium where all $\bar{N}$ non-leaders submit bids with certainty. In this case $q(p), v^*(p)$, and $v(p)$ are characterized by the three equalities in conditions (C1). This is an equilibrium if none of the non-leaders wants to pass and become non-leader at price $p + \bar{N} - 1$ with certainty, which gives us the inequality condition in (C1).

Conditions 1 (C1)**.** $q(p) = 1$, $v^*(p) = v(p + \bar{N})$, and

$$v(p) = \frac{1}{\bar{N}} v^*(p + \bar{N}) + \frac{\bar{N} - 1}{\bar{N}} v(p + N) - c \geq v(p + \bar{N} - 1).$$

Secondly, there could be a stage-game equilibrium where all $\bar{N}$ non-leaders choose to pass. This is characterized by (C2).

Conditions 2 (C2)**.** $q(p) = 0$, $v^*(p) = v - p$, and $v(p) = 0 \geq v^*(p + 1) - c$.

Finally, there could be a symmetric mixed-strategy stage-game equilibrium, where equilibrium, where all $\bar{N}$ non-leaders bid with probability $q \in (0, 1)$. This gives us (C3).

Conditions 3 (C3)**.** $0 < q(p) < 1$,

$$v(p) = \sum_{K=0}^{\bar{N}-1} \binom{\bar{N} - 1}{K} q^K (1 - q)^{\bar{N}-1-K} \left[\frac{1}{K + 1} v^*(p + K + 1) + \frac{K}{K + 1} v(p + K + 1) \right] - c$$

$$= \sum_{K=1}^{\bar{N}-1} \binom{\bar{N} - 1}{K} q^K (1 - q)^{\bar{N}-1-K} v(p + K),$$

$$v^*(p) = (1 - q)^{\bar{N}} (v - p) + \sum_{K=1}^{\bar{N}} \binom{\bar{N}}{K} q^K (1 - q)^{\bar{N}-K} v(p + K).$$

Note that every equilibrium each $q(p)$ must satisfy either (C1), (C2), or (C3) and therefore an equilibrium is recursively characterized. However, nothing is saying that the equilibrium is unique. In Appendix D we have example, where at $p = 2$, each of the three sets of conditions gives different solutions and so there are three different equilibria. Moreover, in (C3) the equation characterizing q is $\bar{N} - 1$'th order polynomial, so it may have up to $\bar{N} - 1$ different solutions which could lead to different equilibria.

Theorem 4.5. *In case $\varepsilon > 0$, there exists a SSSPNE $q : \mathbb{N} \to [0, 1]$, such that q and the corresponding continuation value functions are recursively characterized (C1), (C2), or (C3) at each $p < \tilde{p}$ and $q(p) = 0$ for all $p \geq \tilde{p}$. The equilibrium is not in general unique.*

Proof Case $N + 1 = 2$ is covered by Proposition C.1 in Appendix C and is a special case of the formulation above.

If $N + 1 > 2$, then the formulation above describes the method to find equilibrium q. The conditions (C1), (C2), and (C3) are written so that there are no profitable one-stage deviations. To prove the existence we only have to prove that there is at least one q that satisfies at least one of three sets of conditions.

At each stage, we have a finite symmetric strategic game. Nash (1951) Theorem 2 proves that it has at least one symmetric equilibrium[12]. Since there conditions are constructed so that any mixed or pure

[12]His concept of symmetry was more general — he showed that there is an equilibrium that is invariant under every automorphism (permutation of its pure strategies). Cheng, Reeves, Vorobeychik, and Wellman (2004) point out that in a finite symmetric game this is equivalent to saying that there is a mixed strategy equilibrium where all players play the same mixed strategy. They also offer a simpler proof for this special case as Theorem 4 in their paper.

strategy stage-game Nash equilibrium would satisfy them, there exists at least one such q.

Finally, Appendix D gives a simple example where the equilibrium is not unique. □

Corollary 4.6. *With $\varepsilon > 0$, in any SSSPNE, we can say the following about $\overline{R}$.*

(i) $\overline{R} \leq v$,

(ii) if $q(p) < 1, \forall p$, *then* $\overline{R} = v$,

(iii) In some games in some equilibria $\overline{R} < v$.

Proof

(i) Similarly to the proof of Corollary 3.2, the aggregate expected value to the players must be equal to v minus the aggregate payments, which is the sum of p and costs pc. The revenue to the seller is exactly the sum of all payments, so

$$(N+1)v(0) = v - E(p + pc|p > 0)v - \overline{R}.$$

Players' strategy space includes option of always passing, which gives 0 with certainty. Therefore in any SSSPNE, $v(0) \geq 0$, so $\overline{R} \leq v$.

(ii) If $q(p) < 1$ for all p, then this mixed strategy puts strictly positive probability on the pure strategy where the player never bids. This pure strategy gives 0 with certainty and so $v(0) = 0$.

(iii) If $q(p) = 1$ for some p, then the previous argument does not work, since the player does not put positive probability on never-bidding pure strategy.

To prove the existence claim, it is sufficient to give an example. We already found in previous subsection that in $N + 1 = 2$ player case, if $\tilde{p}$ is odd and $v > 3(c + 1)$, then $q(0) = 1$ and $E(R|p > 0) = 3(c + 1) < v$. Example in Appendix D gives a more complex equilibrium (details are in the Table 3) where $q(0) \in (0, 1), q(1) = 0$, but $q(2) = 1$ and $E(R|p > 0) = 8.62 < 9.1 = v$.

□

The following lemma gives restriction how often the players can pass. It shows that there cannot be two adjacent price levels in $\{1, \ldots, \tilde{p}\}$, where none of the bidders submits a bid. Lemma 4.1 showed that $\tilde{p}$ is the upper bound of the prices where bidders may submit bids. Lemma 4.7 says that at $\tilde{p} - 1$ players always bid with positive probability, so that it is the least upper bound.

Lemma 4.7. *With $\varepsilon > 0$, in any SSSPNE, $\nexists \hat{p} \in \{2, \ldots, \tilde{p}\}$ st $q(\hat{p} - 1) = q(\hat{p}) = 0$. In particular, $q(\tilde{p} - 1) > 0$.*

Proof Suppose $\exists \hat{p} \in \{2, \ldots, \tilde{p}\}$ such that $q(\hat{p} - 1) = q(\hat{p}) = 0$. Since $q(\hat{p}) = 0$, the game ends there with certainty and therefore $v^*(\hat{p}) = v - \hat{p}$.

$q(\hat{p} - 1) = 0$, so the game ends instantly and all non-leaders get 0. By submitting a bid at $\hat{p} - 1$ a non-leader would become leader at price $\hat{p}$ with certainty. So the equilibrium condition at $\hat{p} - 1$ is

$$0 \geq v^*(\hat{p}) - c = v - \hat{p} - c \iff \hat{p} \geq v - c = \tilde{p} + \gamma > \tilde{p}.$$

This is a contradiction with assumption that $\hat{p} \leq \tilde{p}$. Since $q(\tilde{p}) = 0$ by Lemma 4.1, this also implies that $q(\tilde{p} - 1) > 0$. □

The following proposition says that, conditional on the object being sold, very high prices are reached with positive probability. In fact, with relatively weak additional Assumption 4.2, the upper bound of possible prices is reached with positive probability.

Proposition 4.8. *Let $\varepsilon > 0$, fix any SSSPNE where the object is being sold with positive probability, and let p^* be the highest price reached with strictly probability. Then*

(i) $\tilde{p} \leq p^* \leq \max\{\tilde{p} + N - 1, N + 1\}$,

(ii) Under Assumption 4.2, $p^* = \tilde{p} + N - 1$.

Proof

(i) By Corollary 4.3, $p^* \leq \max\{\tilde{p} + N - 1, N + 1\}$.

Since p^* is reached with positive probability and the higher prices are never reached, $q(p^*) = 0$. Equilibrium condition for this is $v^*(p^* + 1) - c \leq 0$. When arriving to any $p > p^*$, the game ends with certainty, so in particular at $p^* + 1$ we have $v^*(p^* + 1) = v - p^* - 1$. This gives $p^* \geq v - c - 1 = \tilde{p} - (1 - \gamma) > \tilde{p} - 1$. Since p^* and $\tilde{p}$ are integers, this implies $p^* \geq \tilde{p}$.

(ii) With Assumption 4.2 Corollary 4.3 gives $p^* \leq \tilde{p} + N - 1$. Suppose by contradiction that $\tilde{p} \leq p^* < \tilde{p} + N - 1$. This can only be true if $Q(p^* - N) > 0$ and $q(p^* - N) > 0$.

First, look at case $q(p^* - N) < 1$. This would mean $Q(()p) > 0$ for all $p \in \{p^* - N, \ldots, p^*\}$. In particular, $Q(\tilde{p} - 1) > 0$ and by Lemma 4.7 $q(\tilde{p} - 1) > 0$, so $Q(\tilde{p} - 1 + N) > 0$, which is contradiction with $p^* < \tilde{p} - 1 + N$. Therefore $q(p^* - N) = 1$, so all non-leaders submit bids, knowing that all others do the same and the price rises to p^* with certainty. This can be an equilibrium action if

$$\frac{1}{N} v^*(p^*) + \frac{N-1}{N} v(p^*) - c \geq v(p^* - 1).$$

Since $p^* \geq \tilde{p}$, the game ends instantly at this price and therefore $v^*(p^*) = v - p^*$ and $v(p^*) = 0$. Finally, $v(p^* - 1) \geq 0$ (since player can always ensure at least 0 payoff by not bidding). This gives the condition

$$v - Nc \geq p^* \geq \tilde{p} = v - c - \gamma \quad \Longleftrightarrow \quad \gamma \geq (N-1)c.$$

This contradicts Assumption 4.2.

□

Corollary 4.9. *When the object is sold and Assumption 4.2 is satisfied,*

(i) $R > v$ *with positive probability,*

(ii) $R < v$ *with positive probability*

So, we have shown in previous Proposition that sometimes the object is sold at very high prices, and in this Corollary that sometimes the seller earns positive profits and sometimes incurs losses. This means that the auction has the stylized properties described in Section A.

Proof

(i) By the previous proposition, there is positive probability that the object is sold at price $p^* \geq \tilde{p} + 1 = v - c + (1 - \gamma)$. Therefore, when object is sold at price p^*, the revenue is

$$R = (c+1)p^* \geq (c+1)[v - c + (1-\gamma)] > v \iff \frac{v}{c+1} > \frac{c - (1-\gamma)}{c},$$

which holds as strict inequality, since $v > c + 1$ and $\gamma < 1$.

(ii) Since $\overline{R} \leq v$ and $R > v$ with strictly positive probability, it must be also $R < v$ with strictly positive probability.

□

With $\varepsilon > 0$ the equilibria are non-trivially related to parameter values. The number of equilibria may increase or decrease as parameter values changes, and the equilibrium outcomes may are generally affected non-monotonically. However, we can make some observations regarding the parameter values in the limits.

When c is very small, then in the limit we would get a version of Dynamic English auction. Perhaps contrary to the intuition this auction generally ends very soon. The reason of this observation is the following. Suppose $N + 1 = 3$, $q(p+1) < 1$, and $q(p+2) < 1$; $v^*(p+1) > 0, v^*(p+2) > 0, v(p+1) = v(p+2) = 0$ and $c \to 0$. Then at price p there is certainly a stage-game equilibrium where $q = 1$ since $v^*(p+2) - c > v(p+1) = 0$. There are no equilibria $q < 1$, since player cannot be indifferent between positive expected value from bid and 0 from no bid. For this reason there will be relatively many prices where $q(p) = 1$. Now, if $q(p) = 1$ then being leader at p is in general worse than being non-leader, so at $p - 1$ the players have lower incentives to bid. In many equilibria this leads to situation where $\overline{R} \ll v$. To put it in the other words, when cost of bid is small, then whenever there is positive expected value from bidding, players compete heavily, which drives down the value to the bidders and therefore there are low incentives to bid in earlier rounds.

If c is nearly the upper bound $v - 1$, then the game gives positive utility to the bidders only if there is exactly one bid. $q = 0$ will not be an equilibrium, since lone bidder would get positive utility. Also, at $p > 0$ no-one bids. Therefore the unique equilibrium is such that $q(0)$ is a very small number and $q(p) = 0$ for all $p > 0$. Then $\overline{R} = v$, but probability of sale is very low. As mentioned above, if $c \geq v - 1$ or equivalently, $1 \geq v - c = \tilde{p} + \gamma$, then $\tilde{p} = 0$ and there can never be any bids. This is obvious, since to get positive payoff one needs to become a leader and minimal possible cost for this is $c + 1$.

Increase in v means that the game is getting longer and this means that there are more states with strategic decisions and generally more possible equilibria and non-trivial effect on strategies and revenue. Decrease in v has the opposite effect and as $v \to c + 1$ we get the case described above.

If N is very large, then $q(p) < 1$ for any p just because if $q(p) = 1$ this would mean that $p + N > v$ and so players cannot get positive value from bidding, whereas they have to incur cost and may ensure 0 by not bidding. Obviously, in $q(p)$ is not always 0, since it would still be good to be a lone bidder. So, in general we would expect to see many p's with low positive (and sometimes 0) values of $q(p)$. Since $q(p) < 1, \forall p$ we would have $\overline{R} = v$.

5 Discussion

In this paper we studied penny auctions, an auction format which leads to unpredictable outcomes in practice. We proposed a tractable model of the auction format and showed that this unpredictability is a property of the auction format. Even under the standard assumptions, i.e. risk-neutral, fully rational bidders, common value, etc, we saw that all symmetric and stationary equilibria of the game must be such that both the highest and the lowest possible equilibrium price is reached with a positive probability. In particular, we showed that in the fixed price penny auctions there can be unboundedly many bids in equilibrium, therefore the (ex-post) revenue of the seller is unbounded. In the increasing price auctions, the upper bound of possible prices is $p^* = \lfloor v - c \rfloor - 1 + N$ and it is reached with a positive probability. This is a very high price where even the winner gets a strictly negative payoff[13] and to reach this price, players had to make many costly bids. Since under some realizations the number of bids is very high, but the expected revenue is always bounded by v, there is also a high probability that the auction ends at low prices. This matches well with stylized facts.

However, the model is unable to explain how real penny auctions can have the average profit margin higher than zero. In penny auctions the objects sold have a well-defined market value, which means that all buyers are likely to value the object more or less equally. However, under common value assumption, the seller's revenue cannot be higher than the value of the object. This is not a property of penny auctions, but a general individual rationality argument—since individuals can always ensure at least 0 value by inactivity, it is impossible to extract on average more than the value they expect to get. To achieve an outcome where the expected revenue is strictly higher than the value of the object, we would need to add something to the model.

This is only the first attempt to characterize penny auctions in a game-theoretic model. Next step would involve extending the model so that it could also explain the paradoxically high average revenue. This would most likely involve deviations from standard assumptions and would benefit from a careful empirical analysis that shows which behaviors or biases are crucial for the outcome. In the following I list a few possible directions how to extend the model to allow the revenue to be higher than the value of the object.

A trivial way to overcome (or ignore) the problem is to say that the value to the seller is some $v_s < v = v_b$. It could be for example that the suggested retail value is much higher than the cost to the seller, and close the value that the customers expect to get. This would obviously mean that there are expected profits, but it does not explain why the seller would not use alternative selling methods, for example selling the object at a posted price v_b.

One explanation promoted by the auctioneers is that it is "Entertainment shopping". This could mean that the bidders get some positive utility from participating, some "entertainment value" v_e in addition to v if winning. Then again $v_b = v + v_e > v_s$. This could be true because winning an auction feels like an accomplishment. In this case this could be an increasing function of N (beating N opponents is great). There are other possible ways to model this entertainment value. (1) For example, modeling it as a lump-sum sum value just from participating or (2) as a positive income that is increasing in the number of bids. (3) Assuming that "saving" money gives some additional happiness. Then instead of $v - p$ the player would have some increasing function $f(v - p)$. If it is linear, it is a simple transformation of previous.

Perhaps one of the most promising explanations would be that the participants of the penny auctions are not risk-neutral, but rather risk loving. Since the auction format leads to highly uncertain outcomes, participating in the auction is from an individual perspective similar to buying a lottery. Risk-loving individuals would be happy to pay more than the expected value of winning.

Another approach would be to consider some boundedly rational behavior in the model. A specific property of penny auctions seems to be that the price increase is marginal for a bidder. We could consider a case where individuals behave (at least for a while) as if that the action is with $\varepsilon = 0$. This would not

[13]The winner has to pay at least $p^* + c$, so her value is at most $v - p^* - c = \gamma + 1 - N < 0$.

be an equilibrium in the game-theoretic sense, but it might be realistic in practice and, as shown in this paper, is computationally easier, since there is a always unique and explicitly characterized equilibrium.

A related question to consider is the reputation of players. Since in real auctions the user name of a bidder is public, this could lead to the reputation effects between the auctions and in a given auction. If a player has built a reputation of being a "tough" bidder in previous auctions, since it is an all-pay auction, it obviously affects the other bidders. Then the first thing to notice is the fact that in this case the equilibrium is in general not symmetric. As we argued in some cases above, there could be (and in some cases are) equilibria, where one bidder always bids and other never bid. This means that there is a reputation-type equilibrium even without any costs of reputation building, just some communication between bidders is enough. Of course, in the long run, it may be profitable to invest in building reputation and therefore there could be some types of behaviors to consider outside of our model.

Finally, the results in this paper and in the related literature point to an interesting research question: perhaps these auctions are good for raising money for public goods. Goeree, Maasland, Onderstal, and Turner (2005) showed that it is better to raise money for public goods by all-pay auctions instead of winner-pay auctions. Carpenter, Holmes, and Matthews (2011) found experimentally that their physical implementation of penny auctions (which they call *bucket auctions*) raised even more money than four alternative all-pay auction formats and attribute the difference to sunk-cost sensitivity.

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A Stylized facts about penny auctions

The data used in this appendix comes from Swoopo, the largest penny auctions site in the beginning of May 2009, when the dataset was collected. Data about 61,153 auctions was collected directly from their website and includes all auctions that had complete data[14] Each auction had information about the auction type, the value of the object (suggested retail price), delivery cost, the winners identity and the number of free and costly bids the winner made (used to calculate "the savings"), and the identities of 10 last bidders with information whether the bid was made using BidButler[15] or not (594,956 observations in total).

All auctions in Swoopo have the same structure as described in this paper, but they have several different types of auctions which imply different parameter values. Their main auction types are the the following[16]. The number of observations and some statistics to compare the orders of magnitude are given in the Table 2.

[14] Auctions that had incomplete data or had not finished were excluded from the dataset.

[15] BidButler is an automatic bidding system where user fixes minimum and maximum price and the number of bids between them and the system makes bids for them according to some semi-public algorithm.

[16] Auctions also differ by the length of timer, i.e. in 20-Second Auction if after the last submitted bid the timer ticks 20 seconds, the auction ends.

1. Regular auction[17] is a penny auction with price increment of \$0.15 and bid cost of \$0.75[18].

2. Penny auction is an auction where price increment is \$0.01 instead of \$0.15.

3. Fixed Price Auction, where at the end of auction the winner pays some pre-announced fixed price instead of the ending price of the auction. The Free Auction (or 100% Off Auction) is a special case of Fixed Price Auction where the winner pays only the delivery charges.[19]Both of these auction have the property that price increment is zero, which means that there is no clear ending point and the auctions could in principle continue infinitely.

4. NailBiter Auction is an auction where BidButlers are not allowed, so that each bid is made by actual person clicking on the bid button.

5. Finally there are some variations regarding restrictions about customers who can participate. If not specified otherwise, everyone who has won less than eight auctions per current calendar month can participate. Beginner Auction is restricted to customers who have never won an auction. Open Auction is an auction where the eight auction limit does not apply, so the participation is fully unrestricted.

Type of auction	Obser-vations	Average value	Average price	Norm. value, v	Norm. cost, c	Avg # of bids
Regular	41760	166.9	46.7	1044	5	242.9
Penny	7355	773.3	25.1	75919.2	75	1098.1
Fixed price	1634	967	64.9	6290.7	5	2007.2
Free	3295	184.5	0	1222	5	558.5
Nailbiter	924	211.5	8.3	1394.1	5	580.1
Beginner	6185	214.5	45.8	1358.5	5	301.6
All auctions	61153	267.6	41.4	10236.3	13.4	420.9

Table 2: General descriptive statistics about the auctions. v and c refer to normalized variables introduced in the next section, the average number of bids can be approximately interpreted as the normalized price p.

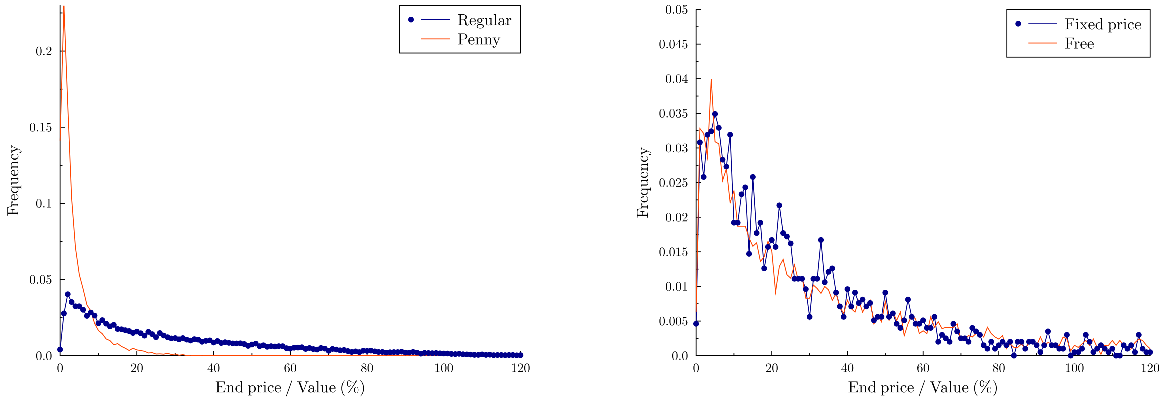
Figure A.1 describes the distribution of the final prices in different auction formats. To be able to compare the prices of objects with different values, the plot is normalized by the value of object. For example 100 means that final price equals the retail price. Most auction formats give very similar distributions with relatively high mass at low values and long tails. Penny auctions are much more concentrated on low values, which is to be expected, since to reach any particular price level, in penny auction the bidders have to make 15 times more bids than in other formats.

The most intriguing fact in the Figure A.1 should be the positive mass in relatively high prices, since the cumulative bid costs to reach to these prices can be much higher than the value of the object. This implies that the profit margins to the seller and winner's payoff are very volatile. Indeed, Figure A.2(a)

[17]In the calculations below, we call the auction regular if it is not any of the other types of the auctions, but the other types are not mutually exclusive. For example auction can be a nailbiter penny auction with fixed price, so it is included in calculations to all three types.

[18]In all auction formats, \$0.75 is the standard price, which is actually the upper bound of bid cost, since bids can be purchased in packages so that they are cheaper and perhaps also sunk. Also, sometimes bids can be purchased at Swoopo auction at uncertain costs.

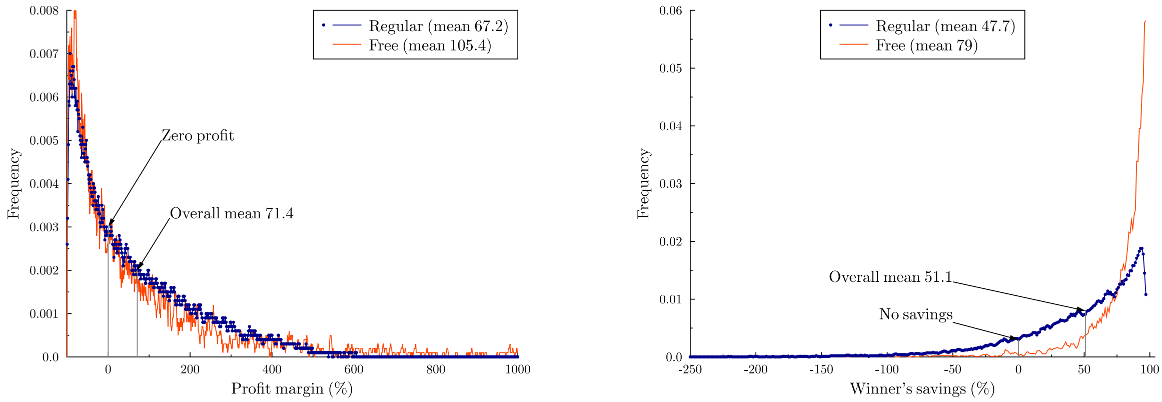
[19]Both Fixed Price Auctions and Free Auctions were discontinued by 2009.



(a) Auctions with increasing prices (b) Auctions with fixed prices

Figure A.1: Distribution of the normalized end prices in different types of auctions

describes the distribution of the profit margin[20][21] and there is positive mass in very high profit margins. The figure is somewhat arbitrarily truncated at 1000%, there also is positive, but small mass at much higher margins. From the auction formats not presented in this figure, Penny auctions have the highest average profit margin (185.8%) and Nailbiter auctions the lowest (25.2%). Note that the profit margin is calculated relative to suggested retail value, so that zero profit margin should be sufficient profit for a retail company, but mean profit margin is positive for all the auctions.



(a) Profit margin (b) Winner's savings

Figure A.2: Distribution of the profit margin and winner's savings in Regular and Free auctions.

Similarly, Figure A.2(b) describes the winner's savings[22][23] from different types of auctions. In this

[20]Profit margin is simply defined as $\frac{\text{End price}+\text{Total bid costs}-\text{Value}}{\text{Value}} \cdot 100$.

[21]To make the plot, we need an approximate for the average bid costs. Official value is \$0.75, but it is possible to get some discounts and free bids, so this would be the upper bound. In the dataset we have the number of free and non-free bids that the winners made and it turns out that about 92.88% of the bids are not free, so we used 92.88% of \$0.75 which is \$0.6966 as the bid cost. The overall average profit margin would be 0 at average bid cost \$0.345, which is about two times smaller than our approximation of the average bid cost.

[22]Defined by Swoopo.com as the difference between the value of the object and winner's total cost divided by the value. Obviously, the losers will not save anything and the winner cannot ensure winning, so the term "savings" can be misleading in ex-ante sense. Note that the reported savings at the website are such that the negative numbers are replaced by 0.

[23]Again, the question is what is the right average bid cost to use. For the winners we know the number of free bids, so this is taken into account precisely, but for the costly bids, the we used the official value \$0.75. True value may be below it, since there could be some quantity discounts, but it does not take into account any other constraints (like cost of time and effort). However, winner's average savings are positive for bid costs up to \$2.485, which is far above the reasonable upper bounds of the bid cost.

plot, 0 would mean no savings compared to retail price, so that on the left of this line even the winner would have gained just by purchasing the object from a retail store. Mostly the winner's savings are highly positive, which is probably the reason why agents participate in the auctions after all. The density of the winnings is increasing in all auctions with mode near 100%, but the auctions differ. Regular auctions have relatively low mean and flattest distribution, whereas Free auctions (and similarly Penny auctions and Fixed price auctions) have highest mean and more mass concentrated near 100%. This is what we would expect, since in these auctions the cost is relatively more equally distributed between the bidders (if the winner was the one making most of the bids, she would win very early).

The final piece of stylized facts we are looking here is the distribution of the number of bids. Figure A.3 shows the distribution of the total number of bids. The frequencies decrease as the number of bids increases, but except in the very low numbers of bids, this decrease is slow. In Penny auctions there are on average much more bids than in other formats. The same is true for Fixed price auctions (on average 2100.6 bids), which is not included in the figure[24] The type where auction ends at relatively low number of bids relatively more often is the nailbiter auction (on average 233.4 bids), where the bidders cannot use automated bidding system.

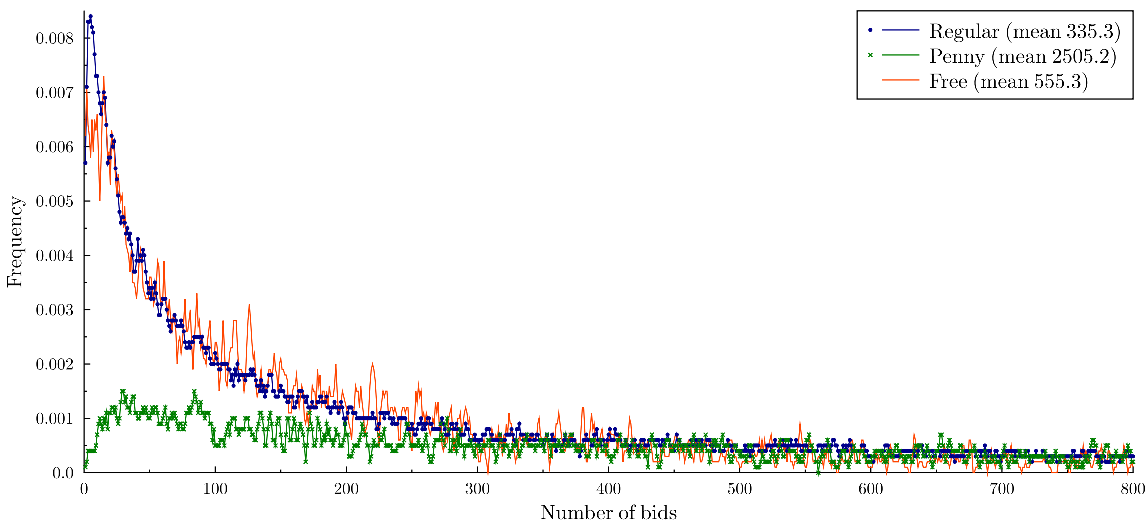


Figure A.3: Distribution of the number of bids submitted in different types of auctions

B Symmetric Stationary Subgame Perfect Nash Equilibrium

In this appendix I introduce formally the equilibrium concept used in this paper, Symmetric Stationary Subgame Perfect Nash Equilibrium (SSSPNE). Let the vector of bids at round t be denoted by $b^t = (b_0^t, \ldots, b_N^t)$, where $b_i^t \in \{0, 1\}$ is 1 if player i submitted a bid at period t. Let's denote the leader after[25] round t by $l^t \in \{0, \ldots, N\}$. The information that each player has when making a choice at time t, or history at t, is $h^t = (b^0, l^0, b^1, l^1, \ldots, b^{t-1}, l^{t-1})$. The game sets some restrictions to the possible histories, in particular to become a leader, one must submit a bid, so $b_{l^t}^t = 1$, and the leader cannot submit a bid, $b_{l^{t-1}}^t = 0$, and h^t is defined only if none of the previous bid vectors b^τ is zero vector. Denote the set of all possible t-stage histories by $\mathcal{H}^t$, and the set of all possible histories, $\mathcal{H} = \bigcup_{t=0}^{\infty} \mathcal{H}^t$.

[24] The fact that in Free auctions and Fixed Price auctions look different in this figure is somewhat surprising and explaining this would probably require more careful empirical analysis. One possibility is that the objects sold are sufficiently different.

[25] That is, the non-leader that submitted a bid at t and became the leader by random draw.

In this game, a pure strategy of player i is $b_i : \mathcal{H} \to \{0,1\}$, where $b_i(h^t) = 1$ means that player submits a bid at h^t and 0 that the player passes. The strategies[26] of the players are $\sigma_i : \mathcal{H} \to [0,1]$, such that $\sigma_i(h^t)$ is the probability that player i submits a bid at history h^t. Note that by the rules of the game, at histories h^t where $l^t = i$, player i is the leader and can only pass.

Def: A strategy profile σ is Symmetric if for all $t \in \{0,1,\dots\}$, for all $i, \hat{\imath} \in \{0,\dots,N\}$, and for all $h^t = (b^\tau, l^\tau)_{\tau=0,\dots,t-1} \in \mathcal{H}^t$, if $\hat{h}^t = (\hat{b}^\tau, \hat{l}^\tau)_{\tau=0,\dots,t-1} \in \mathcal{H}^t$ satisfies

$$\hat{b}_j^\tau = \begin{cases} b_j^\tau & \forall j \notin \{i, \hat{\imath}\}, \\ b_{\hat{\imath}}^\tau & j = \hat{\imath}, \\ b_{\hat{\imath}}^\tau & j = i, \end{cases} \qquad \hat{l}^\tau = \begin{cases} l^\tau & l^\tau \notin \{i, \hat{\imath}\}, \\ i & l^\tau = \hat{\imath}, \\ \hat{\imath} & l^\tau = i, \end{cases} \qquad \forall \tau = \{0,\dots,t-1\},$$

then $\sigma_i(\hat{h}^t) = \sigma_{\hat{\imath}}(h^t)$.

The Symmetry assumption simply states that when we switch the identities of two players, then nothing changes. This means that we could also call it Anonymity assumption. Intuitively, the assumption means that given that other N opponents make exactly the same choices and the uncertainty has realized the same way, different players would behave identically.

Let function L_i be the indicator function that tells whether player i is leader after history h^t or not,

$$L_i(h^t) = \mathbf{1}[i = l^t], \quad \forall i \in \{0,\dots,N\}, \forall h^t \in \mathcal{H}.$$

Let $\mathcal{S}$ be the set of states in the game and $S : \mathcal{H} \to \mathcal{S}$ the function mapping histories to states. In particular, we define these as

(i) If $\varepsilon = 0$, then $\mathcal{S} = \{N+1, N\}$, and

$$S(h^t) = \begin{cases} N+1 & h^t = \emptyset, \\ N & h^t \neq \emptyset. \end{cases}$$

The reason: in infinite game the price does not increase, so the only thing players will condition their behavior is the number of active bidders, which is $N+1$ in the beginning and N at any round after 0.

(ii) If $\varepsilon > 0$, then $\mathcal{S} = \{0,1,\dots\}$, and

$$S(h^t) = \sum_{\tau=0}^{t-1} \sum_{i=0}^{N} b_i^\tau.$$

That is, the total number of bids made so far or equivalently, the normalized price p_t. Note that we do not have to explicitly consider two cases with two different numbers of players, since at $h^t = \emptyset$ we have $S(h^t) = 0$ and at any other history $S(h^t) > 0$.

Def: A strategy profile σ is Stationary if for all $i \in \{0,\dots,N\}$, and for all pairs of histories $h^t = (b^\tau, l^\tau)_{\tau=0,\dots,t-1} \in \mathcal{H}$, $\hat{h}^{\hat{t}} = (\hat{b}^\tau, \hat{l}^\tau)_{\tau=0,\dots,\hat{t}-1} \in \mathcal{H}$ such that $L_i(h^t) = L_i(\hat{h}^{\hat{t}})$, and $S(h^t) = S(\hat{h}^{\hat{t}})$, we have $\sigma_i(h^t) = \sigma_i(\hat{h}^{\hat{t}})$.

[26] The game has perfect recall, so by Kuhn's theorem any mixed strategy profile can be replaced by an equivalent behavioral. Since it makes notation simpler, whenever we are talking about strategies in the text, we mean behavioral strategies.

Stationarity assumption means that the time and particular order of bids are irrelevant. The only two things that affect player's action are current state and the fact whether she is a leader or not.

Def: SPNE strategy profile σ is Symmetric Stationary Subgame Perfect Nash Equilibrium SSSPNE if it is Symmetric and Stationary.

Lemma B.1. *A strategy profile σ is Symmetric and Stationary if and only if it can be represented by $q : \mathcal{S} \to [0,1]$, where $q(s)$ is the probability bidder i bids at state $s \in \mathcal{S}$ for each non-leader $i \in \{0, \ldots, N\}$.*

Proof Since q is only defined on states S and equally for all non-leaders, it is obvious that it is a strategy profile that satisfies Symmetry and Stationarity, so sufficiency is trivially satisfied.

For necessity, take any strategy profile $\sigma = (\sigma_0, \ldots, \sigma_N)$, where $\sigma_i : \mathcal{H} \to [0,1]$, that satisfies Symmetry and Stationarity. Construct functions $q_0, \ldots, q_N$, where $q_i : S \to [0,1]$ by setting

$$q_i(S(h^t)) = \begin{cases} 0 & \forall h^t : L_i(h^t) = 1, \\ \sigma_i(h^t) & \forall h^t : L_i(h^t) = 0, \end{cases} \quad \forall h^t \in \mathcal{H}.$$

Our construction of S and Stationarity ensure that q_i is well-defined function.

We claim that adding Symmetry means that we get $q_i(s) = q(s)$ for all i and $s \in \mathcal{S}$. To see this, fix any i and h^t such that $s = S(h^t)$ and $L_i(h^t) = 0$. By construction, $q_i(s) = q_i(S(h^t)) = \sigma_i(h^t)$.

Now, fix any other non-leader, $\hat{i}$, so that $L_{\hat{i}}(h^t) = 0$. Construct another history $\hat{h}^t$ that is otherwise identical to h^t, but such that i and $\hat{i}$ are swapped. Then $S(\hat{h}^t) = s$ (obvious for both cases) and $L_{\hat{i}}(\hat{h}^t) = 0$. By Symmetry we have $\sigma_i(h^t) = \sigma_{\hat{i}}(\hat{h}^t)$. Therefore

$$q_{\hat{i}}(s) = q_{\hat{i}}(S(\hat{h}^t)) = \sigma_{\hat{i}}(\hat{h}^t) = \sigma_i(h^t) = q_i(s).$$

□

So, if strategy profile satisfies Stationarity and Symmetry, we can greatly simplify its representation. We can replace σ by q that is just defined for all $s \in \mathcal{S}$ instead of full set of histories $\mathcal{H}$. In the following two lemmas we show that at least in the cases considered in this paper the solution method is also simplified by these assumptions, since any SSSPNE can be found simply by solving for stage-game Nash equilibria for each state $s \in \mathcal{S}$ taking into account the solutions to other states and the implied continuation value functions.

Lemma B.2. *With $\varepsilon > 0$, a strategy profile σ is SSSPNE if and only if it can be represented by $q : \mathcal{S} \to [0,1]$ where $q(s)$ is the Nash equilibrium in the stage-game at state s, taking into account the continuation values implied by transitions S.*

Proof Necessity: If σ is SSSPNE, then by Lemma B.1 it can be represented by q and since it is a SPNE, there cannot be profitable one-stage deviations.

Sufficiency: By Corollary 4.3 any auction with $\varepsilon > 0$ ends not later than $\tilde{p} + N$. So, although our game is (by the rules) infinite, it is equivalent in the sense of payoffs and equilibria with a game which is otherwise identical to our initial auction, but where after time $\tilde{p} + N$ the current leader gets the object at the current price. This is finite game and checking one-stage deviations is sufficient condition for SPNE. □

Lemma B.3. *With $\varepsilon = 0$, a strategy profile σ is SSSPNE if and only if it can be represented by $q : \mathcal{S} \to [0,1]$ where $q(s)$ is the Nash equilibrium in the stage-game at state s, taking into account the continuation values implied by transitions S.*

Proof Necessity is identical to Lemma B.2. Sufficiency:[27] Suppose q is Nash equilibrium in the stage-game equilibrium at each state s. To shorten the notation we will use the following notation: $\hat{q}_0 = q(N+1)$, $\hat{q} = q(N)$, $\hat{v}_0$ is the continuation value of the game at state $N+1$, $\hat{v}$ is the continuation value of a non-leader and $\hat{v}^*$ is the continuation value of a leader at state N. By Theorem 3.1 we get $\hat{q} \in (0,1)$, defined by $(1-\hat{q})^N \Psi_N(\hat{q}) = \frac{c}{v}$, $\hat{q}_0 < 1$, $\hat{v} = 0$, and $\hat{v}^* = (1-\hat{q})^N v$. We need to show that there are no profitable unilateral multi-stage deviations from the proposed equilibrium strategy profile.

Take any history $h^t \neq \emptyset$ and individual i who is not the leader at h^t. Let σ_i be the strategy that ensure the highest expected value to player i at history h^t. Denote continuation value using σ_i at history h^τ by $V(h^\tau)$ for all h^τ following h^t. To shorten the notation, denote $\hat{V} = V(h^t)$. Suppose there exists profitable deviation at h^t. Then σ_i must also be profitable deviation and therefore $\hat{V} > \hat{v} = 0$.

Some of the histories h^{t+1} following h^t and i playing $\sigma_i(h^t)$ are such that i is a non-leader. In these situations all the other players use the same mixed strategy in all the continuation paths, so all payoff-relevant details are the same as at h^t. This means that at such histories h^{t+1}, it must be $V(h^{t+1}) = \hat{V}$. It cannot be higher, since $\hat{V}$ is maximum, and it can't be lower, since i could improve $V(h^t)$ by changing strategy starting from this h^{t+1}.

Other histories h^{t+1} following following $h^t, \sigma_i(h^t)$ are the ones where i is the leader. Being the leader at h^{t+1}, two things can happen to i's payoff. First, game may end at h^{t+1} and player i gets v. This happens with probability $(1-\hat{q})^N$ as argued above. Secondly, i can become a non-leader at history h^{t+1} following h^{t+1}. For the same reason as above, $V(h^{t+2}) = \hat{V}$ for all such histories. Therefore in histories h^{t+1} where i is the leader,

$$V(h^{t+1}) = (1-\hat{q})^N v + (1-(1-\hat{q})^N)\hat{V}.$$

The expected value at h^t is the expectation over all the continuation values $V(h^{t+1})$ following mixed action $\sigma_i(h^t)$ minus the expected bid cost. So, we can write

$$\hat{V} = V(h^t) = \sum_{h^{t+1}|h^t,\sigma_i(h^t)} P(h^{t+1}|h^t, \sigma_i(h^t))V(h^{t+1}) - c\sigma_i(h^t)$$

Using the values $V(h^{t+1})$ derived above and the fact that conditional on submitting a bid, the probability of becoming the leader at $t+1$ is $\Psi_N(\hat{q})$. So, the probability of become the leader is $\sigma_i(h^t)\Psi_N(\hat{q})$, which gives us

$$\begin{aligned}\hat{V} &= \sigma_i(h^t)\Psi_N(\hat{q})[(1-\hat{q})^N v + (1-(1-\hat{q})^N)\hat{V}] + [1-\sigma_i(h^t)\Psi_N(\hat{q})]\hat{V} - c\sigma_i(h^t) \\ &= c\sigma_i(h^t) + \sigma_i(h^t)\Psi_N(\hat{q})[1-(1-\hat{q})^N - 1]\hat{V} + \hat{V} - c\sigma_i(h^t) \iff \\ &\sigma_i(h^t)c\frac{\hat{V}}{v} = 0.\end{aligned}$$

By assumptions $c > 0$, $\hat{V} > 0$, and therefore $\sigma_i(h^t) = 0$. What we got is that by not bidding at h^t and at any following h^{t+1} and so on the player can ensure strictly positive expected payoff $\hat{V}$, which is impossible since the only way to get positive value is to be a leader and for this necessary condition is to bid. So there cannot be profitable deviations at any $h^t \neq \emptyset$.

[27] Note that since the game does not satisfy continuity at infinity, checking one-stage deviations may not be sufficient for SPNE.

We showed that at any history that follows h^0, always playing $\hat{q}$ ensures highest possible payoffs. Therefore at round 0 if there is profitable deviation, it must be one-stage deviation. But this is not possible, since we assumed that $\hat{q}_0$ is Nash equilibrium in the stage-game, taking into account the continuation values from $\hat{q}$ in the following periods. □

C Auction with positive price increment: two-player case

The two-player case is very simple, since we have an alternating-move game, where at $t > 0$, one of the players is always leading and the other (non-leader) can choose whether to bid and become leader or pass and end the game. We can simply solve it by backwards induction. To see the intuition, let us start by solving a couple of backward induction steps before stating the result formally.

By Lemma 4.1, at prices $p \geq \tilde{p}$, the non-leader would never bid. Therefore, the continuation values values are $v^*(p) = v - p, v(p) = 0, \quad \forall p \geq \tilde{p}$, and in particular $v^*(\tilde{p}) = v - \tilde{p} = c + \gamma$.

At $p = \tilde{p} - 1$, non-leader will make a bid since $v^*(p+1) - c = v^*(\tilde{p}) - c = \gamma > 0$. Therefore $v^*(\tilde{p} - 1) = v(\tilde{p}) = 0$, $v(\tilde{p} - 1) = \gamma$.

At $p = \tilde{p} - 2 > 0$, non-leader will not make a bid, since continuation value in the next round is 0 which does not cover the cost of bid. Thus $v^*(\tilde{p} - 2) = v - (\tilde{p} - 2) = c + \gamma + 2$, $v(\tilde{p} - 2) = 0$.

We can continue this process for all $t > 0$ and then need to consider the simultaneous decision at stage 0. The following Proposition C.1 characterizes the set of equilibria for two-player case.

Proposition C.1. *Suppose $\varepsilon > 0$ and $N + 1 = 2$. There is a unique SSSPNE and the strategies q are such that*

$$q(p) = \begin{cases} 0 & \forall p \geq \tilde{p} \textit{ and } \forall p = \tilde{p} - 2i > 0, i \in \mathbb{N}, \\ 1 & \forall p = \tilde{p} - (2i+1) > 0, i \in \mathbb{N}, \end{cases}$$

and $q(0)$ is determined for each (v, c) by one of the following cases.

(i) If $\tilde{p}$ is an even integer, then $q(0) = 0$.

(ii) If $\tilde{p}$ is odd integer and $v \geq 3(c+1)$, then $q(0) = 1$.

(iii) If $\tilde{p}$ is odd integer and $v < 3(c+1)$, then $q(0) = 2\frac{v-(c+1)}{v+(c+1)} \in (0,1)$.

Proof As argued above, by Lemma 4.1, $q(p) = 0$ for all $p \geq \tilde{p}$. For $p \in \{1, \ldots, \tilde{p}\}$ we are using backwards induction. In particular, we show that $q(p)$ is optimal at p given that it is optimal for prices higher than p using mathematical induction. Since $q(\tilde{p}) = 0$, at $p = \tilde{p} - 1$ bidding gives $v - (p+1) - c = v - c - \lfloor v - c \rfloor > 0$, so $q(p) = 1$. This gives us induction basis for $i = 0$, since then $\tilde{p} - 2i = \tilde{p}$ and $\tilde{p} - (2i+1) = \tilde{p} - 1$.

Assuming that the claim is true for i, we want to show that it holds for $i + 1$. Since $q(\tilde{p} - 2i) = 0$ the game ends and the leader wins instantly, so

$$v^*(\tilde{p} - 2i) = v - \tilde{p} + 2i = c + \gamma + 2i, \quad v(\tilde{p} - 2i) = 0.$$

Also, $q(\tilde{p} - (2i+1)) = 1$, that is, the price increases by 1 with certainty and the roles are reversed, so

$$v^*(\tilde{p} - (2i+1)) = v(\tilde{p} - 2i) = 0, \quad v(\tilde{p} - (2i+1)) = v - \tilde{p} + 2i - c = 2i + \gamma.$$

Let $p = \tilde{p} - 2(i+1)$. Then $p+1 = \tilde{p} - (2i+1)$, so submitting a bid would give $v^*(\tilde{p} - (2i+1)) - c = -c$ to the non-leader, which is not profitable. Therefore $q(\tilde{p} - 2(i+1)) = 0$ and the leader gets

$$v^*(\tilde{p} - 2(i+1)) = v - \tilde{p} + 2(i+1) = c + \gamma + 2(i+1).$$

Let $p = \tilde{p} - (2(i+1)+1)$, so that $p+1 = \tilde{p} - 2(i+1)$. Then making a bid would give $v^*(\tilde{p} - 2(i+1)) - c = \gamma + 2(i+1) > 0$ to the non-leader, which means that it is profitable to make a bid.

To complete the analysis, we have to consider $t = 0$, where $p = 0$ and both players are non-leaders simultaneously choosing to bid or not. In this stage, there three cases to consider.

First consider the case when $\tilde{p}$ is an even integer, i.e. $\tilde{p} = 2i + 2$ for some $i \in \mathbb{N}$. Then $2 = \tilde{p} - 2i$ and $1 = \tilde{p} - (2i+1)$, so we get the strategic-form stage game in the Figure C.1. In this game both players

		Player 2	
		Bid	Pass
Player 1	Bid	$\frac{1}{2}(2i+\gamma-c), \frac{1}{2}(2i+\gamma-c)$	$-c, 2i+\gamma$
	Pass	$2i+\gamma, -c$	$0, 0$

Figure C.1: Period 0, case when $\tilde{p}$ is even

have strictly dominating strategy to pass, i.e. $q(0) = 0$. That is, the unique SPNE in the case when $\tilde{p}$ is even, is the one where the seller keeps the object.

Suppose now that $\tilde{p}$ is odd number, i.e. $\tilde{p} = 2i + 1$, so that $1 = \tilde{p} - 2i$ and $2 = \tilde{p} - (2i - 1)$. Then we get strategic form in the Figure C.2

		Player 2	
		Bid	Pass
Player 1	Bid	$\frac{\gamma}{2}+i-1-c, \frac{\gamma}{2}+i-1-c$	$2i+\gamma, -c$
	Pass	$0, 2i+\gamma$	$0, 0$

Figure C.2: Period 0, case when $\tilde{p}$ is odd

Note that $2i + \gamma = \tilde{p} - 1 + \gamma = v - c - 1$, so $\frac{1}{2}(2i + \gamma - 2) - c = \frac{1}{2}(v - 3(c+1))$. The sign of this expression is not determined by assumptions, so we have to consider two cases.

If $v \geq 3(c+1)$, then bidding at round 0 is dominating strategy for both players, i.e. $q(0) = 1$. Both players will submit a bid at round 0, and the one who will be the non-leader will submit another bid after that. This means that in total players make 3 bids and the price ends up to be 3. This is where the condition $v \geq 3(c+1)$ comes from.

If $v < 3(c+1)$, then there is a symmetric MSNE[28], where both bidders bid with probability $q \in (0,1)$, where q is determined by

$$q\left(\frac{1}{2}(2i+\gamma-2) - c\right) + (1-q)(2i+\gamma) = 0 \iff$$

$$q(0) = \frac{2(2i+\gamma)}{2c+2+2i+\gamma} = 2\frac{v-(c+1)}{v+(c+1)} \in (0,1).$$

□

Observation C.2. Some observations regarding the SSSPNE in the two-player case.

[28]There are also two asymmetric pure-strategy NE in the subgame, (P, B) and (B, P), where one player makes exactly one bid, so the revenue is $c+1$ and the value for this bidder is $v - (c+1)$.

(i) Equilibrium outcomes are very sensitive to seemingly irrelevant detail — is $\tilde{p}$ even or odd.

(ii) For realistic parameter values $v \gg 3(c+1)$. Then the equilibrium collapses in a sense that $\overline{R} = 3(c+1) \ll v$ or the object is not sold.

(iii) In a special case when $\tilde{p}$ is an odd integer and $v < 3(c+1)$, we get the results similar to $\varepsilon = 0$ case: $P(p > 0) \in (0,1)$, $E(R|p > 0) = v$, $v(0) = 0$.

In this equilibrium the players submit bids with positive probabilities and hope that the other does not submit a bid. But if she does, players actually prefer to be non-leaders, since at price $p = 2$, non-leader submits one more bid and the game ends at $p = 3$. Therefore $P(0) > 0, P(1) > 0, P(2) = 0, P(3) > 0, P(p) = 0, \forall p \geq 4$.

D A penny auction with multiple equilibria

Let $N + 1 = 3, v = 9.1, c = 2, \varepsilon > 0$. In this case, there are three SSSPNE, in Tables 3, 4, and 5 (which differ by actions at $p = 2$).

p	$q(p)$	$v^*(p)$	$v(p)$	$\overline{Q}(p)$	Qpp
0	0.509		0	0.1183	
1	0	8.1	0	0.3681	0.4175
2	1	0	0	0	0
3	0.6996	0.5504	0	0.0119	0.0135
4	0	5.1	0	0.4371	0.4958
5	0.4287	1.3381	0	0.0211	0.0239
6	0.0645	2.7129	0	0.0277	0.0314
7	0	2.1	0	0.0157	0.0178
8	0	1.1	0	0.0001	0.0001
9	0	0.1	0	0	0

Table 3: Equilibrium with $q(2) = 1$

p	$q(p)$	$v^*(p)$	$v(p)$	$\overline{Q}(p)$	$Q(p)$
0	0.5266		0	0.1061	
1	0	8.1	0	0.354	0.3961
2	0.7249	0.5371	0	0.0298	0.0333
3	0.6996	0.5504	0	0.0273	0.0306
4	0	5.1	0	0.3344	0.3741
5	0.4287	1.3381	0	0.0484	0.0542
6	0.0645	2.7129	0	0.0636	0.0711
7	0	2.1	0	0.036	0.0403
8	0	1.1	0	0.0003	0.0003
9	0	0.1	0	0	0

Table 4: Equilibrium with $q(2) = 0.7249 \in (0,1)$

p	$q(p)$	$v^*(p)$	$v(p)$	$\overline{Q}(p)$	$Q(p)$
0	0		0	1	
1	0.7473	0.5174	0	0	
2	0	7.1	0	0	
3	0.6996	0.5504	0	0	
4	0	5.1	0	0	
5	0.4287	1.3381	0	0	
6	0.0645	2.7129	0	0	
7	0	2.1	0	0	
8	0	1.1	0	0	
9	0	0.1	0	0	

Table 5: Equilibrium with $q(2) = 0$