Collegio Carlo Alberto

Dependence Calibration and Portfolio Fit with
Factor-Based Time Changes

Elisa Luciano

Marina Marena

Patrizia Semeraro

No. 307

October 2013 (Revised, January 2015)

Carlo Alberto Notebooks

www.carloalberto.org/research/working-papers

Dependence Calibration and Portfolio Fit with Factor-Based Time Changes

Elisa Luciano¹ Marina Marena² and Patrizia Semeraro³

¹Department of Economics and Statistics, University of Torino, Collegio Carlo Alberto and Netspar.

²Department of Economics and Statistics, University of Torino

³Politechnical School of Torino

Abstract

The paper explores the fit properties of a class of multivariate Lévy processes, which are characterized as time-changed correlated Brownian motions. The time-change has a common and an idiosyncratic component, to reflect the properties of trade, which it represents. The resulting process may provide Variance-Gamma, Normal-Inverse-Gaussian or Generalized-Hyperbolic margins. A non-pairwise calibration to a portfolio of ten US daily stock returns over the period 2009-2013 shows that fit of the Hyperbolic specification is very good, in terms of marginal distributions and overall correlation matrix. It succeeds in explaining the return distribution of both long-only and long-short random portfolios better than competing models do. Their tail behavior is well captured also by the Variance-Gamma specification.

Journal of Economic Literature Classification: G12, G13

Keywords: Lévy processes, multivariate subordinators, dependence, correlation, multivariate asset modelling, multivariate time-changed processes, factor-based time changes.

Introduction

A number of Lévy processes, which extend the classical Black-Scholes benchmark, have been adopted to represent stock returns. Among those, the Variance-Gamma, Normal-Inverse-Gamma, as well as the Generalized-Hyperbolic model have nice analytical tractability as well as fit properties, which explain their popularity, especially for derivative pricing. These models have been adopted to represent single-asset returns and have been extended in various ways to represent returns on several assets, or multivariate returns. All of them can be represented as time-changed Brownian motions. As a consequence, one could think of obtaining a multivariate model which has still Variance-Gamma, Normal-Inverse-Gamma, or Hyperbolic marginal distributions, by applying the same time-change to a multivariate Brownian motion. The technique is straightforward, but the economic implication is strong, because the time change represents the transformation from business time, under which returns may be Gaussian, thanks to a central-limit property of market returns, to calendar time (Clark [7]). The change of time represents trade and arrival of information. The higher is trade, the faster economic time goes with respect to calendar time. And there is a long-standing empirical literature, which shows that trade may be significantly different across assets (Harris [13]). Because recent empirical evidence shows that trade in different financial assets presents at least one common factor (Lo and Wang [16]), a natural extension of time-changed processes from a univariate to a multivariate setting, which preserves the marginal distributions, is through a multivariate change of time, with at least one common factor (see for instance Semeraro, [24]). The factor structure allows to fully and analytically characterize the return process, and may reproduce a single change of time if the common component is the predominant one. Alternative constructions, as such the ones which keep a distinct time change for each marginal distribution and correlate the Brownian motions, without providing a full process construction and characterization, are an appealing, straightforward alternative. Eberlein and Madan [9], for instance, correlate the unit-time random variables resulting from the marginal laws.

Calibration of the models which maintain the marginal distributions Variance-Gamma, Normal-Inverse-Gaussian or Hyperbolic is not always efficient in terms of computational time or satisfactory in terms of fit. A notable exception is indeed the model by Eberlein and Madan, which is calibrated in two steps, one for the marginals and one for the correlation matrix. In this paper we show that the factor-based processes not only preserve the marginal distributions and have a full analytical characterization, but are simple to calibrate and provide a very good fit of single assets and portfolios. To this end, we consider a class of factor-based processes, first introduced by Luciano and Semeraro ([18] and [19]), and we perform an extensive, not-pairwise calibration on MSCI data from January 2009 to May 2013. The calibration shows that the Hyperbolic specification fits marginal distributions and sample correlations with greater accuracy than both a Black-Scholes model and a model à la Eberlein and Madan. We then compare their performance

on randomly-weighted portfolios. We still find that all Lévy multivariate specifications of the factor-based model outperform the multivariate Gaussian model. They also outperform a model à la Eberlein and Madan, extended from Variance-Gamma marginals to Normal-Inverse-Gamma and Generalized-Hyperbolic, when evaluated on both long-only and long-short portfolios. In terms of overall distribution fit, as expressed by the Kolmogorov distance, both the Variance-Gamma and Hyperbolic distributions provide a good performance, the latter specification being better than the former for long-short portfolios.

The outline of the paper is as follows. Section 1 presents background and motivation. Section 2 introduces the factor-based multivariate time-changed processes, providing a construction which is less cumbersome than previous ones. Section 3 shows that, under suitable parameter restrictions, the model can be specified so as to have Variance-Gamma, Normal Inverse Gaussian and Generalized-Hyperbolic marginal distributions. Section 4 illustrates the approach to correlate time-changed Brownian motions adopted by Eberlein and Madan. Section 5 is devoted to the two model calibrations. We first illustrate the calibration procedure, then the fit. Section 5.4 shows the portfolios fit. The final Section summarizes and concludes.

1 Background and motivation

The construction of multivariate Lévy processes by means of a unique time change acting on several assets dates back at least to Luciano and Schoutens ([17]), Leoni and Schoutens ([15]). Both use a single subordinator, i.e. a single non-negative process which represents the change of time; the former paper has independent Brownian motions, while the second has correlated Brownian motions, to incorporate the idea that returns may be correlated over and above the trade effect incorporated in time changes. The shift from the first to the second model reflects the attempt to better capture the high correlation frequently observed in the data, that independent Brownian motions did not reproduce.

The construction technique by means of a multivariate – instead of a single – change of time is due to Barndorff-Nielsen [5], who builds a multivariate extension of the Variance-Gamma model. Semeraro [24] introduces a factor-based time change, with a component common to all assets and an idiosyncratic component, applied to independent Brownian motions. She constructs a multivariate Variance-Gamma process, named α -Variance-Gamma (αVG). Luciano and Semeraro [18] extend the αVG process and improve the correlation flexibility. First, they extend the technique to other marginal processes, including the compound Poisson, Normal-Inverse-Gaussian [2] and Carr-Geman-Madan-Yor [6]. Second, they generalize the correlation structure by using correlated Brownian motions. Luciano and Semeraro [19] use the factor-based time-change with correlated Brownian motions to specify Generalized-Hyperbolic and Hyperbolic marginal distribu-

tions (Eberlein and Prause [10]). Also Ballotta and Bonfiglioli [1] use a common and an idiosyncratic component both for the time-change and for the returns.

The calibrated behavior of the previous models has not been tested massively, especially beyond the pairwise case. As a partial exception, Wallmeier and Diethelm [26] study the calibration performance of the multivariate model of Leoni and Schoutens and the αVG model on a large dataset of reverse convertibles with three underlying stocks, traded on the Swiss market from January 2009 to May 2013. Both models provide a significant improvement over the multivariate Gaussian model, in terms of smile fitting. The αVG model provides a better fit, since the kurtosis parameter can freely vary across stocks, while it cannot when the subordinator is unique. However, for a large portion of the reverse convertible dataset, both models fail to match the three by three empirical correlation matrix of underlyings, the Leoni and Schoutens model fitting a higher percentage of the sample. Wallmeier and Diethelm conclude that there is the need for more flexible approaches. The same conclusions on the αVG model are reached by Guillaume [12] using the weekly quotes of four major stocks included in the S&P 500 index, from June 2nd, 2008 to October 20, 2009¹. Ballotta and Bonfiglioli [1] calibrate a multivariate Lévy model with fixed marginal distributions to option prices on three stocks belonging to the S&P100 index, considering the Variance-Gamma, Normal-Inverse-Gaussian, Merton and Kou jump-diffusion specifications. Calibration is performed in two steps. First, marginal distributions are calibrated. In the second step, the parameters of the idiosyncratic and common component are determined, by fitting the sample correlations and enforcing the convolution conditions necessary to recover the desired marginal distributions. All specifications are able to match the market correlations up to a good level of accuracy. However, the second step of the calibration procedure becomes increasingly challenging from a numerical point of view, as the number of assets taken into consideration grows.

As a result, an extensive calibration of time-changed Brownian motions which preserves the Variance-Gamma, Normal Inverse Gaussian and Generalized-Hyperbolic nature of the marginal distributions either has not been attempted, or has given quite a poor fit, because it was based on a model not flexible enough, or cannot be easily performed, because of the model construction. Consistently, our main goal consists in selecting a model which permits easy calibration with a large number of assets, is flexible enough to capture high correlation, and show that it may provide good fit of the marginal distributions, of the correlation matrix and, last but not least, of portfolio returns, which – opposite to the correlation matrix – reflects linear and non-linear dependence.

---

¹To improve the correlation fit, Guillaume proposed to relax some constraints to the marginal processes. In this paper, we restrict our attention to fixed marginals belonging to one of the classes specified above.

2 Factor-based time-changed processes

In this section we recall the construction of factor-based time-changed Lévy processes and provide their characteristic function, which is used to derive the distribution of portfolio returns, in closed form. Afterwards, we present different specifications of the model, which have Variance-Gamma (VG), Normal Inverse Gaussian (NIG) and Generalized-Hyperbolic marginal distributions (GH). The first two processes have been introduced in Luciano and Semeraro [18], the third one follows from a construction in Luciano and Semeraro [19].

Let us subordinate a Brownian motion, defined below, using a multivariate subordinator. To this end, let $\mathbf{X} = \{(X_1(t), \dots, X_n(t)), t \geq 0\}$ and $Z = \{Z(t), t \geq 0\}$ be independent subordinators, with zero drift and let $\mathbf{X}$ have independent components. They represent the idiosyncratic and common factors. The Lévy process $\mathbf{G} = \{\mathbf{G}(t), t \geq 0\}$ defined by:

$$\mathbf{G}(t) = (X_1(t) + \alpha_1 Z(t), \dots, X_n(t) + \alpha_n Z(t)), \quad \alpha_j > 0, j = 1, \dots, n, \quad (2.1)$$

is a multidimensional factor-based subordinator, with zero drift. Let $\mathbf{B}$ be a Brownian motion with independent components and Lévy triplet $(\boldsymbol{\mu}, \Sigma, 0)$:

$$\Sigma = \text{diag}(\sigma_1^2, \dots, \sigma_n^2) := \begin{pmatrix} \sigma_1^2 & 0 \dots & 0 \\ 0 & \sigma_2^2 \dots & 0 \\ 0 & 0 \dots & \sigma_n^2 \end{pmatrix}, \quad \boldsymbol{\mu} = (\mu_1, \dots, \mu_n), \quad (2.2)$$

and let $\mathbf{B}(\mathbf{s})$ be the corresponding multi-parameter Brownian Motion, defined as follows. Consider a multiparameter $\mathbf{s} = (s_1, \dots, s_n)^T \in \mathbb{R}_+^n$ and the partial order on $\mathbb{R}_+^n$:

$$\mathbf{s}^1 \preceq \mathbf{s}^2 \Leftrightarrow s_j^1 \leq s_j^2, j = 1, \dots, n.$$

The multi-parameter Brownian motion $\{\mathbf{B}(\mathbf{s}), \mathbf{s} \in \mathbb{R}_+^n\}$ is defined by (Barndorff-Nielsen et al., [5])

$$\mathbf{B}(\mathbf{s}) = \{(B_1(s_1), \dots, B_n(s_n)), \mathbf{s} \in \mathbb{R}_+^n\}$$

The following definition constructs a process $\mathbf{Y}$ as the sum of the Brownian motions in $\mathbf{B}$, each time-changed by a specific component of X , and some other Brownian motions, independent of $\mathbf{B}$, and consequently of X and Z , collected in a vector $\mathbf{B}^\rho$:

Definition 2.1. Let $\mathbf{B}(\mathbf{s})$ be a multi-parameter Brownian Motion and let $\mathbf{B}^\rho(t) = (B_1^\rho(t), \dots, B_n^\rho(t))$ be a multivariate Brownian motion, independent of $\mathbf{B}$, with Lévy triplet $(\boldsymbol{\mu}^\rho, \Sigma^\rho, 0)$, where

$$\boldsymbol{\mu}^\rho = (\mu_1 \alpha_1, \dots, \mu_n \alpha_n),$$

and

$$\Sigma^\rho := \begin{pmatrix} \sigma_1^2 \alpha_1 & \rho_{12} \sigma_1 \sigma_2 \sqrt{\alpha_1 \alpha_2} & \cdots & \rho_{1n} \sigma_1 \sigma_n \sqrt{\alpha_1 \alpha_n} \\ \rho_{12} \sigma_1 \sigma_2 \sqrt{\alpha_1 \alpha_2} & \sigma_2^2 \alpha_2 & \cdots & \rho_{2n} \sigma_2 \sigma_n \sqrt{\alpha_2 \alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1n} \sigma_1 \sigma_n \sqrt{\alpha_1 \alpha_n} & \rho_{2n} \sigma_2 \sigma_n \sqrt{\alpha_2 \alpha_n} & \cdots & \sigma_n^2 \alpha_n \end{pmatrix}, \quad (2.3)$$

with

$$\boldsymbol{\mu} \in \mathbb{R}^n, \quad \alpha_j > 0, \sigma_j > 0, j = 1, \dots, n.$$

Define an $\mathbb{R}^n$ -valued time-changed process $\mathbf{Y} = \{\mathbf{Y}(t), t > 0\}$ as:

$$\mathbf{Y}(t) = \begin{pmatrix} Y_1^I(t) + Y_1^\rho(t) \\ \dots \\ Y_n^I(t) + Y_n^\rho(t) \end{pmatrix} = \begin{pmatrix} B_1(X_1(t)) + B_1^\rho(Z(t)) \\ \dots \\ B_n(X_n(t)) + B_n^\rho(Z(t)) \end{pmatrix}, \quad (2.4)$$

where X_j and Z are independent subordinators, independent from $\mathbf{B}$. This is a **factor-based time-changed Brownian motion**.

The process $\mathbf{Y}$ is constructed by time-changing the multi-parameter process $\mathbf{B}(\mathbf{s})$ with independent changes of time and by time-changing the correlated multidimensional Brownian motion $\mathbf{B}^\rho(t)$ with a unique change of time. This construction has an important economic interpretation: each marginal Lévy process is obtained as a sum of an idiosyncratic and a systematic component. While the stock-specific components are independent, the systematic components are correlated. The following important property holds: the single components of $\mathbf{Y}$ are distributed as a Brownian motion time-changed by the corresponding component of $\mathbf{G}$. Indeed, the following equality in law $\mathcal{L}$ holds:²

$$\mathcal{L}(Y_j(t)) = \mathcal{L}(\mu_j G_j(t) + \sigma_j W(G_j(t))), \quad (2.5)$$

where W is a standard Brownian Motion, $G_j(t)$ is defined in (2.1). The marginal laws of $\mathbf{Y}$ are therefore one-dimensional time-changed Brownian motions, and the parameters of $\mathbf{Y}$ may be specified so to have VG, NIG and GH marginal distributions.

Applying Theorem 3.3 in Barndorff-Nielsen et al. [5] and its univariate version (Theorem 30.1 in Sato [23]) to $\mathbf{Y}^I(t) = (B_1(X_1(t)), \dots, B_1(X_1(t)))$ and $\mathbf{Y}^\rho(t) = (B_1^\rho(Z(t)), \dots, B_n^\rho(Z(t)))$, we find the characteristic function $\psi_{\mathbf{Y}(t)}$ of $\mathbf{Y}(t)$:

$$\begin{aligned} \psi_{\mathbf{Y}(t)}(\mathbf{u}) &= \psi_{\mathbf{Y}^I(t)}(\mathbf{u}) \psi_{\mathbf{Y}^\rho(t)}(\mathbf{u}) = \\ &= (t \sum_{j=1}^n l_{X_j}(\log(\psi_{B_j}(u_j)))) \exp(t l_Z(\log(\psi_{\mathbf{B}^\rho}(\mathbf{u}))). \end{aligned} \quad (2.6)$$

where l_{X_j} and l_Z are the Laplace exponents of the subordinators.³ Thanks to the representation in the definition above, we have the characteristic function of $\mathbf{Y}$, while equality in law permits us to reproduce specific marginal distributions, as we do now.

²See theorem 5.1 in Luciano and Semeraro [18].

³This means that, for any $w \in \mathbb{C}$ with $\text{Re}(w) \leq 0$, $j = 1, \dots, n$, $l_{X_j}(w) = \int_{\mathbb{R}_+} (e^{\langle w, s \rangle} - 1) \nu_{X_j}(ds)$ and $l_Z(w) = \int_{\mathbb{R}_+} (e^{\langle w, s \rangle} - 1) \nu_Z(ds)$, where ν_{X_j} and ν_Z are the Lévy measures of X_j and Z respectively.

Obviously, whenever all the parameters ρ_{ij} collapse to 0 across different components, i.e. $\rho_{ij} = 0$, for $i \neq j$, $\rho_{ij} = 1$, for $i = j$, we have a version of the model in which the Brownian motions are independent. We will denote this as the α version of the time-changed model, with the general version being the $\rho\alpha$ version.

3 Specifications

We now specify the process $\mathbf{Y}$ so that it has VG, NIG and GH marginal distributions, as in [18] and [19]. This allows us to introduce the notation and parameter conditions which will be used in the calibration procedure. The different specifications are obtained using different subordinators. To do that, using the correspondence between Lévy processes and their law at time 1, it is sufficient to specify the distributions of the time-1 subordinators. Let

$$X_j := X_j(1), Z := Z(1) \text{ and } G_j := G_j(1). \quad (3.1)$$

In the next subsections, we specify the distribution of G_j , i.e. the distribution of X_j , $j = 1, \dots, n$, the distribution of Z and the parameters $\mu_j, \sigma_j, \alpha_j$, $j = 1, \dots, n$ so that $\mu_j G_j + \sigma_j W(G_j)$, $j = 1, \dots, n$ have VG, NIG, and GH marginal distributions.

For each specification, we also provide the characteristic function.

3.1 Variance-Gamma marginal distributions

Recall that *Variance-Gamma* univariate process, introduced by Madan and Seneta [20], is a real Lévy process $L_{VG} = \{L_{VG}(t), t \geq 0\}$ which can be obtained as a Brownian motion time-changed by a Gamma process $\{G(t), t \geq 0\}$. Let $\sigma > 0$ and μ be real parameters, then the process L_{VG} is defined as

$$L_{VG}(t) = \mu G(t) + \sigma B(G(t)),$$

where B is a standard Brownian motion. Its characteristic function is

$$\psi_{VG}(u) = \left(1 - iu\mu\alpha + \frac{1}{2}\sigma^2\alpha u^2\right)^{-\frac{t}{\alpha}}. \quad (3.2)$$

We now specify $\mathbf{G}$ to have Gamma marginal distributions. Let X_j and Z be distributed according to Gamma laws:

$$\mathcal{L}(X_j) = \Gamma\left(\frac{1}{\alpha_j} - a, \frac{1}{\alpha_j}\right) \quad \text{and} \quad \mathcal{L}(Z) = \Gamma(a, 1).$$

To have marginal distributions with non negative parameters, the parameters α_j and a must satisfy the constraints

$$0 < \alpha_j < \frac{1}{a} \quad j = 1, \dots, n. \quad (3.3)$$

In this case the random variables G_j defined in (3.1) are Gamma-distributed too:

$$\mathcal{L}(G_j) = \Gamma\left(\frac{1}{\alpha_j}, \frac{1}{\alpha_j}\right), \quad j = 1, \dots, n.$$

The subordinator $\mathbf{G}$ defined in equation 3.1 has Gamma marginal processes at all times t :

$$\mathcal{L}(G_j(t)) = \Gamma\left(\frac{t}{\alpha_j}, \frac{1}{\alpha_j}\right), \quad j = 1, \dots, n.$$

It follows from the factor-based time-changed process $\mathbf{Y}$ defined in (2.4) has VG marginal processes with parameters $\mu_j, \alpha_j, \sigma_j$ - denoted as $VG(\mu_j, \alpha_j, \sigma_j)$ - i.e.

$$\mathcal{L}(Y_j) = \mathcal{L}(\mu_j G_j(t) + \sigma_j W(G_j(t))). \quad (3.4)$$

We name $\mathbf{Y}$ a $\rho\alpha$ *Variance-Gamma* process, shortly $\rho\alpha VG$. The $\alpha\rho VG$ model has a total of $1 + 3n + \frac{n(n-1)}{2}$ parameters:

- one common parameter a ;
- three marginal parameters $\mu_j, \alpha_j, \sigma_j$ for each marginal distribution, $j = 1, \dots, n$;
- as many additional parameters as the Brownian motions correlations ρ_{ij} , $i, j = 1, \dots, n$.

The characteristic function of $\mathbf{Y}$ at time 1 is:

$$\psi_{\mathbf{Y}}(\mathbf{u}) = \prod_{j=1}^n \left(1 - \alpha_j(i\mu_j u_j - \frac{1}{2}\sigma_j^2 u_j^2)\right)^{-\left(\frac{1}{\alpha_j} - a\right)} \left(1 - \left(i\mathbf{u}^T \boldsymbol{\mu}^\rho - \frac{1}{2}\mathbf{u}^T \Sigma^\rho \mathbf{u}\right)\right)^{-a}. \quad (3.5)$$

By imposing $\rho_{ij} = 0$, for $i \neq j$, $\rho_{ij} = 1$, for $i = j$, we find as a subcase of the current model the αVG process introduced in Semeraro [24].

3.2 Normal-Inverse-Gaussian marginal distributions

The *normal inverse Gaussian* (NIG) process has been constructed by subordination by Barndorff-Nielsen [4]. A NIG process with parameters $\gamma > 0$, $-\gamma < \beta < \gamma$, $\delta > 0$ is a Lévy process $L_{NIG} = \{L_{NIG}(t), t \geq 0\}$ with characteristic function

$$\psi_{NIG}(u) = \exp\left(-\delta t \left(\sqrt{\gamma^2 - (\beta + iu)^2} - \sqrt{\gamma^2 - \beta^2}\right)\right). \quad (3.6)$$

It can be constructed by subordinating a Brownian motion with an inverse Gaussian distribution. An *inverse Gaussian* (IG) process with parameters (a, b) is a Lévy process with the following characteristic function:

$$\psi_{IG}(u) = \exp\left(-a \left(\sqrt{-2iu + b^2} - b\right)\right). \quad (3.7)$$

In order for the time-changed setting $\mathbf{Y}$ to have NIG marginal distributions, we specify the subordinator $\mathbf{G}$ in (2.1) to have IG marginal distributions, by defining

$$\begin{aligned} X_j &\sim IG\left(1 - a\sqrt{\alpha_j}, \frac{1}{\sqrt{\alpha_j}}\right), \quad j = 1, \dots, n \\ Z &\sim IG(a, 1). \end{aligned}$$

Using the closure properties of the IG distribution, we obtain that the process $\alpha_j Z$ is $\alpha_j Z \sim IG\left(a\sqrt{\alpha_j}, \frac{1}{\sqrt{\alpha_j}}\right)$ and that its sum with X_j is still IG (see Luciano and Semeraro [18]):

$$X_j + \alpha_j Z \sim IG\left(1, \frac{1}{\sqrt{\alpha_j}}\right).$$

In order for the marginal distributions to have non negative parameters, the following constraints must be satisfied:

$$0 < a < \frac{1}{\sqrt{\alpha_j}}, \quad j = 1, \dots, n. \quad (3.8)$$

Let $\mathbf{G}$ be as in (2.1), with X_j and Z as defined above. The marginal time-change processes $G_j(t)$ are IG:

$$\mathcal{L}(G_j(t)) = IG\left(t, \frac{1}{\sqrt{\alpha_j}}\right), \quad j = 1, \dots, n.$$

Suitable constraints on the parameters make the subordinated process have NIG marginal distributions. Let $\gamma_j, \beta_j, \delta_j$ be such that

$$\gamma_j > 0, \quad -\gamma_j < \beta < \gamma_j, \quad \delta_j > 0.$$

Further, let

$$\frac{1}{\sqrt{\alpha_j}} = \delta_j \sqrt{\gamma_j^2 - \beta_j^2}. \quad (3.9)$$

If we set $\mu_j = \beta_j \delta_j^2$ and $\sigma_j = \delta_j$ in (2.3) the process $\mathbf{Y}$ defined in (2.4) has NIG marginal processes, i.e.

$$\mathcal{L}(Y_j) = \mathcal{L}(\beta_j \delta_j^2 G_j(t) + \delta_j W(G_j(t))) \quad (3.10)$$

We name $\mathbf{Y}$ a $\rho\alpha$ Normal Inverse Gaussian process, shortly $\rho\alpha$ NIG. By imposing $\rho_{i,j} = 0$, for $i \neq j$, we have the α NIG model ([18]). Note that the process has a total of $1 + 3n + \frac{n(n-1)}{2}$ parameters:

- a is a common parameter;
- $\gamma_j, \beta_j, \delta_j, j = 1, \dots, n$ are marginal parameters;
- $\rho_{ij}, i, j = 1, \dots, n$, are the $\mathbf{B}^\rho$ correlations.

Its characteristic function at time one is:

$$\begin{aligned} \psi_{\mathbf{Y}}(\mathbf{u}) = \exp \left\{ - \sum_{j=1}^n \left(1 - \frac{a}{\zeta_j} \right) \left(\sqrt{-2(i\beta_j \delta_j^2 u_j - \frac{1}{2} \delta_j^2 u_j^2) + \zeta_j^2} - \zeta_j \right) \right. \\ \left. - a \left(\sqrt{-2(i\mathbf{u}^T \boldsymbol{\mu}^\rho - \frac{1}{2} \mathbf{u}^T \Sigma^\rho \mathbf{u}) + 1} - 1 \right) \right\} \end{aligned} \quad (3.11)$$

where $\zeta_j = \delta_j \sqrt{\gamma_j^2 - \beta_j^2}$.

3.3 Generalized-Hyperbolic marginal distribution

Recall that the Generalized-Hyperbolic process is generated by the Generalized-Hyperbolic distribution. The latter distribution, shortly GH($\gamma, \beta, \delta, \lambda$), has been introduced by Barndorff-Nielsen [2] and has been first applied to finance by Eberlein and Keller [8] and Eberlein and Prause [10]. Its characteristic function is:

$$\psi_{GH}(u) = \left(\frac{\gamma^2 - \beta^2}{\gamma^2 - (\beta + iu)^2} \right)^{\lambda/2} \frac{K_\lambda \left(\delta \sqrt{\gamma^2 - (\beta + iu)^2} \right)}{K_\lambda \left(\delta \sqrt{\gamma^2 - \beta^2} \right)}. \quad (3.12)$$

The GH distribution can be defined as a normal mean-variance mixture with a *generalized inverse Gaussian* (GIG) as mixing distribution; as a consequence the GH process can be constructed by time-changing a Brownian motion with a GIG subordinator. A GIG distribution with parameters $\lambda \in \mathbb{R}$, $a, b \in \mathbb{R}_+$, shortly $GIG(\lambda, a, b)$, is a three-parameter distribution defined on the positive half line. It is an infinitely divisible distribution and it generates a GIG subordinator. Its characteristic function is

$$\psi_{GIG}(u) = \frac{1}{K_\lambda(ab)} \left(1 - \frac{2iu}{b^2} \right)^{-\frac{\lambda}{2}} K_\lambda \left(ab \sqrt{1 - 2iub^{-2}} \right), \quad x > 0, \quad (3.13)$$

where $K_\lambda(x)$ denotes the modified Bessel function of the third kind with index λ .

If $G \sim GIG(\lambda, a, b)$ (positive distribution) and W is standard normal, independent of G , then $\sqrt{G}W + \mu G$ has a GH distribution, with parameters $\gamma, \beta, \delta, \lambda$ where:

$$\begin{aligned} a &= \delta \\ \mu &= \beta \\ b &= \sqrt{\gamma^2 - \beta^2}. \end{aligned} \quad (3.14)$$

The parameters $\lambda, \beta \in \mathbb{R}$, $\gamma, \delta \in \mathbb{R}_+$ have to satisfy the following constraints:

$$\begin{aligned} \delta &\geq 0, \quad |\beta| < \gamma \text{ if } \lambda > 0 \\ \delta &> 0, \quad |\beta| < \gamma \text{ if } \lambda = 0 \\ \delta &> 0, \quad |\beta| \leq \gamma \text{ if } \lambda = 0 \end{aligned} \quad (3.15)$$

To obtain GH marginal returns in the factor-based context, we specify a subordinator with marginal GIG laws. The main difficulty in the construction is that the GIG distribution is not closed under convolution. However, under a proper choice of the parameters, the convolution of a Gamma and a GIG distribution is itself GIG distributed. We adopt the device of defining the change of time by means of a Gamma distributed common component. This means that we do not recover as a limit case the multidimensional GH process analyzed by Eberlein and Prause [10], and discussed in McNeil et al. [10]. If the idiosyncratic component degenerates instead we find the VG process. The peculiarity of this model is then that both the distribution and the process are generalizations of the $\rho\alpha$ VG model.

Let G_j be as in (2.1), where $X_j = R_j + V_j$, R_j and V_j are independent and

$$L(R_j) = GIG\left(-\lambda, \delta_j, \frac{1}{\sqrt{\alpha_j}}\right), L(V_j) = \Gamma\left(\lambda - a, \frac{1}{2\alpha_j}\right) \text{ and } L(Z) = \Gamma\left(a, \frac{1}{2}\right).$$

If the parameters satisfy the following constraints:

$$\lambda > 0, \alpha_j > 0, 0 < a < \lambda, \quad (3.16)$$

and $\delta_j, \frac{1}{\sqrt{\alpha_j}}$ are both nonnegative and not simultaneously zero, the following equality in law holds:

$$\mathcal{L}(X_j + \alpha_j Z) = GIG\left(\lambda, \delta_j, \frac{1}{\sqrt{\alpha_j}}\right),$$

where $\alpha_j > 0$ (see Barndorff-Nielsen et al. [2]). Notice that the law of X_j - the idiosyncratic component - is the convolution of a GIG and a Gamma distribution.

Now, let $\mathbf{B}$ and $\mathbf{B}^\rho$ have marginal laws with mean β_j and standard deviation 1, and mean $\beta_j \alpha_j$ and standard deviation $\sqrt{\alpha_j}$, respectively. It follows from the construction of the GH distribution that the process $\mathbf{Y}$ has GH marginal distributions with parameters $\gamma_j, \beta_j, \delta_j, \lambda$, where:

$$\frac{1}{\sqrt{\alpha_j}} = \sqrt{\gamma_j^2 - \beta_j^2}. \quad (3.17)$$

We name the above process $\rho\alpha$ *Generalized-Hyperbolic*, shortly $\rho\alpha$ GH. By imposing $\rho_{i,j} = 0$, for $i \neq j$, we have the α GH model ([19]). We emphasize that this construction holds for a subclass of Generalized-Hyperbolic marginal distributions, namely, those with parameter $\lambda > 0$. Therefore, the NIG marginal distribution is not a subcase of this construction, since it corresponds to $\lambda = -1/2$. The process has a total of $2 + 3n + \frac{n(n-1)}{2}$ parameters:

- λ and a are common parameters;
- $\gamma_j, \beta_j, \delta_j, j = 1, \dots, n$, are marginal parameters;
- $\rho_{ij}, i, j = 1, \dots, n$, are the $\mathbf{B}^\rho$ correlations.

Note also that λ is the only common parameter of the marginal returns. As a consequence, if we consider the subcase of Hyperbolic marginal distributions introduced by Barndorff-Nielsen [3], obtained setting $\lambda = 1$, marginal returns do not have common parameters.

The $\rho\alpha$ GH characteristics function at time 1 is

$$\begin{aligned} \psi_{\mathbf{Y}}(\mathbf{u}) &= \prod_{j=1}^n \left(1 - 2 \left(\frac{\delta_j}{\zeta_j} \right)^2 \left(i u_j \beta_j - \frac{1}{2} u_j^2 \right) \right)^{-\frac{\lambda}{2} + a} \frac{K_{\lambda} \left(\zeta_j \sqrt{1 - 2 \left(\frac{\delta_j}{\zeta_j} \right)^2 \left(i u_j \beta_j - \frac{1}{2} u_j^2 \right)} \right)}{K_{\lambda}(\zeta_j)} \\ &\quad \cdot \left(1 - 2 \left(i \mathbf{u}^T \boldsymbol{\mu}^{\rho} - \frac{1}{2} \mathbf{u}^T \Sigma^{\rho} \mathbf{u} \right) \right)^{-a}. \end{aligned}$$

where, again, $\zeta_j = \delta_j \sqrt{\gamma_j^2 - \beta_j^2}$.

3.4 Linear correlation

The class of processes introduced so far has both linear correlation and non-linear dependence. Only linear correlation coefficients will be used in the calibration below. The linear correlation coefficients of a factor-based time changed Brownian motion $\mathbf{Y}$ are:

$$\begin{aligned} \rho_{\mathbf{Y}}(i, j) &= \frac{\text{Cov}(B_i^{\rho}, B_j^{\rho}) E[Z] + E[B_i^{\rho}] E[B_j^{\rho}] V(Z)}{\sqrt{V(Y_i) V(Y_j)}}, \\ &= \frac{\rho_{ij} \sigma_i \sigma_j \sqrt{\alpha_i} \sqrt{\alpha_j} E[Z] + \mu_i \mu_j \alpha_i \alpha_j V(Z)}{\sqrt{V(Y_i) V(Y_j)}}. \end{aligned}$$

In the $\rho\alpha$ VG case they become:

$$\rho_{\mathbf{Y}_{\rho}}(i, j) = \frac{(\mu_i \alpha_i \mu_j \alpha_j + \rho_{ij} \sigma_i \sqrt{\alpha_i} \sigma_j \sqrt{\alpha_j})}{\sqrt{(\sigma_i^2 + \mu_i^2 \alpha_i)(\sigma_j^2 + \mu_j^2 \alpha_j)}} a. \quad (3.18)$$

They are increasing in a , and, under (3.3), must satisfy the constraint

$$0 < a < \min_j \left(\frac{1}{\alpha_j} \right).$$

By exploiting (3.9), the linear correlations of the $\rho\alpha$ NIG case are:

$$\rho_{\mathbf{Y}}(i, j) = \frac{\left(\beta_i \frac{\delta_i^2}{\zeta_i^2} \beta_j \frac{\delta_j^2}{\zeta_j^2} + \rho_{ij} \frac{\delta_i}{\zeta_i} \frac{\delta_j}{\zeta_j} \right)}{\sqrt{\left(\gamma_i^2 \delta_i (\gamma_i^2 - \beta_i^2)^{-\frac{3}{2}} \right) \left(\gamma_j^2 \delta_j (\gamma_j^2 - \beta_j^2)^{-\frac{3}{2}} \right)}} a. \quad (3.19)$$

They are increasing in a , and, under (3.8) and (3.9), must satisfy the constraint

$$0 < a < \min_j \zeta_j.$$

Enforcing (3.9), the linear correlation coefficients of the $\rho\alpha$ GH case are:

$$\rho_{\mathbf{Y}}(i, j) = \frac{\left(4\beta_i \frac{\delta_i^2}{\zeta_i^2} \beta_j \frac{\delta_j^2}{\zeta_j^2} + 2\rho_{ij} \frac{\delta_i}{\zeta_i} \frac{\delta_j}{\zeta_j}\right)}{\sqrt{V(Y_i)V(Y_j)}} a,$$

where the marginal variance is given by

$$V(Y_i) = \frac{\delta_i^2}{\zeta_i} \frac{K_{\lambda+1}(\zeta_i)}{K_{\lambda}(\zeta_i)} + \beta_i^2 \frac{\delta_i^4}{\zeta_i^2} \left(\frac{K_{\lambda+2}(\zeta_i)}{K_{\lambda}(\zeta_i)} - \frac{K_{\lambda+1}^2(\zeta_i)}{K_{\lambda}^2(\zeta_i)} \right). \quad (3.20)$$

The linear correlation coefficients are increasing in a . Under (3.16), the common parameter a must satisfy the constraint

$$0 < a < \lambda.$$

The α version of the models can be recovered by setting the correlation coefficients between the Brownian components to zero. We notice that the $\rho\alpha$ models introduce an additional term in the expression of the return correlations with respect to the α version. By so doing, they widen the range of achievable linear dependence.

While the constraint on the common parameter a for the GH process is independent of the marginal parameters, in the VG and NIG cases the constraints turn out to be closely related to the maximum kurtosis level of the marginal processes. Actually, the kurtosis increases with the parameter α_j in the VG case and decreases with ζ_j in the NIG case. While low empirical return correlations may always be accounted for by choosing a sufficiently low a , high correlations may be challenging to achieve if there is at least one asset with very high kurtosis. On the other hand, if all assets show a moderate level of kurtosis and the correlation range is not too large in absolute value, then the constraint is not likely to limit the model ability of matching sample correlations within the VG and NIG subclasses.

4 An alternative approach: correlating marginal processes

Before calibrating the $\rho\alpha$ - models introduced above, we recall here the approach followed by Eberlein and Madan [9] to correlate the unit-time random variables resulting from univariate time-changed Brownian motions. Although they explored the VG specification, their model of dependence applies also to marginal processes of the NIG and GH types.

Let us model individual returns as one dimensional time changed Brownian motions:

$$Y_j(t) = \mu_j G_j(t) + \sigma_j W_j(G_j(t)), \quad j = 1, \dots, n, \quad (4.1)$$

and assume that the subordinators are independent. Eberlein and Madan consider introducing dependence between returns at unit time by merely correlating the Brownian

motions and keeping the time changing subordinators independent. Therefore

$$\mathcal{L}(Y_j) = \mathcal{L}(\mu_j G_j + \sigma_j \sqrt{G_j} W_j), \quad j = 1, \dots, n, \quad (4.2)$$

where W_j are standard normal variates with correlations ρ_{ij}^W . Under this model the correlation $\rho_{\mathbf{Y}}^{EM}(i, j)$ between returns becomes

$$\rho_{\mathbf{Y}}^{EM}(i, j) = \frac{\text{cov}(Y_i, Y_j)}{\sqrt{V(Y_i)V(Y_j)}} = \frac{\sigma_i \sigma_j E[\sqrt{G_i}] E[\sqrt{G_j}] \rho_{ij}^W}{\sqrt{V(Y_i)V(Y_j)}}. \quad (4.3)$$

The correlation between the Brownian motions implied by the model can be estimated by

$$\rho_{ij}^W = \frac{\text{cov}(Y_i, Y_j)}{\sigma_i \sigma_j E[\sqrt{G_i}] E[\sqrt{G_j}]},$$

where $\text{cov}(Y_i, Y_j)$ is estimated via sample covariance.

For the VG specification (VG-EM) $\sqrt{V(Y_i)V(Y_j)}$ is provided in equation (3.18) and (see [9])

$$E[\sqrt{G_i}] = \frac{\sqrt{\alpha_i} \Gamma((\frac{1}{\alpha_i}) + \frac{1}{2})}{\Gamma(\frac{1}{\alpha_i})}.$$

For the NIG specification (NIG-EM) $\sqrt{V(Y_i)V(Y_j)}$ is provided in equation (3.19) and

$$E[\sqrt{G_i}] = \int_{\mathbb{R}^+} \sqrt{x} \frac{1}{\sqrt{2\pi}} \exp(\frac{1}{\sqrt{\alpha_i}}) x^{-3/2} \exp(-\frac{1}{2}(x^{-1} + \frac{x}{\alpha_i})) dx.$$

For the HYP specification (HYP-EM) the marginal variance $V[Y_i]$ is given by (3.20) and

$$E[\sqrt{G_i}] = \int_{\mathbb{R}^+} \sqrt{x} \frac{\frac{1}{\sqrt{\alpha_i} \delta_i}}{2K_1(\frac{\delta_i}{\sqrt{\alpha_i}})} \exp(-\frac{1}{2}(\delta_i^2 x^{-1} + \frac{x}{\alpha_i})) dx.$$

5 Calibration and fit

Define an n -dimensional price process, $\mathbf{S} = \{\mathbf{S}(t), t \geq 0\}$, by:

$$\mathbf{S}(t) = \mathbf{S}(0) \exp(\mathbf{c}t + \mathbf{Y}(t)), \quad \mathbf{c} \in \mathbb{R}^n, \quad (5.1)$$

where $\mathbf{c}$ is the drift term (equivalently, $S_j(t) = S_j(0) \exp(c_j t + Y_j(t))$, $t \geq 0, j = 1, 2, \dots, n$). The different specifications listed above of the factor-based time-changed process $\mathbf{Y}$ can then be fitted to stock-market return data. We perform the calibration under the historical measure, since we want to assess how the previous models describe historically observed returns. The calibration is performed in two steps. The first consists in fitting the marginal parameters from marginal return data; the second in selecting the common parameters by matching the historical return correlation matrix. We calibrate the marginal return parameters by maximum likelihood (MLE), and then we calibrate

the common parameters, i.e. the common-time-change parameter a and the correlation coefficients of the $\mathbf{B}^p$ component (the other Brownians in $\mathbf{B}$ being independent), by minimizing the distance between model and empirical correlations. We use as a distance the Frobenius norm. We include the Gaussian case [shortly, G], as the benchmark model.

We are interested both in the marginal fit, as measured by the Kolmogorov-Smirnov and Anderson-Darling statistics, and in the correlation fit of the different $\rho\alpha$ - Lévy models provided above. Last but not least, we explore the joint fit – which depends on linear and non-linear correlation – by testing the ability of the model distribution of portfolio returns to describe their sample distribution, in the spirit of [9] and [27]. We consider long-short and long-only portfolios. Long-short portfolio weights are generated from independent normal variates, rescaled by the sum of their squares. For the long-only case, we take the absolute value of the standard normal variates and rescale them by their sum. For each vector $\mathbf{w}$ of randomly selected portfolio weights, we compute the sample portfolio return from the whole time series data. The model portfolio return R_P is given by

$$R_P = \mathbf{w} \cdot (\mathbf{c} + \mathbf{Y}(1)).$$

and its characteristic function is

$$\psi_{R_P}(u) = \exp(\mathbf{c} \cdot u\mathbf{w})\psi_{\mathbf{Y}}(u\mathbf{w}).$$

For both dependence models we compare the performance of the different specifications in explaining portfolio returns.

We perform the Kolmogorov-Smirnov test, deriving the density function of model portfolio returns by FRFT inversion of their characteristic function. For the purpose of comparison, we consider also the VG, NIG and HYP specifications of the Eberlein and Madan model [9]. Since the characteristic function of the portfolio returns is not available under the latter approach, we rely on Monte Carlo simulation of 100,000 portfolio returns to derive their empirical cumulative distribution functions⁴.

5.1 Data and summary statistics

We consider daily logreturns on MSCI US **Investable** Market Indices from January 2, 2009 to May 31st, 2013. Specifically, we look at the following sector indices: consumer discretionary (CD), consumer staples (CS), energy (EN), financials (FN), health care (HC), industrials (IN), information technology (IT), materials (MT), telecommunications (TC), and utilities (UT). We have 10 indices, with a total of 1109 observations.

Sample moments and correlations are reported in Table 1 and Table 2.

[Insert Table 1]

[Insert Table 2]

---

⁴For testing purposes, we also derive the cumulative distribution functions of portfolio returns for the $\rho\alpha$ - models by Monte Carlo simulation, obtaining the same plots as in Fig. 1.

Because of the model properties explained in section 3.4 above, matching market correlations appears to be particularly challenging within our framework, since the dataset includes assets with high kurtosis together with high sample correlations. For instance, the Financials Index has a sample kurtosis equal to 11.946 and several correlation coefficients are above 0.8, with the correlation coefficient between the Consumer Discretionary Index and the Industrials Index as high as 0.933.⁵

5.2 Calibration of the marginal distributions

We use MLE to estimate the marginal return distribution on each stock individually. The density function is recovered applying the Fractional Fast Fourier Transform (FRFT) [11]. Initial conditions are chosen according to the method of moments.⁶ The calibrated marginal parameters are reported in Tables 3, 4, 5 and 6 for the different model specifications, including the G model as a benchmark. In all cases, the parameters in $\mathbf{c}$ are the drift terms in equation (3.4). Obviously, these parameters will apply both to our joint model and to the Eberlein and Madan one, since in both cases the two-step procedure is permitted.

[Insert Table 3]

[Insert Table 4]

[Insert Table 5]

[Insert Table 6]

The goodness of fit is evaluated by the Kolmogorov-Smirnov (KS) test, whose results are reported in Table 7. For all Lévy models and all indices, we cannot reject the null hypothesis that the sample comes from the model distribution, at the 5% level of significance. On the contrary, the G model provides a poor fit for all assets. Obviously, there are nuances across assets and models, in particular with reference to assets with high kurtosis, such as the Financials Index.

[Insert Table 7]

Table 8 shows the Anderson and Darling (AD) distance between the model and the sample cumulative distribution functions. This statistics puts more weight on the tails than the Kolmogorov-Smirnov test. Again, the poor fit of the Gaussian distribution is reflected by the high values of the AD statistic, while all Lévy models provide a good fit. Overall, none of them seems to be superior to the others, considering the whole basket of assets. Differences in fit across models exist, with the NIG outperforming alternative specifications.

[Insert Table 8]

---

⁵Considering, e.g. the VG case, if $\alpha_j \rightarrow \infty$ for one j , $a \rightarrow 0$ and $\rho_{\mathbf{Y}} \rightarrow 0$. This explains why the model is not flexible enough to match high correlations when one of the assets has high kurtosis.

⁶We refer to Seneta [25] and Prause [22] for details.

5.3 Calibration of the dependence structure

Given the marginal parameters, we jointly calibrate the common parameter a , and the correlation between the Brownian components, by fitting sample return correlations. Specifically, we minimize the root-mean-squared error between the empirical and the $\rho\alpha$ -model return correlations

$$RMSE(a, \rho) = \sqrt{\frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j>i}^n (\rho_{\mathbf{Y}}^{emp}(i, j) - \rho_{\mathbf{Y}}(i, j; a, \rho_{ij}))^2}$$

where $\rho = \{\rho_{ij}, i = 1, \dots, n, j = 2, \dots, n\}$ are the correlation coefficients between the Brownian components collected in the $\mathbf{B}^\rho$, $\rho_{\mathbf{Y}}^{emp}(i, j)$ and $\rho_{\mathbf{Y}}(i, j)$ are the sample and model return correlations, respectively. Since all the marginal parameters are fixed from step 1, the correlation coefficients $\rho_{\mathbf{Y}}(i, j)$ depend on a and ρ_{ij} only, $\rho_{\mathbf{Y}}(i, j) = \rho_{\mathbf{Y}}(i, j; a, \rho_{ij})$.

In case cross-return correlations result in a matrix that is not symmetric positive semidefinite, a regularization procedure is applied. We follow the one proposed by Higham [14], which allows to find the symmetric positive-semidefinite matrix which is closest to an arbitrary real matrix (in our case the one formed by the $\rho_{\mathbf{Y}}^{emp}(i, j)$) in the Frobenius norm. Note that this calibration step can be easily implemented. Moreover, the algorithm convergence is fast and robust with respect to the initial conditions.

We provide both joint and pairwise calibration results for the different Lévy $\rho\alpha$ -subclasses. As concerns the joint calibration, the upper bound of the common parameter a are given in Table 9 for different submodels. The optimal value of the common parameter a is equal to its maximal value in all submodels for the 10 asset portfolio, which is consistent with high correlations and their range. We emphasize that this is not the case when subportfolios are considered. The intuition for this is again the behavior of the correlation submatrix and the range of its components. The calibrated correlation coefficients between the Brownian components for the $\rho\alpha$ -models are reported in Tables 10, and 11. In the NIG case, all correlations are equal to one. In all cases, we applied the Higham procedure [14] in order to enforce a positive-semidefinite matrix.

[Insert Table 9]

[Insert Table 10]

[Insert Table 11]

Tables 12 and 13 provide the RMSE and the maximum absolute errors between the sample and model return correlations for both α and $\rho\alpha$ -models. They clearly show that moving from the α -models to the $\rho\alpha$ -models allows to mitigate the correlation problem in the VG and NIG cases, although in both models the constraint on the common parameter a turns out to be binding and several correlation coefficients between the Brownian components are close to 1. In the HYP case, while the α -model cannot

recover market correlations, the $\rho\alpha$ - model achieves an almost perfect matching across all pairs.

[Insert Table 12]

[Insert Table 13]

The calibration errors in matching correlations for the $\rho\alpha$ - models are shown in Tables 14, 15, and 16, for the VG, NIG and HYP models, respectively.

[Insert Table 14]

[Insert Table 15]

[Insert Table 16]

We remark that, in the $\rho\alpha$ VG and $\rho\alpha$ NIG cases, the constraint on a does not allow to accurately reproduce the sample correlations of our dataset. The restriction on the common parameter a limits the ability of these models to match high market correlations. On the contrary, the HYP model proves to be able to reproduce even very high correlation coefficients. This happens, in the sample, because the idiosyncratic component has mean and variance close to zero, while the common component has not. This shows that the factor-based model captures high correlation, simply by playing on the relative importance of the two factors, without relying ex ante on a single change of time. The latter would affect also the marginal distributions and would be less flexible in their calibration (for instance, it would not allow a two-step fit). In addition, it would not fit low correlations.

Let us consider now the pairwise calibration, which is quite common in the Lévy literature. We expect it to give a good performance also for those models that are relatively unable to fit joint distributions with all assets, both for analytical reasons and given our previous numerical explorations. Indeed, the $\rho\alpha$ VG and $\rho\alpha$ NIG models, though not able to generate the sample correlation in the joint calibration of this portfolio, since it is large and has critical features, still provide a good matching across most pairs, when the calibration is performed pairwise. Furthermore, a pairwise calibration confirms that, in the $\rho\alpha$ VG and $\rho\alpha$ NIG models, the only correlations that cannot be satisfactorily matched involve either assets with a very high kurtosis or very high correlation levels. Calibration errors under the pairwise optimization procedure are given in Tables 17 and 18.

[Insert Table 17]

[Insert Table 18]

5.4 Fit of portfolio returns

For each model and portfolio type, we compute the empirical complementary distribution $F(p)$ of the p-values across our sample of 1000 randomly generated portfolios, and we provide the proportion of portfolios with p-value greater than p . The higher is the complementary function, the better is the model ability to capture the laws of the randomly generated portfolios. Figure 1 shows the results obtained for all the specifications of $\rho\alpha$ - models and for the VG-EM specification. We do not plot all other EM specifications, since they are rejected at 1% level of significance. On the left we present long-only portfolios, while on the right we have long-short portfolios. We make the following remarks on the ability of the models under consideration in describing sample portfolio returns of our dataset.

[Insert Figure 1]

- All Lévy multivariate models outperform the multivariate Gaussian model, since the $F(p)$ of the Gaussian is close to zero at almost all p-values, while the $F(p)$ of the Lévy models stay high.
- In terms of overall distribution fit, as expressed by the Kolmogorov distance, the VG and HYP subclasses provide a good performance on both long-only and long-short portfolios, the HYP specification being superior to VG one for long-short portfolios.
- In the case of long-short portfolios, the VG specification of the Eberlein and Madan model shows a performance comparable to the VG specification of the $\rho\alpha$ - model. As noticed in [9], we find a departure of long-only portfolios from the model. All other specifications, which indeed were not examined by Eberlein and Madan, have p-values very close to zero, for both long-only and long-short portfolios.

Figure 2 provides the Gaussian kernel density estimator of the Kolmogorov distance and Figure 3 shows the Gaussian kernel density estimator of the Anderson and Darling statistics for the Lévy specifications.

- Figure 2 confirms the relatively good fit of all the Lévy models, in the KS sense. The VG subclass provides a slightly better description of sample portfolio returns for long-only portfolios, despite the fact that it is not able to match all sample correlations. An exception is the NIG, consistently with the results of Figure 1.
- In terms of fitting of the tails, as reflected by the Anderson and Darling statistics in Figure 3, the HYP subclass provides the best description of sample portfolio returns for both long-only and long-short portfolios.

[Insert Figure 2]

[Insert Figure 3]

The three Figures above and the underlying tests show that, once evaluated on portfolios, which reflect not only the linear dependence, but also the non-linear one, the Lévy models are more flexible than a simple superposition of linear dependence on marginal distributions. Here the marginal distributions of the EM specifications and the factor-model ones are indeed the same, with the same parameter values. The non-factor-based construction, although appealing for its conceptual simplicity and similar to the factor-based construction in calibration, seems to be less prone to reproduce non-linear dependence. Non-linear dependence depends on the common component of the subordinator. Not surprisingly, the specifications of the factor-model in which the common component is stronger (VG and HYP) have a better portfolio fit, both with respect to the other factor-based construction (NIG) and with respect to the EM dependence structure. This happens on long-only portfolios. With long-short portfolios, the simple construction of VG-EM gains in competitiveness.

Last, we explore the presence of a size effect in the ability of different model subclasses in explaining portfolio returns by comparing the subportfolios of two (EN and FN) and five indices (EN, FN, HC, IN and IT) with the overall portfolio of 10 indices. Results are reported in Figure 4, 5, and 6 for the Lévy specifications, and in Figure 7 for the VG-EM model. Among the Lévy models, in our dataset we find a significant size effect only for the NIG specification, which is most evident for long-only portfolios. We remark that the NIG specification has the best marginal fit on the FN index, included in the two asset portfolio. Furthermore, it shows an improvement of the correlation fit with respect to the 10 assets case, although it has the highest absolute error if compared with the other submodels. Nevertheless, it provides the best fit for long-only portfolios.

[Insert Figure 4]

[Insert Figure 5]

[Insert Figure 6]

[Insert Figure 7]

6 Summary and further research

This paper explores the fit properties of a class of multivariate Lévy processes, named factor-based multivariate time-changed Brownian motions. The model permits to stay in the Lévy setting and may be specified so that it includes either Variance-Gamma or Normal-Inverse-Gaussian and Generalized-Hyperbolic marginal distributions.

A joint - as opposite to pairwise - calibration to the level and linear correlation of a portfolio of ten US daily stock-market returns over the period 2009-2013 shows that several specifications of the model provide a strong improvement with respect to the

traditional, Gaussian returns hypothesis. The fit of the Hyperbolic specification very good, both in terms of marginal distributions and dependence. We then evaluate the performance of the joint calibrated Lévy models on a sample of randomly generated portfolios. We find that the calibrated Lévy models outperform the multivariate Gaussian model and the model by Eberlein and Madan, which shows an adequate performance only for the VG specification and long-short portfolios. In terms of overall distribution fit, as expressed by the Kolmogorov distance, our VG and HYP subclasses provide a good performance on both long-only and long-short portfolios, the HYP specification being better than the VG's one for long-short portfolios. In terms of fitting of the tails, as reflected by the Anderson and Darling statistics, the VG subclass provides a good description of sample portfolio returns, despite the fact that it doesn't match perfectly sample correlations.

Overall, the model - which clarifies and extends similar attempts in the same direction - seems to be flexible and general enough to incorporate empirical behavior of asset returns without computational efforts and keeping run times very limited. The intuition is that the model is able to capture also non linear dependence. Further research on the dependence structure, based on the computation of the implied copula and its behaviour over time, or on the moments of the joint distribution of order higher than two, is needed, even though it is out of the scope of the present paper.

References

- [1] Ballotta, L., Bonfiglioli, E. (2012). Multivariate Asset Models Using Levy Processes and Applications. Available at SSRN: <http://ssrn.com/abstract=1695527>.
- [2] Barndorff-Nielsen, O.E. (1977). Exponentially decreasing ditributions for the logarithm of particle size. *Proc. R. Soc. A* 353, 401-419.
- [3] Barndorff-Nielsen, O.E. (1978). Hyperbolic distribution and distributions on hyperbolae. *Scand. J. Statist.* 34, 67-69.
- [4] Barndorff-Nielsen, O.E.(1995). Normal Inverse Gaussian distributions and the modeling of stock returns. *Research report* no. 300, Department of Theoretical Statistics, Aarhus University.
- [5] Barndorff-Nielsen, O.E., Pedersen, J. Sato, K.I. (2001). Multivariate Subordination, Self-Decomposability and Stability. *Adv. Appl. Prob.* 33, 160-187.
- [6] Carr, P. Geman, H. Madan, D. H., Yor, M. (2002) The fine structure of asset returns: an empirical investigation. *Journal of Business* 75, 305-332.
- [7] Clark, P.K. (1973). A subordinated stochastic process model with finite variance for speculative prices. *Econometrica* 41, 135-155.

- [8] Eberlein, E., Keller, U. (1995). Hyperbolic distributions in finance. *Bernoulli* 1, 281-299.
- [9] Eberlein, E. and Madan, D.B. (2010). On correlating Lévy processes. *Journal of Risk* 13(1), 3-16.
- [10] Eberlein, E. and Prause, K. (2002). The Generalized-Hyperbolic model: financial derivatives and risk measures. *Mathematical Finance - Bachelier Congress (Paris)*, Springer Finance, pp. 245-267.
- [11] Chourdakis, K. (2005). Option pricing using the fractional FFT, *Journal of Computational Finance* 8 (2) 1 18.
- [12] Guillaume, F. (2012). Sato two-factor models for multivariate option pricing. *The Journal of Computational Finance* 15(4), 159-192.
- [13] Harris, L. (1986). Cross-security tests of the mixture distribution. *The Journal of Financial and Quantitative Analysis* 21, 39-46.
- [14] Higham, Nicholas J. (2002). Computing the Nearest Correlation Matrix - A Problem from Finance. *IMA Journal of Numerical Analysis*, 22, 329–343.
- [15] Leoni, P. and Schoutens, W. (2008). Multivariate Smiling. *Wilmott Magazine*
- [16] Lo, A. W., Wang, J. (2000). Trading volume: definitions, data analysis, and implications of portfolio theory. *Review of Financial Studies*, 13, 257-300.
- [17] Luciano, E., Schoutens, W. (2006). A multivariate Jump-Driven Financial Asset Model. *Quantitative Finance* , 6 (5), 385-402.
- [18] Luciano, E., Semeraro, P. (2010a). Multivariate Time Changes for Lévy Asset Models: characterization and calibration. *Journal of Computational and Applied Mathematics*, 233 (8) 1937-1953.
- [19] Luciano, E., Semeraro, P.(2010b). A Generalized Normal Mean Variance Mixture for Return Processes in Finance. *International Journal of Theoretical and Applied Finance*, 13 (3): 415-440.
- [20] Madan, D. B., Seneta, E. (1990). The v.g. model for share market returns. *Journal of Business* 63, 511-524.
- [21] Mc Neil, A.J., Frey, R. and Embrechts, P. (2005) .Quantitative Risk Management: Concepts, Techniques, and Tools. *Princeton University Press*.
- [22] Prause, K. (1999). The Generalized-Hyperbolic model: Estimation, financial derivatives, and risk measures. Doctoral thesis, University of Freiburg.

- [23] Sato, K.I. (2003). Lévy processes and Infinitely divisible distributions Cambridge studies in advanced mathematics. *Cambridge University Press*.
- [24] Semeraro, P. (2008). A multivariate Variance-Gamma model for financial application. *International Journal of Theoretical and Applied Finance*, 11, 1-18.
- [25] Seneta, E. (2004). Fitting the Variance-Gamma Model to Financial Data. *Journal of Applied Probability*, Vol. 41, pp. 177-187.
- [26] Wallmeier, M., Diethelm, M., K.I. (2012). Multivariate downside risk: Normal versus Variance-Gamma. *Journal of Futures Markets*, 13 (5), 431-458.
- [27] Wang, J. (2009). The multivariate Variance-Gamma Process and its applications in multi-asset option pricing. PHD dissertation,, available at <http://hdl.handle.net/1903/9920>.

Index	Mean	Std. Dev.	Skewness	Kurtosis
CD	0.0009	0.0146	-0.1663	6.0003
CS	0.0005	0.0084	-0.3873	5.4783
EN	0.0004	0.0168	-0.3566	5.4490
FN	0.0004	0.0224	0.1573	11.9466
HC	0.0006	0.0106	-0.4573	6.5265
IN	0.0005	0.0158	-0.2487	6.1286
IT	0.0007	0.0139	-0.0972	5.4574
MT	0.0006	0.0172	-0.3907	5.1771
TC	0.0003	0.0113	-0.1627	6.2099
UT	0.0002	0.0100	-0.3457	6.0961

Table 1: Sample moments.

	CD	CS	EN	FN	HC	IN	IT	MT	TC	UT
CD	1.0000									
CS	0.7985	1.0000								
EN	0.8214	0.7404	1.0000							
FN	0.8267	0.6781	0.7707	1.0000						
HC	0.7975	0.8338	0.7566	0.6985	1.0000					
IN	0.9337	0.7946	0.8690	0.8403	0.8089	1.0000				
IT	0.9052	0.7429	0.8244	0.7886	0.7728	0.8926	1.0000			
MT	0.8822	0.7462	0.8952	0.7989	0.7661	0.9165	0.8698	1.0000		
TC	0.7393	0.7391	0.6939	0.6921	0.6832	0.7352	0.7226	0.7012	1.0000	
UT	0.7247	0.7705	0.7457	0.6503	0.7343	0.7552	0.6863	0.7089	0.7022	1.0000

Table 2: Sample correlation matrix.

Index	c	σ
CD	0.0010	0.0146
CS	0.0005	0.0084
EN	0.0005	0.0168
FN	0.0006	0.0224
HC	0.0006	0.0106
IN	0.0007	0.0158
IT	0.0008	0.0139
MT	0.0007	0.0172
TC	0.0004	0.0113
UT	0.0003	0.0100

Table 3: G model. Calibrated parameters.

Index	c	μ	σ	α
CD	0.0021	-0.0013	0.0144	0.9026
CS	0.0013	-0.0009	0.0083	0.7602
EN	0.0018	-0.0014	0.0166	0.7348
FN	0.0005	-0.0001	0.0206	1.2848
HC	0.0019	-0.0013	0.0105	0.8420
IN	0.0018	-0.0013	0.0156	0.9973
IT	0.0016	-0.0009	0.0139	0.9103
MT	0.0025	-0.0019	0.0170	0.7812
TC	0.0019	-0.0017	0.0111	0.7293
UT	0.0015	-0.0013	0.0099	0.6646

Table 4: VG model. Calibrated parameters.

Index	c	γ	β	δ
CD	0.0020	51.7708	-5.0441	0.0112
CS	0.0014	108.3392	-12.8277	0.0076
EN	0.0021	54.9486	-6.0927	0.0155
FN	0.0012	22.7119	-1.7045	0.0113
HC	0.0021	82.5935	-13.7078	0.0090
IN	0.0019	45.0711	-5.2494	0.0115
IT	0.0015	57.3094	-4.3395	0.0114
MT	0.0028	54.3748	-7.4708	0.0159
TC	0.0018	81.6045	-12.0085	0.0101
UT	0.0010	97.9514	-7.5590	0.0098

Table 5: NIG model. Calibrated parameters.

Index	c	γ	β	δ
CD	0.0019	100.6824	-5.0872	0.0022
CS	0.0014	180.4814	-13.5324	0.0024
EN	0.0020	90.2830	-5.6782	0.0053
FN	0.0014	71.5114	-2.6797	0.0000
HC	0.0020	142.6616	-13.1937	0.0025
IN	0.0020	92.0004	-5.8528	0.0012
IT	0.0016	103.4022	-4.8038	0.0018
MT	0.0026	86.9305	-6.8379	0.0045
TC	0.0020	138.4737	-13.9333	0.0039
UT	0.0012	157.8498	-9.4632	0.0043

Table 6: HYP model. Calibrated parameters.

Index	p-value				KS dist.			
	G	VG	NIG	HYP	G	VG	NIG	HYP
CD	0.0000	0.8143	0.9509	0.8191	0.0815	0.0189	0.0154	0.0188
CS	0.0003	0.9961	0.9996	0.9999	0.0628	0.0122	0.0105	0.0093
EN	0.0001	0.9048	0.9378	0.9479	0.0682	0.0169	0.0159	0.0155
FN	0.0000	0.2074	0.8720	0.1055	0.1181	0.0314	0.0168	0.0354
HC	0.0000	0.5645	0.8765	0.6599	0.0842	0.0235	0.0176	0.0218
IN	0.0000	0.9179	0.9121	0.8729	0.0843	0.0165	0.0167	0.0177
IT	0.0000	0.9963	0.9844	0.9984	0.0731	0.0121	0.0136	0.0114
MT	0.0001	0.9635	0.9325	0.9866	0.0662	0.0149	0.0161	0.0134
TC	0.0006	0.9692	0.9969	0.9830	0.0602	0.0146	0.0119	0.0128
UT	0.0004	0.6943	0.6615	0.7422	0.0619	0.0203	0.0218	0.0203

Table 7: KS test at the 5% significance level. The critical value is 0.0406.

Index	G	VG	NIG	HYP
CD	1.6075	0.0780	0.0428	0.0721
CS	0.6727	0.0597	0.0482	0.0495
EN	0.5621	0.0364	0.0355	0.0336
FN	132.4952	0.2194	0.0568	0.3901
HC	1.0028	0.0629	0.0507	0.0678
IN	1.9102	0.0693	0.0439	0.0733
IT	1.7122	0.0444	0.0447	0.0413
MT	0.6557	0.0332	0.0411	0.0299
TC	4.4743	0.1769	0.0933	0.1642
UT	1.5534	0.0498	0.0439	0.0409

Table 8: Anderson and Darling distance.

Lévy model	a max
VG	0.7784
NIG	0.2555
HYP	1.0000

Table 9: Maximal values of the common parameter a .

	CD	CS	EN	FN	HC	IN	IT	MT	TC	UT
CD	1.0000									
CS	0.9866	1.0000								
EN	0.9999	0.9847	1.0000							
FN	0.9627	0.9055	0.9656	1.0000						
HC	0.9830	0.9998	0.9809	0.8967	1.0000					
IN	0.9989	0.9930	0.9984	0.9492	0.9904	1.0000				
IT	0.9969	0.9963	0.9960	0.9385	0.9944	0.9995	1.0000			
MT	0.9999	0.9847	1.0000	0.9656	0.9809	0.9984	0.9960	1.0000		
TC	0.9948	0.9981	0.9936	0.9301	0.9966	0.9984	0.9997	0.9936	1.0000	
UT	0.9915	0.9994	0.9901	0.9194	0.9985	0.9965	0.9987	0.9901	0.9996	1.0000

Table 10: $\rho\alpha$ VG model. Correlation matrix between the Brownian components.

	CD	CS	EN	FN	HC	IN	IT	MT	TC	UT
CD	1.0000									
CS	0.8611	1.0000								
EN	0.8942	0.8298	1.0000							
FN	0.8441	0.7156	0.8185	1.0000						
HC	0.8489	0.9165	0.8364	0.7279	1.0000					
IN	0.9615	0.8440	0.9320	0.8462	0.8477	1.0000				
IT	0.9392	0.7962	0.8898	0.8003	0.8179	0.9131	1.0000			
MT	0.9414	0.8235	0.9852	0.8379	0.8343	0.9646	0.9260	1.0000		
TC	0.8115	0.8365	0.7908	0.7445	0.7601	0.7940	0.7888	0.7860	1.0000	
UT	0.8127	0.8944	0.8684	0.7145	0.8406	0.8350	0.7657	0.8188	0.8290	1.0000

Table 11: $\rho\alpha$ HYP model. Correlation matrix between the Brownian components.

Lévy subclass	α -model	$\rho\alpha$ -model
VG	1.0970	0.2094
NIG	1.0941	0.5645
HYP	1.0919	0.0026

Table 12: RMSE between empirical and model asset correlations under different model specifications.

Lévy subclass	α -model	$\rho\alpha$ -model	$\rho\alpha$ -model (pairwise)
VG	0.9287	0.3056	0.0315
NIG	0.9270	0.5957	0.2593
HYP	0.9275	0.0107	0.0000

Table 13: Maximum absolute error between empirical and model asset correlations under different model specifications.

	CD	CS	EN	FN	HC	IN	IT	MT	TC	UT
CD	0.0000									
CS	-0.1624	0.0000								
EN	-0.1876	-0.1676	0.0000							
FN	-0.0223	0.0159	-0.0421	0.0000						
HC	-0.1307	-0.2114	-0.1564	0.0232	0.0000					
IN	-0.1961	-0.1216	-0.2038	-0.0065	-0.1027	0.0000				
IT	-0.2020	-0.0980	-0.1904	0.0000	-0.0960	-0.1515	0.0000			
MT	-0.2288	-0.1555	-0.3056	-0.0490	-0.1468	-0.2307	-0.2163	0.0000		
TC	-0.1117	-0.1610	-0.1286	0.0034	-0.0754	-0.0731	-0.0899	-0.1177	0.0000	
UT	-0.1270	-0.2177	-0.2075	0.0076	-0.1529	-0.1239	-0.0822	-0.1536	-0.1606	0.0000

Table 14: $\rho\alpha$ VG model. Differences between model and sample correlations.

	CD	CS	EN	FN	HC	IN	IT	MT	TC	UT
CD	0.0000									
CS	-0.4290	0.0000								
EN	-0.4588	-0.4334	0.0000							
FN	-0.1425	-0.0734	-0.1763	0.0000						
HC	-0.4089	-0.5026	-0.4312	-0.1174	0.0000					
IN	-0.4646	-0.3924	-0.4738	-0.1413	-0.3857	0.0000				
IT	-0.4902	-0.3913	-0.4790	-0.1657	-0.4039	-0.4537	0.0000			
MT	-0.5226	-0.4414	-0.5957	-0.2593	-0.4463	-0.5362	-0.5304	0.0000		
TC	-0.3714	-0.4268	-0.3871	-0.1407	-0.3554	-0.3462	-0.3754	-0.3958	0.0000	
UT	-0.3823	-0.4817	-0.4620	-0.1351	-0.4315	-0.3931	-0.3632	-0.4273	-0.4145	0.0000

Table 15: $\rho\alpha$ NIG model. Differences between model and sample correlations.

	CD	CS	EN	FN	HC	IN	IT	MT	TC	UT
CD	0.0000									
CS	0.0001	0.0000								
EN	0.0025	0.0002	0.0000							
FN	-0.0001	0.0000	-0.0003	0.0000						
HC	-0.0001	0.0000	-0.0002	0.0000	0.0000					
IN	0.0006	0.0000	0.0020	-0.0001	0.0000	0.0000				
IT	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000			
MT	-0.0031	-0.0002	-0.0107	0.0004	0.0002	-0.0025	-0.0001	0.0000		
TC	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	-0.0001	0.0000	
UT	-0.0006	0.0000	-0.0021	0.0001	0.0000	-0.0005	0.0000	0.0027	0.0000	0.0000

Table 16: $\rho\alpha$ HYP model. Differences between model and sample correlations.

	CD	CS	EN	FN	HC	IN	IT	MT	TC	UT
CD	0.0000									
CS	0.0000	0.0000								
EN	0.0000	0.0000	0.0000							
FN	0.0000	0.0000	0.0162	0.0000						
HC	0.0000	0.0000	0.0000	0.0000	0.0000					
IN	0.0000	0.0000	0.0106	0.0000	0.0000	0.0000				
IT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000			
MT	0.0000	0.0000	0.0000	0.0223	0.0000	0.0315	0.0000	0.0000		
TC	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
UT	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 17: Pairwise calibration of the $\rho\alpha$ VG model. Differences between model and sample correlations.

	CD	CS	EN	FN	HC	IN	IT	MT	TC	UT
CD	0.0000									
CS	0.0000	0.0000								
EN	0.0000	0.0000	0.0000							
FN	0.1425	0.0734	0.1763	0.0000						
HC	0.0000	0.0000	0.0000	0.1174	0.0000					
IN	0.0000	0.0000	0.0721	0.1413	0.0000	0.0000				
IT	0.0000	0.0000	0.0000	0.1657	0.0000	0.0075	0.0000			
MT	0.0674	0.0000	0.0000	0.2593	0.0000	0.1496	0.0033	0.0000		
TC	0.0000	0.0000	0.0000	0.1407	0.0000	0.0000	0.0000	0.0000	0.0000	
UT	0.0000	0.0000	0.0000	0.1351	0.0000	0.0250	0.0000	0.0000	0.0000	0.0000

Table 18: Pairwise calibration of the $\rho\alpha$ NIG model. Differences between model and sample correlations.

Figure 1: P-values complementary distribution function for the KS test.

Figure 2: Gaussian kernel density estimator of the Kolmogorov distance for the Lévy specifications.

Figure 3: Gaussian kernel density estimator of the Anderson and Darling distance for the Lévy specifications.

Figure 4: Lévy VG model. P-values complementary distribution function for the KS test. Subportfolios of 2, 5 and 10 indices.

Figure 5: Lévy NIG model. P-values complementary distribution function for the KS test. Subportfolios of 2, 5 and 10 indices.

Figure 6: Lévy HYP model. P-values complementary distribution function for the KS test. Subportfolios of 2, 5 and 10 indices.

Figure 7: VG-EM model. P-values complementary distribution function for the KS test. Subportfolios of 2, 5 and 10 indices.