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Preferences With Grades of Indecisiveness☆

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Abstract

Departing from the traditional approach to modeling an agent who finds it difficult to make clear-cut comparisons between alternatives, we introduce the notion of *graded preferences*: Given two alternatives, the agent reports a number between 0 and 1, which reflects her inclination to prefer the first option over the second or, put differently, how confident she is about the superiority of the first one. In the classical framework of uncertainty, we derive a representation of a graded preference by a measure of the set of beliefs that rank one option better

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than the other. Our model is a refinement of Bewley's (1986) model of Knightian uncertainty: It is based on the same object of representation — the set of beliefs — but provides more information about how the agent compares alternatives. We also define and characterize, in terms of the representation, the notion of one agent being "more decisive" than another.

Keywords: incomplete preferences, Knightian uncertainty, graded preferences, confidence, decisiveness

1. Introduction

In their seminal works, Aumann (1962) and Bewley (1986) argue that the completeness assumption in Utility Theory is likely the most disputable rationality tenet: It seems reasonable, on both descriptive and normative grounds, that an individual might not always be able to decide which of two alternatives she prefers. Since these pioneering contributions, several authors have investigated the nature of incomplete preferences in various decision-theoretic settings.[1]

The focus of the existing research on incomplete preferences has been on answering two questions: *when* a decision maker is able to make clear-cut comparisons among alternatives, and what her choice is *if* she can make the comparison. The analysis traditionally ends as soon as the decision maker is found unable to compare two particular alternatives: In this situation, the alternatives at hand are deemed "incomparable" (and the decision maker "indecisive"). In particular, this approach does not distinguish the case in which the decision maker is just slightly indecisive between alternatives from the case in which she is entirely puzzled and unable to decide. Rather, the two situations are treated equally.

[1]See Section 4 for an overview of the literature on incomplete preferences.

Nevertheless, even when the decision maker is indecisive, she may still be able to answer questions about the extent to which she is predisposed towards one alternative over another. In fact, questions that allow the respondent to be unsure but, nevertheless, to express her predisposition are often found in consumer surveys and quality-assurance questionnaires. They may be asked in the following way: "On a scale from 1 to 10, how confident are you that A is better than B?"[2]

The objective of this paper is to formally model a decision maker who is allowed to convey whether she is *inclined* to prefer one alternative over another and, at the same time, to express how *confident* she is about the superiority of the indicated alternative. The primitive that we adopt to capture this behavior is a function μ that assigns a number between zero and one to any given (ordered) pair of alternatives f and g. The two extreme points of the scale capture the standard situations in which the agent is decisive: Number one corresponds to the situation in which f is surely at least as good as g, whereas point zero reflects the circumstance in which f is (strictly) worse than g.[3] If the decision maker does not have a clear-cut preference, she reports a number strictly between zero and one, which reflects the *extent* to which she is sure of her preference for f over g. We use the term "graded preference relation" to refer to such a function μ.

We study graded preferences in the context of decision making under Knightian uncertainty. Since the seminal work of Bewley (1986), it has been well

[2]In psychology, Anderson and Whaler (1960), Suydam and Myers (1962), and Zakay (1984), among others, have used rating scales to measure preferences. For marketing literature discussing those types of questionnaires, see, e.g., Hauser and Shugan (1980) and Novemsky, Dhar, Schwarz and Simonson (2007). Scale-based surveys are also used to measure individuals' happiness in both the psychology and, more recently, economic literature (see, e.g., Frey and Stutzer, 2002).

[3]Consistently, indifference between f and g is expressed by assigning one to both ordered pairs (f, g) and (g, f).

understood that the main source of indecisiveness in this framework is the lack of information to formulate a single prior. In the main result of our paper, we show the equivalence between a certain set of behavioral assumptions imposed on μ and the existence of the following representation:

$$\mu(f,g) = \pi\left(\left\{p \in \mathcal{M} : \int_{\Omega} (u \circ f)\, dp \geq \int_{\Omega} (u \circ g)\, dp\right\}\right), \tag{1}$$

where f and g are payoff profiles (or "acts") mapping a state space Ω to a convex outcome space X; u is a von Neumann-Morgenstern utility function; $\mathcal{M}$ is a set of subjective probability distributions on Ω; and π is a capacity (i.e., non-additive) measure on $\mathcal{M}$. In Representation (1), the decision maker has in mind a set $\mathcal{M}$ of plausible priors; for given acts f and g, she considers the priors in $\mathcal{M}$ for which the expected utility of f exceeds that of g; and then, she takes the measure of this subset of priors as her level of confidence in the superiority of f. The more priors that ascribe higher expected utility to f, the more confident the agent is that f is the better option; and when f is better than g according to measure one of her priors, she becomes certain of her preference for f over g.

Our representation result provides a refinement of Bewley's (1986) seminal work on incomplete preferences under Knightian uncertainty. In this model, the agent holds a set of probability measures and adopts a decision rule based on unanimity: An act f is preferred to another act g if and only if, according to each plausible prior, the expected utility of f exceeds that of g. Therefore, two acts are incomparable whenever the relevant priors do not agree in ranking them. Representation (1) is based on the same object of representation — the set of beliefs — but provides more information about the decision process: When the representation à la Bewley renders the alternatives simply incomparable, our representation delivers the decision maker's grade of indecisiveness.

The representation that we introduce allows us to perform comparative statics exercises. We propose a behavioral definition of comparative decisiveness — that is, when we can say that one decision maker is more decisive than another — and characterize it in terms of our functional form of preferences.

Although our model provides additional insights into how the decision maker thinks about alternatives when they are difficult to compare, like other models of incomplete preferences, it does not prescribe how the agent will choose in situations of indecisiveness. Therefore, in order to gain predictive power in these cases, one needs to complement the model by specifying some decision rule that translates the intermediate grades of preferences into choices. In Section 3, we illustrate this point by proposing two decision rules that one may adopt: a Random Choice rule that is related to the *random expected utility* theory of Gul and Pesendorfer (2006), and a Maxmin Expected Utility rule related to the *maxmin preferences* of Gilboa and Schmeidler (1989). This exercise sheds some additional light on our approach. Indeed, by investigating the evaluation stage that precedes the actual choice, one can have a better understanding of the rationality behind the act of choice *per se*. In this respect, graded preferences can be interpreted as revealing how convinced a decision maker is of her eventual choice — something that is hard to infer by examining solely the choice data.

Before introducing the formal model, a few remarks are in order. First, we need to clarify the connection between μ and observable behavior. Indeed, as opposed to revealed preference theory, graded preferences may not be revealed by choice behavior alone. However, we believe that μ can still be elicited if one considers not only choice data but also the possibility that the agent directly reports verbal statements about her preferences. That is, the agent makes introspective evaluations and communicates her level of confidence in the superiority of one of the options to the external observer. Resorting to verbal statements seems to be necessary whenever the one-to-one mapping between choice and preferences breaks down.[4] Indeed, Gilboa, Maccheroni, Marinacci and Schmeidler (2008) argue for the need to extend observable behavior to incorporate the possibility of preferences being stated in order to recover an incomplete preference relation. Similarly, Rustichini (2008) promotes the use of statements to measure the strength of preferences. In our framework of graded preferences, suitable scale-based questionnaires, as the one suggested before, provide a natural tool to recover μ through verbal statements. Further comments on the observability of μ are provided in Subsection 2.6.

Our second related remark is about methodology. In this paper, we have chosen to adopt a *cardinal* notion of graded preferences: For each pair of alternatives, f and g, the agent reports a number, $\mu(f, g)$, that represents her disposition and confidence. Alternatively, one may follow a purely ordinal approach, in which the decision maker's graded preferences are described by a binary relation $\gtrsim$ over *pairs* of alternatives: Instead of reporting her degree of confidence $\mu(f, g)$ in a numeric form, the agent can state her preferences in the form "my confidence that f is superior to g is greater than my confidence that h is superior to t", where h and t are some other alternatives. That is, $(f, g) \gtrsim (h, t)$. It is worth noting that replacing cardinal comparisons with ordinal ones does not change the intuition that we develop here; therefore, our analysis is robust to these two alternative ways of modeling graded preferences. For instance, a binary relation over pairs can be represented by a functional form similar to (1) that computes the measure of the set of priors that assign a higher expected utility to the first option in the pair relative to the second.[5]

The paper is organized as follows. After introducing the framework, Section 2 presents our main results: Subsection 2.3 contains the representation theorem;

[4] As a caveat, however, we should mention that Eliaz and Ok (2006) provide choice-theoretic foundations of incomplete preferences.

[5] Preliminary results in the ordinal setting are available from the authors upon request.

Subsection 2.4 relates our findings to the Bewley model; and Subsection 2.5 provides a comparative attitude exercise in the class of graded preferences that we axiomatize. Section 3 discusses the possible choice behavior that can arise from graded preferences by suggesting a few decision rules that the decision maker may adopt. In Section 4, we discuss the related literature and make concluding remarks. All proofs and related material are contained in the Appendices.

2. The model

2.1. Setup

We adopt the standard Anscombe-Aumann setting (Anscombe and Aumann, 1963). Let Ω be the set of states of the world, endowed with an algebra Σ of events, and X be the set of consequences. We assume that X is a convex subset of a metric space — for example, the set of all lotteries over an underlying set of prizes. We denote by $\Delta(\Omega)$ the set of all finitely additive probabilities on (Ω, Σ) endowed with the weak*-topology: A net $\{p_\alpha\}_{\alpha \in D}$ in $\Delta(\Omega)$ converges to $p \in \Delta(\Omega)$ if and only if $p_\alpha(S) \to p(S)$ for all $S \in \Sigma$.

An act is a mapping from Ω to X that attaches a consequence to each possible state. Formally, an act is a Σ-measurable function $f : \Omega \to X$ that takes finitely many values. We denote by $\mathcal{F}$ the set of all acts endowed with the sup-norm. With the usual abuse of notation, $x \in \mathcal{F}$ is a constant act that yields $x \in X$ for every $\omega \in \Omega$. Mixtures of acts are defined pointwise: For every $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ yields $\alpha f(\omega) + (1-\alpha)g(\omega) \in X$ for every $\omega \in \Omega$.

We occasionally use the notation $B_0(\Omega, \Sigma, \mathbb{R})$ to denote the set of all Σ-measurable simple functions from Ω to $\mathbb{R}$, i.e., functions taking finitely many values. Observe that, for any given utility index $u : X \to \mathbb{R}$, $u \circ f \in B_0(\Omega, \Sigma, \mathbb{R})$. Note also that our topology on $\Delta(\Omega)$ is the weakest topology such that the functions $\Delta(\Omega) \to \mathbb{R}$ defined as $p \mapsto \int \varphi \, dp$ are continuous for all $\varphi \in B_0(\Omega, \Sigma, \mathbb{R})$.

The decision maker's *preferences* are described by a *graded* (or, equivalently, *fuzzy*) preference relation $\mu : \mathcal{F} \times \mathcal{F} \to [0,1]$. Given two acts f and g, we interpret $\mu(f,g) \in [0,1]$ as the decision maker's degree of confidence that f is at least as good as g. The extreme points of the scale correspond to situations in which the decision maker is perfectly decisive. More precisely, if $\mu(f,g) = 1$, then she is sure that f is at least as good as g; whereas if $\mu(f,g) = 0$, she is sure that f is strictly worse than g. If the decision maker is indifferent between f and g, then $\mu(f,g) = 1$ and $\mu(g,f) = 1$. Finally, if she is indecisive, $\mu(f,g) \in (0,1)$ reflects how confident she is that f is better than g.

Occasionally, we will refer to standard binary relations $\succsim \subseteq \mathcal{F} \times \mathcal{F}$ as *crisp* binary relations. In the context of uncertainty, the benchmark model for the representation of a crisp binary relation $\succsim$ is the Subjective Expected Utility (SEU) model, which ranks payoff profiles according to the mapping $f \mapsto \int_\Omega (u \circ f)\, dp$, where $u : X \to \mathbb{R}$ is an affine utility function and $p \in \Delta(\Omega)$.

2.2. Axioms

This section introduces the behavioral assumptions that we impose on a graded preference μ. The first set of axioms (Reflexivity, Weak Transitivity, Monotonicity, C-Completeness, and Independence) represent translations of standard properties into our setting of graded preferences.

Axiom A1 (Reflexivity)**.** For all $f \in \mathcal{F}$, $\mu(f,f) = 1$.

In words, for any act f, the agent should be sure that f is as good as itself.

Next, we introduce a novel axiom dubbed Weak Transitivity.

Axiom A2 (Weak Transitivity)**.** For all $f,g,h \in \mathcal{F}$, if $\mu(f,g) = 1$ then $\mu(f,h) \geq \mu(g,h)$.

This axiom imposes a minimal rationality requirement on μ: If the agent is sure that f is at least as good as g, then she should also find f at least as attractive as g compared to any 'cut-off act' h. To better illustrate the appeal of this property, a couple of remarks are in order. First, if the agent is indifferent between f and g, then her level of confidence in the superiority of f over any other act h should be the same as her level of confidence in the superiority of g over h. Second, this axiom directly implies a standard transitivity property on crisp comparisons — i.e., for any f, g, and h, if $\mu(f,g) = 1$ and $\mu(g,h) = 1$, then $\mu(f,h) = 1$. Thus, Reflexivity and Weak Transitivity, together, impose a very basic structure on μ requiring its sub-relation of sure comparisons to be a preorder.

Axiom A3 (Monotonicity)**.** For all $f, g \in \mathcal{F}$, if $\mu(f(\omega), g(\omega)) = 1$ for all $\omega \in \Omega$, then $\mu(f,g) = 1$.

Monotonicity captures the standard postulate that improving an act in a state-by-state manner brings an overall improvement, or, put differently, a more desirable outcome improves the entire act regardless of the state in which it is offered.

Axiom A4 (C-Completeness)**.** For all $x, y \in X$, $\mu(x,y) = 1$ or $\mu(y,x) = 1$.

C-Completeness maintains that μ restricted to $X \times X$ induces a complete preorder and, therefore, postulates that the agent does not have any difficulty in comparing acts that involve risk but not uncertainty. As in Bewley (1986), this assumption restricts the type of incompleteness that the agent is allowed to exhibit by prescribing that indecisiveness may arise only due to the presence of uncertainty and not due to the presence of risk.

Axiom A5 (Independence)**.** For all $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1]$,

$$\mu(f,g) = \mu(\alpha f + (1-\alpha)h, \alpha g + (1-\alpha)h).$$

As in the standard framework of crisp relations, Independence postulates that the decision maker's ranking of mixtures $\alpha f + (1-\alpha)h$ and $\alpha g + (1-\alpha)h$ is the same as her ranking of f and g. Notice that while the preceding axioms referred only to situations in which the decision maker is decisive, Independence imposes a certain structure that covers comparisons of acts of all possible confidence levels.

Axiom A6 (Reciprocity)**.** For all $f, g \in \mathcal{F}$, if $\mu(f,g) \in [0,1)$, then $\mu(f,g) = 1 - \mu(g,f)$.

Reciprocity imposes a certain regularity on the decision maker's reports about the comparison of f vs. g and g vs. f within each pair (f,g) of acts. The normative appeal of this axiom rests on two presumptions: 1) a perceptive decision maker should consider questions about f vs. g and g vs. f as two faces of the same decision problem; and 2) whenever the decision maker becomes more confident that f is better than g, she cannot simultaneously become more confident that g is better than f unless indifference comes into play. Moreover, our formalization of indifference as $\mu(f,g) = 1 = \mu(g,f)$ explains the antecedent $\mu(f,g) \in [0,1)$ in the statement of the axiom.

Notice that Reciprocity turns out to be our only axiom that is sensitive to the numeric values of confidence levels and puts restrictions on the *scale* of the decision maker's reports.[6] In particular, Reciprocity requires that whenever $\mu(f,g) = \mu(g,f) < 1$, it must be that $\mu(f,g) = \frac{1}{2}$, and the range of μ must be symmetric around the point of $\frac{1}{2}$.

Although Reciprocity adds a lot of structure to a graded preference relation, the power of our other axioms makes it not so important for pinning down the representation. As we elaborate after stating the main result (Theorem 1), the same representation of graded preferences can be derived without this axiom at the expense of heavier notation. Moreover, it is worth noting that this axiom does not impose any additional restrictions on the decision maker's perception of indifferences.

[6]All other axioms essentially refer to the minimal and the maximal values in the range of μ.

Finally, we impose two standard technical assumptions.

Axiom A7 (Continuity)**.** For all $f, g, h \in \mathcal{F}$, the mappings $\alpha \mapsto \mu(\alpha f + (1-\alpha)g, h)$ and $\alpha \mapsto \mu(h, \alpha f + (1-\alpha)g)$ are upper semicontinuous on $[0,1]$.

The Continuity axiom adapts the standard Mixture (or Archimedean) Continuity to our framework.[7] We also note that the presence of Reciprocity makes the continuity properties of the mappings $\alpha \mapsto \mu(\alpha f + (1-\alpha)g, h)$ and $\alpha \mapsto \mu(h, \alpha f + (1-\alpha)g)$ stronger than what our Continuity axiom postulates directly: These mappings are, in fact, continuous for all $f, g, h \in \mathcal{F}$ and $\alpha \in [0,1]$ except at points of indifference — i.e., when $\mu(\alpha f + (1-\alpha)g, h) = 1 = \mu(h, \alpha f + (1-\alpha)g)$ (see Lemma 8).

Axiom A8 (Nondegeneracy)**.** $\mu(f, g) = 0$ for some $f, g \in \mathcal{F}$.

In words, Nondegeneracy requires that there exist two acts such that one is surely worse than the other.

2.3. Main Result

This section presents our main result, which derives a representation of the graded preferences that satisfy the aforementioned axioms.

Before stating the main result, let us introduce a few preliminary definitions. We use the term *capacity measure* for a monotone and normalized set function that is not necessarily additive: If $\mathfrak{S}$ is a collection of subsets of some arbitrary set Y such that $\varnothing \in \mathfrak{S}$ and $Y \in \mathfrak{S}$, then a function $\pi : \mathfrak{S} \to [0,1]$ is a capacity if $\pi(\varnothing) = 0$, $\pi(Y) = 1$, and $\pi(S) \geq \pi(S')$ for all $S, S' \in \mathfrak{S}$ such that $S \supseteq S'$.

For any given set $\mathcal{M}$ of finitely additive probabilities in $\Delta(\Omega)$, we will denote by $\mathscr{B}(\mathcal{M})$ the Borel algebra of the space $\mathcal{M}$ endowed with the relative topology.

[7] Indeed, it can be equivalently stated as follows: For all $f, g, h \in \mathcal{F}$ and $\gamma \in [0,1]$, the sets $\{\alpha \in [0,1] : \mu(\alpha f + (1-\alpha)g, h) \geq \gamma\}$ and $\{\alpha \in [0,1] : \mu(h, \alpha f + (1-\alpha)g) \geq \gamma\}$ are closed — compare with Axiom 2 of Herstein and Milnor (1953).

The next definition introduces the notion of continuity for a capacity measure that will be used in our main result.

Definition 1. Let $\mathcal{M}$ be a nonempty subset of $\Delta(\Omega)$, and $\mathcal{L}$ be the set of affine functionals $\mathcal{M} \to \mathbb{R}$ defined as $\mathcal{L} := \{p \mapsto \int_\Omega \varphi \, dp \mid \varphi \in B_0(\Omega, \Sigma, \mathbb{R})\}$. We say that a capacity measure π on $\mathscr{B}(\mathcal{M})$ is *linearly continuous* if the mapping $\alpha \mapsto \pi(L^{-1}([\alpha, \infty)))$ is continuous on $\mathbb{R}$ for all $L \in \mathcal{L}$ that are not constant on $\mathcal{M}$.

Finally, we will use the following notion of full support for a capacity measure.

Definition 2. We say that a capacity measure π on $\mathscr{B}(\mathcal{M})$, where $\mathcal{M}$ is a nonempty, closed, and convex subset of $\Delta(\Omega)$, has *linear full support* if $\pi(\{p \in \mathcal{M} : \int_\Omega \varphi \, dp \geq 0\}) < 1$ for any $\varphi \in B_0(\Omega, \Sigma, \mathbb{R})$ such that $\int_\Omega \varphi \, dp < 0$ for at least one $p \in \mathcal{M}$.

We are now ready to state our main representation result.

Theorem 1. *Let μ be a graded preference relation on $\mathcal{F}$. Then, μ satisfies Reflexivity, Weak Transitivity, Monotonicity, C-Completeness, Independence, Reciprocity, Continuity, and Nondegeneracy if and only if there exist a nonconstant affine function $u : X \to \mathbb{R}$, a nonempty, convex, and closed set $\mathcal{M}$ of probabilities in $\Delta(\Omega)$, and a linearly continuous capacity measure $\pi : \mathscr{B}(\mathcal{M}) \to [0,1]$ with linear full support such that*

(i) representation

$$\mu(f, g) = \pi\left(\left\{p \in \mathcal{M} : \int_\Omega (u \circ f) \, dp \geq \int_\Omega (u \circ g) \, dp\right\}\right) \tag{2}$$

holds for all $f, g \in \mathcal{F}$;

(ii) $\pi\left(\left\{p \in \mathcal{M} : \int_\Omega (u \circ f) \, dp \geq \int_\Omega (u \circ g) \, dp\right\}\right) = 1 - \pi\left(\left\{p \in \mathcal{M} : \int_\Omega (u \circ g) \, dp \geq \int_\Omega (u \circ f) \, dp\right\}\right)$ *for all $f, g \in \mathcal{F}$ such that $\int_\Omega (u \circ f) \, dp \neq \int_\Omega (u \circ g) \, dp$ for some $p \in \mathcal{M}$.*

According to Representation (2), the decision maker holds a set $\mathcal{M}$ of priors that she considers plausible. Given two acts f and g, she compares the expected utility of f with that of g for each prior p in $\mathcal{M}$. Then, the decision maker grades the extent to which she is sure that f is at least as good as g by measuring the set of priors in $\mathcal{M}$ that rank f (weakly) better than g.[8] This representation suggests the following interpretation of the decision process: The set $\mathcal{M}$ can be thought of as the collection of the decision maker's multiple selves that have clear but different opinions on the ranking of f and g. In turn, the capacity measure π provides a way to aggregate these opinions. Besides aggregation, π also reflects the extent to which the conflict among the decision maker's selves is resolved. Therefore, Representation (2), as a whole, provides insight into the decisiveness of the decision maker. A similar interpretation of Representation (2) consists of viewing the priors in $\mathcal{M}$ as reflecting the opinions of the members of a committee that has to choose a course of action. Under this interpretation, π captures the share of votes in favor of one resolution.

Condition (ii) of the theorem clearly ensures that if the graded preference relation μ admits Representation (2), then it satisfies Reciprocity. As our short note Minardi and Savochkin (2013b) shows, only minor adjustments are required to derive Representation (2) without imposing Reciprocity; however, in the present paper, we keep Reciprocity for the benefit of a clearer exposition.

We will use the following short name to refer to a graded preference relation that satisfies the axioms listed in Theorem 1.

Definition 3. A graded preference relation μ on $\mathcal{F}$ is called *affine* if it satisfies the Reflexivity, Weak Transitivity, Monotonicity, C-Completeness, Independence,

[8] It is possible to provide a strengthening of Representation (2) in which the decision maker uses (up to an increasing transformation) a standard probability measure instead of a non-additive measure to aggregate her priors. This result is available from the authors upon request.

Reciprocity, Continuity, and Nondegeneracy axioms.

By Theorem 1, we can represent affine graded preferences by a triple $(u, \mathcal{M}, \pi)$. Henceforth, we will refer to such triple as a representation of an affine graded preference relation μ. Next, we establish the uniqueness properties of this representation.

Proposition 2. *Let μ be an affine graded preference with representation $(u, \mathcal{M}, \pi)$. Then, u is unique up to positive affine transformations; $\mathcal{M}$ is unique in the class of closed and convex sets; and the restriction of the function π to the collection $\left\{\left\{p \in \mathcal{M} : \int_\Omega (u \circ f)\, dp \geq \int_\Omega (u \circ g)\, dp\right\} \mid f, g \in \mathcal{F}\right\}$ is unique.*

We conclude this subsection with an important remark about the nature of indecisiveness modeled in this paper. Our model does not allow an agent to be indifferent between two alternatives with some intermediate level of confidence: If she is indifferent between two options f and g, it must be that she is sure about it; therefore, she will report $\mu(f, g) = \mu(g, f) = 1$. The normative reason for this restriction is that the situations in which an agent is indifferent between alternatives can be considered knife-edge cases.[9] Therefore, if $\mu(f, g) \in [0, 1)$, then the extent to which f is *weakly* preferred to g coincides with the extent to which f is *strictly* preferred to g. The purpose of the next result is precisely to formalize this point.

Proposition 3. *Let μ be an affine graded preference with representation $(u, \mathcal{M}, \pi)$. For arbitrary $f, g \in \mathcal{F}$, let $S := \left\{p \in \mathcal{M} : \int_\Omega (u \circ f)\, dp \geq \int_\Omega (u \circ g)\, dp\right\}$ and $S_0 :=$*

[9] According to the categorization of indifference and indecisiveness developed by Eliaz and Ok (2006), a decision maker is indifferent between two options if any marginal improvement of one of the alternatives is sufficient to prompt a strict preference for it. Therefore, if our decision maker does not display 100-percent confidence in her preference between two alternatives, it is reasonable to assume that her confidence will not jump up to 100 percent from marginal changes, and, hence, those cases should be regarded as indecisiveness rather than indifference.

$\{p \in \mathcal{M} : \int_\Omega (u \circ f)\, dp > \int_\Omega (u \circ g)\, dp\}$. *Then, either* $\pi(S) = \pi(S_0)$, *or* $\pi(S) = 1$ *and* $\pi(S_0) = 0$.

In words, Proposition 3 states that, for any acts f and g in $\mathcal{F}$, either the extent to which f is weakly preferred to g coincides with the extent to which f is strictly preferred to g, or the agent is indifferent between these two acts.[10]

2.4. Relationship with Bewley's (1986) model of Knightian Uncertainty

This section relates our result to the well-known model of Bewley (1986). In his work, the agent's preferences are described by an incomplete preference relation $\succ$ on $\mathcal{F}$. He proposes a representation of the agent's behavior through a set of priors and a unanimity rule. In his original work, Bewley posits a strict preference and assumes that the state space is finite. In the present paper, following the approach of Gilboa, Maccheroni, Marinacci and Schmeidler (2010), we consider a weak preference $\succsim$ on $\mathcal{F}$ and an arbitrary state space. Gilboa et al. propose the following representation of $\succsim$, which is equivalent, in spirit, to that of Bewley (1986):

$$f \succsim g \quad \Leftrightarrow \quad \int_\Omega (u \circ f)\, dp \geq \int_\Omega (u \circ g)\, dp \quad \forall p \in \mathcal{C},$$

where $\mathcal{C}$ is a convex subset of the set $\Delta(\Omega)$ of all probability measures. This set $\mathcal{C}$ is interpreted as a set of plausible priors held by the agent. Knightian uncertainty is captured in the model through the multiplicity of priors and the partial (incomplete) decision rule.

To provide a behavioral characterization of this class of preferences, Gilboa et al. (2010) impose the following axioms on $\succsim$:

[10] It is also worth noting that this result does not hinge on Reciprocity, but is mainly due to Independence and Continuity. Indeed, Proposition 3 also remains valid if μ does not satisfy Reciprocity — see the parallel result in our short note Minardi and Savochkin (2013b).

Preorder: $\succsim$ is reflexive and transitive.

Monotonicity: If $f, g \in \mathcal{F}$ and $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, then $f \succsim g$.

Archimedean Continuity: If $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\}$ are closed in $[0,1]$.

Nontriviality: $f \succ g$ for some $f, g \in \mathcal{F}$.

C-Completeness: For any $x, y \in X$, $x \succsim y$ or $y \succsim x$.

Independence: If $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$, $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$.

We are now ready to state our results. The next definition plays a key role in Propositions 4 and 5.

Definition 4. Let μ be a graded preference and $\succsim$ be a binary relation on $\mathcal{F}$. We say that μ is a *refinement* of $\succsim$, and $\succsim$ is a *coarsening* of μ, if

$$f \succsim g \quad \Leftrightarrow \quad \mu(f,g) = 1 \quad \text{for all } f, g \in \mathcal{F}.$$

Definition 4 points out the richness of our primitive μ in comparison with the traditional crisp binary relations $\succsim$. Indeed, μ captures the same information as $\succsim$ in all situations in which the agent is willing to compare alternatives in a crisp way. Additionally, μ provides information about the agent's grade of indecisiveness in all circumstances in which $\succsim$ would simply render the alternatives incomparable.

The following result shows that if $\succsim$ is a coarsening of μ, then the set of behavioral assumptions that we impose on μ implies that $\succsim$ satisfies all the Bewley axioms and, therefore, admits a representation à la Bewley.

Proposition 4. *Let μ be an affine graded preference and $\succsim$ be a coarsening of μ. Then, $\succsim$ satisfies the axioms of Bewley's model: Preorder, Monotonicity, Archimedean Continuity, Nontriviality, C-Completeness, and Independence.*

The next result shows that our functional form is a refinement of the functional form of Bewley.[11] In other words, the representation à la Bewley of $\succsim$ can be deduced from our affine graded representation of μ. In particular, both μ and $\succsim$ can be represented by the same utility function u and the same set of priors $\mathcal{M}$.

Proposition 5. *Let μ be an affine graded preference with representation $(u, \mathcal{M}, \pi)$, and suppose that $\succsim$ is a coarsening of μ. Then, for all $f, g \in \mathcal{F}$,*

$$f \succsim g \quad \Leftrightarrow \quad \int_{\Omega} (u \circ f)\, dp \geq \int_{\Omega} (u \circ g)\, dp \quad \forall p \in \mathcal{M}. \tag{3}$$

Let us note in passing that, as is well-known, a special case of the Bewley model is the Subjective Expected Utility (SEU) benchmark. In a similar vein, we can show that the SEU paradigm is a special case of our model and can be derived by imposing the Completeness axiom on an affine graded preference μ.

Axiom (Completeness)**.** For any $f, g \in \mathcal{F}$, $\mu(f, g) = 1$ or $\mu(g, f) = 1$.

Proposition 6. *A graded preference relation μ is an affine graded preference that satisfies Completeness if and only if there exist a nonconstant affine function $u : X \to \mathbb{R}$, and a probability measure $q \in \Delta(\Omega)$ such that, for all $f, g \in \mathcal{F}$,*

$$\mu(f, g) = \begin{cases} 1 & \text{if } \int_{\Omega} (u \circ f)\, dq \geq \int_{\Omega} (u \circ g)\, dq, \\ 0 & \text{if } \int_{\Omega} (u \circ f)\, dq < \int_{\Omega} (u \circ g)\, dq. \end{cases} \tag{4}$$

Moreover, if $\tilde{u} : X \to \mathbb{R}$ and $\tilde{q} \in \Delta(\Omega)$ also satisfy (4), then $\tilde{u}$ is a positive affine transformation of u and $\tilde{q} = q$.

2.5. Comparative Attitudes

This section introduces a comparative notion of indecisiveness, which enables us to say when one agent is more decisive than another. In a similar spirit to the

[11] It is worth noting that the notion of refinement in terms of the primitive is conceptually different from the notion of refinement in terms of the representation. In general, μ might be a refinement of $\succsim$, and, yet, its representation might be unrelated to the representation of $\succsim$.

notion of comparative risk aversion (or comparative ambiguity aversion[12]), the following definition provides a basis for performing comparative statics exercises involving graded preferences.

Definition 5. Given two graded preference relations μ_1 and μ_2 that satisfy the Reciprocity axiom, we say that μ_1 is *more decisive than* μ_2 if, for all $f, g \in \mathcal{F}$,

$$\mu_2(f,g) \geq \mu_2(g,f) \quad \Rightarrow \quad \mu_1(f,g) \geq \mu_2(f,g). \tag{5}$$

In words, consider two agents, identified by μ_1 and μ_2. We would like to say that the first agent is more decisive than the second. Suppose that $\mu_2(f,g) \geq \mu_2(g,f)$ for some $f, g \in \mathcal{F}$. Under Reciprocity, this means that $\mu_2(f,g) \geq \frac{1}{2}$ so that the second agent is inclined to prefer f over g. Then, if the first agent is more decisive than the second, she should be at least as confident as the second agent in her preference for f over g, and, therefore, we should have $\mu_1(f,g) \geq \mu_2(f,g)$. Next, we state the main result of this subsection, which provides a characterization of comparative indecisiveness in terms of the representation. For that, we need to introduce the following notation.

Notation 6. Given a representation $(u_i, \mathcal{M}_i, \pi_i)$ for an affine graded preference μ_i, let

$$\mathfrak{H}_i := \left\{ \left\{ p \in \mathcal{M}_i : \int_\Omega (u_i \circ f)\, dp \geq \int_\Omega (u_i \circ g)\, dp \right\} \,\middle|\, f, g \in \mathcal{F} \right\}.$$

Theorem 7. *Given two affine graded preferences μ_1 and μ_2 with representations $(u_1, \mathcal{M}_1, \pi_1)$ and $(u_2, \mathcal{M}_2, \pi_2)$, μ_1 is more decisive than μ_2 if and only if the following conditions hold:*

(i) u_1 is a positive affine transformation of u_2;

(ii) $\mathcal{M}_1 \subseteq \mathcal{M}_2$;

[12] For a notion of comparative ambiguity aversion, see Epstein (1999) and Ghirardato and Marinacci (2002).

(iii) for all $B \in \mathfrak{H}_2$ *such that* $\pi_2(B) \geq \frac{1}{2}$, $\pi_1(\mathcal{M}_1 \cap B) \geq \pi_2(B)$.

According to Condition (i), two agents have comparable decisiveness only if they have the same attitude towards pure risk (i.e., their utility functions over risk are cardinally equivalent). Therefore, Theorem 7 shows that the comparative decisiveness attitude for affine graded preferences is determined by the set $\mathcal{M}$ and the capacity measure π. As in many models featuring multiple priors, the size of the set $\mathcal{M}$ reflect the amount of ambiguity that is perceived by the agent. Since indecisiveness is related to ambiguity, the smaller the set $\mathcal{M}$, the more decisive the agent. This is the content of Condition (ii).

Now, consider two acts f and g, and suppose that one agent assigns a measure of at least $1/2$ to the set of priors that rank f better than g. According to Condition (iii), a more decisive agent should assign an even bigger measure to her corresponding set of priors for which f is better than g. That is, more decisive agents should report greater numerical values when comparing a "better" option to a "worse" one. Naturally, for a more decisive agent, the capacity measure gravitates more towards the extreme values of zero and one.

In passing, observe that our Definition 5, as well as Theorem 7, can be viewed as refinements of, respectively, the concept of one (crisp) binary relation being "more complete" than another and its characterization in terms of the inclusion of the set of priors.[13]

To illustrate our notion of comparative decisiveness and Theorem 7, we present two examples of the extreme cases of decisiveness.

Example 1 (Subjective expected utility preferences are maximally decisive)**.** We say that an affine graded preference μ is *maximally decisive* if there does not exist another affine graded preference relation $\tilde{\mu} \neq \mu$ such that $\tilde{\mu}$ is more decisive than

[13] For a related discussion in the domain of crisp preferences, see, e.g., Ghirardato, Maccheroni and Marinacci (2004).

μ. Our claim is that a graded preference μ that falls into the special case of subjective expected utility preferences of Proposition 6 is maximally decisive.

Indeed, in this case, μ has an affine graded representation $(u, \{q\}, \pi)$ for some nonconstant and affine function $u : X \to \mathbb{R}$, a probability measure $q \in \Delta(\Omega)$, and a capacity measure π satisfying the conditions of Theorem 1. In this case, the algebra $\mathscr{B}(\mathcal{M})$ is $\{\varnothing, \mathcal{M}\}$, and, by normalization, the only capacity measure on this algebra is trivial: $\pi(\varnothing) = 0$ and $\pi(\mathcal{M}) = 1$. By Completeness, for any $f, g \in \mathcal{F}$, either $\mu(f, g) = 1$ or $\mu(f, g) = 0$. Now, let $\tilde{\mu}$ be a graded preference relation that is more decisive than μ. Then, by definition, $\tilde{\mu}(f, g) = 1$ for any $f, g \in \mathcal{F}$ such that $\mu(f, g) = 1$, and $\tilde{\mu}(f, g) = 0$ for any $f, g \in \mathcal{F}$ such that $\mu(f, g) = 0$. Hence, we conclude that $\tilde{\mu} = \mu$.

Example 2 (Decreasing Decisiveness)**.** Let Ω be a finite set that has at least two elements, $\mathcal{M}$ be some closed and convex subset of the set of probability measures on Ω (which is a simplex in a Euclidean space), and ν be a uniform distribution on $\mathcal{M}$. Consider a sequence of graded preferences $(\mu_n)_{n=1}^{\infty}$ that admit representation (2) with the same utility function, same set $\mathcal{M}$, and capacities $\pi_n = \varphi_n \circ \nu$, where $\varphi_n : [0, 1] \to [0, 1]$ are some continuous and strictly increasing functions such that $\varphi_n(0) = 0$, $\varphi_n(\frac{1}{2}) = \frac{1}{2}$, $\varphi_n(1) = 1$, and $\varphi_n(t) + \varphi_n(1 - t) = 1$ for all $t \in [0, 1]$ and all $n \in \mathbb{N}$. Moreover, suppose that the sequence $(\varphi_n)_{n=1}^{\infty}$ converges pointwise to the function φ_∞ defined as $\varphi_\infty(0) = 0$, $\varphi_\infty(1) = 1$, and $\varphi_\infty(t) = \frac{1}{2}$ for all $t \in (0, 1)$ (see Figure 1). By Theorem 7, it follows that each affine graded preference μ_{n+1} is less decisive than μ_n.

Now we can define $\mu_\infty : \mathcal{F} \times \mathcal{F} \to [0, 1]$ as the pointwise limit of the sequence

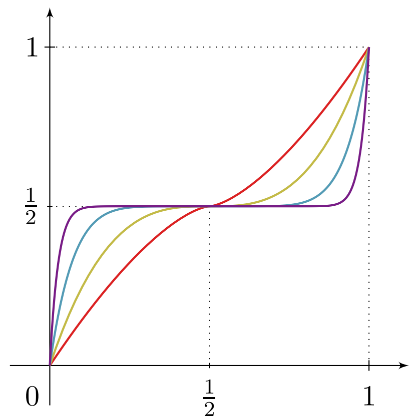

Figure 1: A sequence of functions φ_n manifesting decreasing decisiveness

μ_n, and observe that

$$\mu_\infty(f,g) = \begin{cases} 1, & \text{if } \int (u \circ f)\, dp \geq \int (u \circ g)\, dp \text{ for all } p \in \mathcal{M}, \\ 0, & \text{if } \int (u \circ f)\, dp \leq \int (u \circ g)\, dp \text{ for all } p \in \mathcal{M} \text{ and} \\ & \quad \int (u \circ f)\, dp < \int (u \circ g)\, dp \text{ for at least one } p \in \mathcal{M}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

In words, an agent with a graded relation μ_∞ reports the confidence level of one if all her priors unanimously agree that f is at least as good as g, zero if the priors agree that g is at least as good as f, and $\frac{1}{2}$ otherwise. Naturally, such an agent can be viewed as minimally decisive. At the same time, note that μ_∞ does not satisfy our definition of an *affine graded* preference relation for technical reasons — it violates the Continuity axiom.

2.6. A Discussion on Observability

We close this section with a comment on the question about how an analyst may observe graded preferences, and whether such observations can be trusted.[14]

[14] A related question about the use of graded preferences is addressed in the next section.

The difficulties here — stemming from our reliance of non-choice data with the aim of gaining a better understanding of agents' final choices and, eventually, better ability to predict them — are far from being entirely new, and have been discussed in various other contexts.[15] In the context of uncertainty, Gilboa et al. (2008, p. 19) advocate that "[e]xtending the notion of observability beyond pure choice data seems essential for the discussion of incomplete preference, as well as the process by which preferences are generated," and conclude that the observability of such preferences "includes the possibility of preferences being stated, not only revealed through action." Next, we discuss a few possible sources of data on graded preferences.

The primary method for eliciting a graded preference is to ask the agent to fill an appropriate questionnaire that, as discussed in the Introduction, may be formulated in a scale-based (cardinal) form or in the (ordinal) form of pairwise comparisons. Working on a related problem of modeling the strength of preferences, Rustichini (2008, p. 38) suggests to elicit it in a similar manner: "she introspectively evaluates the strength and communicates it to the experimenter, with words, not with choice." The challenge of increasing the trustworthiness of this kind of data amounts to providing suitable incentives for subjects' responses. From this perspective, eliciting *choice* is a simple problem: Giving the subject the chosen object is the simplest and most reliable way of supplying incentives. However, incentives do not always have to be so direct, and the experimental literature uses a number of indirect schemes for eliciting components of the decision process that are not observable directly, such as proper scoring rules for eliciting beliefs. We believe that similar methods can be developed to elicit grades of indecisiveness and leave this investigation as a possible venue for future research.

[15]For arguments why non-choice data should not be dismissed outright, see, e.g., Dekel and Lipman (2010).

Besides the use of questionnaires, other complementary methods may certainly come to mind. In particular, one may envisage a connection between incompleteness and the time needed to make a choice in binary comparisons.[16] To illustrate a possible connection between grades of indecisiveness and response times, consider a hypothetical experiment in which the decision maker is asked to make choices from pairs of objects. The experimental data would consist of both the selected alternative for each comparison and the corresponding length of time taken to reach a decision. To elicit a graded preference relation, the experimenter could specify a continuous and decreasing function converting, for any pair of alternatives, response times into levels of confidence. This approach would rely on the intuitive presumption that the more indecisive the agent is, the longer the time needed to report a choice.[17]

Finally, another way to measure indecisiveness based on choice might be offered by stochastic choice data. If the agent has a clear-cut preference over a pair of alternatives, she should choose her preferred option with probability one; the more indecisive the agent is, the closer to $\frac{1}{2}$ the empirical frequency should be. Naturally, we are far from suggesting a one-to-one mapping between indecisiveness, on one hand, and the response time or stochastic volatility of choice, on the other. However, vast experimental evidence finds that difficult choices — i.e., comparisons between options which do not clearly dominate one another — are associated with both longer reaction times and higher volatility.[18]

[16] Response times have been extensively studied in psychology (see, e.g., Luce, 1986). In economics, they have received less attention, with notable exceptions such as Rubinstein (2007).

[17] A related experiment is done by Danan and Ziegelmeyer (2006), who test for the presence of incomplete preferences by offering subjects the option to postpone the time of choice at a small cost.

[18] For an early experiment combining response times and probabilistic choice, see Cartwright (1941) — we refer to Rustichini (2008) for a recent analysis of that work. In a series of games, Rubinstein (2007) finds that choices which require higher cognitive abilities are associated with

3. Extension: Decision Rules

Similar to many other models of incomplete preferences, our model of graded preferences does not address what an agent does if she is indecisive but needs to make a choice. This section provides two examples describing choices that can eventually arise from affine graded preferences. The presentation of our examples is based on the idea of a *decision rule* — an independent model of how decision maker's evaluation of two prospects with an incomplete preference relation translates into her final choices. In the tandem of a graded preference relation and a decision rule, the former represents the agent's "introspective preferences" that capture her judgments, and the latter is the description of her choice behavior.[19]

As we will show, adopting different decision rules for any single graded preference relation results in stark changes of observable choices: One of the decision rules described below results in stochastic choices (that are related to the random choice rule of Gul and Pesendorfer, 2006); the other one is deterministic, but displays another type of behavior that is well recognized in the literature — namely, sensitivity to ambiguity and uncertainty aversion.

We now turn to describing our examples of decision rules.

3.1. Random Choice Decision Rule

This subsection describes a decision rule by which the agent *randomizes* between the offered alternatives if she is not completely sure about the superiority

longer response times. Furthermore, Agranov and Ortoleva (2013) find evidence supporting the hypothesis that stochastic choice may result from a deliberate action. They also find a positive correlation between stochastic variation and response times. As a potential interpretation of their results, they advance the possibility that probabilistic choice may, indeed, reflect the presence of an underlying incomplete preference.

[19]The distinction between introspective preferences and choice behavior is explored, for instance, by Mandler (2005), Danan (2008), and Gilboa et al. (2010).

of one of them. Formally, for any doubleton set $\{f, g\}$ of acts, a random choice rule $\rho^{\{f,g\}}$ is a probability distribution over $\{f, g\}$ that describes the probabilistic prediction of the agent's choice from this set.[20] In particular, $\rho^{\{f,g\}}(f)$ denotes the probability that the agent chooses f from the pair $\{f, g\}$ of acts. Now, let μ be an affine graded preference with representation $(u, \mathcal{M}, \pi)$ and fix some random variable ξ taking values in $[0, 1]$. Then, we define a random choice rule ρ as follows:

$$\rho^{\{f,g\}}(f) := \begin{cases} \xi, & \text{if } \mu(f, g) = 1 = \mu(g, f), \\ \mu(f, g), & \text{otherwise.} \end{cases} \tag{6}$$

This expression implies that the decision maker chooses f over g with probability one if she has a clear strict preference for f over g — i.e., if $\mu(f, g) = 1$ and $\mu(g, f) = 0$. Similarly, she chooses g with probability one if she has a clear strict preference for g over f. If she is indifferent — i.e., if $\mu(f, g) = 1 = \mu(g, f)$ — she chooses according to the realization of the random variable ξ. Finally, if the agent is caught in indecision, the probability of choosing f over g is determined by her level of confidence that f is really better. Note that, although the graded preference μ is directly translated into ρ whenever the agent is not indifferent between acts, these two functions describe different stages of the decision process. Indeed, μ reflects the agent's introspective preferences, whereas ρ captures her (probabilistic) choice behavior.

The combination of our model of affine graded preferences and the random choice rule described here produces a model of choice that is related to the random expected utility model of Gul and Pesendorfer (2006). They study choices among lotteries over a finite set of prizes and obtain a representation in which the

[20] In the literature on random choice, the emergence of probabilities is motivated as arising from the presence of stochastic factors in the agent's decisions (in which case the probabilities can be linked to frequencies if the agent is observed facing the same decision problem repeatedly) or from the aggregation of the choices of a population of individuals with heterogeneous tastes.

agent has a set of von Neumann-Morgenstern utility functions and a probability measure over them; the probability of choosing a particular option from a set can, then, be computed as the probability that this option is a maximizer of one of the utilities in her set.[21]

The aforementioned decision rule further highlights the differences between the approach of this paper and that of the literature on random choice — capturing grades of indecisiveness versus frequencies. For instance, similar to other models of incomplete preferences, our model maintains a sharp distinction between indifference and indecisiveness. However, these two states of the decision maker's mind remain indistinguishable if the analyst inspects only the frequency data.

We now introduce a very different decision rule.

3.2. Ambiguity-Sensitive Decision Rule

This subsection proposes a decision rule which relies on the agent's graded judgements to determine the *value* of an act — the placement of the act on the scale that consists of certain outcomes (elements of X) ordered from worse to better. The evaluation of an act depends on both the agent's *confidence* in comparing it to constant acts and her *tolerance* for the inability to make clear-cut comparisons. Then, in any set of alternatives, the agent chooses deterministically by selecting the alternative with the highest value.

In order to introduce this rule formally, we begin with a piece of notation and a preliminary definition. Let $\breve{\succsim}$ denote a (crisp) binary relation on X defined as

$$x \mathrel{\breve{\succsim}} y \quad \Leftrightarrow \quad \mu(x, y) = 1 \quad \text{for all } x, y \in X.$$

[21] McFadden and Richter (1990) and McFadden (2005) originally proposed a random utility model in which the objects of choice are deterministic alternatives. To the best of our knowledge, random choice models in the Knightian uncertainty framework have not been developed yet.

Note that $\breve{\succsim}$ is a complete and transitive binary relation if μ satisfies the axioms that we imposed earlier (i.e., μ is affine graded).

Definition 7. Let γ be a real number in $(0,1]$. For any $f \in \mathcal{F}$, we denote by x_f^γ an element in X such that $x_f^\gamma \in \max_{\breve{\succsim}}\{x \in X : \mu(f,x) \geq \gamma\}$.[22]

In words, given any act f, we consider the set of all constant acts x for which the agent is γ-sure that f is at least as good as x. Then, x_f^γ denotes the best ($\breve{\succsim}$-maximal) element in this set.

Now, for a fixed $\gamma \in (0,1]$, the decision rule that we propose is a complete preference relation $\succsim^\gamma$ on $\mathcal{F}$ defined by

$$f \succsim^\gamma g \quad \Leftrightarrow \quad x_f^\gamma \,\breve{\succsim}\, x_g^\gamma. \tag{7}$$

According to this rule, the key step to compare two acts f and g is to compute their "certainty equivalents" — the constant acts x_f^γ and x_g^γ.[23] Given an act f, the certainty equivalent is found by assessing the level of confidence that f is superior to a constant act x, and comparing that level to the tolerance parameter γ. If $\gamma = 1$, the confidence threshold is maximal, and the agent's tolerance to the lack of perfect decisiveness is minimal. The lower value the parameter γ takes, the less confidence is needed for her to switch away from constant acts, and the higher is her valuation of nonconstant ones.

As we show in-depth in our companion paper, Minardi and Savochkin (2013a), if μ is affine graded, then the preference relation $\succsim^\gamma$ defined in (7) is continuous, monotone, and satisfies the Certainty Independence axiom of Gilboa and Schmeidler (1989); therefore, it admits a generalized α-maxmin representation as

[22] Generally, there may be more than one maximal element in this set. In this case, x_f denotes one of them — in what follows, the choice of the maximal element will be immaterial.

[23] As can be easily seen, our definition implies that any act f is equivalent (with respect to the binary relation $\succsim^\gamma$) to x_f^γ, so our use of the term "certainty equivalent" is justified.

in Ghirardato et al. (2004). In other words, choices that are compatible with the generalized α-maxmin representation can arise as a rational response to indecisiveness and lack of confidence.[24] We further show that decision makers who are more decisive also exhibit less aversion to ambiguity.

4. Related Literature and Concluding Remarks

The present work seeks to contribute, primarily, to the literature on incomplete preferences under uncertainty, which studies agents who may not necessarily be able to rank all possible alternatives. As our analysis in Subsection 2.4 shows, our paper has a particularly close relationship with Bewley's (1986) seminal work. Our model of graded preferences can be thought of as a *refinement* of his model: They have certain similarities in their behavioral assumptions and functional form of the representation; at the same time, ours is able to capture finer distinctions of the decision maker's attitudes towards the alternatives that are offered to her.

Our paper studies one particular model of graded preferences that assumes the Independence axiom. However, our general approach of modeling introspection of an agent with graded preferences can be extended to other classes of incomplete preferences, such as the ones studied by Faro (2011), Lehrer and Teper (2011), and Hill (2011). Similarly, while we do this exercise in the Anscombe-Aumann framework, our analysis can also, in principle, be repeated in other frameworks. The most natural extension of this sort can be done in the framework of choice under risk, in which Dubra, Maccheroni and Ok (2004) and Kochov (2007) have developed multi-utility models that are parallel to Bewley's multi-prior model.[25]

[24]Moreover, if, in addition, μ satisfies a transitivity condition (appropriately defined), then $\succsim^\gamma$ will also satisfy the Uncertainty Aversion axiom of Schmeidler (1989) and admit a maxmin representation as in Gilboa and Schmeidler (1989).

[25]In addition to that, Seidenfeld, Schervish and Kadane (1995), Nau (2006), Ok, Ortoleva

From the perspective of linking an agent's indecisiveness under uncertainty to her confidence, our paper is related to the works of Nau (1992) and Hill (2011). Nau (1992) studies incomplete preferences that are invariant to uniform increases of payoffs but otherwise violate the Independence axiom,[26] and connects the relations "prefer with confidence c," where $c \in \mathbb{R}$, to the ranges of probabilities that the decision maker assigns to events. Hill's (2011) model provides a generalization of Bewley's model by allowing the set of priors that an agent is using to vary with the "stakes" of the decision problem; the collection of these sets is, then, interpreted as a hierarchy of the decision maker's levels of confidence. Both these papers infer the agent's confidence from her choices; in contrast, we assume that the analyst can learn her confidence level directly by presenting a suitable questionnaire. As we discuss in Section 3, the observed confidence can be linked to choices by extensions of our model — the decision rules. Our setting also allows us to characterize the comparative statics notion of increasing confidence in terms of the objects of the representation.

The key object of analysis in our paper — a graded preference relation — has a counterpart in the mathematical literature on fuzzy sets originated by Zadeh (1965). In the language of that theory, a graded preference relation μ on $\mathcal{F}$ can be thought of as a fuzzy subset of $\mathcal{F} \times \mathcal{F}$, or as a *fuzzy binary relation*.[27] In the literature on fuzzy binary relations, Orlovsky (1978) starts a non-axiomatic dis-

and Riella (2012), and Galaabaatar and Karni (2013) study representations that admit multiple utilities and multiple priors.

[26]Preferences with similar properties are studied in the standard Anscombe-Aumann framework by Faro (2011).

[27]Our interpretation of a graded preference relation differs from the conventions assumed in the study of fuzzy sets. Suppose, for instance, that $\mu(a,b) = \frac{3}{4}$ and $\mu(b,a) = \frac{1}{4}$. We interpret these numbers as expressing the fact that a is strictly better than b with 75% confidence and that b is strictly better than a with 25% confidence, as well as ruling out the possibility that a is equally good as b. The fuzzy set literature interprets this situation as the fact that a is strictly better than b with the degree of 50% (Orlovsky, 1978) or 75% (Ovchinnikov, 1981) and that a

cussion about the procedures that a decision maker with fuzzy preferences may use to make choices; and Basu (1984) discusses WARP and various ways in which a fuzzy choice function can be rationalizable by a fuzzy preference relation.[28] In contrast with those papers, we develop a very specific model that can accommodate behavioral traits such as confidence and decisiveness; and we perform a decision-theoretic exercise through which we establish an equivalence between a certain set of axioms and a representation.

A graded preference relation also resembles objects that are studied in the literature on random choice and on "preference intensities." In the first of these branches of literature, the closest to this paper are the works of Fishburn (1973) and Gul and Pesendorfer (2006). Fishburn (1973) studies interval representations of binary relations P_λ that indicate the fact that one alternative is chosen over the other with the probability greater than λ, a parameter that captures decisiveness. The relationship between our work and the work of Gul and Pesendorfer (2006) is discussed in Subsection 3.1. There, we summarize the differences between our theory and models of random choice in terms of objectives — we study introspection that may result in deterministic choice — as well as the general approach to modeling (including the setup, handling of indifferences, and so on).

The second of these branches of literature (including, for instance, Krantz, Luce, Suppes and Tversky, 1971, and Dyer and Sarin, 1979) seeks to accommodate the concept of intensity of preferences — the preference for one million dollars over nothing may be stronger than the preference for one dollar over noth-

is indifferent to b with the degree of 25%, as well as ruling out the possibility that b is strictly better than a.

[28] A number of papers also study general properties that one may want to impose on a fuzzy binary relation — e.g., possible extensions of transitivity (Ovchinnikov, 1981 and Ok, 1994), and reciprocity (Basu, 1987). We do not impose transitivity, and our reciprocity axiom is weaker than the proposed one.

ing — without relying on the presence of risk or uncertainty. In contrast, our decision maker reports the same level of confidence (one) in her preference for the first option in both of these scenarios. The level of confidence other than zero or one can arise in our model only in comparisons that are difficult to make, and in which the superiority of one alternative over the other is not absolutely clear — in particular, because the objects of evaluation are multi-dimensional, do not dominate each other, and, thus, present a trade-off.

Appendix

Let $B_0(\Omega, \Sigma, \mathbb{R})$ denote the set of all real-valued, Σ-measurable functions taking only finitely many values (simple functions) endowed with the sup-norm, and let $B_0(\Omega, \Sigma, K)$ denote the set of functions in $B_0(\Omega, \Sigma, \mathbb{R})$ taking values in the interval $K \subseteq \mathbb{R}$. For a function $\varphi \in B_0(\Omega, \Sigma, \mathbb{R})$ and a measure $p \in \Delta(\Omega)$, let $\langle \varphi, p \rangle$ denote $\int_\Omega \varphi \, dp$.

We will also use the short-hand notation $L_{f,g} : \Delta(\Omega) \to \mathbb{R}$, where $f, g \in \mathcal{F}$, for the affine functional $\Delta(\Omega) \to \mathbb{R}$ defined as $L_{f,g}p := \langle (u \circ f) - (u \circ g), p \rangle$ for a given utility function $u : X \to \mathbb{R}$.

A. Proof of Theorem 1

Lemma 8. *Suppose that μ is a graded preference relation on $\mathcal{F}$ that satisfies the Reciprocity and Continuity axioms. Then, the mappings $\alpha \mapsto \mu(\alpha f + (1 - \alpha)g, h)$ and $\alpha \mapsto \mu(h, \alpha f + (1 - \alpha)g)$ are continuous for all $f, g, h \in \mathcal{F}$ and $\alpha \in [0, 1]$ except where $\mu(\alpha f + (1 - \alpha)g, h) = 1 = \mu(h, \alpha f + (1 - \alpha)g)$.*

Proof. Let $f, g, h \in \mathcal{F}$ and $\alpha \in [0, 1]$ be such that either $\mu(\alpha f + (1 - \alpha)g, h) < 1$ or $\mu(h, \alpha f + (1 - \alpha)g) < 1$, and assume without loss of generality that $\mu(\alpha f + (1 - \alpha)g, h) < 1$. Consider the sequence $\{\alpha_n\}_{n=1}^{\infty}$ of numbers in $[0, 1]$ such that $\alpha_n \to \alpha$ as $n \to \infty$. By the Continuity axiom, $\limsup_{n\to\infty} \mu(\alpha_n f + (1 - \alpha_n)g, h) \leq \mu(\alpha f + (1 - \alpha)g, h)$. Then, $\mu(\alpha_n f + (1 - \alpha_n)g, h) < 1$ for sufficiently large n, and we can apply the Reciprocity axiom to obtain $\liminf_{n\to\infty} \mu(\alpha_n f + (1 - \alpha_n)g, h) =$

$1 - \limsup_{n\to\infty} \mu(h, \alpha_n f + (1-\alpha_n)g)$. In turn, by the Continuity and Reciprocity axioms, $1 - \limsup_{n\to\infty} \mu(h, \alpha_n f + (1-\alpha_n)g) \geq 1 - \mu(h, \alpha f + (1-\alpha)g) = \mu(\alpha f + (1-\alpha)g, h)$, which completes the proof that $\lim_{n\to\infty} \mu(\alpha_n f + (1-\alpha_n)g, h) = \mu(\alpha f + (1-\alpha)g, h)$. By Reciprocity, we also have $\lim_{n\to\infty} \mu(h, \alpha_n f + (1-\alpha_n)g) = \mu(h, \alpha f + (1-\alpha)g)$. □

Lemma 9. *Suppose that μ is a graded preference relation on $\mathcal{F}$ that satisfies the Reflexivity, Weak Transitivity, Reciprocity, Monotonicity, Independence, C-Completeness, Continuity, and Nondegeneracy axioms. Then, there exists a nonconstant affine function $u : X \to \mathbb{R}$ and a nonempty, convex, and closed set $\mathcal{M} \subseteq \Delta(\Omega)$ such that*

(i) for all $f, g \in \mathcal{F}$, $\mu(f,g) = 1 \Leftrightarrow \forall_{p\in\mathcal{M}} L_{f,g}p \geq 0$;

(ii) the closed and convex set $\mathcal{M}$ satisfying Part (i) is unique, and u is unique up to a positive affine transformation;

(iii) if $(u\circ f) - (u\circ g) = \lambda((u\circ f') - (u\circ g'))$ for some $\lambda > 0$ and $f, g, f', g' \in \mathcal{F}$, then $\mu(f,g) = \mu(f',g')$;

Proof. Step 1. An auxiliary crisp binary relation. Let $\gtrsim$ be a crisp binary relation on $\mathcal{F}$ defined as $f \gtrsim g \Leftrightarrow \mu(f,g) = 1$.

As immediately follows from Reflexivity and Weak Transitivity, $\gtrsim$ is a preorder. It also has the monotonicity, independence, and nontriviality properties (as defined in Subsection 2.4) due to the Monotonicity, Independence, and Nontriviality axioms. The Continuity axiom implies that, for any $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \mu(\alpha f + (1-\alpha)g, h) = 1\}$ and $\{\alpha \in [0,1] : \mu(h, \alpha f + (1-\alpha)g) = 1\}$ are closed, which, in turn, means that the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \gtrsim h\}$ and $\{\alpha \in [0,1] : h \gtrsim \alpha f + (1-\alpha)g\}$ are closed as well.

Step 2. The von Neumann-Morgenstern utility function u. C-Completeness implies that the restriction of $\gtrsim$ to X is a complete preorder. Therefore, by the Mixture Space Theorem (Herstein and Milnor, 1953), there exists an affine function $u : X \to \mathbb{R}$ such that $x \gtrsim y$ if and only if $u(x) \geq u(y)$. Moreover, Nondegeneracy implies that u is nonconstant.

Step 3. The set $\mathcal{M}$ of priors. Define a binary relation $\hat{\gtrsim}$ on $B_0(\Omega, \Sigma, u(X))$ as $(u\circ f)\hat{\gtrsim}(u\circ g) \Leftrightarrow f \gtrsim g$ for all $f, g \in \mathcal{F}$. This binary relation is well defined, i.e., if $f', g' \in \mathcal{F}$ are such that $u\circ f = u\circ f'$ and $u\circ g = u\circ g'$, then $f \gtrsim g \Leftrightarrow f' \gtrsim g'$

by the Monotonicity axiom: Indeed, if $u \circ f = u \circ f'$ and $u \circ g = u \circ g'$, then $\mu(f(\omega), f'(\omega)) = 1 = \mu(f'(\omega), f(\omega))$ and $\mu(g(\omega), g'(\omega)) = 1 = \mu(g'(\omega), g(\omega))$ for all $\omega \in \Omega$, and, therefore, $f \sim f'$ and $g \sim g'$. Given the properties of $\gtrsim$, it is easy to check that $\hat{\gtrsim}$ is a preorder that satisfies the nondegeneracy, archimedean continuity, monotonicity, and independence conditions of Gilboa et al. (2010, Appendix B). By their Corollary 1, there exists a nonempty, closed, and convex set $\mathcal{M} \subseteq \Delta(\Omega)$ such that

$$f \gtrsim g \quad \Leftrightarrow \quad \langle u \circ f, p \rangle \geq \langle u \circ g, p \rangle \text{ for all } p \in \mathcal{M};$$

moreover,

$$\mathcal{M} = \left\{ p \in \Delta(\Omega) : L_{f,g} p \geq 0 \text{ for all } f, g \in \mathcal{F} \text{ such that } f \gtrsim g \right\}.$$

Claim (i) is now proven.

Claim (ii). Suppose that u' is a nonconstant affine function $X \to \mathbb{R}$ and $\mathcal{M}'$ is a closed and convex subset of $\Delta(\Omega)$ such that Claim (i) holds. By Nondegeneracy, $\mathcal{M}' \neq \varnothing$. Then, for any $x, y \in X$, $\mu(x, y) = 1 \Leftrightarrow u'(x) - u'(y) \geq 0$. Therefore, as follows from Herstein and Milnor (1953, Theorem 7), u' is a positive affine transformation of u. Finally, $\mathcal{M} = \mathcal{M}'$ by the uniqueness part of the same Corollary 1 of Gilboa et al. (2010).

Claim (iii). Suppose, first, that $f, f' \in \mathcal{F}$ are such that $u \circ f = u \circ f'$, and fix an arbitrary $g \in \mathcal{F}$. Our objective is to prove that $\mu(f, g) = \mu(f', g)$ and $\mu(g, f) = \mu(g, f')$. By the construction of u, we have $\mu(f(\omega), f'(\omega)) = \mu(f'(\omega), f(\omega)) = 1$ for all $\omega \in \Omega$. By Monotonicity, this implies that $\mu(f, f') = \mu(f', f) = 1$. In turn, by Weak Transitivity, we have $\mu(f, g) \geq \mu(f', g) \geq \mu(f, g)$, which means that $\mu(f, g) = \mu(f', g)$. If $\mu(f, g) = \mu(f', g) < 1$, then the equality $\mu(g, f) = \mu(g, f')$ immediately follows from Reciprocity. Otherwise, Reciprocity rules out the case in which $\mu(g, f) \in (0, 1)$ or $\mu(g, f') \in (0, 1)$ — that is, $\mu(g, f)$ and $\mu(g, f')$ can only be zero or one. Since Weak Transitivity implies $\mu(g, f) = 1 \Leftrightarrow \mu(g, f') = 1$, we have $\mu(g, f) = \mu(g, f')$.

Now, consider the general case: Suppose that $f, g, f', g' \in \mathcal{F}$ and $\lambda > 0$ are such that $(u \circ f) - (u \circ g) = \lambda((u \circ f') - (u \circ g'))$. Let $k \in int\, u(X)$ be chosen arbitrary, and let $x \in X$ be such that $u(x) = k$ and $\varphi := (u \circ f) - (u \circ g)$. Then, one can find a sufficiently small $\varepsilon > 0$ such that $\lambda\varepsilon < 1$, and such that $\psi := \frac{1}{1-\varepsilon} u(x) - \frac{\varepsilon}{1-\varepsilon}(u \circ g)$ and $\psi' := \frac{1}{1-\lambda\varepsilon} u(x) - \frac{\lambda\varepsilon}{1-\lambda\varepsilon}(u \circ g')$ satisfy $\psi, \psi' \in u(X)^{\Omega}$. Let $h, h' \in \mathcal{F}$ be such that

$\psi = u \circ h$ and $\psi' = u \circ h'$. We observe that

$$\begin{aligned} u \circ \big((1-\varepsilon)h + \varepsilon f\big) &= k + \varepsilon\varphi, & u \circ \big((1-\varepsilon)h + \varepsilon g\big) &= k, \\ u \circ \big((1-\lambda\varepsilon)h' + \lambda\varepsilon f'\big) &= k + \varepsilon\varphi, & u \circ \big((1-\lambda\varepsilon)h' + \lambda\varepsilon g'\big) &= k. \end{aligned}$$

Therefore, as follows from the claim proven in the preceding paragraph, $\mu((1-\varepsilon)h + \varepsilon f, (1-\varepsilon)h + \varepsilon g) = \mu((1-\lambda\varepsilon)h' + \lambda\varepsilon f', (1-\lambda\varepsilon)h' + \lambda\varepsilon g')$; at the same time, $\mu((1-\varepsilon)h+\varepsilon f, (1-\varepsilon)h+\varepsilon g) = \mu(f,g)$ and $\mu((1-\lambda\varepsilon)h' + \lambda\varepsilon f', (1-\lambda\varepsilon)h' + \lambda\varepsilon g') = \mu(f', g')$ by Independence, and we can conclude that $\mu(f,g) = \mu(f',g')$. □

Lemma 10. *Suppose that μ is a graded preference relation on $\mathcal{F}$ that satisfies the Reflexivity, Weak Transitivity, Reciprocity, Monotonicity, Independence, C-Completeness, Continuity, and Nondegeneracy axioms. Furthermore, suppose that the function $u : X \to \mathbb{R}$ and a set $\mathcal{M} \subseteq \Delta(\Omega)$ are as described in Lemma 9. Then, for any $f, g, f', g' \in \mathcal{F}$, $\{p \in \mathcal{M} : L_{f,g}p \geq 0\} \subseteq \{p \in \mathcal{M} : L_{f',g'}p \geq 0\}$ implies that $\mu(f,g) \leq \mu(f',g')$.*

Proof. Step 1. Suppose, first, that $f, g, f', g' \in \mathcal{F}$ are such that $\{p \in \mathcal{M} : L_{f,g}p \geq 0\} \cap \{p \in \mathcal{M} : L_{f',g'}p \leq 0\} = \varnothing$. Fix an arbitrary $k \in \mathit{int}\, u(X)$, and find a sufficiently small $\varepsilon > 0$ such that, for

$$M := \max_{\omega\in\Omega} \max\{|u(f(\omega))|, |u(g(\omega))|, |u(f'(\omega))|, |u(g'(\omega))|\},$$

we have $[k - 2\varepsilon M, k + 2\varepsilon M] \subseteq u(X)$. Let $\varphi, \psi \in B(\Omega, \Sigma, \mathbb{R})$ be defined as $\varphi := (u \circ f) - (u \circ g)$ and $\psi := (u \circ f') - (u \circ g')$.

Step 2. We claim that there exists $\alpha \in (0,1)$ such that $\langle (1-\alpha)\varphi - \alpha\psi, p\rangle \leq 0$ for all $p \in \mathcal{M}$. Indeed, suppose, by contradiction, that, for each $\alpha \in (0,1)$, there exists $p \in \mathcal{M}$ such that $\langle (1-\alpha)\varphi - \alpha\psi, p\rangle > 0$. Let $C_1 := \{\xi \in B_0(\Omega, \Sigma, \mathbb{R}) : \langle \xi, p\rangle \leq 0 \text{ for all } p \in \mathcal{M}\}$ and C_2 be the convex hull of 0, φ, and $-\psi$. Note that C_1 and C_2 are closed and convex sets; moreover, C_1 has a nonempty interior: The ball centered at the constant -1 of radius $\frac{1}{2}$ is contained in C_1 entirely. As follows from our assumption, $C_1 \cap \{(1-t)\varphi - t\psi \mid t \in (0,1)\} = \varnothing$, and, therefore, $\mathit{int}\, C_1 \cap C_2 = \varnothing$. Then, by the Interior Separating Hyperplane theorem (Aliprantis and Border, 2006, Theorem 5.67), there exists a nonzero continuous linear functional L^0 on $B_0(\Omega, \Sigma, \mathbb{R})$ such that $L^0\xi \leq 0$ for all $\xi \in C_1$ and $L^0\xi \geq 0$ for all $\xi \in C_2$. This L^0 can be represented as $L^0\xi = \langle \xi, p\rangle$ for all $\xi \in B_0(\Omega, \Sigma, \mathbb{R})$, where p is some nonzero, bounded, and finitely additive function $\Sigma \to \mathbb{R}$ (Aliprantis and Border, 2006, Lemma 14.31). Notice that $p(E) \geq 0$ for all $E \in \Sigma$ because $-\mathbb{1}_E \in C_1$; therefore, we can assume without loss of generality that $p \in \Delta(\Omega)$. Then, in fact, $p \in \mathcal{M}$, as follows from Step 3 in the proof of Lemma 9. This means that we have found $p \in \mathcal{M}$ such that $\langle \varphi, p \rangle \geq 0$ and $\langle \psi, p \rangle \leq 0$, a contradiction to the assumption made in Step 1.

Step 3. Let $f^*, g^*, h^* \in \mathcal{F}$ be such that $u \circ f^* \equiv k$, $u \circ g^* = k + \varepsilon(1-\alpha)\varphi - \varepsilon\alpha\psi$, $u \circ h^* = k - \varepsilon\alpha\psi$, and note that such acts exist by the choice of ε. By the result of the Step 2, we have $L_{f^*,g^*}p \geq 0$ for all $p \in \mathcal{M}$, and, therefore, $\mu(f^*, g^*) = 1$. By Weak Transitivity, this implies that $\mu(f^*, h^*) \geq \mu(g^*, h^*)$. Now, observe that $(u \circ f^*) - (u \circ h^*) = \varepsilon\alpha\psi = \varepsilon\alpha[(u \circ f') - (u \circ g')]$ and $(u \circ g^*) - (u \circ h^*) = \varepsilon(1-\alpha)\varphi = \varepsilon(1-\alpha)[(u \circ f) - (u \circ g)]$. By Part (iii) of Lemma 9, we have $\mu(f, g) = \mu(g^*, h^*)$ and $\mu(f^*, h^*) = \mu(f', g')$, which proves (under the assumption of Step 1) that $\mu(f, g) \leq \mu(f', g')$.

Step 4. Now, let f, g, f', g' be arbitrary acts such that $\{p \in \mathcal{M} : L_{f,g}p \geq 0\} \subseteq \{p \in \mathcal{M} : L_{f',g'}p \geq 0\}$. Fix an arbitrary $k \in int\, u(X)$, let $x_0, x_1 \in X$ be such that $u(x_0) = k$ and $u(x_1) > u(x_0)$, and $f'', g'' \in \mathcal{F}$ be defined as $f'' := \frac{1}{2}x_0 + \frac{1}{2}f'$ and $g'' := \frac{1}{2}x_0 + \frac{1}{2}g'$, and observe that $(u \circ f'') - (u \circ g'') = \frac{1}{2}[(u \circ f') - (u \circ g')$, and, therefore, $\{p \in \mathcal{M} : L_{f'',g''}p \geq 0\} = \{p \in \mathcal{M} : L_{f',g'}p \geq 0\}$. Finally, let $f''_t := \frac{1}{2}(1-t)x_0 + \frac{1}{2}tx_1 + \frac{1}{2}f'$ for all $t \in [0,1]$, and note that $f''_0 = f''$.

We claim that $\{p \in \mathcal{M} : L_{f'',g''}p \geq 0\} \subseteq \{p \in \mathcal{M} : L_{f''_t,g''}p \geq 0\}$ and $\{p \in \mathcal{M} : L_{f,g}p \geq 0\} \cap \{p \in \mathcal{M} : L_{f''_t,g''}p \leq 0\} = \varnothing$ for all $t \in (0,1]$. Indeed, for all $p \in \Delta(\Omega)$, we have $L_{f''_t,g''}p = L_{f'',g''}p + \frac{1}{2}t[u(x_1) - u(x_0)]$, and, therefore, $L_{f'',g''}p \geq 0$ implies $L_{f''_t,g''}p > 0$.

Given the above no-intersection property, we have $\mu(f, g) \leq \mu(f''_t, g'')$ for all $t \in (0,1]$, as proven earlier. By Continuity, this implies $\mu(f, g) \leq \mu(f'', g'')$. Since $\mu(f'', g'') = \mu(f', g')$ by Part (iii) of Lemma 9, the proof of the lemma is now complete. □

Proof of Theorem 1. *Only if* part. Assume that μ is a graded preference relation on $\mathcal{F}$ that satisfies the Reflexivity, Weak Transitivity, Monotonicity, C-Completeness, Independence, Reciprocity, Continuity, and Nondegeneracy axioms.

Step 1. Let function $u : X \to \mathbb{R}$ and set $\mathcal{M} \subseteq \Delta(\Omega)$ be as described in Lemma 9. Let $K := u(X)$, $\mathfrak{H} := \{\{p \in \mathcal{M} : L_{f,g}p \geq 0\} \mid f, g \in \mathcal{F}\}$, and let $\pi : \mathfrak{H} \to [0,1]$ be defined as $\pi(\{p \in \mathcal{M} : L_{f,g}p \geq 0\}) := \mu(f, g)$. By Lemma 10, such π is well defined:

If $f, g, f', g' \in \mathcal{F}$ are such that $\{p \in \mathcal{M} : L_{f,g}p \geq 0\} = \{p \in \mathcal{M} : L_{f',g'}p \geq 0\}$, then it must be that $\mu(f,g) = \mu(f',g')$. By the same lemma, π is monotone, and it is normalized by construction.

Step 2. By the Reciprocity axiom, we have $\pi(\{p \in \mathcal{M} : L_{f,g}p \geq 0\}) + \pi(\{p \in \mathcal{M} : L_{g,f}p \geq 0\}) = 1$ for all $f, g \in \mathcal{F}$ such that $\mu(f,g) < 1$ or $\mu(g,f) < 1$. By Part (i) of Lemma 9, such $f, g \in \mathcal{F}$ are exactly those for which $L_{f,g}p \neq 0$ for at least one $p \in \mathcal{M}$: Indeed, $L_{f,g}p \geq 0$ for all $p \in \mathcal{M}$ is equivalent to $\mu(f,g) = 1$, and $L_{f,g}p \leq 0$ for all $p \in \mathcal{M}$ is equivalent to $\mu(g,f) = 1$ because $L_{g,f} = -L_{f,g}$. Therefore, Part (ii) of the theorem is proven.

Step 3. To prove the linear continuity of π, consider an arbitrary nonconstant $L \in \mathcal{L}$, and suppose that $Lp = \langle \varphi, p \rangle$, where $\varphi \in B_0(\Omega, \Sigma, \mathbb{R})$.

Let $k \in int\, K$ be chosen arbitrary, and let $\varepsilon > 0$ be a sufficiently small number such that $[k - 3\varepsilon\|\varphi\|, k + 3\varepsilon\|\varphi\|] \subset K$. Let $\xi := k + \varepsilon\varphi$, and note that $Lp = \frac{1}{\varepsilon}\langle \xi - k, p\rangle$ for all $p \in \mathcal{M}$. Our objective is to prove that the function $F : \mathbb{R} \to \mathbb{R}$ defined as $F(\alpha) := \pi(\{p \in \mathcal{M} : \langle \xi - k, p \rangle \geq \alpha\})$ is continuous.

First, let $\underline{\alpha} := \sup\{\alpha \in \mathbb{R} : \forall_{p \in \mathcal{M}} \langle \xi - k, p \rangle \geq \alpha\}$, $\overline{\alpha} := \inf\{\alpha \in \mathbb{R} : \forall_{p \in \mathcal{M}} \langle \xi - k, p \rangle < \alpha\}$, and observe that F is monotone, $F(\alpha) = 1$ for all $\alpha < \underline{\alpha}$, and $F(\alpha) = 0$ for all $\alpha > \overline{\alpha}$.

Second, we note that $\underline{\alpha} > -2\varepsilon\|\varphi\|$ and $\overline{\alpha} < 2\varepsilon\|\varphi\|$, and, therefore, for any $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, we have $\xi - \alpha \in K^\Omega$ by the choice of ε. Hence, there exist $x \in X$ and $f, g \in \mathcal{F}$ such that $k = u(x)$, $\xi - \underline{\alpha} = u \circ f$, and $\xi - \overline{\alpha} = u \circ g$. Let $f_\alpha := \frac{\overline{\alpha} - \alpha}{\overline{\alpha} - \underline{\alpha}} f + \frac{\alpha - \underline{\alpha}}{\overline{\alpha} - \underline{\alpha}} g$, and note that the mapping $\alpha \mapsto f_\alpha$ is continuous on $[\underline{\alpha}, \overline{\alpha}]$. It is easy to verify that $u \circ f_\alpha = \xi - \alpha$, and, thus, $F(\alpha) = \pi(\{p \in \mathcal{M} : \langle \xi, p \rangle - \alpha - k \geq 0\}) = \mu(f_\alpha, x)$ for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.

Finally, we observe that there is no $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ such that $\mu(f_\alpha, x) = 1 = \mu(x, f_\alpha)$: Otherwise, as follows from Claim (i) of Lemma 9, we would have $\langle \varepsilon\varphi - \alpha, p \rangle = 0$ for all $p \in \mathcal{M}$, which contradicts to L being nonconstant. Therefore, $\mu(f_\alpha, x)$ is continuous in α by Lemma 8, and the continuity of F is proven.

Step 4. To prove that π has linear full support, suppose that $\varphi \in B_0(\Omega, \Sigma, \mathbb{R})$ is such that $\langle \varphi, q \rangle < 0$ for some $q \in \mathcal{M}$. Let $k \in int\, K$ be arbitrary, and $\varepsilon > 0$ be a sufficiently small number such that $k + \varepsilon\varphi \in K^\Omega$. Finally, let $x \in X$ and $f \in \mathcal{F}$ be such that $u(x) = k$ and $u \circ f = k + \varepsilon\varphi$. By Claim (i) of Lemma 9, $L_{f,x}q < 0$ implies that $1 > \mu(f,x) = \pi(\{p \in \mathcal{M} : L_{f,x} \geq 0\}) = \pi(\{p \in \mathcal{M} : \langle \varphi, p \rangle \geq 0\})$.

Step 5. Now, we extend π from $\mathfrak{H}$ to $\mathscr{B}(\mathcal{M})$. For any $\mathcal{S} \in \mathscr{B}(\mathcal{M})$, let

$\hat{\pi}(\mathcal{S}) := \sup\{\pi(\mathcal{S}') \mid \mathcal{S}' \in \mathfrak{H}, \mathcal{S}' \subseteq \mathcal{S}\}$. As follows from the monotonicity of π, $\hat{\pi}$ agrees with π on the intersection of their domains: $\hat{\pi}(S) = \pi(S)$ for all $\mathcal{S} \in \mathfrak{H}$. Therefore, Representation (2) holds for $\hat{\pi}$, as well. Monotonicity and normalization of $\hat{\pi}$ follow from the monotonicity of π. Finally, linear continuity and linear full support of $\hat{\pi}$ hold immediately because these properties operate with the values of $\hat{\pi}$ on sets that always belong to $\mathfrak{H}$.

If part. Assume that there exist a nonconstant affine function $u : X \to \mathbb{R}$, a nonempty, convex, and closed set $\mathcal{M} \subseteq \Delta(\Omega)$, and a capacity $\pi : \mathscr{B}(\mathcal{M}) \to [0,1]$ such that Representation (2) holds. As directly follows from (2), μ satisfies Reflexivity, Monotonicity, C-Completeness, and Independence. Nondegeneracy of μ follows from u being nonconstant, and Reciprocity follows from Condition (ii) of the theorem.

Next, we prove Weak Transitivity. Suppose $f, g \in \mathcal{F}$ are such that $\mu(f,g) = 1$, and let $h \in \mathcal{F}$ be arbitrary. Since π has full support, it follows that $\langle u \circ f, p\rangle \geq \langle u \circ g, p\rangle$ for all $p \in \mathcal{M}$. Consequently, $\{p \in \mathcal{M} : L_{f,h}p \geq 0\} \supseteq \{p \in \mathcal{M} : L_{g,h}p \geq 0\}$, and, by the monotonicity of π, we have $\mu(f,h) \geq \mu(g,h)$.

Finally, we prove Continuity. Fix arbitrary $f, g, h \in \mathcal{F}$, $\alpha \in [0,1]$, and consider a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of numbers in $[0,1]$ such that $\alpha_n \to \alpha$. Our goal is to prove that $\limsup_{n\to\infty} \mu(\alpha_n f + (1-\alpha_n)g, h) \leq \mu(\alpha f + (1-\alpha)g, h)$. Let $M := \sup_{\omega\in\Omega}|u(f(\omega)) - u(g(\omega))|$, and notice that $M < \infty$ because f and g take only finitely many values. Then, $\left\|\big(u \circ (\alpha_n f + (1-\alpha_n)g)\big) - \big(u \circ (\alpha f + (1-\alpha)g)\big)\right\| = \|(\alpha_n - \alpha)(u \circ f) - (\alpha_n - \alpha)(u \circ g)\| \leq |\alpha_n - \alpha| \cdot M$. Therefore, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $L_{\alpha_n f + (1-\alpha_n)g,h}p \leq L_{\alpha f + (1-\alpha)g,h}p + \varepsilon$ for all $p \in \Delta(\Omega)$. Hence, $\{p \in \mathcal{M} : L_{\alpha_n f + (1-\alpha_n)g,h}p \geq 0\} \subseteq \{p \in \mathcal{M} : L_{\alpha f + (1-\alpha)g,h}p \geq -\varepsilon\}$ for all $n > N$, which implies that $\pi(\{p \in \mathcal{M} : L_{\alpha_n f + (1-\alpha_n)g,h}p \geq 0\}) \leq \pi(\{p \in \mathcal{M} : L_{\alpha f + (1-\alpha)g,h}p \geq -\varepsilon\})$ for all $n > N$ by the monotonicity of π. Therefore, $\limsup_{n\to\infty} \mu(\alpha_n f + (1-\alpha_n)g, h) \leq \pi(\{p \in \mathcal{M} : L_{\alpha f + (1-\alpha)g,h}p \geq -\varepsilon\})$. As follows from the linear continuity of π, $\lim_{\varepsilon\to 0+} \pi(\{p \in \mathcal{M} : L_{\alpha f + (1-\alpha)g,h}p \geq -\varepsilon\}) = \pi(\{p \in \mathcal{M} : L_{\alpha f + (1-\alpha)g,h}p \geq 0\})$ regardless of whether $L_{\alpha f + (1-\alpha)g,h}$ is constant on $\mathcal{M}$ or not, and the claimed inequality is now proven. The symmetric inequality, $\limsup_{n\to\infty} \mu(h, \alpha_n f + (1-\alpha_n)g) \leq \mu(h, \alpha f + (1-\alpha)g)$, can be proven similarly. ☐

B. Proofs of the Remaining Results

Proof of Proposition 2. We already established the uniqueness of u and $\mathcal{M}$ in Claim (ii) of Lemma 9. Given that, the value of π on any set of the form $\{p \in \mathcal{M} : \int_\Omega (u \circ f)\, dp \geq \int_\Omega (u \circ g)\, dp\}$, where $f, g \in \mathcal{F}$, is pinned down uniquely by the value of $\mu(f, g)$. □

Proof of Proposition 3. Let $f, g \in \mathcal{F}$ be arbitrary, and let S and S_0 be the sets defined as in the claim of the proposition.

Suppose, first, that $L_{f,g}$ is nonconstant on $\mathcal{M}$. For any $\varepsilon > 0$, we have $\pi(\{p \in \mathcal{M} : L_{f,g}p \geq \varepsilon\}) \leq \pi(S_0) \leq \pi(S)$ by the monotonicity of π. Taking the limit as $\varepsilon \to 0$ and using the linear continuity of π, we obtain $\pi(S) = \pi(S_0)$.

Second, suppose that $L_{f,g}p = c$ for all $p \in \mathcal{M}$, and $c \in \mathbb{R}\backslash\{0\}$. In this case, it is immediate that $S = S_0$, and $\pi(S) = \pi(S_0)$.

Finally, suppose that $L_{f,g}p = 0$ for all $p \in \mathcal{M}$. Then, $S = \mathcal{M}$ and $S_0 = \varnothing$, and, in turn, $\pi(S) = 1$ and $\pi(S_0) = 0$. □

Proof of Proposition 4. This proposition is proven in Step 1 of Lemma 9. □

Proof of Proposition 5. This proposition is proven by Claim (i) of Lemma 9. □

Proof of Proposition 6. Let $\gtrsim$ be a binary relation on $\mathcal{F}$ defined as $f \gtrsim g \Leftrightarrow \mu(f, g) = 1$. By Proposition 4, $\gtrsim$ satisfies the Bewley axioms. Since μ satisfies the Completeness axiom, $\gtrsim$ is a complete preference relation and, therefore, admits an expected utility representation $f \mapsto \int_\Omega (u \circ f)\, dq$ for some nonconstant affine function $u : X \to \mathbb{R}$ and some $q \in \Delta(\Omega)$ (see, e.g., Fishburn, 1970, Theorem 13.3). By Reciprocity, $\mu(f, g) < 1$ implies $\mu(f, g) = 0$ for any $f, g \in \mathcal{F}$. Therefore, Representation (4) holds. Moreover, by the uniqueness part of the Anscombe-Aumann Theorem, it follows that u is unique up to positive affine transformations and q is unique.

Conversely, given Representation (4), the Completeness axiom follows immediately. □

Proof of Theorem 7. Let $\gtrsim_i$, where $i = 1, 2$, be preorders on $\mathcal{F}$ defined as $f \gtrsim_i g \Leftrightarrow \mu_i(f, g) = 1$. Since μ_1 is more decisive than μ_2, we have $\gtrsim_2 \subseteq \gtrsim_1$. By Proposition 5, preorders $\gtrsim_1$ and $\gtrsim_2$ admit Bewley-type representation (3). Then, Claims (i) and (ii) follow from the proof of Proposition 6 of Ghirardato et al. (2004).

To prove Claim (iii), consider $B \in \mathfrak{H}_2$ such that $\pi_2(B) \geq \frac{1}{2}$. By the definition of $\mathfrak{H}_2$, there exist $f, g \in \mathcal{F}$ such that $B = \{p \in \mathcal{M}_2 : \int (u_2 \circ f)\, dp \geq \int (u_2 \circ g)\, dp\}$. We observe that $\mu_2(f,g) \leq \mu_1(f,g) = \pi_1\left(\{p \in \mathcal{M}_1 : \int (u_1 \circ f)\, dp \geq \int (u_1 \circ g)\, dp\}\right)$. The latter set is equal to $\{p \in \mathcal{M}_1 : \int (u_2 \circ f)\, dp \geq \int (u_2 \circ g)\, dp\}$ by Claim (i), and, in turn, to $\mathcal{M}_1 \cap B$ by Claim (ii). Therefore, we have $\pi_2(B) \leq \pi_1(\mathcal{M}_1 \bigcap B)$.

Conversely, suppose that Conditions (i), (ii), and (iii) hold. Fix arbitrary $f, g \in \mathcal{F}$ such that $\mu_2(f,g) \geq \frac{1}{2}$, and observe that $\pi_2\left(\{p \in \mathcal{M}_2 : \int (u_2 \circ f)\, dp \geq \int (u_2 \circ g)\, dp\}\right) \leq \pi_1\left(\{p \in \mathcal{M}_1 : \int (u_2 \circ f)\, dp \geq \int (u_2 \circ g)\, dp\}\right)$ by Conditions (ii) and (iii). The latter expression is, in turn, equal to $\pi_1\left(\{p \in \mathcal{M}_1 : \int (u_1 \circ f)\, dp \geq \int (u_1 \circ g)\, dp\}\right)$ by Condition (i). Therefore, we have $\mu_2(f,g) \leq \mu_1(f,g)$. □

References

M. Agranov, P. Ortoleva, Stochastic choice and hedging, 2013. Mimeo, California Institute of Technology.

C.D. Aliprantis, K.C. Border, Infinite Dimensional Analysis: A Hitchhiker's Guide, 3rd ed., Springer-Verlag, 2006.

N.H. Anderson, R.E. Whaler, Likelihood judgments and sequential effects in a two-choice probability learning situation, Journal of Experimental Psychology 2 (1960) 111–120.

F.J. Anscombe, R.J. Aumann, A definition of subjective probability, The Annals of Mathematical Statistics 34 (1963) 199–205.

R.J. Aumann, Utility theory without the completeness axiom, Econometrica 30 (1962) 445–462.

K. Basu, Fuzzy revealed preference theory, Journal of Economic Theory 32 (1984) 212–227.

K. Basu, Axioms for a fuzzy measure of inequality, Mathematical Social Sciences 14 (1987) 275–288.

T. Bewley, Knightian uncertainty theory: Part I, Cowles Foundation Discussion Paper No. 807 (1986).

D. Cartwright, The relation of the decision time to the categories of response, American Journal of Psychology 54 (1941) 174–196.

E. Danan, Revealed preference and indifferent selection, Mathematical Social Sciences 55 (2008) 24–37.

E. Danan, A. Ziegelmeyer, Are preferences complete? an experimental measurement of indecisiveness under risk, Working Paper Max Planck Institute of Economics (2006).

E. Dekel, B.L. Lipman, How (not) to do decision theory, Annual Review of Economics 2 (2010) 257–282.

J. Dubra, F. Maccheroni, E.A. Ok, Expected utility theory without the completeness axiom, Journal of Economic Theory 115 (2004) 118–133.

J.S. Dyer, R.K. Sarin, Measurable multiattribute value functions, Operations Research 27 (1979) 810–822.

K. Eliaz, E.A. Ok, Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences, Games and Economic Behavior 56 (2006) 61–86.

L. Epstein, A definition of uncertainty aversion, Review of Economic Studies 66 (1999) 579–608.

J.H. Faro, Variational Bewley Preferences, Ibmec Working Papers 258, Insper Working Paper, Insper Instituto de Ensino e Pesquisa, 2011.

P.C. Fishburn, Utility Theory for Decision Making, Wiley, New York, 1970.

P.C. Fishburn, Binary choice probabilities: on the varieties of stochastic transitivity, Journal of Mathematical Psychology 10 (1973) 327–352.

B.S. Frey, A. Stutzer, What can economists learn from happiness research?, Journal of Economic Literature 40 (2002) 402–435.

T. Galaabaatar, E. Karni, Subjective expected utility with incomplete preferences, Econometrica 81 (2013) 255–284.

P. Ghirardato, F. Maccheroni, M. Marinacci, Differentiating ambiguity and ambiguity attitude, Journal of Economic Theory 118 (2004) 133–173.

P. Ghirardato, M. Marinacci, Ambiguity made precise: A comparative foundation, Journal of Economic Theory 102 (2002) 251–289.

I. Gilboa, F. Maccheroni, M. Marinacci, D. Schmeidler, Objective and Subjective Rationality in a Multiple Prior Model, Carlo Alberto Notebooks 73, Collegio Carlo Alberto, 2008.

I. Gilboa, F. Maccheroni, M. Marinacci, D. Schmeidler, Objective and subjective rationality in a multiple prior model, Econometrica 78 (2010) 755–770.

I. Gilboa, D. Schmeidler, Maxmin expected utility with non-unique prior, Journal of Mathematical Economics 18 (1989) 141–153.

F. Gul, W. Pesendorfer, Random expected utility, Econometrica 74 (2006) 121–146.

J.R. Hauser, S.M. Shugan, Intensity measures of consumer preference, Operations Research 28 (1980) 278–320.

I. Herstein, J. Milnor, An axiomatic approach to measurable utility, Econometrica 21 (1953) 291–297.

B. Hill, Incomplete preferences and confidence, 2011. Mimeo, GREGHEC, HEC Paris.

A.S. Kochov, Subjective states without the completeness axiom, 2007. URL: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.164.647, mimeo, University of Rochester.

D.H. Krantz, R.D. Luce, P. Suppes, A. Tversky, Foundations of Measurement Volume I: Additive and Polynomial Representations, Academic Press, New York, 1971.

E. Lehrer, R. Teper, Justifiable preferences, Journal of Economic Theory 146 (2011) 762–774.

R.D. Luce, Response Times: Their Role in Inferring Elementary Mental Organization, Oxford University Press, Oxford, New York, 1986.

M. Mandler, Incomplete preferences and rational intransitivity of choice, Games and Economic Behavior 50 (2005) 255–277.

D.L. McFadden, Revealed stochastic preference: a synthesis, Economic Theory 26 (2005) 245–264.

D.L. McFadden, M.K. Richter, Stochastic rationality and revealed stochastic preference, in: J. Chipman, D. McFadden, K. Richter (Eds.), Preferences, Uncertainty, and Optimality, Boulder, CO: Westview Press, 1990, pp. 161–186.

S. Minardi, A. Savochkin, A confidence-based decision rule and ambiguity attitudes, 2013a. Working paper, http://sites.carloalberto.org/savochkin/ConfidenceDecisionRule.pdf.

S. Minardi, A. Savochkin, A note on preferences with grades of indecisiveness without reciprocity, 2013b. Working paper, http://sites.carloalberto.org/savochkin/GradedPreferencesNR.pdf.

R. Nau, The shape of incomplete preferences, The Annals of Statistics 34 (2006) 2430–2448.

R.F. Nau, Indeterminate probabilities on finite sets, The Annals of Statistics 20 (1992) 1737–1767.

N. Novemsky, R. Dhar, N. Schwarz, I. Simonson, Preference fluency in choice, Journal of Marketing Research 44 (2007) 347–356.

E.A. Ok, On the approximation of fuzzy preferences by exact relations, Fuzzy Sets and Systems 67 (1994) 173–179.

E.A. Ok, P. Ortoleva, G. Riella, Incomplete preferences under uncertainty: Indecisiveness in beliefs versus tastes, Econometrica 80 (2012) 1791–1808.

S.A. Orlovsky, Decision making with a fuzzy preference relation, Fuzzy Sets and Systems 1 (1978) 155–167.

S.V. Ovchinnikov, Structure of fuzzy binary relations, Fuzzy Sets and Systems 6 (1981) 169–195.

A. Rubinstein, Instinctive and cognitive reasoning: A study of response times, Economic Journal 117 (2007) 1243–1259.

A. Rustichini, Neuroeconomics: Formal models of decision making and cognitive neuroscience, in: P.W. Glimcher, C.F. Camerer, E. Fehr, R.A. Poldrack (Eds.), Neuroeconomics, Decision Making and the Brain, Elsevier, 2008, pp. 33–46.

D. Schmeidler, Subjective probability and expected utility without additivity, Econometrica 57 (1989) 571–587.

T. Seidenfeld, M.J. Schervish, J.B. Kadane, A representation of partially ordered preferences, The Annals of Statistics 23 (1995) 2168–2217.

M.M. Suydam, J.L. Myers, Some parameters of risk-taking behavior, Psychological Reports 10 (1962) 559–562.

L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

D. Zakay, The relationship between preference strength and relative frequency of choice, Applied Psychological Measurement 8 (1984) 207–211.