Mean-variance target-based optimisation in DC plan with stochastic interest rate

Francesco Menoncin
Elena Vigna

No. 337
December 2013
Mean-variance target-based optimisation in DC plan with stochastic interest rate

Francesco Menoncin *  Elena Vigna†

December 2013

Abstract

We solve a mean-variance optimisation problem of a defined contribution pension scheme in the accumulation phase. The financial market consists of: (i) the risk-free asset, (ii) a risky asset following a GBM, and (iii) a bond driven by a stochastic interest rate following the Vasicek [1977] dynamics. We find a closed-form solution for both the optimal investment strategy and the portfolio efficient frontier. We show that the mean-variance approach is equivalent to a "user-friendly" target-based approach optimisation problem which minimises a quadratic loss function. We show that the ruin probability can be kept under control through the choice of the target level. Numerical applications show that the proportions of bond and risky asset decrease when retirement approaches.

Keywords. Mean-variance approach, efficient frontier, stochastic interest rate, defined contribution pension scheme, portfolio selection, risk aversion, ruin probability.

JEL classification: C61, D81, D90, G11, G22.

1 Introduction

Defined contribution (DC) pension schemes are becoming increasingly important in the pension systems of most industrialised countries. The ageing population problem, coupled with late entry in the workforce, threatens the sustainability of PAYG public pension systems. As a consequence, the second and third pillars of pension provision (i.e. private pension funds and individual investment) play a crucial role in building individual pensions. Most pension funds nowadays are set in a DC scheme and are increasingly taking the place of defined benefit (DB) schemes that were more frequent in the past. It is well

*University of Brescia, Email: menoncin at eco.unibs.it.
†University of Torino, Collegio Carlo Alberto and CeRP, Email: elena.vigna at unito.it.
known that the financial risk is mainly faced by the sponsor in DB schemes and by the member in DC schemes. Therefore, the analysis of the investment strategy in the accumulation phase of a DC plan is of the utmost importance.

Previous actuarial literature on DC schemes has focused on the investment risk borne by the member. The optimal investment strategy in the accumulation phase of a DC plan has been derived with a variety of both objective functions and financial market structures. The most common problem to be solved is maximisation of expected utility (belonging to the hyperbolic absolute risk aversion family) of final wealth or other functions of wealth (such as pension rate or replacement ratio). Papers belonging to this stream of research are, among others, Boulier et al. [2001], Vigna and Haberman [2001], Haberman and Vigna [2002], Deelstra et al. [2003], Devolder et al. [2003], Battocchio and Menoncin [2004], Cairns et al. [2006], Xiao et al. [2007], Battocchio et al. [2007], Gao [2008], Di Giacinto et al. [2011].

The mean-variance approach for portfolio selection of a DC scheme has been adopted in Vigna [2013] in the simple Black-Scholes financial market with a riskless and a risky asset. Vigna [2013] makes a comparison between the expected utility approach and the mean-variance approach. Among other results, she provides a useful interpretation of the seminal results of Zhou and Li [2000], by showing that the mean-variance approach is equivalent to minimising a quadratic target-driven loss function on the terminal wealth of the type \( (X(T) - \gamma)^2 \) where \( X(T) \) is the wealth at the optimisation financial horizon and \( \gamma \) is the target. In other words, there is a one-to-one correspondence between the points of the efficient frontier and the targets selected by an investor whose preferences are described by the above mentioned quadratic loss function.

The link between the target and the corresponding point on the efficient frontier proves useful in the selection of the subjective profile risk/reward. Indeed, it looks much easier for any individual to reason in terms of targets to be reached rather than in terms of some abstract index of risk aversion. This is particularly true in the case of DC pension schemes. Traditionally, DB schemes provided and earning-related pension and frequently this implied some guarantee on the replacement ratio (i.e. the ratio between pension rate and last salary). The replacement ratio is considered one of the most common measures of welfare of a pensioner (see also Cairns et al., 2006). While there cannot be any guarantee on a pre-determined replacement ratio in DC schemes (as it was the case in DB plans), every member knows that a good total replacement ratio lies between 70% and 80% (where “total” stands for any possible source of income after retirement). This is where targets come at help in DC plans. The evident loss in welfare experienced by new members with respect to old ones in the passage from DB to DC can be counterbalanced by the possibility of selecting a wealth-target at retirement that might make them able to reach the desired replacement ratio.

Conscious of the importance of targets for members of DC schemes, and of the relevance of the mean-variance approach for portfolio selection, here we extend the work of Vigna [2013] to a market with a stochastic interest rate, and select the Vasiček [1977] model. We accordingly have three asset classes:
riskless asset, bond and stock. The result is a model that is more realistic for pension fund management, and is easy to implement, thanks to the closed-form investment strategy provided. A problem similar to ours was solved in Bajeux-Besnainou and Portait [1998] in the context of portfolio selection (without contribution in the fund). While they solved it with the martingale approach, we tackle it with the dynamic programming approach. Through dynamic programming (and the corresponding Hamilton-Jacobi-Bellman equation – HJB), we obtain both the optimal investment strategy and the portfolio efficient frontier in closed form. Based on the results of Zhou and Li [2000], we further show that there exists a one-to-one correspondence between points of the efficient frontier and target-based optimisation problems. The link between risk profile and selection of the target is highlighted, as well as the closed-form relationship between the ruin probability and the target. Analytical investigation (when possible) and sensitivity analysis illustrate how the efficient frontier depends on the model parameters. A numerical application shows the composition of the optimal portfolio over time with different levels of risk aversion.

The remainder of the paper is organized as follows. In Section 2 we outline the financial market and derive the wealth equation. In Section 3 we define and solve the mean-variance optimisation problem. In Section 4 we derive analytically the efficient frontier of portfolios. In Section 5 we show the equivalence between the mean-variance approach and the target-based approach. Section 6 contains a numerical application that illustrates the dependence of the efficient frontier on the model parameters, and the composition of the optimal portfolio with different risk profiles. Section 7 concludes.

2 The framework

2.1 The financial market

We consider the portfolio selection problem for a representative member of a DC pension plan in the accumulation phase or the investor of a saving scheme. The member/investor is assumed to join the scheme at time 0 and retire at time $T$ (without any mortality risk). The financial market is complete and frictionless. The risk is described by two standard and independent Brownian motions, $W_r(t)$ and $W_s(t)$, $t \in [0, T]$, defined on the complete filtered probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$, where $\mathcal{F} = \{\mathcal{F}(t)\}_{t \in [0, T]}$ is the filtration generated by the two Brownian motions and $\mathbb{P}$ is the real-world probability measure. As usual $\mathcal{F}(t)$ can be interpreted as the information set available at time $t$.

We assume that under the historical probability measure $\mathbb{P}$ the instantaneous riskless interest rate follows the Ornstein-Uhlenbeck process (see Vasiček, 1977):

$$dr(t) = a(b - r(t))dt + \sigma dr(t),$$

(1)

with $a > 0$, $b > 0$, $\sigma > 0$. The initial value $r(0) = r_0$ is known. The choice of the Vasiček model is mainly motivated by: (i) its analytical tractability (because
of its Gaussian distribution), (ii) the possibility of having a closed-form optimal investment strategy.

We recall that the solution of (1) is

\[ r(t) = b + (r_0 - b) e^{-at} + \int_0^t e^{-a(t-s)} \sigma_r dW_r(s), \]

and

\[
\int_0^T r(t) \, dt = \int_0^T \left( b + (r_0 - b) e^{-at} \right) dt + \int_0^T \left( \int_0^s e^{-a(t-s)} \sigma_r dW_r(s) \right) \, dt \\
= \int_0^T \left( b + (r_0 - b) e^{-at} \right) dt + \int_0^T \left( \int_s^T e^{-a(t-s)} \sigma_r \, dt \right) dW_r(s) \\
= bT + (r_0 - b) \frac{1 - e^{-aT}}{a} + \sigma_r \int_0^T \frac{1 - e^{-a(T-s)}}{a} dW_r(s). \tag{2}
\]

The investment manager can invest in three assets: cash (or a riskless asset), a zero-coupon bond and a stock (which does not pay dividends). The price of the cash asset follows the dynamics:

\[ dS_0(t) = r(t) S_0(t) \, dt. \tag{3} \]

In the two following subsections we present the dynamics of the bond and the stock, respectively.

### 2.1.1 The dynamics of the bond

In an arbitrage free and complete financial market, the price of a zero-coupon bond is equal to the expected value, under the (unique) risk-neutral probability measure \( (\mathbb{Q}) \), of the discount factor, i.e.

\[ B(t,T) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_0^T r(s) \, ds} \right], \]

where \( \mathbb{E}^{\mathbb{Q}}[\cdot] \) is the expected value operator under the probability \( \mathbb{Q} \) and the information set (sigma-algebra) at time \( t \) (i.e. \( r(t) \) is known).

If \( r(s) \) solves (1), it is well known (see Vasićek, 1977) that the price of the bond can be written as

\[ B(t,T) = e^{f(t,T)-g(t,T)r(t)}, \tag{4} \]

where

\[
f(t,T) = \left( \frac{1 - e^{-a(T-t)}}{a} - (T-t) \right) \left( b - \frac{\sigma_r \xi_r}{a} - \frac{1}{2} \frac{\sigma_r^2}{a^2} \right) - \frac{\sigma_r^2}{4a^3} \left( 1 - e^{-a(T-t)} \right)^2, \tag{5} \]

\[
g(t,T) = \frac{1 - e^{-a(T-t)}}{a}, \tag{6} \]

\[ B(t,T) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_0^T r(s) \, ds} \right], \]
and $\xi_r$ is the market price of the interest rate risk, assumed to be constant, and satisfying (see Girsanov’s theorem)

$$dW^Q_r(t) = \xi_r dt + dW_r(t). \tag{7}$$

**Remark 1.** Since the market price of risk $\xi_r$ is constant, the stochastic process (1) keeps its statistical properties unchanged under both $P$ and $Q$, in fact, given (7),

$$dr(t) = a(b - r(t)) dt + \sigma_r dW_r(t)$$

$$= a\left(b \frac{\sigma_r \xi_r}{a} - r(t)\right) dt + \sigma_r dW^Q_r(t),$$

where just the long-term (constant) mean of the process is changed.

When the bond price (4) is differentiated with respect to $r(t)$ through Itô’s lemma, we obtain

$$\frac{dB(t,T)}{B(t,T)} = r(t) dt + \frac{1}{B(t,T)} \frac{\partial B(t,T)}{\partial r(t)} \sigma_r dW^Q_r(t)$$

$$= r(t) dt - g(t,T) \sigma_r (\xi_r dt + dW_r(t))$$

$$= (r(t) - g(t,T) \sigma_r \xi_r) dt - g(t,T) \sigma_r dW_r(t). \tag{8}$$

Since it is evident that $g(t,T)$ is a decreasing function of time $t$, then the volatility of the zero-coupon bond decreases through time, and this may alter the optimal portfolio allocation. In order to avoid such a distortion, we consider a constant time-to-maturity zero-coupon bond (see, for instance, Boulier et al., 2001, Battocchio and Menoncin, 2004) whose price is

$$B_K(t) = \mathbb{E}_t^Q\left[e^{-\int_t^T r(s) ds}\right] = e^{f(0,K) - g(0,K)r(t)},$$

and whose dynamics is

$$\frac{dB_K(t)}{B_K(t)} = (r(t) - g(0,K) \sigma_r \xi_r) dt - g(0,K) \sigma_r dW_r(t), \tag{8}$$

where the volatility is now constant through time.\(^1\)

### 2.1.2 The dynamics of the stock

We assume that the stock price is driven by both the risk source of the interest rate ($W_r(t)$) and a risk source of its own ($W_s(t)$). Under the risk-neutral probability the dynamics of the stock price is

$$\frac{dS(t)}{S(t)} = r(t) dt + \sigma_{sr} dW^Q_r(t) + \sigma_s dW^Q_s(t), \tag{9}$$

\(^1\)Note from (8) that, since the instantaneous drift of the bond return must exceed $r(t)$, the price of risk $\xi_r$ is negative.
with $\sigma_{sr}$ and $\sigma_s$ both constant.

Now, if the market price of the stock risk $\xi_s$ solves

$$dW_s^Q(t) = \xi_s dt + dW_s(t),$$

and because of (7), we can write the stock price dynamics as

$$dS(t) = \left( r(t) + \xi_s \sigma_{sr} + \xi_s \sigma_s \right) dt + \sigma_{sr} dW_r(t) + \sigma_s dW_s(t).$$

### 2.2 The wealth process

We can summarise the dynamics of the stock and the bond through the following matrix stochastic equation:

$$\begin{bmatrix}
ddB_K(t) \\
\frac{dB_K(t)}{dS(t)} \\
\frac{dS(t)}{S(t)}
deS(t)
\end{bmatrix} = \begin{bmatrix}
r(t) - g(0, K) \sigma_r \xi_r & -g(0, K) \sigma_r & 0 \\
r(t) + \xi_s \sigma_{sr} + \xi_s \sigma_s & \sigma_{sr} & \sigma_s \\
\mu(t, r) & \Sigma(t, r) & \omega(t, r) \end{bmatrix} dt + \begin{bmatrix}
dW_r(t) \\
dW_s(t)
deW(t)
\end{bmatrix},$$

with the only state variable (the interest rate) following

$$dr(t) = a(b - r(t)) dt + [\sigma_r \ 0] dW(t).$$

**Remark 2.** Note that the vector of the market prices of risks

$$\xi = \begin{bmatrix} \xi_r \\ \xi_s \end{bmatrix},$$

solves

$$\Sigma(t, r) \xi = \mu(t, r) - r(t) 1,$$

where 1 is a vector of 1’s.

We denote with $X(t)$ the total wealth at time $t$. The contribution paid into the fund is constant and equal to $c > 0$ per unit time. Let us call $w_r(t)$ and $w_s(t)$ the dollar amount invested at time $t$ in the bond and in the stock, respectively, then the dynamic behaviour of wealth is

$$dX(t) = (X(t) - w_r(t) - w_s(t)) \frac{dS_0(t)}{S_0(t)} + w_r(t) \frac{dB_K(t)}{B_K(t)} + w_s(t) \frac{dS(t)}{S(t)} + c dt$$

$$= (X(t) - w_r(t) - w_s(t)) r(t) dt + \begin{bmatrix} w_r(t) & w_s(t) \end{bmatrix} \begin{bmatrix} \frac{dB_K(t)}{B_K(t)} \\
\frac{dS(t)}{S(t)}
\end{bmatrix} + c dt$$

and, after denoting

$$w(t)^\top = \begin{bmatrix} w_r(t) & w_s(t) \end{bmatrix},$$
where the superscript $\top$ means transposition, we obtain

$$dX(t) = \left( X(t) r(t) + c + w(t) \top (\mu(t, r) - r(t) \mathbf{1}) \right) dt + w(t) \top \Sigma(t, r) dW(t).$$  

(14)

The vector $w(t)$ will be called a portfolio. The problem faced by the representative member of the pension fund consists in selecting the portfolio $w(t)$ at any time $t$, according to some optimal rule.

3 Mean-variance optimisation problem

In this section we define and solve the mean-variance optimisation problem.

3.1 Problem definition

We assume that the member chooses the mean-variance approach to solve her portfolio selection problem. She wants to minimise the variance of the final wealth, given a certain expectation. Before defining properly the mean-variance problem, let us introduce the notion of admissible portfolios.

Definition 3. A portfolio $w(\cdot)$ is said to be admissible if

$$w(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^2) \quad \text{and} \quad \mathbb{E} \left( \int_0^T |w(t)|^2 \right) < +\infty.$$

The classical mean-variance problem can be now formulated (we will use $\mathbb{V}[\cdot]$ to indicate the variance operator).

Definition 4. The mean-variance optimisation problem is defined as

$$\min_{\mathbb{P}_\alpha} J(w(\cdot)) \equiv \mathbb{V}[X(T)] - \mathbb{E}[X(T)],$$

(15)

with $\alpha > 0$, over the set of admissible strategies.

Let $X^{w(\cdot)}(T)$ be the fund at time $T$ when the investment strategy $w(\cdot)$ is adopted, then an admissible strategy $w^*(\cdot)$ is called an efficient strategy if there exists no admissible strategy $w(\cdot)$ such that

$$\mathbb{E} \left[ X^{w^*(\cdot)}(T) \right] \geq \mathbb{E} \left[ X^{w(\cdot)}(T) \right], \quad \mathbb{V} \left[ X^{w^*(\cdot)}(T) \right] \leq \mathbb{V} \left[ X^{w(\cdot)}(T) \right],$$

and at least one of the inequalities holds strictly. In this case, the point $(\mathbb{V} \left[ X^{w^*(\cdot)}(T) \right], \mathbb{E} \left[ X^{w^*(\cdot)}(T) \right]) \in \mathbb{R}^2$ is called an efficient point and the set of all efficient points is called the efficient frontier.

Notice that $\alpha > 0$ is a measure of the risk aversion of the individual: the higher $\alpha$ the higher her risk aversion.

It is well known that it is not straightforward to tackle problem (15) with standard stochastic control techniques. This is due to the fact that while the
expectation operator possesses the “smoothing” property, the variance operator does not. However, Zhou and Li [2000] show that problem (15) can be approached by solving the standard linear quadratic (LQ) control problem:

$$\min J(w(\cdot)) \equiv E \left[ \alpha X(T)^2 - \beta X(t) \right],$$

over the set of admissible strategies. Indeed, they show that if \( w^*(\cdot) \) is a solution of (15), then it is a solution of (16) with

$$\beta = 1 + 2\alpha E[X^*(T)],$$

where \( X^*(T) \) is the wealth under optimal control. In solving the standard LQ control problem (16) we follow the approach presented in Zhou and Li [2000] and set:

$$\gamma = \frac{\beta}{2\alpha}.$$ 

It turns out that problem (16) is equivalent to:

$$\min J(w(\cdot)) \equiv E \left[ \frac{\alpha}{2} (X(T) - \gamma)^2 \right],$$

over the set of admissible strategies.

Since the actual value of \( J(w(\cdot)) \) is mainly immaterial, we will replace problem (19) with the equivalent problem

$$\{P_\gamma\} \min J(w(\cdot)) \equiv E \left[ \frac{1}{2} (X(T) - \gamma)^2 \right],$$

and we will refer to it in the rest of the paper.

### 3.2 The target-based approach

Problem \( \{P_\gamma\} \) can be interpreted as a target-based approach where \( \gamma \) plays the role of a target. Decision-making based on the achievement of targets has become widely accepted in the economic, financial and actuarial theory. The importance of benchmarks and reference points in decision theory was introduced by the seminal paper on Prospect Theory by Kahneman and Tversky [1979]. Use of benchmark tracking in portfolio selection and management can be found in Gaivoronski et al. [2005] and Lioui and Poncet [2013]. Similarly, He and Zhou [2011] and Jin and Zhou [2013] use reference points in portfolio selection and analyse the dependence of the optimal risky exposure on the reference point. Penalty costs and benchmark targets in pension funds optimisation appear in Geyer and Ziemba [2008] and Blake et al. [2013].

Moreover, in Section 1 we have already argued on the appropriateness of the use of targets in DC pension funds nowadays. The shift from DB to DC plans that occurred in the last decades has increased the inequity among pension fund members belonging to different cohorts, favouring older generations with respect to younger ones. The possibility of selecting a suitable wealth-target \( \gamma \)
to be reached at the time of retirement might enable the member of the DC plan to get close to a desired replacement ratio that, in a DC fund, cannot be guaranteed by definition.

Notice, in addition, that the optimisation of a quadratic loss or utility function is a typical approach in the actuarial literature on pension funds. Examples of this approach for defined benefit pension funds can be found, among others, in Boulier et al. [1996, 1995], Cairns [2000], Haberman and Sung [1994], Josa-Fombellida and Rincón-Zapatero [2012]. As for defined contributions pension schemes, some examples are Haberman and Vigna [2002], Gerrard et al. [2004, 2006], Emms [2010], Gerrard et al. [2012], Di Giacinto and Vigna [2012].

While it is commonly accepted that the mean-variance approach is the cornerstone of modern portfolio theory, it may be not easy for the less financial educated investors to find the couple \( \left( \sqrt{V[X^*(T)]}, \mathbb{E}[X^*(T)] \right) \) on the efficient frontier that represents her preferences. Even less easy is the task of selecting her own coefficient \( \alpha > 0 \) identifying that point on the efficient frontier. Empirical economics provides little guidance for the task of measuring one’s own risk aversion (see for instance Holt and Laury, 2002). On the contrary, the target approach seems to be more “user-friendly”, since the financial meaning of a target is evident to almost any individual.

For this reason, investigating the link between the two approaches would help bridging the gap between the theory and the application. Such a link we have just outlined, will be thoroughly investigated in Section 5 where the equivalence between the two approaches will be proved.

3.3 Dynamic programming approach

We tackle the standard stochastic optimal control problem (20) with the dynamic programming approach. To this aim, let us define the value function:

\[
V(t, x, r) = \inf_{w(t)} \mathbb{E} \left[ \frac{1}{2} (X(T) - \gamma)^2 \right],
\]

where \( X(t) = x \), \( r(t) = r \), together with the boundary condition

\[
V(T, x, r) = \frac{1}{2} (x - \gamma)^2.
\]

The Hamilton-Jacobi-Bellman (HJB) equation of the problem is

\[
\frac{\partial V(t, x, r)}{\partial t} + \inf_{w(t)} A V(t, x, r) = 0,
\]

where \( A \) is the Dynkin’s infinitesimal generator. Therefore, we have (in the following we will omit sometimes the variables \( t \) and \( r \) for notation convenience):

\[
A V(t, x, r) = V_x (rx + c + w^\top (\mu - r 1)) + V_r \mu_r + \frac{1}{2} V_{xx} w^\top \Sigma \Sigma^\top w + \frac{1}{2} V_{r} \omega^\top \omega + V_{x r} w^\top \Sigma \omega,
\]

where the subscripts on function \( V \) denote partial derivatives.
By first order condition, the optimal control is
\[ w^* = -\frac{V_x}{V_{xx}} \left( \Sigma \Sigma^\top \right)^{-1} (\mu - r) 1 - \frac{V_{xr}}{V_{xx}} \left( \Sigma \Sigma^\top \right)^{-1} \Sigma \omega. \quad (22) \]

Plugging \( w^* \) into the HJB equation we have the following PDE for the value function (we have also used (13)):
\[ 0 = V_t + V_x (rx + c) - \frac{1}{2} \frac{V_x^2}{V_{xx}} \xi^\top \xi + V_t \mu_r - \frac{V_{xr} V_x}{V_{xx}} \omega^\top \xi \]
\[ - \frac{1}{2} \frac{V_x^2}{V_{xx}} \omega^\top \omega + \frac{1}{2} V_{rr} \omega^\top \omega. \quad (23) \]

3.3.1 Guess function

As standard, we find the solution by guessing the value function:
\[ V(t, x, r) = \frac{(x - D(t, r))^2}{2A(t, r)}, \]

with boundary conditions
\[ D(T, r) = \gamma, \]
\[ A(T, r) = 1. \]

Plugging the partial derivatives of \( V(t, x, r) \) into (23) and collecting the terms multiplying \((x - D)\) and \((x - D)^2\) we end up with the following two PDEs:
\[ 0 = A_t + A_r \left( \mu_r - 2 \omega^\top \xi \right) + \frac{1}{2} A_{rr} \omega^\top \omega - A \left( 2r - \xi^\top \xi \right), \quad (24) \]
\[ 0 = D_t + D_r \left( \mu_r - \omega^\top \xi \right) + \frac{1}{2} D_{rr} \omega^\top \omega - rD - c. \quad (25) \]

Remark 5. If Itô’s lemma is applied to \( D(t, r) \), its differential is
\[ dD = \left( D_t + D_r \left( \mu_r - \omega^\top \xi \right) + \frac{1}{2} D_{rr} \omega^\top \omega \right) dt + D_r \omega^\top dW^Q (t), \]
and, because of (25), it can be written as
\[ dD = (Dr + c) dt + D_r \omega^\top dW^Q (t) \]
\[ = (Dr + c + D_r \omega^\top \xi) dt + D_r \omega^\top dW (t). \quad (26) \]

The solutions of both PDEs (24) and (25) can be expressed through the Feynman-Kaç representation theorem (see, for instance, Duffie, 2001), but under two different probabilities. While \( D(t, r) \) can be written as an expected value under \( Q \), the function \( A(t, r) \) can be expressed under a probability such that
the drift of (1) is \( \mu_r - 2\omega^\top \xi \). Such a new probability \( Q_2 \) is defined through the following relationship:

\[
\mu_r dt + \omega^\top dW(t) = (\mu_r - 2\omega^\top \xi) dt + \omega^\top dW_Q^2(t),
\]

\[
dW_Q^2(t) = 2\xi dt + dW(t).
\]

Thus, the two functions \( A(t, r) \) and \( D(t, r) \) can be written as the following expected values:

\[
D(t, r) = \mathbb{E}_t^Q \left[ \int_t^T -ce^{-\int_u^t \omega^\top dW(u)} \, ds + \gamma e^{-\int_u^t \omega^\top dW(u)} \, du \right]
= -c \mathbb{E}_t^Q \left[ \int_t^T e^{-\int_u^t \omega^\top dW(u)} \, ds + \gamma e^{-\int_u^t \omega^\top dW(u)} \, du \right]
= -c \mathbb{E}_t^Q \left[ \int_t^T B(t, s) \, ds + \gamma B(t, T) \right],
\]

\[
A(t, r) = \mathbb{E}_t^Q \left[ e^{-\int_t^T \omega^\top z(u) \, du} \right] = e^{\tilde{r}^\top (2(\tilde{r}^\top r - (D-r)a) - (T-t)\tilde{r}) - 2g(T-t)\tilde{r}^\top z(t) \, du}.
\]

The dynamic equation for \( z(\cdot) \equiv 2r(\cdot) \) under the new probability \( Q_2 \) is

\[
dz(t) = 2a \left( b - \frac{1}{2} z(t) \right) dt + 2\sigma_r dW_r(t)
= 2a \left( b - \frac{1}{2} z(t) \right) dt + 2\sigma_r (dW_r(t) - 2\xi dt)
= a \left( 2b - \frac{4\sigma_r \xi}{a} - z(t) \right) dt + 2\sigma_r dW_r(t).
\]

Accordingly, we obtain

\[
A(t, r) = e^{\tilde{r}^\top (2(\tilde{r}^\top r - (D-r)a) - (T-t)\tilde{r}) - 2g(T-t)\tilde{r}^\top z(t) \, du}.
\]

\[
f_2(t, T) = 2 \left( b - 2\frac{\sigma_r \xi}{a} - \frac{\sigma_r^2}{a^2} \right) \left( \frac{1 - e^{-a(T-t)}}{a} - (T-t) \right) - \sigma_r^2 \left( \frac{1 - e^{-a(T-t)}}{a^3} \right)^2.
\]

### 3.4 The optimal portfolio

By plugging partial derivatives of the guess function \( V(t, x, r) \) into (22) we get the optimal investment strategy:

\[
w^* = (D - x) (\Sigma \Sigma^\top)^{-1} (\mu - r \mathbf{1}) + \left( D_r - (D - x) \frac{A_r}{A} \right) (\Sigma^\top)^{-1} \omega.
\]

Given (28), we can easily compute

\[
\frac{A_r}{A} = -2g(t, T).
\]
The optimal portfolio can be finally rewritten as

\[
\begin{bmatrix}
w^*_s(t) \\
w^*_r(t)
\end{bmatrix} = \begin{bmatrix}
\frac{D(t,x) - x(t)(-\xi_r \sigma_r^2 + \xi_g \sigma_g^2 - 2g(t,T))}{y(0,r)} - \frac{D_r(t,r) x(t)}{y(0,r)} \\
(D(t,r) - x(t) \frac{\xi_r}{\xi_r})
\end{bmatrix}.
\]  

(30)

4 Efficient frontier

In order to find the efficient frontier of portfolios, it is necessary to compute first the evolution of the fund when the optimal investment strategy (30) is adopted, then the mean and the variance of the optimal wealth at time \(T\).

4.1 Evolution of the optimal fund \(X^*(t)\), mean and variance of \(X^*(T)\)

Once the optimal portfolio (30) is plugged into the wealth equation (14), the optimal wealth becomes

\[
dX^* = (X^* r + c - (X^* - D) \xi^\top \xi + (D_r - (X^* - D) 2g) \xi^\top \omega) dt \\
+ \left(- (X^* - D) \xi^\top + (D_r - (X^* - D) 2g) \omega^\top \right) dW(t).
\]

(31)

For the calculation of both the mean and the variance of the final wealth, it turns out convenient to calculate the dynamics of \(X^* - D\). By using (26) we obtain

\[
d \left(\frac{X^* - D}{X^* - D}\right) = r dt + (-\xi^\top \xi - 2g \xi^\top \omega) dt + (-\xi^\top - 2g \omega^\top) dW(t).
\]

(32)

Remark 6. Observe from (32) that the process \(X^*(t) - D(t,r)\) follows a time-inhomogeneous geometric Brownian motion, and therefore it never crosses zero. Its sign depends on the sign of the initial observation point \(x_0 - D(0,r)\). In addition, from (30) we see that the optimal amount \(w^*_s\) invested in the stock is always negative/positive, depending on the sign of \(x_0 - D(0,r)\). It is therefore of the utmost importance the determination of this sign. We will return to this crucial point later, after having calculated the mean of the optimal fund at time \(T\). We will show that the relevant sign is strictly connected to the sign of \(\alpha\).

Remark 7. Under the risk-neutral probability (32) becomes

\[
d \left(\frac{X^* - D}{X^* - D}\right) = r dt + (-\xi^\top - 2g \omega^\top) dW^Q(t),
\]

which is an obvious result, since the drifts of \(X^*(t,r)\) and \(D(t,r)\) depend on \(c\) in the same way under \(Q\).
After applying Itô’s lemma to \( \ln (X^* - D) \), the solution of (32) is

\[
\frac{X (T)^* - D (T, r)}{X (0)^* - D (0, r)} = e^{\Lambda (0, T)},
\]

where, recalling (2),

\[
\Lambda (0, T) = \int_0^T \left( r (t) - \frac{3}{2} \xi^T \xi - 4g (t, T) \omega^T \xi - 2g^2 (t, T) \omega^T \omega \right) dt + \int_0^T \left( -\xi^T - 2g (t, T) \omega^T \right) dW (t)
\]

\[
\frac{W (t)}{\Lambda (0, T)} = \frac{r (t) - \frac{3}{2} \xi^T \xi - 4g (t, T) \omega^T \xi - 2g^2 (t, T) \omega^T \omega}{\Lambda (0, T) + \frac{3}{2} \xi^T \xi + 4g^2 (t, T) \omega^T \omega}
\]

\[
= \left( b - \frac{3}{2} \xi^T \xi \right) T + (r_0 - b) \frac{1 - e^{-aT}}{a} - 2 \int_0^T (2g (t, T) \omega^T \xi + g^2 (t, T) \omega^T \omega) dt
\]

\[
- \int_0^T (\xi^T + g (t, T) \omega^T) dW (t),
\]

with

\[
\mathbb{E}_0 [\Lambda (0, T)] = \left( b - \frac{3}{2} \xi^T \xi \right) T + (r_0 - b) \frac{1 - e^{-aT}}{a} - 2 \int_0^T (2g (t, T) \omega^T \xi + g^2 (t, T) \omega^T \omega) dt
\]

\[
= \left( b - \frac{3}{2} \xi^T \xi \right) T + (r_0 - b) \frac{1 - e^{-aT}}{a} - \frac{(4a^2 \sigma_r \xi_r + 2a \sigma^2) T - 4a \sigma_r \xi_r - 3 \sigma^2 + (4a \sigma_r \xi_r + 4 \sigma^2) e^{-aT} - \sigma^2 e^{-2aT}}{a^3},
\]

\[
\mathbb{V}_0 [\Lambda (0, T)] = \int_0^T (\xi^T + g (t, T) \omega^T) (\xi + g (t, T) \omega) dt
\]

\[
= \xi^T \xi T + \int_0^T (2g (t, T) \omega^T \xi + g^2 (t, T) \omega^T \omega) dt
\]

\[
= \xi^T \xi T + \frac{1}{2a^3} \left[ (4a^2 \sigma_r \xi_r + 2a \sigma^2) T - 4a \sigma_r \xi_r - 3 \sigma^2 + (4a \sigma_r \xi_r + 4 \sigma^2) e^{-aT} - \sigma^2 e^{-2aT} \right].
\]

Remark 8. The stochastic variable \( \Lambda (0, T) \) is normally distributed since both \( W (t) \) and \( r (t) \) are normal variables. Furthermore, it is easy to demonstrate that

\[
\mathbb{E}_0^Q \left[ e^{\Lambda (0, T)} e^{-\int_0^T r(u) du} \right] = 1.
\]

Intuitively, \( e^{\Lambda (0, T)} \) can be interpreted as a compound factor which compensates the discount factor \( e^{-\int_0^T r(u) du} \). It is well known that the expected value of a product can be written as the product of two expected values by changing the probability (and the numéraire of the economy). In this case the suitable numéraire is the zero-coupon bond and, accordingly, the corresponding probability is the so-called forward probability \( \mathbb{F}_T \) (see Björk [2009]), whose relationship with the risk neutral probability is given by the Girsanov theorem:

\[
dW (t) = -g (t, T) \omega dt + dW (t)^{\mathbb{F}_T}.
\]
The relationship with the physical probability is, of course
\[ dW(t) = (-g(t, T) \omega - \xi) \, dt + dW(t)^\mathcal{F}_T. \]  
(37)

Thus, the previous expected value can be simplified as
\[ \mathbb{E}_T^\mathcal{F} \left[ e^{\Lambda(0,T)} \right] \mathbb{E}_0^\mathcal{Q} \left[ e^{-\int_0^T r(u) \, du} \right] = 1, \]
\[ \mathbb{E}_0^\mathcal{F} \left[ e^{\Lambda(0,T)} \right] = \frac{1}{B(0,T)}, \]
and, since \( \Lambda(0,T) \) is normally distributed:
\[ e^{\mathbb{E}_0^\mathcal{F} \left[ \Lambda(0,T) \right]} + \frac{1}{2} \mathbb{V}_0[\Lambda(0,T)] = 1, \]
(38)

where we know that the variance is not affected by the probability change.

Furthermore, by using (37), it is easy to show that
\[ e^{\mathbb{E}_0^\mathcal{F} \left[ \Lambda(0,T) \right]} e^{\mathbb{E}_0^\mathcal{Q} \left[ e^{-T r(u)} \, du \right]} = 1 = B(0,T), \]
(39)

Recalling that \( X^* (0) = x_0 \) and \( D(T,r) = \gamma \), the optimal final wealth, its mean and its variance are
\[ X^*(T) = \gamma - (D(0,r_0) - x_0) e^{\Lambda(0,T)}, \]
\[ \mathbb{E}_0 \left[ X^*(T) \right] = \gamma - (D(0,r_0) - x_0) e^{\mathbb{E}_0^\mathcal{Q} \left[ e^{-T r(u)} \, du \right]} e^{\mathbb{E}_0^\mathcal{Q} \left[ e^{-s r(u)} \, du \right]}, \]
\[ \mathbb{V}_0 \left[ X^*(T) \right] = (D(0,r_0) - x_0)^2 e^{2 \mathbb{E}_0^\mathcal{Q} \left[ e^{-T r(u)} \, du \right]} (e^{\mathbb{V}_0^\mathcal{Q} \left[ e^{-s r(u)} \, du \right]} - e^{\mathbb{V}_0^\mathcal{Q} \left[ e^{-T r(u)} \, du \right]}), \]
where we have used the property of the log-normal distribution of \( e^{\Lambda(0,T)} \). These equations can be simplified through (38) and (39) as follows:
\[ \mathbb{E}_0 \left[ X^*(T) \right] = \gamma - \frac{D(0,r_0) - x_0}{B(0,T)} e^{-\mathbb{V}_0[\Lambda(0,T)]}, \]
(40)
\[ \mathbb{V}_0 \left[ X^*(T) \right] = \left( \frac{D(0,r_0) - x_0}{B(0,T)} \right)^2 e^{\mathbb{V}_0^\mathcal{Q} \left[ e^{-s r(u)} \, du \right]} - 1. \]
(41)

Remark 9. The quantity \( D(0,r_0) - x_0 \), that affects both the mean and the variance of the final wealth, has an important financial interpretation. It can be rewritten as
\[ D(0,r_0) - x_0 = \gamma \mathbb{E}_0^\mathcal{Q} \left[ e^{-\int_0^T r(v) \, dv} \right] - \left( x_0 + c \int_0^T \mathbb{E}_0^\mathcal{Q} \left[ e^{-\int_0^T r(v) \, dv} \right] ds \right) \]
\[ = \gamma B(0,T) - \left( x_0 + c \int_0^T B(0,s) ds \right), \]
(42)
i.e. it is the expected present value of the target \( \gamma \) minus the expected present value of initial wealth and future contributions; such a sum can be interpreted as what would be needed today to reach the final target \( \gamma \) should the portfolio and the contributions grow at the return \( r(\cdot) \).
4.2 Efficient frontier

Now, if we merge (40) and (41) via isolation of $\gamma$ (considering also (42)), we easily obtain the efficient frontier in the mean-standard deviation plan:

$$
E_0 [X^* (T)] = \frac{x_0 + c \int_0^T B(0, s) \; ds}{B(0, T)} + \sqrt{V_0 [\Lambda(0, T)]} - 1 \sqrt{V_0 [X^*(T)]}.
$$

(43)

Remark 10. The intercept of the semi-line (43) has a clear financial interpretation. The numerator is the sum between the initial wealth and the expected present value of all the future contributions. The denominator is the price of a zero-coupon bond, i.e. a discount factor. When a given amount of money is divided by the discount factor, it is moved forward in time. Accordingly, the intercept of (43) is the compounded value in $T$ of the total discounted wealth (gross of contributions) computed at time zero. In what follows, we thus define

$$
\chi_T \equiv \frac{x_0 + c \int_0^T B(0, s) \; ds}{B(0, T)}.
$$

As expected, the efficient frontier (43) coincides with that of Vigna [2013] in a world where the interest rate is constant over time (i.e. $r = 0$ and $a = 0$). In this case, the function $\Lambda(0, T)$ becomes (noting that also $\xi_r = 0$)

$$
\Lambda(0, T)|_{r=a=0} = \left( r_0 - \frac{3}{2} \xi_s^2 \right) T - \int_0^T \xi_s dW_s(t),
$$

and

$$
V_0 [\Lambda(0, T)|_{r=a=0}] = \xi_s^2 T.
$$

5 Mean-variance versus Target-based

Similarly to Vigna [2013], in the framework outlined in Section 2.1 we use results shown in Zhou and Li [2000] to further refine the equivalence between the target-based approach ($P_\gamma$), given by (20), and the mean-variance approach ($P_\alpha$), given by (15). We provide the exact correspondence between the target $\gamma$ and the coefficient of risk aversion $\alpha$.

5.1 Equivalence between the mean-variance ($P_\alpha$) – problem and the target-based ($P_\gamma$) – problem

According to Zhou and Li [2000], a solution of Problem ($P_\alpha$) given by (15), with $\alpha > 0$ is a solution of Problem ($P_\gamma$) given by (20), with

$$
\gamma = \frac{1}{2\alpha} + E_0 [X^* (T)].
$$

We have used the result $D(0, r_0) > x_0$ that will be shown in Section 5.1.
By plugging $\gamma$ into (40) we obtain

$$\frac{1}{2\alpha} = \frac{D(0,r_0) - x_0}{B(0,T)} e^{-\nu_0[A(0,T)]}. \quad (45)$$

Therefore, we have the important result:

$$\alpha > 0 \iff D(0,r_0) > x_0. \quad (46)$$

Given that the mean-variance optimisation problem is characterised by $\alpha > 0$, the inequality (46), coupled with Remark 6, gives rise to the following proposition on the positivity of the amount invested in the stock under optimal rules.

**Proposition 11.** In the financial market outlined in Section 2.1, the optimal amount invested in the stock for the mean-variance problem defined by (15) is strictly positive at any time $0 \leq t \leq T$.

**Proof.** This follows from (30), Remark 6 and inequality (46). \qed

The inequality (46) can be rewritten in terms of $\gamma$ as follows:

$$\alpha > 0 \iff \gamma > \chi_T. \quad (47)$$

The inequality (47) illustrates the link between $\alpha-$problems and $\gamma-$problems. In particular, a $(P_\alpha)$ problem with $\alpha \in (0, +\infty)$ is associated to a unique $(P_\gamma)$ problem with $\gamma \in (\chi_T, +\infty)$.

The relationship between the risk aversion $\alpha$ and the target $\gamma$ can be found working out $\alpha$ from (45):

$$\alpha = \frac{e^{\nu_0[A(0,T)]}}{2(\gamma - \chi_T)}. \quad (48)$$

Notice that, under the constraints (46), $\alpha$ and $\gamma$ are inversely proportional to each other. The higher the risk aversion $\alpha$, the lower the target $\gamma$ and vice versa.

We are now ready to provide a proposition that establishes the link between mean-variance and target-based problems. This proposition extends Theorem 2.3 of Vigna [2013] to a financial market with a stochastic interest rate.

**Proposition 12.** Assume that the financial market and the wealth equation are as described in Section 2.1. Then, there is a one-to-one correspondence between portfolios on the efficient frontier (43), identified by $\alpha > 0$, and optimal solutions of target-based problems (20), identified by $\gamma > \chi_T$. The relationship between the corresponding $\alpha$ and $\gamma$ is given by (48).

**Proof.** The proof follows from all the results of the present section and the results in Zhou and Li [2000]. First we show that the solution of a target-based problem is mean-variance efficient. Then we show that each point on the efficient frontier is associated to a target-based problem.

Consider a target-based problem $(P_\gamma)$ defined by (20) with a given target $\hat{\gamma} > \chi_T$. Due to Zhou and Li [2000] the optimal investment strategy associated
to it, given by (30), is also the optimal investment strategy of the mean-variance problem (15) with coefficient of risk aversion given by

$$\hat{\alpha} = \frac{e^{V_0[\Lambda(0,T)]}}{2(\hat{\gamma} - \chi_T)}.$$  \hspace{1cm} (49)

Vice versa, consider a mean-variance problem \((P_\alpha)\) defined by (15) with a given coefficient of risk aversion \(\hat{\alpha} > 0\). Then, again due to Zhou and Li [2000], its solution coincides with the solution of the associated \((P_\gamma)\) problem, with target given by

$$\hat{\gamma} = \chi_T + \frac{e^{V_0[\Lambda(0,T)]}}{2\hat{\alpha}}.$$  \hspace{1cm} (50)

5.2 Choice of the target and risk aversion

Proposition 12 provides a mapping between \((P_\alpha)\)−problems, with \(\alpha \in (0, +\infty)\) and \((P_\gamma)\)−problems, with \(\gamma \in (\chi_T, +\infty)\). The parameter \(\alpha\) is a measure of the risk aversion of the mean-variance optimiser. The higher \(\alpha\) the higher her risk aversion, and vice versa. Similarly, \(\gamma\) is a measure of the risk tolerance of the target-based optimiser. The higher \(\gamma\) the higher her risk tolerance, and vice versa.

Due to (50), when the risk aversion \(\alpha\) tends to infinity the target \(\gamma\) tends to the point \(\chi_T\). The lowest possible target can be interpreted as the \(T\)−value of initial wealth and all the contributions when the portfolio grows at the rate \(r(\cdot)\). Intuitively, the target aimed by the investor cannot be less than what could be achieved by investing at the interest rate \(r(\cdot)\). Whenever the interest rate is constant the threshold for \(\gamma\) collapses to the value given in Vigna [2013]: in a Black and Scholes financial market the final target must exceed the wealth achievable by investing the portfolio in the riskless asset.

On the contrary, from (50) we see that when the risk aversion \(\alpha\) tends to 0, the target chosen tends to infinity, that is also intuitive.

Reversing the arguments and using (49), one can see that the higher the target \(\gamma\), the lower the risk aversion \(\alpha\) and vice versa. An infinite target implies null risk aversion, while the lowest possible target for \(\gamma\) implies infinite risk aversion.

The link between quadratic utility function and M-V approach in continuous-time models highlighted in Proposition 12 is not new: it was mentioned by Bielecki et al. [2005] in a more general model. Differently from them we provide, in a simpler model, the exact expected return and variance of the optimal portfolio via optimisation of the quadratic loss function, i.e. the exact point on the efficient frontier of portfolios.

Proposition 12 has important practical implications. In fact, it allows the scheme’s member to identify her own risk aversion parameter \(\alpha\), and therefore the corresponding point on the efficient frontier, just by selecting a final target \(\gamma\) to be reached. This property can be used in the implementation of the model.
by financial advisors of DC pension funds. The natural way to do it would be to show to members/investors the distribution of final income relative to the selection of different targets. This could be done by showing different tables with the percentiles of final wealth obtained by selecting different targets. This way, the advisor should underline that a higher target $\gamma$ is associated to a higher riskiness/variability of the distribution of outcomes, and vice versa. The member or investor could then select the target by choosing the table of outcomes that she prefers. In the practical implementation of the model, it would also be important to underline to members that the “target” of this model is an upper bound for the achievable wealth, in that, by construction, it can never be reached. On the other hand, the mean or the median or the mode of the final wealth could act as pursued targets. A complete consciousness and understanding of the meaning of the target (as an upper threshold for the desired final wealth) would then help the member decide what is the best description of her preferences and needs.

5.3 Ruin probability and risk aversion

One of the drawbacks of the mean-variance approach is that, despite the constant inflow $c > 0$ in the fund, the optimal wealth can become negative with positive probability. This is due to the fact that, as shown by (33), the distance of the final wealth from the target (the quantity $\gamma - X^*(T)$) has a log-normal distribution. Thus, the domain of the density is $\gamma - X^*(T) > 0$, i.e. $X^*(T) \in (-\infty, \gamma)$.\footnote{Indeed, $-X^*(T)$ has a shifted lognormal distribution.} From (33) the ruin probability is

$$
P \{ X^*(T) < 0 \} = P \left\{ \gamma + (x_0 - D(0, r_0)) \, e^{\Lambda(0,T)} < 0 \right\} = P \left\{ \Lambda(0,T) > \ln \frac{\gamma}{D(0, r_0) - x_0} \right\}. $$

It is convenient to parametrise the target $\gamma$ as a multiple of the lowest threshold $\chi_T$:

$$\gamma = \kappa \cdot \chi_T,$$

with $\kappa \geq 1$. It is then straightforward to see that the ruin probability becomes

$$
P \{ X^*(T) < 0 \} = P \left\{ \Lambda(0,T) > - \ln B(0,T) + \ln \left( 1 + \frac{1}{\kappa - 1} \right) \right\}. \quad (51)$$

Notice that $\Lambda(0,T)$ is normally distributed with mean $E_0[\Lambda(0,T)]$ given by (35) and variance $V_0[\Lambda(0,T)]$ given by (36), therefore the calculation of the ruin probability (51) is immediate for every choice of $\kappa \geq 1$. In particular, in the previous section we have seen that the risk aversion is measured by the choice of $\gamma$, that in turn is driven by the choice of $\kappa$. The extreme cases are:

1. infinite risk aversion: $\gamma = \chi_T \to \kappa = 1$; in this case the ruin probability (51) is null, as is the probability that the normal random variable $\Lambda(0,T)$ is greater than infinity;
2. null risk aversion: $\gamma \to +\infty \Rightarrow \kappa = +\infty$; in this case the ruin probability (51) is equal to

$$\Pr \{ X^* (T) < 0 \} = \Phi \left( \frac{\ln B(0,T) + E_0 [\Lambda (0,T)]}{\sqrt{V_0 [\Lambda (0,T)]}} \right),$$  \hspace{1cm} (52)$$

where $\Phi (\cdot)$ is the cumulative distribution function of the standard normal random variable.

Clearly, 0 and the probability in (52) are the minimum and the maximum ruin probability by application of the mean-variance approach. With infinite risk aversion the member chooses the target equal to what she could reach by investing initial wealth and contribution in the riskless asset ($\chi_T$): in this case ruin has probability 0. In all the other cases, if the target chosen is greater than the threshold $\chi_T$, the event of ruin has strictly positive probability. In other words, the price to be paid in order to be mean-variance efficient with a target higher than the threshold $\chi_T$ is to accept a positive probability of ruin. On the other hand, the ruin probability is controllable and can/should indeed affect the choice of the target. Thanks to the closed-form (51), it is possible to choose $\kappa \geq 1$ in order to set the ruin probability at any desired level in the interval

$$\left[ 0, \Phi \left( \frac{\ln B(0,T) + E_0 [\Lambda (0,T)]}{\sqrt{V_0 [\Lambda (0,T)]}} \right) \right).$$  \hspace{1cm} (53)$$

A common rule in investment management is to select the “safety-first-portfolio”, found by maximisation of the expected wealth under the constraint that the ruin probability is lower than a sufficiently low level (e.g. $10^{-5}$, $10^{-3}$ etc.). In the numerical application we will compare different levels of risk aversion by the selection of targets (i.e. $\kappa$-values) that are driven by the safety-first-portfolio rule with different safety levels.

It is worth noting that in the Black-Scholes financial market with constant interest rate $r$ and constant price of risk $\xi$, the range (53) for the ruin probability is simply $\left[ 0, \Phi \left(-\frac{3}{2} \sqrt{\xi^2 T} \right) \right]$.

### 6 Numerical application

The section is devoted to the numerical application of the model presented so far. First, we fix a base scenario by calibrating the model parameters. Then, we investigate two aspects:

1. efficient frontier and its dependence on the model’s parameters;
2. evolution of the optimal efficient investment strategy over time and its dependence on the risk aversion of the individual.
6.1 Base scenario

All the parameters have been estimated from three time series for the period between January 1st 1962 to January 1st 2007 (thus we take out the turbulence of the sub-prime crisis):

- the 3-month US Treasury Bill interest rate (on secondary market): for calibrating the parameters of \( r(t) \) process;
- the 10-year US Bond interest rate (on secondary market): for calibrating the parameters of \( B_K(t) \) process (with \( K = 10 \));
- S&P 500: for calibrating the parameters of \( S(t) \).

The parameters of the riskfree interest rate process are as follows:

\[
\begin{align*}
    a &= 0.1777, \\
    b &= 0.0596, \\
    \sigma_r &= 0.0158.
\end{align*}
\]

The initial value of interest rate is set to its long term equilibrium value (i.e. \( r_0 = b \)). The average return on 10-year bonds is about 7.1%. Thus, if we solve

\[
\mathbb{E}_t [d \ln B_K(t)] = 0.071dt,
\]

i.e.

\[
r(t) - g(0, 10) \sigma_r \xi_r - \frac{1}{2} g(0, 10)^2 \sigma_r^2 = 0.071,
\]

where, instead of \( r(t) \) we use the long term equilibrium value \( b \), we obtain

\[
\xi_r = -0.1912.
\]

The variance of the log-return of S&P is

\[
\mathbb{V} [d \ln S(t)] = (\sigma_{sr}^2 + \sigma_s^2) dt = 0.0223dt,
\]

and its average return is

\[
\mathbb{E} [d \ln S(t)] = \left( r(t) + \xi_r \sigma_{sr} + \xi_s \sigma_s - \frac{1}{2} \sigma_{sr}^2 - \frac{1}{2} \sigma_s^2 \right) dt = 0.067dt,
\]

where \( r(t) \) will be substituted with the long term equilibrium \( b \).

The covariance between the S&P log-return and the return on the 10-year bonds is

\[
\mathbb{C} [d \ln S(t), d \ln B_K(t)] = -g(0, K) \sigma_r \sigma_{sr} dt = -0.0004552dt.
\]

Thus, we have to solve the following system

\[
\begin{align*}
    \sigma_{sr}^2 + \sigma_s^2 &= 0.0223, \\
    b + \xi_r \sigma_{sr} + \xi_s \sigma_s - \frac{1}{2} \sigma_{sr}^2 - \frac{1}{2} \sigma_s^2 &= 0.067, \\
    -g(0, 10) \sigma_r \sigma_{sr} &= -0.0004552.
\end{align*}
\]
Table 1: Parameters for the base scenario, calibrated on the S&P 500, 3-month Treasury Bills, and 10-year Bonds time series (between January 1st 1962 to January 1st 2007)

<table>
<thead>
<tr>
<th>Interest rate</th>
<th>Stock</th>
<th>Bond</th>
<th>Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 0.1775$</td>
<td>$\sigma_s = 0.1492$</td>
<td>$K = 10$</td>
<td>$x_0 = 1$</td>
</tr>
<tr>
<td>$b = 0.0595$</td>
<td>$\sigma_{sr} = 0.006162$</td>
<td>$c = 0.1$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_r = 0.0158$</td>
<td>$\xi_s = 0.1322$</td>
<td>$T = 20$</td>
<td></td>
</tr>
<tr>
<td>$\xi_r = -0.1913$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

which has two solutions for $\sigma_s$ (one positive and one negative); we take the positive value:

$$
\begin{align*}
\sigma_{sr} &= 0.006162, \\
\sigma_s &= 0.1492, \\
\xi_s &= 0.1322.
\end{align*}
$$

Finally, the initial wealth is set to $x_0 = 1$, the contribution rate to $c = 0.1$ and the time-horizon to $T = 20$. The values of all the parameters are summarised in Table 1.

### 6.2 The efficient frontier: comparative statics

In this section we investigate the dependence of the efficient frontier on the model parameters.

Considering (43), (4) and (36), we deduce the following:

1. both the rate of contribution $c$ and the initial wealth $x_0$ affect only the intercept $\chi_T$ of the efficient frontier, and in a positive way; a higher rate of contribution or a higher initial wealth lead to a higher intercept and, in other words, for any level of standard deviation, the investor will have a higher return;

2. the same result holds for the parameters $r_0$ and $b$: they only affect the intercept of the efficient frontier but not its slope; the intercept $\chi_T$ positively depends on both $r_0$ and $b$, that is intuitive because a higher initial interest rate or a higher long term mean of interest rate produces a higher wealth achievable by investing in $r(t)$; the fact that $\chi_T$ positively depends on $r_0$ is also consistent with the fact that the duration of a zero-coupon bond (that is the opposite of the semi-elasticity of the bond price with respect to the interest rate) is always greater than that of a coupon-bond having the same maturity;

3. the diffusion terms $\sigma_s$ and $\sigma_{sr}$ do not affect the efficient frontier;
4. the stock market price of risk $\xi_s$ affects only the slope of the efficient frontier and in a positive way; this is also intuitive, as $\xi_s$ measures the goodness of the risky asset, so with high $\xi_s$ it is possible to reach a given level of expected wealth with lower risk;

5. both the intercept and the slope of the frontier increase when $T$ increases; this is intuitive: on the one hand, with a longer time horizon the wealth achievable with investment in the riskless asset increases, and on the other hand it is possible to obtain a higher expected wealth with a lower risk level;

6. the risk premium of the bond ($\xi_r$), the volatility of the interest rate ($\sigma_r$) and the speed of return to the long term mean of the interest rate ($a$) affect both the intercept and the slope of the efficient frontier; in particular, $\chi_T$ negatively depends on $\xi_r$;

7. on the other hand, the effect of $\xi_r$, $\sigma_r$ and $a$ on the slope of the efficient frontier and the effect of $\sigma_r$ and $a$ on the intercept $\chi_T$ cannot be determined; the comparison between the values of $-\xi_r$ and $\sigma_r$ determines the impact of $\xi_r$, $\sigma_r$ and $a$ on the slope and the intercept; in particular

- the condition
  $$-\xi_r > \sigma_r \frac{1 - e^{-aT}}{a},$$
  (54)
  is sufficient for negative dependence of the slope on $\xi_r$ and $\sigma_r$, and positive dependence on $a$;

- the condition
  $$-\xi_r > \frac{\sigma_r}{a},$$
  (55)
  is sufficient for positive dependence of the intercept $\chi_T$ on $\sigma_r$;

- on the other hand, if $-\xi_r$ is small enough and/or $\sigma_r$ is large enough (surely if $\sigma_r \geq 1$), the results are reversed and the slope has positive dependence on $\xi_r$ and $\sigma_r$, and negative dependence on $a$, and $\chi_T$ has positive dependence on $\sigma_r$.

- the dependence of the intercept $\chi_T$ on $a$ is quite hard to establish.

It is therefore convenient to analyse the dependence of the efficient frontier on the parameters $\xi_r$, $\sigma_r$ and $a$ via a numerical analysis or comparative statics. For each of the parameters $\xi_r$, $\sigma_r$, $a$, $b$, $\xi_s$ and $T$ we have realised three different scenarios by taking the base scenario of Section 6.1 (Table 1) and halving and doubling the value of the relevant parameter. Table (2) reports in the eighteen scenarios constructed the value of the intercept $\chi_T$ of the frontier and its slope. We observe what follows.

- The sufficient conditions (54) and (55) are both satisfied in the base scenario, therefore there is negative dependence of the slope on $\xi_r$ and $\sigma_r$ and positive dependence on $a$, and there is positive dependence of the intercept $\chi_T$ on $\sigma_r$. 

Table 2: Values of the intercept ($\chi_T$) and slope of the efficient frontier by changing six parameters of the model, taken from the base scenario ($\chi_T = 8.43$ and slope 0.99) and halved and doubled

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.08</td>
<td>8.89</td>
<td>0.36</td>
<td>7.93</td>
<td>1.13</td>
</tr>
<tr>
<td>$b$</td>
<td>0.03</td>
<td>5.90</td>
<td>-0.1</td>
<td>7.63</td>
<td>0.69</td>
</tr>
<tr>
<td>$T$</td>
<td>10</td>
<td>3.40</td>
<td>0.06</td>
<td>8.43</td>
<td>0.70</td>
</tr>
<tr>
<td>$20$</td>
<td>8.43</td>
<td>0.99</td>
<td>$\xi_o$</td>
<td>0.1322</td>
<td>8.43</td>
</tr>
<tr>
<td>$40$</td>
<td>40.44</td>
<td>1.57</td>
<td>$\xi_o$</td>
<td>0.26</td>
<td>8.43</td>
</tr>
</tbody>
</table>

- The intercept $\chi_T$ negatively depends on $a$.
- The slope and the intercept of the efficient frontier are affected by the two parameters $a$ and $\sigma_r$ in opposite way:
  - when $\sigma_r$ increases, the intercept increases and the slope decreases. This is intuitive. Indeed, a higher volatility $\sigma_r$ enhances the mean value of the wealth achievable with investment at rate $r(t)$; on the other hand, given that the higher $\sigma_r$ the higher the bond risk, every level of expected wealth $E[X^*(T)]$ can be reached facing a higher risk;
  - opposite conclusions can be drawn about $a$: when $a$ increases, the intercept decreases and the slope increases. Indeed, a higher $a$ implies a quicker return to the mean rate $b$ and is equivalent to a lower risk of the interest rate $r(t)$, and vice versa. Therefore, the conclusions for $a$ are opposite to those for $\sigma_r$.
- The effect on the intercept of changes in the parameter $T$ is remarkably larger than the effect of changes of the other parameters. Indeed, doubling $T$ the intercept increases by a factor 5, and halving $T$ it decreases by a factor between 2 and 3. This underlines the importance of the time horizon in building wealth.
- The largest variability for the slope of the frontier occurs in correspondence of changes of the interest rate risk premium $\xi_r$. 

23
Differently from all the other parameters, the parameters $a$ and $\sigma_r$ affect intercept and slope in opposite way. Thus, the efficient frontiers of different scenarios cross over (around the level $E[X^*(T)] = 11.5$) – differently from the frontiers corresponding to changes in all other parameters. In Figure 1 we show how the efficient frontier changes when either $a$ or $\sigma_r$ change.

6.3 Optimal portfolio behaviour over time with different risk profiles

This section investigates the behaviour of the optimal portfolio over time by means of Monte Carlo simulations. We have considered the base scenario as in Section 6.1 and found the efficient frontier. As reported in Table 2, the wealth that could be reached by investing the initial wealth and all future contributions in the risk-less asset (i.e. the intercept of the frontier) is $\gamma_T = 8.43$. The potential applicability of the target-based approach consists in allowing the member to choose a target $\gamma = \kappa \gamma_T$. Applying (53), in the base scenario the ruin probability lies in the range

\[ [0, 10.8\%], \]

depending on the value of $\kappa \geq 1$. The ruin probability can be set equal to 0, by setting $\kappa$ equal to 1. At the other extreme, with an infinite target ($\kappa = +\infty$) the ruin probability equals 10.8%. Obviously, with intermediates values of $\kappa$ the ruin probability lies in the range reported above. Notice that if the interest rate were constant and the price of risk were $\xi_s = 0.33$ (like in Vigna, 2013), with the same time horizon of 20 years the range for the ruin probability would
Figure 2: The efficient frontier in the base scenario (with $\chi_T = 8.43$ and slope 0.99) with three optimal portfolios corresponding to three different risk aversion (i.e. three different ruin probability). With $\gamma = 1.15\chi_T$ the ruin probability is 0.01%, with $\gamma = 1.28\chi_T$ the ruin probability is 0.1% and with $\gamma = 1.5\chi_T$ the ruin probability is 0.5%.

be $[0, 1.34\%)$. The introduction of a stochastic interest rate has inflated the maximum ruin probability by a factor 8.

In order to show applicability of our model, we place ourselves in the base scenario described in Section 6.1 and select three values for $\kappa$, to test three different levels of risk aversion. Our choice is dictated by the safety-first-portfolio rule for the first two risk profiles, and by a common value for the multiplier for the third risk profile.

We have selected:

1. high risk aversion with ruin probability 0.01% $\Rightarrow$ $\kappa = 1.15$;
2. medium risk aversion with ruin probability 0.1% $\Rightarrow$ $\kappa = 1.28$;
3. low risk aversion with $\kappa = 1.5$ $\Rightarrow$ ruin probability 0.5%.

Figure 2 reports the efficient frontier with the three different efficient portfolios associated to the choice of the three different risk profiles.

By means of Monte Carlo simulations, for each risk profile (i.e. for each value of $\kappa$) the optimal portfolio (as percentage of wealth) has been derived in 10,000 different scenarios (see Figure 3). Time is on the abscissa. In particular, for each risk profile:

- the top-left graph reports the average proportion of portfolio invested in bond over time;
- the bottom-left graph reports the average proportion of portfolio invested in cash over time;
Figure 3: Optimal portfolios and wealth for three different risk profiles: top-left the optimal bond share, top-right the optimal stock share, bottom-left the optimal cash share, bottom-right both the optimal wealth and the target.

- the top-right graph reports the average proportion of portfolio invested in stock over time;
- the bottom-right graph reports the average behaviour of optimal wealth, as compared to the target $\gamma$ over time.

We observe what follows.

- For any risk profile, at $t = 0$ the optimal portfolio is heavily invested in bonds and significantly invested in equities, at the price of heavy negative quantities invested in the cash, meaning high level of borrowing.

- For any risk profile, as time passes the percentage invested in both bonds and equities declines, whereas the quote used for borrowing decreases (in other words, the quote invested in cash – negative – increases): this is similar to the so-called "lifestyle strategy" largely adopted in investment management in defined contribution pension schemes in UK.

- In all risk profiles and at any time $0 \leq t \leq T$, the percentage invested in equities is on average restricted between 0 and 1, but this is not the case for bonds and cash; however, with higher values of $\kappa$ also the quote invested in equities would exceed 1; the introduction of constraints on the control variable makes the problem significantly harder to solve (see for instance Di Giacinto et al. [2011]).

- For any risk profile the average optimal wealth increases and approaches the target (reported as the horizontal coloured lines in the bottom-right graph) and never reaches it (as expected by the model).
Figure 4: Comparison of the final wealth \((X^*(T))\) distribution in three scenarios: with a high risk aversion \((\gamma = 1.15\chi_T)\) in the top graph, with a medium risk aversion \((\gamma = 1.28\chi_T)\) in the middle graph, and with a low risk aversion \((\gamma = 1.5\chi_T)\) in the lower graph.

- The comparison between the three risk profiles provides intuitive patterns: the quoted invested in bonds and equities is highest for the low risk aversive (red lines), intermediate for the medium risk aversive (green line) and lowest for the high risk aversive (blue line). In particular, for the low risk aversive the share invested in equities decreases from about 90% to about 10%, for the medium risk aversive from about 60% to about 8%, and for the high risk aversive from about 30% to about 5%.

- The last result is clear indication of the fact that the bond is treated by the strategy just as a milder risky asset: despite being less risky than the stock, and therefore being an intermediate asset between cash and stock, the optimal investment in bond over time is very similar to that of the stock and very different from that of the cash.

Finally, for the three risk profiles we have analysed the distribution of the final wealth and compared it with the target. Figure 4 reports the three histograms with the distribution of final wealth. The mean and the median of the final wealth are also reported, together with the target \(\gamma\).

As expected, in all the cases the distribution of final wealth is a shifted log-normal and concentrated on the left of the target \(\gamma\). With higher risk aversion the distribution is much more concentrated near \(\gamma\), while it is much more spread out when risk aversion decreases. This is intuitive and is also consistent with the fact that the investment strategy is riskier with a lower risk aversion and vice versa.
Clearly, lower risk aversion leads to final wealth in general larger than that of higher risk aversion:

- for the high risk averse, the final wealth is on average equal to 9.06 and in 75% of the cases it lies between 8.89 and 9.69;
- for the medium risk averse, the final wealth is on average equal to 9.61 and in 75% of the cases it lies between 9.28 and 10.79;
- for the low risk averse, the final wealth is on average equal to 10.54 and in 75% of the cases it lies between 10 and 12.65.

According to these statistics, one could be tempted to choose *a priori* the low risk aversion profile. However, the price to be paid in order to be richer “on average” is the higher probability of ruin: 0.01% for the high risk averse, 0.1% for the medium risk averse, and 0.5% for the low risk averse. Furthermore, a higher ruin probability is also associated to a longer left tail of the distribution of wealth, i.e. to worse results in the bad scenarios. More precisely, out of the 10,000 Monte Carlo simulations run, the minimum final wealth was -2.52 for the high risk averse (unique case of negative wealth), -2.92 for the medium risk averse (5 cases of negative wealth), -19.82 for the low risk averse (38 cases of negative wealth). Even if this is indeed a quite rare event, investors should be aware of the possibility of starting with initial wealth equal to 1 and ending up with a final wealth of -19. Therefore, in a practical application of the model, empirical data on the distribution of final wealth should be clearly disclosed to the pension fund member, to help her deciding what is the subjective level of trade-off between expected wealth and risk that better describes her preferences.

7 Final remarks

In this paper we have solved a mean-variance optimisation problem in a defined contribution pension scheme in the presence of a stochastic interest rate following the Vasiček [1977] dynamics. The financial market consists of three assets: the riskless one, the bond and the stock. Using the approach introduced by Zhou and Li [2000] we have transformed the problem into a standard linear-quadratic control problem, and we have tackled it with standard dynamic programming. We have provided a link between the class of mean-variance problems and the class of target-based problems characterised by minimisation of a quadratic loss driven by a target. We have shown a one-to-one correspondence between risk aversion coefficients and targets to be achieved, providing financial interpretation for the link between risk aversion and target.

The interpretation of the mean-variance problem as a target-based problem should make it easier to apply the model to real investment management in DC pension funds. The selection of the correct trade-off between the desired value of the final wealth and the risk that the member is willing to accept should be facilitated by the clear disclosure of tables reporting the distribution of final wealth and the probability of ruin. In particular, many risk averse members
would be probably driven by the desire of accepting a sufficiently small probability of ruin: in this model this is possible, because the ruin probability can be kept under any desired level by selecting properly the final target. Finally, the closed-form solution for the investment strategy further facilitates the practical implementation of the model.

References


