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to model asset returns

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Abstract

This paper constructs a class of multivariate Gaussian marked Poisson processes to model asset returns. The model proposed accommodates the cross section properties of trades, allows for returns to be correlated conditional on trading activity, and preserves the economic intuition of normality of returns conditional on trading activity. We prove that the new class of processes are in law subordinated Brownian motions and we provide their characteristic function and correlation matrix in closed form. As a first application we specify a process of variance gamma type and show that, under suitable conditions, we find as subcases some of the well known multivariate variance gamma processes recently introduced in the financial literature.

Journal of Economic Literature Classification: G12, G13

Keywords: marked Poisson processes, subordinated Lévy processes, multivariate Poisson random measure, multivariate subordinators, multivariate asset modelling, multivariate variance gamma process.
Introduction

The idea of a stochastic change of time, interpreted as a measure of trading activity, dates back to Clark (1973) who was the first to link the deviation from normality of asset prices to the changes in the number of market orders in different time periods. Since then, time changed Brownian motion was introduced to model asset returns, having the change of time captured by a subordinator. Madan and Seneta (1990) introduced the variance gamma process using a gamma subordinator, Barndorff-Nielsen (1995) proved that the normal inverse Gaussian process is, in law, a Brownian motion time changed by an inverse Gaussian subordinator, and Geman et al. (2001) proved that the CGMY process, introduced by Carr et al. (2002) can be derived by time changing a Brownian motion with a Meixner subordinator.

The first subordinated multivariate model was constructed by considering a common time change to all assets represented by a univariate subordinator (see Madan and Seneta (1990) and Luciano and Schoutens (2006)). Unfortunately, the resulting models exhibited several shortcomings including the lack of independence between asset returns and a limited span of linear correlations. Furthermore, in the empirical domain, the work of Harris (1986), which investigated the cross sectional properties of trades and rejected equality of daily trade distributions across different assets, posed the basis for the intuition of different assets having different change of times. From the theoretical perspective multivariate subordination which allowed different changes of times to different assets was introduced in the work of Barndorff-Nielsen et al. (2001). Even though both empirical evidence existed and theoretical tools were available, time changed Lévy processes with multivariate subordinators appeared only recently. For instance, Eberlein and Madan (2009) considered independent changes of time, while Semeraro (2008) and Luciano and Semeraro (2010) introduced a multivariate subordinator composed of a common component and an idiosyncratic component, named factor based subordinator. However, to preserve the intuition of economic time, each asset return distribution is time changed by a one-dimensional subordinator. Due to this constraint, the models fail to include the dependence of marginal asset return from cross information. In addition, returns conditional to trading activity are uncorrelated.

The introduction of factor based subordinators allowed for the independence of asset returns and was consistent with the new findings of Lo and Wang (2000), who explored the cross section properties of trades and empirically found a significant common factor in trading volume in the USA market. Using available volume data for individual securities from the Center for Research in Security Prices they found that one factor explains between 70% and 85% of the variance, two explain over 90% of it.

The main purpose of this paper is to propose a model of asset returns which accommodates the cross section properties of trade, allows returns to be correlated conditional to trading activity and preserves the economic intuition of normality of returns conditional on trading activity. To this aim, we introduce a class of Lévy multivariate marked Poisson processes. Intuitively, a marked Poisson process is constructed by attaching to the atoms of a Poisson random measure a collection of random variables, marks, conditionally independent of the random measure. In this framework, the Poisson random measure is a measure of the trading activity on the collection of assets up to time \( t \), and marks represent returns conditional on the trading activity. We specify marks to have a multivariate Gaussian distribution in order to have normality of asset returns conditional on the trading activity, and, at the same time, we specify the Poisson measure to recover a one
factor structure of trade information, according to Lo and Wang (2000) findings. We prove that the Poisson random measure defines a factor based multivariate subordinator and the class of models introduced generalizes the time changed Brownian motions in Semeraro (2008) and Luciano and Semeraro (2010) and spans a wider range of linear correlations. By further specifying the random measure we could obtain processes of variance gamma, normal inverse Gaussian and generalized hyperbolic types. However, for our first application we focus on the variance gamma specification. The new class of processes is fully characterized through its Lévy triplet, and the characteristic function is given in closed form.

The paper is organized as follows. Section 1 recalls the notions of the marked Poisson process, Lévy process, their relationship and the preliminary results needed to introduce the model. Section 2 introduces the class of Lévy marked Poisson models and their link with subordinated Lévy processes. Section 3 specifies a flexible class of Lévy marked Poisson processes suitable to model stock returns. The characteristic function is provided in closed form as well as the linear correlation coefficient. Section 4 specifies the marks and Poisson measure to find a multivariate version of the variance gamma processes that includes as special cases some of the processes recently proposed in the literature to model stock returns. The conclusion follows.

1 Preliminaries

This section introduces marked Poisson processes, Lévy processes and their relationship. A good reference for this connection is Çinlar (2011). Here, we report the main results used for the construction and characterization of the class of marked Poisson processes we studied and we start by recalling the notion of Poisson random Measure.

A random measure $\Pi$ on $(E,\mathcal{E})$ with mean measure $\lambda$, i.e. $\lambda(A) = \mathbb{E}[\Pi(\omega, A)]$, $A \in \mathcal{E}$, is called Poisson random measure if for every $A \in \mathcal{E}$, $\Pi(\omega, A)$ has the Poisson distribution with mean $\lambda(A)$, and whenever $A_1, \ldots, A_n \in \mathcal{E}$ and disjoint, the random variables $\Pi(\omega, A_1), \ldots, \Pi(\omega, A_n)$ are independent, this being true for every $n \geq 2$. If $\Pi$ is a Poisson random measure on $(E,\mathcal{E})$, there exists a countable collection of random variables $\{\Pi_i, i \in I\}$ on a probability space $(\Omega, \mathcal{F}, P)$ taking values in $(E,\mathcal{E})$, say atoms of $\Pi$, such that

$$\Pi(\omega, A) = \sum_{i \in I} 1_A(\Pi_i(\omega)), \omega \in \Omega, A \in \mathcal{E}$$

(1.1)

By slight abuse of notation, with $\Pi$ we indicate both the random measure and the collection of its atoms, $\Pi = \{\Pi_i, i \in I\}$.

A càdlàg stochastic process $L = \{L(t), t \geq 0\}$ with values in $\mathbb{R}^n$ such that $L(0) = 0$ is called a Lévy process if it has independent and stationary increments and it is stochastically continuous, i.e. $\forall \varepsilon > 0, \lim_{h \to 0} P(\|L(t+h) - L(t)\| \geq \varepsilon) = 0$. We recall that the characteristic function of a Lévy process $L(t)$ admits the following Lévy-Khinchin representation

$$\psi_{L(t)}(z) = \mathbb{E}[e^{i \langle z, L(t) \rangle}] = e^{t \Psi(z)}, \quad z \in \mathbb{R}^n,$$
where
\[ \Psi_L(z) = -\frac{1}{2}(z, \Sigma_L z) + i\langle \gamma_L, z \rangle + \int_{\mathbb{R}^n} \left( e^{iz}\psi(x) - 1 - i(z, x)1_{|x| \leq 1} \right) \nu_L(dx), \]

is the characteristic exponent, \( \Sigma_L \in \mathbb{R}^{n \times n} \) and is a nonnegative definite matrix, called Gaussian covariance matrix, \( \gamma_L \in \mathbb{R}^n \) and \( \nu_L \) is a Lévy measure on \( \mathbb{R}^n \). The triplet \((\gamma_L, \Lambda_L, \nu_L)\) is called the Lévy triplet of the process.

We focus on pure jump zero drift \(^1\) Lévy processes and in this case the process \( L(t) \) can be written as the sum of its jumps
\[ L(t) = \sum_{s \in D} \Delta L(s), \quad (1.3) \]
where \( D = \{ t > 0 : \Delta L(t) \neq 0 \} \). The Lévy measure \( \nu_L \) satisfies
\[ \nu_L(A) = \mathbb{E} \left[ \sum_{s \in \mathcal{D} \cap (0, 1]} 1_A(\Delta L(s)) \right], \quad \text{for any } A \subset \mathbb{R}^n. \quad (1.4) \]

The link between Poisson random measures and Lévy processes are jumps. On the one side, the jumps of a Lévy process are governed by a Poisson random measure. Formally if \( L(t) \) is a Lévy process we can define the random measure \( M_L(A) := \sum_{t \in D} 1_A(t, \Delta L(t)) \), where by \( A \) we now denote a Borel subset of \( \mathbb{R}_+ \times \mathbb{R}^n \). The measure \( M_L \), whose atoms are the jumps of \( L(t) \), is a Poisson random measure on \( \mathbb{R}_+ \times \mathbb{R}^n \) with mean measure of the type \( Leb \times \nu_L \). On the other side, it can be proved that the atoms of a Poisson random measure \( M \) on \( \mathbb{R}_+ \times \mathbb{R}^n \) with mean measure of the type \( Leb \times \nu_M \), where \( \nu_M \) is a Lévy measure such that \( \int_{\mathbb{R}^n}(|x| \wedge 1)\nu_M(dx) < \infty \) defines (the jump part of) a Lévy process. In fact the process
\[ L_M(t) = \int_{(0, t] \times \mathbb{R}^n} xM(ds, dx) \quad (1.5) \]
is a pure jump Lévy process in \( \mathbb{R}^n \) with Lévy measure \( \nu_M \). Jump times of \( L_M(t) \) with corresponding jump sizes are the atoms of the Poisson random measure \( M \).

### 1.1 Marked Poisson process

Here we introduce marked Poisson processes, which are constructed by attaching a random variable to each atom of a random measure \( \Pi = \{\Pi_i, i \in I\} \). Formally, let \( Z = \{Z_i, i \in I\} \) be a family of random variables (marks) on a measurable space \((F, \mathcal{F})\) indexed by the same countable set \( I \). Assume that \( \mu_\Pi \) is the mean measure of \( \Pi \) and the variables \( Z_i \) are conditionally independent given \( \Pi \) with the respective distributions \( Q(\Pi, \cdot) \), where \( Q(s, B) \) is a transition probability kernel from \( E \times \mathcal{F} \) into \( \mathbb{R}_+ \). We recall that a transition probability kernel from \((E, \mathcal{E})\) into \((F, \mathcal{F})\) is a map from \( E \times \mathcal{F} \) into \( \mathbb{R}_+ \), such that the mapping \( x \mapsto Q(x, B) \) is \( \mathcal{E} \)-measurable for every set \( B \in \mathcal{F} \), the mapping \( B \mapsto Q(x, B) \) is a measure on \( (F, \mathcal{F}) \) for every \( x \in E \), and \( Q(x, F) = 1 \) for every choice

\(^1\) Notice if \( L(t) \) is a zero drift pure jump process then
\[ \gamma_L = \int_B x\nu_L(dx), \quad \Sigma_L = 0 \quad (1.2) \]
where \( B \) is the unit ball in \( \mathbb{R}^n \).
of \( x \in E \). Each variable \( Z_i \) can be considered as an indicator of some property associated with the atom \( \Pi_i \). Then, as proved in Çinlar (2011) (Theorem 3.2), \( M := (\Pi, Z) \) forms a Poisson random measure on \((E \times F, \mathcal{E} \otimes \mathcal{F})\) with mean \( \mu_{\Pi} \times Q \), where \( (\mu_{\Pi} \times Q)(dx, dy) = \mu_{\Pi}(dx)Q(x, dy) \) The new measure \( M \) is called marked Poisson random measure.

### 1.2 Subordinated Lévy processes

Our main result establishes a link between marked Poisson measures and subordinated Lévy processes. Let us recall that the subordination of a one dimensional Lévy process \( L = \{L(t), t \geq 0\} \) by a subordinator \( \tau(t) \), i.e. a Lévy process on \( \mathbb{R}_+ \) with increasing trajectories, defines a new process \( X = \{X(t), t \geq 0\} \) by the composition \( X(t) := L(\tau(t)) \) (see Sato (1999)). This construction can be extended to define a multidimensional Lévy process \( \{X(t), t \geq 0\} \) by subordination of a multidimensional Lévy process \( L = \{L(t), t \geq 0\} \) through one dimensional subordinator \( \pi(t) \). Each component of \( L \) is subordinated by \( \pi(t) \), i.e. \( X \) is defined by the composition \( X(t) = (L_1(\pi(t)), \ldots, L_n(\pi(t)))^T \). By so doing, the time change is common to each component. Theorem 30.1 in Sato (1999) characterizes the subordinated process \( X \) in terms of Lévy triplet. Barndorff-Nielsen et al. (2001) generalize the above construction by allowing the introduction of multivariate subordinators, i.e. a Lévy process on \( \mathbb{R}_+^n = [0, \infty)^n \), whose trajectories are increasing in each coordinate. For purposes of introduction of multivariate subordination, the notion of \( \mathbb{R}_+^n \)-parameter process, as introduced for instance in Barndorff-Nielsen et al. (2001), is required.

Consider the multi-parameter \( s = (s_1, \ldots, s_n) \in \mathbb{R}_+^n \) and the partial order on \( \mathbb{R}_+^n \)

\[
s^1 \preceq s^2 \iff s_j^1 \leq s_j^2, \quad j = 1, \ldots, n.
\]

Let now

\[
L(s) = (L_1(s), L_2(s), \ldots, L_N(s))^T
\]

be a process with parameters in \( \mathbb{R}_+^n \) and values in \( \mathbb{R}^N \). It is called an \( \mathbb{R}_+^n \)-parameter Lévy process on \( \mathbb{R}^N \) if the following holds

1. for any \( m \geq 3 \) and for any choice of \( s^1 \preceq \ldots \preceq s^m \), \( L(s^j) - L(s^{j-1}) \), \( j = 2, \ldots, m \), are independent;
2. for any \( s^1 \preceq s^2 \) and \( s^3 \preceq s^4 \) satisfying \( s^2 - s^1 = s^4 - s^3 \), \( L(s^2) - L(s^1) \preceq L(s^4) - L(s^3) \);
3. \( L(0) = 0 \) almost surely;
4. almost surely, \( L(s) \) is right continuous with left limits in \( s \) in the partial ordering of \( \mathbb{R}_+^n \).

Notice that if \( N = n \) and \( L(t) = (L_1(t), \ldots, L_n(t))^T \) is a Lévy process with independent components, then the process \( \{L(s), s \in \mathbb{R}_+^n\} \) defined as

\[
L(s) = (L_1(s_1), \ldots, L_n(s_n))^T
\]

is a multi-parameter process that we name independent multi-parameter Lévy process.
Let \( \{L(s), s \in \mathbb{R}^n_+\} \) be a multi-parameter Lévy process on \( \mathbb{R}^N \) with Lévy triplet \( (\gamma_L, \Sigma_L, \nu_L) \), and let \( \tau(t) \) be a \( n \) dimensional subordinator independent of \( L(s) \) having Lévy triplet \( (\gamma_\tau, 0, \nu_\tau) \). The subordinated process \( X = \{X(t), t \geq 0\} \) defined by

\[
X(t) := (L(\tau(t))), \ t \geq 0
\]

is a Lévy process, as proved in Theorem 4.7 in Barndorff-Nielsen et al. (2001), which also provide its characteristic function \( \psi_X(t) = \exp(t\Psi_x(\log \psi_L(z))) \), \( z \in \mathbb{R}^n_+ \), where \( \lambda^x = \mathcal{L}(X(s)) \) and for any \( w = (w_1, ..., w_n)^T \in \mathbb{C}^n \) with \( Re(w_j) \leq 0, \ j = 1, ..., n \), \( \Psi_x(w) = \int_{\mathbb{R}^n_+} (e^{w,s}) - 1)\nu_\tau(ds) \) and

\[
\log(\psi_L(z)) = (\log(\psi_1(z)), ..., \log(\psi_1(z)))^T
\]

where \( \psi_i \) is the characteristic function of \( L(\delta_j) \) and \( \delta_j = (\delta_{j1}, ..., \delta_{jn})^T \) where \( \delta_{ij} \) is Kronecker’s delta.

Theorem 4.7 in Barndorff-Nielsen et al. (2001) also characterizes \( X \) in terms of its Lévy triplet. Since the construction proposed here uses subordinators with zero drift we assume here that \( \tau(t) \) has zero drift and the Lévy triplet \( X \) becomes

\[
\begin{align*}
\gamma_X &= \int_{\mathbb{R}^n_+} \nu_\tau(ds) \int_{|x| \leq 1} x\lambda^x(dx), \\
\Sigma_L &= 0, \\
\nu_X(B) &= \int_{\mathbb{R}^n_+} \lambda^x(B)\nu_\tau(ds), \ B \in B(\mathbb{R}^n_+). \\
\end{align*}
\]

(1.6)

The multivariate model we introduce in the following section is proved to be subordinated multi-parameter Lévy process defined on the same space of the subordinator, i.e. \( N = n \). Its marginal processes are subordinated one-dimensional multiparameter process, i.e. \( N = 1 \).

## 2 The model: Lévy Marked Poisson processes

Here we construct a Marked Poisson process of Lévy type.

Let \( \Pi \) be a Poisson random measure on \( (\mathbb{R}_+ \times \mathbb{R}^n_+, \mathcal{B}_{n+1}) \) with mean measure \( \mu_\Pi = \text{Leb} \times \nu_\Pi \), where \( \nu_\Pi \) is a Lévy measure. The process defined by

\[
\pi(t) := \int_{[0,t] \times \mathbb{R}^n_+} x\Pi(ds, dx),
\]

(2.1)

is a zero drift multivariate subordinator with Lévy measure \( \nu_\Pi \). The atoms of \( \Pi \) are family of random variables \( \Pi = \{ (\Pi_1, \Pi_2) = \{ (\Pi_{1i}, \Pi_{2i}), i \in I \} \) on \( \mathbb{R}_+ \times \mathbb{R}^n_+ \), where \( \Pi_{1i} \) are the jump times and \( \Pi_{2i} \) are the jump sizes.

Since

\[
\pi(t) = \sum_{s \in D \cap [0,t]} \Delta \pi(s),
\]

(2.2)

where \( D = \{ t > 0 : \Delta \pi(t) \neq 0 \} \), we can write

\[
\pi(t) = \sum_{i \in I} \Pi_{2i} 1_{[0,t]}(\Pi_{1i}).
\]

(2.3)
If $L(s)$ is a multiparameter process and $\lambda^s = \mathcal{L}(L(s))$, and $B$ is any set belonging to $\mathcal{B}_n$, we introduce the transition probability kernel $Q$ defined by $Q(s, B) = \lambda^s(B)$, i.e.

$$Q(0, B) := P(L(0) \in B) = 1_B(0)$$

$$Q(s, B) := P(L(s) \in B)$$

and name it multiparameter Lévy kernel. The first expression of equation (2.4) is a consequence of $L(0) = 0$ with probability one. Let us have $Z = \{Z_i, i \in I\}$ be a family of marks of $\Pi = \{(\Pi_{1i}, \Pi_{2i}), i \in I\}$ on $(\mathbb{R}^n, \mathcal{B}_n)$, with distribution $Q(\Pi_{2i}, \cdot)$. By so doing, we regard each $Z_i$ as a property associated with the atom size $\Pi_{2i}$. The following proposition, whose proof is in the Appendix A.1, proves that we can introduce a marked version of $\Pi$ by using the collection $Z$.

**Proposition 2.1.** The family $N = (\Pi_1, Z)$ forms a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B})$ with mean measure $\mu_N(dt, dy) = dt \int_{\mathbb{R}^n} v_1(ds)Q(s, dy)$.

Here, we assume $\int_{\mathbb{R}^n} (|y| \wedge 1) \int_{\mathbb{R}_+} v_1(ds)Q(s, dy) < \infty$ which implies that the process $Y$ is of bounded variation. In this case, we define the process $Y(t)$ as

$$Y(t) := \int_{[0,t] \times \mathbb{R}^n} yN(ds, dy).$$

Here we state our main result, which we prove in the Appendix A.2.

**Theorem 2.1.** The process $Y(t)$ in (2.5) is (in law) a subordinated Lévy process constructed by subordination of a multi-parameter Lévy process $L(s)$ such that $\mathcal{L}(L(s)) = Q(s, \cdot)$, with the subordinator $\pi(t)$. The Lévy triplet $(\gamma_{Y}, \Sigma_{Y}, \nu_{Y})$ of $Y$ is

$$\gamma_{Y} = \int_{\mathbb{R}^n} v_1(ds) \int_{|x| \leq 1} xQ(s, dx)$$

$$\Sigma_{Y} = 0$$

$$\nu_{Y}(B) = \int_{\mathbb{R}_+} v_1(ds)Q(s, B).$$

By Theorem 2.1 it holds

$$Y(t) = \mathcal{L} L(\pi(t)) := \left( \begin{array} {c} L_1(\pi_1(t), \ldots, \pi_n(t)) \\ \vdots \\ L_n(\pi_1(t), \ldots, \pi_n(t)) \end{array} \right).$$

We name the process $Y$ introduced in (2.5) it Lévy marked Poisson process (LmPP) and the corresponding random measure $N$ a Lévy marked Poisson random measure.

The process $Y$ is a pure jump zero drift -since $Y$ has bounded variations it follows straightforward from the expression of $\gamma_{L}$- Lévy process and can be written as the sum of its jumps

$$Y(t) = \sum_{s \in D} \Delta Y(s),$$

where $D = \{t > 0 : \Delta Y(t) \neq 0\}$. The measure $N$ satisfies

$$N((0, t] \times A) = \sum_{s \in D \cap (0, t]} 1_A(\Delta Y(s)), $$

$$ \int_{\mathbb{R}_+} v_1(ds)Q(s, B).$$
thus $\mathbf{N}((0, t] \times A)$ measures the number of jumps in $(0, t]$ of size belonging to $A$ and its mean measure is $Leb \times \nu_Y$, where $\nu_Y$ is exactly the Lévy measure of $Y$. Intuitively, Theorem 2.1 states that Lévy marked Poisson random measures define subordinated Lévy motions.

Since we proved that $Y$ is a subordinated Lévy process, its characteristics function follows by applying Theorem 4.7 in Barndorff-Nielsen et al. (2001)

$$E[e^{i\langle z, Y(t) \rangle}] = \exp(t\Psi_\pi(\log \psi_L(z))), \quad z \in \mathbb{R}^n_+,$$

(2.9)

where for any $w = (w_1, ..., w_n)^T \in \mathbb{C}^n$ with $Re(w_j) \leq 0, \ j = 1, ..., n$

$$\Psi_\pi(w) = \int_{\mathbb{R}^n_+} (e^{\langle w, s \rangle} - 1)\nu_\Pi(ds), \quad \log(\psi_L(z)) = (\log(\psi_1(z)), ..., \log(\psi_n(z)))^T$$

$$\psi_i$$ being the characteristic function of $L(\delta_j), \ \delta_j = (\delta_{j1}, ..., \delta_{jn})^T$.

In this framework $\Pi(t, \cdot)$ measures the trading activity of the collection of $n$ assets up to time $t$. In addition, the measure $\Pi$ defines the multivariate subordinator $\pi(t)$. The distribution of marks defines the distribution of $Y$ conditional to $\Pi$, i.e. the distribution of asset returns conditional to trading activity at given time $t$.

Before specifying the marks distribution and the Poisson measure in the following section, we report below two subcases, which lead to two class of time changed Lévy processes widely used in financial application.

a. **One dimensional Poisson random measure.** In this case the Lévy marked Poisson random measure is in law a subordinated Lévy process with a one dimensional subordinator, i.e.

$$Y(t) = L(\pi(t)) := (L_1(\pi(t)), L_2(\pi(t)), ..., L_1(\pi(t)))^T.$$

The proof is similar to the proof of Theorem 2.1 thus omitted.

b. **Conditional independence of marks.** Assume that the kernel $Q$ is the product of $n$ i.d. kernels from $(\mathbb{R}_+, \mathcal{B})$ into $(\mathbb{R}^n, \mathcal{B}_n)$, i.e. for each $s \in \mathbb{R}^n_+$ let $Q(s, \cdot) = \prod_{i=1}^n Q_i(s_i, \cdot)$. It holds $Y(t) \overset{d}{=} L(\pi(t))$, where $L(s)$ is the unique independent multiparameter Lévy process such that for any $s \in \mathbb{R}^n_+$, $L(L(s)) = \prod_{i=1}^n Q_i(s_i, \cdot)$ and $\pi(t)$ is the subordinator with Lévy measure $\nu_\Pi$. In this case,

$$Y(t) = L(\pi(t)) := (L_1(\pi_1(t)), L_2(\pi_2(t)), ..., L_1(\pi_n(t)))^T,$$

therefore the process $Y$ is, in law, a subordinated Lévy motion, where each component has its own one-dimensional subordinator.

### 3 The asset return Gaussian marked factor-based Poisson process

In this section, we specify a class of Lévy marked multivariate Poisson process to model asset returns. Firstly, we specify the marks distribution to accommodate normality of returns conditional on trading activity (see Ané and Geman (2000)). Secondly, to measure trading activity we introduce a class of Poisson random measures based on the low factor structure of trading volume (see Lo and Wang (2000)).
3.1 Gaussian-marked Poisson processes

We start by introducing the class of Gaussian marked Poisson processes, i.e. Lévy marked Poisson processes with Gaussian marks. We recall that a Gaussian kernel $G$ is a transition probability kernel from $(\mathbb{R}_+, B)$ into $(\mathbb{R}^n, B_n)$ defined by

$$G(0, B) := P(B^\rho(0) \in B) = 1_B(0)$$
$$G(t, B) := P(B^\rho(t) \in B),$$

where $B \in B_n$, $B^\rho(t)$ is a multidimensional Brownian motion, having drift $\mu^\rho = (\mu^\rho_1, \ldots, \mu^\rho_n)^T$ and Gaussian covariance matrix $\Sigma^\rho = (\sigma^\rho_i \sigma^\rho_j \rho_{ij})_{n \times n}$.

In order to mark multidimensional Poisson measures with Gaussian marks, we need the definition of multi-parameter Gaussian kernel, defined by means of the law of a multi-parameter Brownian motion, which we introduce below.

Let $B^\rho(t)$ be a multivariate Brownian motion having drift $\mu^\rho$ and Gaussian covariance matrix $\Sigma^\rho$, as above, and let $B(t)$ be a Brownian motion with independent components, drift $\mu$ and Gaussian covariance matrix $\Sigma = \text{diag}(\sigma^2_j)_{n \times n}$.

Now, let there be a matrix $A = (a_{ij})_{n \times n}$ such that $\Sigma^\rho = A \Sigma A^T$ and we choose $\mu$ so that $\mu^\rho = A\mu$, then

$$B^\rho(t) = (B^\rho_1(t), \ldots, B^\rho_n(t))^T = AB(t). \quad (3.1)$$

Define the process

$$B^\rho(s) := AB(s),$$

where $\{B(s), s \in \mathbb{R}^n_+\}$ is the independent multi-parameter Lévy process defined from a Brownian motion $B(t)$ with independent components and Lévy triplet $(\mu, \Sigma, 0)$. Semeraro (2008) proved that $\{B^\rho(s), s \in \mathbb{R}^n\}$ is an $\mathbb{R}^n$-parameter Lévy process.

At this point we can define the multi-parameter Gaussian kernel.

**Definition 3.1.** A multi-parameter Gaussian kernel $G$ is a multiparameter Lévy kernel from $(\mathbb{R}_+, B_n)$ into $(\mathbb{R}^n, B_n)$ such that

$$G(0, B) := P(B^\rho(0) \in B) = 1_B(0) \quad (3.2)$$
$$G(s, B) := P(B^\rho(s) \in B)$$

where $B \in B_n$, and $B^\rho(s)$ as defined above, is a multi-parameter Brownian motion with drift $\mu^\rho$ and Gaussian covariance matrix $\Sigma^\rho$.

**Definition 3.2.** A Gaussian marked multivariate Poisson process $Y$ is a Lévy marked multivariate Poisson process, where the conditional distribution of marks is defined by a multi-parameter Gaussian kernel.

We end this section by giving the characteristic function of a Gaussian-marked Poisson process derived in the Appendix B.1

$$\mathbb{E}[e^{it^T Y(t)}] = \exp\{t \Psi_\pi(\log \psi_B(z))\} \quad (3.3)$$

$$= \exp \left\{ t \Psi_\pi \left( i \mu_1 \sum_{i=1}^n a_{i1} z_i - \frac{1}{2} \sigma^2_1 \left( \sum_{i=1}^n a_{i1} z_i \right)^2, \ldots, i \mu_n \sum_{i=1}^n a_{n1} z_i - \frac{1}{2} \sigma^2_n \left( \sum_{i=1}^n a_{n1} z_i \right)^2 \right) \right\},$$

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where $\Psi_\pi$ is provided in (2.10). Hence, the characteristic function of the $k$-th asset log-return process is then
\[
\mathbb{E}[e^{iz_k Y_k(t)}] = \exp\{t \Psi_\pi(i\alpha_k \mu_1 z_k - \frac{1}{2} \alpha_k^2 \sigma_k^2 z_k^2, \ldots, i\alpha_k \mu_n z_k - \frac{1}{2} \alpha_k^2 \sigma_n^2 z_k^2)\}. \tag{3.4}
\]

### 3.2 Factor-based Poisson measure

We now introduce a factor Poisson random measure by assuming that trading activity has a common component and an idiosyncratic component.

Consider a set $A \in B(\mathbb{R}^n \setminus \{0\})$ and $\Delta_\alpha = \{(\alpha_1 s, \ldots, \alpha_n s)^T : s \in \mathbb{R}_+\}$ and $A_\alpha^0 = \text{Pr}_j(A \cap \Delta_\alpha)$, having $\text{Pr}_j$ be the projection of $A$ on the $j$-th coordinate axes. Since $\frac{A_\alpha^0}{\alpha_j} = \{s \in \mathbb{R} : \alpha_j s \in A_\alpha^0\}$, and by construction $\frac{A_\alpha^0}{\alpha_j} = \frac{A_\alpha^0}{\alpha_k}$, we define $A_\Delta := \frac{A_\alpha^0}{\alpha_j}$ for each $j, k = 1, \ldots, n$. Finally, let $A_j := A \cap D_j$ having $D_j = \{x \in \mathbb{R}^n : x_k = 0, k \neq j, k = 1, \ldots, n\}$.

**Definition 3.3.** A factor Poisson random measure $\Pi$ on $(\mathbb{R}_+ \times \mathbb{R}^n, B(\mathbb{R}_+ \times \mathbb{R}^n))$ is a Poisson random measure with mean $\text{Leb} \times \nu_\Pi$. The Lévy measure is as follows
\[
\nu_\Pi(A) = \sum_{j=1}^{n} \nu_j^f(A_j) + \nu_j^C(A_\Delta), \quad A \in B(\mathbb{R}^n \setminus \{0\}). \tag{3.5}
\]

The Lévy measure $\nu_\Pi$ has been firstly introduced in Semeraro (2008) as the Lévy measure of a factor based subordinator $\pi(t) = \pi^f(t) + \pi^C(t)$, defined by
\[
\pi(t) = (\pi^f_1(t) + \alpha_1 \pi^C(t), \ldots, \pi^f_n(t) + \alpha_n \pi^C(t)),
\]
where $\pi^f_j(t)$ and $\pi^C(t)$, for $j = 1, \ldots, n$, are independent subordinators with Lévy measures $\nu^f_j$ and $\nu^C$ respectively.

We recall here that for any $w = (w_1, \ldots, w_n)^T \in \mathbb{C}^n$ with $\Re(w_j) \leq 0$, $j = 1, \ldots, n$ $\Psi_\pi(t)$ is given by
\[
\Psi_\pi(w) = \sum_{j=1}^{n} \Psi^f_{\pi_j}(w_j) + \Psi_{\pi^C}\left(\sum_{j=1}^{n} \alpha_j w_j\right) \tag{3.6}
\]
where for any $w \in \mathbb{C}$ with $\Re(w) \leq 0$, $j = 1, \ldots, n$, $\Psi^f_{\pi_j}(w) = \int_{\mathbb{R}_+} (e^{(w,s)} - 1) \nu^f_{\pi_j}(ds)$ and $\Psi_{\pi^C}(w) = \int_{\mathbb{R}_+} (e^{(w,s)} - 1) \nu_{\pi^C}(ds)$.

Now we can define the Gaussian-marked factor-based Poisson process.

**Definition 3.4.** A Gaussian-marked factor-based Poisson process is a Gaussian-marked multivariate Poisson process $Y$, where the multivariate Poisson random measure $\Pi$ is a factor-based Poisson random measure.

Theorem 2.1 applies to Gaussian-marked factor-based Poisson processes. As a consequence they are multivariate subordinated Brownian motions. In fact it holds
\[
Y(t) = L_B^\rho(\pi(t)) := \left(\begin{array}{c} B^\rho_1(\pi_1(t), \ldots, \pi_n(t)) \\
\vdots \\
B^\rho_n(\pi_1(t), \ldots, \pi_n(t)) \end{array}\right) \tag{3.7}
\]
where $\pi(t)$ represents trading activity at time $t$, measured by $\Pi$. Each marginal return depends on the entire trading activity, since the returns conditional to trading activity are multiparameter Brownian motions, thus they are not uncorrelated.

By equation (3.3), where $\Psi_\pi$ is deduced by (3.6) we can easily derive the characteristic function of a Gaussian-marked factor-based Poisson process, which is

$$
\psi_{\pi(t)}(z) = \exp\{t \Psi_\pi(\log \psi_{B}(z))\} = \prod_{j=1}^{n} \Psi_{\pi_j(t)}\left(i \sum_{i=1}^{n} a_{ij}\mu_{j}z_{i} - \frac{1}{2}\sigma_{j}^{2}\left(\sum_{i=1}^{n} a_{ij}z_{i}\right)^{2}\right) \Psi_{\pi_C(t)}\left(\sum_{j=1}^{n} \alpha_{j} \left[i \sum_{i=1}^{n} a_{ij}\mu_{j}z_{i} - \frac{1}{2}\sigma_{j}^{2}\left(\sum_{i=1}^{n} a_{ij}z_{i}\right)^{2}\right]\right).
$$

The marginal processes, which model individual asset returns, are multiparameter process defined on $\mathbb{R}^n$, in fact the $k$-th log-return is modeled by

$$
B_{k}^{\pi}(\pi_{1}(t), \ldots, \pi_{n}(t)) = \sum_{i=1}^{n} a_{ki}B_{i}(\pi_{i}(t)),
$$

having the marginal characteristic function to be

$$
\psi_{Y_{k}(t)}(z_{k}) = \prod_{j=1}^{n} \Psi_{\pi_j(t)}\left(i a_{kj}\mu_{j}z_{k} - \frac{1}{2}a_{kj}^{2}\sigma_{j}^{2}z_{k}^{2}\right) \Psi_{\pi_C(t)}\left(\sum_{j=1}^{n} \alpha_{j} \left[i a_{kj}\mu_{j}z_{k} - \frac{1}{2}a_{kj}^{2}\sigma_{j}^{2}z_{k}^{2}\right]\right).
$$

Notice that the marginal distributions of returns depend on the joint distribution of $\pi(t)$. The dependence of marginal returns from the trading activity of the entire collection of assets is now evident.

### 3.2.1 Linear correlation

Using the representation of the Gaussian-marked factor-based Poisson processes as subordinated Brownian motions, we also find the return correlations. The correlation matrix $\rho = (\rho_{m,l})_{n \times n}$ can be derived, as presented in the Appendix, by using the total covariance formula and has entries

$$
\rho_{m,l}^{Y(t)} = \frac{\sum_{i=1}^{n} a_{mi}a_{lj}\sigma_{i}^{2}\mathbb{E}[\pi_{i}(t)] + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{mi}a_{lj}\mu_{i}\mu_{j}\alpha_{i}\alpha_{j}\mathbb{V}[\pi_{i}^{C}(t)]}{\sqrt{\mathbb{V}[Y_{m}(t)]\mathbb{V}[Y_{l}(t)]}},
$$

where

$$
\mathbb{V}[Y_{l}(t)] = \sum_{j=1}^{n} a_{kj}^{2}\mathbb{V}[\pi_{j}^{C}(t)] + \sum_{j=1}^{n} a_{k}\alpha_{j}\sigma_{j}^{2}\mathbb{V}[\pi_{j}^{C}(t)] + \sum_{j=1}^{n} a_{k}\alpha_{j}\sigma_{j}^{2}\mathbb{E}[\pi_{j}(t)].
$$

Notice that, by infinite divisibility $\mathbb{E}[\pi_{i}(t)] = t\mathbb{E}[\pi_{i}(1)]$, $\mathbb{V}[\pi_{i}^{C}(t)] = t\mathbb{V}[\pi_{i}^{C}(1)]$ and $\mathbb{V}[Y_{l}(t)] = t\mathbb{V}[Y_{l}(1)]$, thus $\rho_{m,l}^{Y(t)}$ is independent from $t$. The model correlations are flexible, since we can move independently return correlations and subordinator correlations. Furthermore, returns correlations are not bounded in absolute value from marks correlations neither from the subordinator correlations, as shown by considering the following limit cases.
a. Consider the limit case of conditional independence of marks and positively correlated subordinators. In this case $\rho_{Z_{m,l}} = 0$ and $\rho_{Y(t)} > 0$, where $\rho_{Z_{m,l}}$ is the correlation between the marks $Z_m$ and $Z_n$, conditional to $\Pi_2$. Thus $\rho_{Y(t)}$ is not bounded from $\rho_{Z_{m,l}}$. The case of negatively correlated marks is similar.

b. Consider the case of independence of the subordinator, i.e. the subordinator correlations $\rho_{\pi_{m,l}} = 0$ for any $m, l = 1, \ldots, n$, and positively correlated Gaussian marks, we have

$$\rho_{Y(t)} = \frac{\sum_{i=1}^{n} a_m a_l \sigma_i^2 E[\pi_i(1)]}{\sqrt{\text{Var}[Y_m(1)]} \sqrt{\text{Var}[Y_l(1)]}},$$

which is positive. Thus $\rho_{Y(t)} > \rho_{\pi_{m,l}}$. The case of negatively correlated Gaussian marks is similar.

4 The variance gamma specification

This section introduces a Gaussian-marked factor-based Poisson process of variance gamma ($VG$) type. We leave it to further research to introduce processes of normal inverse Gaussian ($NIG$), Carr Geman Madan and Yor ($CGMY$) and generalized hyperbolic ($GH$) type. Those different specifications could be obtained extending the definition of $Y$ to the unbounded variation case, using different Poisson random measures and Brownian motion parameters.

The univariate $VG$ process, $Y_{VG}$, introduced by Madan and Seneta (1990), is the basis of our multivariate generalization. Let $\{B(t), t \geq 0\}$ be a standard Brownian motion, in addition, let $\{\pi(t), t \geq 0\}$ be a gamma process with parameters $(\frac{1}{\alpha}, \frac{1}{\alpha})$, such that $E[\pi(t)] = t$ according to the intuition of economic time. The variance gamma process, $Y_{VG}$, defined as

$$Y_{VG}(t) := \mu \pi(t) + \sigma B(\pi(t)), \quad \mu \in \mathbb{R}, \quad \sigma > 0$$

is a process whose paths are of infinite activity and finite variation, having the characteristic function to be

$$\psi_{Y_{VG}}(z) = (1 - iz\mu\alpha + \frac{1}{2}\alpha\sigma^2 z^2)^{-\frac{1}{2}}. \quad (4.1)$$

We now specify the factor-based Poisson measure to be of gamma type. Let us assume that the factor based Poisson measure $\Pi$ has Lévy measure $\nu_\Pi$ defined in (3.5) such that $\nu_f$ are gamma measures with parameters $(\gamma_i, \beta_i)$ and $\nu_f^C$ is a gamma measure with parameters $(\gamma, \beta)$. Considering the properties of gamma distribution, the subordinator $\pi(t)$ has $i$-th marginal distribution to be gamma distribution if $\beta_i = \frac{\beta}{\alpha_i}$. In this case, $i$-th marginal distribution of $\pi(1)$ becomes a gamma distribution with parameters $(\gamma_i + \gamma, \frac{\beta}{\alpha_i})$. Under these assumptions the process $Y$ defined in (3.7) is named multivariate variance gamma marked Poisson ($mVGmP$) process. Note that interpretation of $\pi(t)$ as trading activity allows to omit the assumption $E[\pi_i(t)] = t$ and consider a Poisson random measure of gamma type without parameter constraints pertinent to this condition. This gives an additional degree of freedom.
We recall that for any $\boldsymbol{w} = (w_1, ..., w_n)^T \in \mathbb{C}^n$ with $\Re(w_j) \leq 0$, $j = 1, ..., n$ it easily follows from (3.6)

$$\Psi_{\pi}(\boldsymbol{w}) = \sum_{j=1}^{n} \log \left( 1 - \frac{w_j \alpha_j}{\beta} \right)^{-\gamma_j} + \log \left( 1 - \frac{\sum_{j=1}^{n} w_j \alpha_j}{\beta} \right)^{-\gamma}.$$  

(4.2)

By means of equations (3.8), where $\Psi_{\pi}$ is deduced by (4.2), the characteristic function for the $mVGmP$ process is,

$$\psi_{Y(t)}^{VG}(\boldsymbol{z}) = \prod_{j=1}^{n} \left[ 1 - \alpha_j \left( \frac{i \mu_j \sum_{i=1}^{n} a_{ij} z_i - \frac{1}{2} \sigma_j^2 (\sum_{i=1}^{n} a_{ij} z_i)^2}{\beta} \right)^{-\gamma_j t} + \left( 1 - \frac{\sum_{j=1}^{n} a_{ij} \alpha_j}{\beta} \right)^{-\gamma t} \right].$$  

(4.3)

Hence, the $k$-th marginal characteristic function is

$$\psi_{Y_k(t)}^{VG}(z_k) = \prod_{j=1}^{n} \left[ 1 - \frac{\alpha_k (i a_{kj} \mu_j z_k - \frac{1}{2} a_{kj}^2 \sigma_j^2 z_k^2)}{\beta} \right]^{-\gamma_j t}.$$  

(4.4)

and we observe that the marginal distributions depend on the common parameter $\gamma$, since they depend on the joint distribution of trading volume.

In the variance gamma case the asset returns correlation matrix $\rho = (\rho_{m,l})_{n \times n}$ has entries

$$\rho_{m,l}^{VG} = \frac{\sum_{i=1}^{n} a_{mi} a_{li} \sigma_i^2 \gamma_i \sigma_l^2 \gamma_l}{\sqrt{\Var[Y_1(1)]} \sqrt{\Var[Y_1(1)]}},$$

where $\Var[Y_1(1)] = t \left( \sum_{j=1}^{n} a_{kj}^2 \gamma_j \sigma_j^2 + \sum_{j=1}^{n} a_{kj} \gamma_j \sigma_j^2 \gamma_j \sigma_k^2 + \sum_{j=1}^{n} a_{kj}^2 \gamma_j \sigma_k^2 \gamma_j \sigma_k^2 \right)$. We now consider some subcases of interest which are obtained by considering the main assumptions on trading activity made in the literature. For instance, we assume that the $\pi(t)$ is a vector of change of times, thus $\mathbb{E}[\pi_j(t)] = t$, and we make the following choice of parameters: $\gamma_j = \frac{\beta_j}{\alpha_j} - \gamma$, where $j \in \{1, 2, \ldots, n\}$.

Common change of time to all assets. By considering the limit case $\gamma \to \frac{\beta_j}{\alpha_j}$ for each index $j$, the idiosyncratic components of $\pi(t)$ degenerate and we find the model with only one real subordinator introduced in Leoni and Schoutens (2007).

Independent time changes. We assume that trading activities of assets are independent, thus the common component of the Poisson measure degenerate by having $a = 0$. In addition let $\beta = 1$. Then, we have

$$\psi_{Y(t)}^{VG}(\boldsymbol{z}) = \prod_{j=1}^{n} \left[ 1 - \alpha_j \left( i \mu_j \sum_{i=1}^{n} a_{ij} z_i - \frac{1}{2} \sigma_j^2 (\sum_{i=1}^{n} a_{ij} z_i)^2 \right)^{-t/\alpha_j} \right].$$
Hence, the $k$-th marginal characteristic function is

$$
\psi_{\mathcal{V}G}^{V(t)}(z_k) = \prod_{j=1}^{n} \left[ 1 - \alpha_j \left( i\mu_j a_{kj} z_k - \frac{1}{2} \sigma_j^2 a_{kj}^2 z_k^2 \right) \right]^{-t/\alpha_j}.
$$

In Eberlein and Madan (2009) a similar framework is proposed. They introduced correlation by merely correlating the unit time random variables resulting for the marginal laws, leaving the subordinators to be independent and without constructing a joint process. More specifically, in the proposed model each return distribution depends only on the corresponding distribution of the time change whereas in our model each marginal return depends on the subordinator joint distribution.

**Independence of assets conditional to the time change.** Independence of assets conditional to the change of time is modelled by assuming conditionally independent Brownian marks. By assuming $a_{ij} = 0$ if $i \neq j$ and $a_{ii} = 1$ otherwise, the matrix $A$ in (3.1) becomes an identity matrix. In this case we recover the $\alpha$-variance gamma process as in Semeraro (2008).

**Common economic time and independence of assets conditional to the change of time.** Finally, by considering independent Brownian motions and the limit case $\gamma \to \frac{\beta}{\alpha_j}$, for each $j = 1, \ldots, n$ we find the model introduced in the literature by Madan and Seneta (1990) for the symmetric case and calibrated by Luciano and Schoutens (2006).

### 5 Conclusion

The class of Lévy marked Poisson processes to model asset returns was introduced. Intuitively, the Poisson random measure is a measure of trading activity on a collection of assets up to time $t$ and marks represent returns conditional on the trading activity. The class introduced allows the capture of the cross section properties of trades and, in addition, allows for returns correlations conditional on the trading activity. It was proved that the model is in law a Lévy motions constructed by subordinating a multiparameter Lévy process by a multivariate subordinator. In particular, it was proved that the Poisson random measure defines the multivariate subordinator, which represents economic time, and the marks are linked with the subordinand, which represents the distribution of returns conditional on trading activity at a given time $t$. The new class of processes is fully characterized through its Lévy triplet and the characteristic function is given in the closed form.

The economic intuition of normality of returns conditional on the trading activity was preserved by specifying normally distributed marks. According to the empirical findings in Lo and Wang (2000) of a strong common factor in trades, we specify a factor based Poisson measure, which defines a factor based multivariate subordinator. By so doing, we specify a subclass of models which generalize, for instance, the time changed Brownian motions in Semeraro (2008) and Luciano and Semeraro (2010) and span a wider range of linear correlations. As the first application, the variance gamma specification, obtained by choosing a Poisson measure of gamma type, was considered.

Further research aims at exploring different specifications of the Poisson measure and marks which may lead to processes of $NIG, GH, CGMY$ types.
Appendices

A Appendix

A.1 Proof of the Proposition 2.1

Let $\hat{Q}$ be a transition probability kernel from $(\mathcal{E} := \mathbb{R}^n 	imes \mathbb{R}^n, \mathcal{F} := B_{n+1})$ into $(\mathbb{R}^n, B_n)$, so that for each $t \in \mathbb{R}_+$ we have $\hat{Q}(t, s, dy) = Q(s, dy)$. From Theorem 3.2 in Çinlar (2011) it follows that the pair $T := (\Pi, Z)$ forms a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^n, B \otimes B_n)$ with mean $\mu_T = Leb \times \nu_\Pi \times \hat{Q}$, i.e. $\mu_T(dt, ds, dy) = dt \nu_\Pi(ds) \hat{Q}(t, s, dy)$.

Let now $N := (\Pi_1, Z)$, i.e. $N = \sum_{i \in I} 1_{[0, t] \times \mathcal{A}}(\Pi_i, Z_i)$. Then $N$ forms a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^n, B)$ with mean the margin on $\mathbb{R}_+ \times \mathbb{R}^n$ of the measure $T$, i.e. $\mu_N(dt, dy) = dt \int_{\mathbb{R}_+} \nu_\Pi(ds)Q(s, dy)$.

A.2 Proof of Theorem 2.1.

Let $L(s)$ be a multiparameter process with $L(L(s)) = Q(s, \cdot)$ and let it be independent from the subordinator $\pi(t)$. Now, let us consider the process $X(t) = L(\pi(t))$. The Theorem 4.7 in Barndorff-Nielsen et al. (2001) applies and $X(t)$ is a Lévy process with the following Lévy triplet

$$
\gamma_X = \int_{\mathbb{R}_+^n} \nu_\Pi(ds) \int_{|x| \leq 1} x \lambda^x(dx),
$$
$$
\Sigma_X = 0,
$$
$$
\nu_X(B) = \int_{\mathbb{R}_+} \nu_\Pi(ds)Q(s, B).
$$

By Proposition 2.1 the Poisson random measure $N$ has mean measure $\mu_N = Leb \times \nu_X$. Since $\nu_X$ is a Lévy measure the process $Y$ defined in equation (2.5) it is a pure jump Lévy process, i.e. $\Sigma_Y = 0$, with mean measure $\nu_Y = \nu_X$ and with zero drift, thus

$$
\gamma_Y = \int_{\mathbb{R}_+^n} \nu_\Pi(ds) \int_{|x| \leq 1} x Q(s, dx).
$$

Therefore $Y$ has the same Lévy triplet of $X$ and the assert is proved.

B Appendix

B.1 Derivation of the Equation (3.3)

Since

$$
B^\rho(s) := \begin{pmatrix}
B_1^\rho(s_1, \ldots, s_n) \\
\vdots \\
B_n^\rho(s_1, \ldots, s_n)
\end{pmatrix}
= \begin{pmatrix}
a_{11}B_1(s_1) + \cdots + a_{1n}B_n(s_n) \\
\vdots \\
a_{n1}B_1(s_1) + \cdots + a_{nn}B_n(s_n)
\end{pmatrix}
$$

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we have

\[ B^\theta(\delta_j) = B^\theta(0, \ldots, \frac{1}{j}, \ldots, 0) = \begin{pmatrix} a_{1j}B_j(1) \\ \vdots \\ a_{nj}B_j(1) \end{pmatrix} \]

thus for \( j = 1, \ldots, n \)

\[ \psi_j(z) = \mathbb{E}[\exp\{<B^\theta(\delta_j), z>\}] = \mathbb{E}\left[ \exp\left\{ i\sum_{i=1}^{n} a_{ij}B_j(1)z_i \right\} \right] \]

\[ = \mathbb{E}\left[ \exp\left\{ iB_j(1)\sum_{i=1}^{n} a_{ij}z_i \right\} \right] = \exp\left\{ i\mu_j \sum_{i=1}^{n} a_{ij}z_i - \frac{1}{2}\sigma_j^2 \left( \sum_{i=1}^{n} a_{ij}z_i \right)^2 \right\} \]

hence

\[ \log(\psi_{B^\theta}(z)) = (\log \psi_1(z), \ldots, \log \psi_n(z)) \]

\[ = \left( i\mu_1 \sum_{i=1}^{n} a_{i1}z_i - \frac{1}{2}\sigma_1^2 \left( \sum_{i=1}^{n} a_{i1}z_i \right)^2, \ldots, i\mu_n \sum_{i=1}^{n} a_{in}z_i - \frac{1}{2}\sigma_n^2 \left( \sum_{i=1}^{n} a_{in}z_i \right)^2 \right) \]

giving

\[ \psi_Y(t)(z) = \exp\left\{ t\Psi(\log \psi_{B^\theta}(z)) \right\} \]

\[ = \exp\left\{ t\Psi\left( i\mu_1 \sum_{i=1}^{n} a_{i1}z_i - \frac{1}{2}\sigma_1^2 \left( \sum_{i=1}^{n} a_{i1}z_i \right)^2, \ldots, i\mu_n \sum_{i=1}^{n} a_{in}z_i - \frac{1}{2}\sigma_n^2 \left( \sum_{i=1}^{n} a_{in}z_i \right)^2 \right) \right\}. \]

**B.2 Derivation of the Equation (3.10).**

Given the law of total covariance we have

\[ \text{Cov}[B^\theta_m(\pi(t)), B^\theta_l(\pi(t))] = \mathbb{E}[\text{Cov}[B^\theta_m(\pi(t)), B^\theta_l(\pi(t)) \mid \pi(t)))]

\[ + \text{Cov} [\mathbb{E}(B^\theta_m(\pi(t)) \mid \pi(t)), \mathbb{E}(B^\theta_l(\pi(t)) \mid \pi(t))]. \]

Since we have

\[ \mathbb{E}[B^\theta_m(\pi(t)) \mid \pi(t)] = \mathbb{E}\left[ \sum_{j=1}^{n} a_{mj}B_j(\pi_j(t)) \mid \pi(t) \right] \]

\[ = \sum_{j=1}^{n} a_{mj}\mathbb{E}[B_j(\pi_j(t)) \mid \pi(t)] = \sum_{i=1}^{n} a_{mi}\mu_i\pi_i(t) \]

then

\[ \text{Cov} [\mathbb{E}[B^\theta_m(\pi(t)) \mid \pi(t)), \mathbb{E}[B^\theta_l(\pi(t)) \mid \pi(t)]] \]

\[ = \text{Cov} \left[ \sum_{i=1}^{n} a_{mi}\mu_i\pi_i(t), \sum_{i=1}^{n} a_{li}\mu_i\pi_i(t) \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{mi}a_{lj}\mu_i\mu_j\text{Cov}[\pi_i(t), \pi_j(t)]. \]
Now for a given realization of $\pi(t) = s$ we have

$$\mathcal{L}(B^{\rho}(\pi(t)) \mid \pi(t) = s) \sim \mathcal{N}(\mu^{\rho}(s), \Sigma^{\rho}(s)),$$

where $\Sigma^{\rho}(s) = A \Sigma(s) A^{T}$, hence

$$\text{Cov}[B^{\rho}_{m}(\pi(t)), B^{\rho}_{l}(\pi(t)) \mid \pi(t)] = \Sigma^{\rho}(\pi(t))_{m,l}$$

i.e. the $m, l$-th entry of $\Sigma^{\rho}(\pi(t))$ matrix.

Hence

$$\mathbb{E}[\Sigma^{\rho}(\pi(t))_{m,l}] = \sum_{i=1}^{n} a_{mi} a_{li} \sigma_{i}^{2} \mathbb{E}[\pi_{i}(t)].$$

Finally,

$$\text{Cov}[B^{\rho}_{m}(\pi(t)), B^{\rho}_{l}(\pi(t))] = \sum_{i=1}^{n} a_{mi} a_{li} \sigma_{i}^{2} \mathbb{E}[\pi_{i}(t)] + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{mi} a_{lj} \mu_{i} \mu_{j} \text{Cov}[\pi_{i}(t), \pi_{j}(t)]$$

giving

$$\rho_{m,l}^{Y_{(t)}} = \frac{\sum_{i=1}^{n} a_{mi} a_{li} \sigma_{i}^{2} \mathbb{E}[\pi_{i}(t)] + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{mi} a_{lj} \mu_{i} \mu_{j} \text{Cov}[\pi_{i}(t), \pi_{j}(t)]}{\sqrt{\text{Var}[Y_{m}(t)]} \sqrt{\text{Var}[Y_{l}(t)]}}$$

and therefore

$$\rho_{m,l}^{Y_{(t)}} = \frac{\sum_{i=1}^{n} a_{mi} a_{li} \sigma_{i}^{2} \mathbb{E}[\pi_{i}(t)] + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{mi} a_{lj} \mu_{i} \mu_{j} \alpha_{i} \alpha_{j} \text{Var}[\pi_{C}(t)]}{\sqrt{\text{Var}[Y_{m}(t)]} \sqrt{\text{Var}[Y_{l}(t)]}}.$$
References


