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ABSTRACT

Intuition and leading equilibrium models are at odds with the empirical evidence that expected returns are weakly related to volatility at the market level. This paper proposes a closed-form general equilibrium model, which connects the investors’ expectations of fundamentals with those of market returns, as documented by survey data. Forecasts suggest that investors feature pro-cyclical optimism and, then, overestimate the persistence of aggregate risk. The forward-looking component of stock volatility offset the transient risk and leads to a weak risk-return relation, in line with survey data about market returns. The model mechanism is robust to many features of financial markets.

Keywords: risk-return trade off · survey expectations · general equilibrium · optimism · asset pricing puzzles · heterogeneous preferences · closed-form expression

JEL Classification: D51 · D53 · D83 · G11 · G12

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I. Introduction

Standard equilibrium consumer models fail to explain the main properties of asset prices and their connection with the macroeconomy. Most of these models rely on the assumption that market participants rationally maximize their expected utility under the true probabilities of uncertain economic states. However, economic fundamentals are not directly observable: a mounting evidence suggests the presence in the financial markets of traders with biased beliefs and a growing literature studies if such a presence has a price impact or can be neglected. Instead, one aspect that has attracted little investigation, so far, concerns the role of the investors’ survey expectations in the context of equilibrium asset pricing. Namely, the literature has usually disregarded how survey expectations –that is, the actual expectations computed by the investors over time– relate with the economic conditions and which effects they produce on the equilibrium dynamics of asset prices.\(^1\)

The aim of this paper is to question whether a stylized representation of the investors’ survey expectations about fundamentals can help to explain in equilibrium a stylized representation of the investors’ survey expectations of market returns, as well as some standard asset pricing puzzles. In particular, I focus on the key relation between market risk and its expected remuneration, that is the risk-return trade off and its dynamics.

Survey data from professional forecasters (SPF) suggest that investors’ beliefs about economic growth (i.e. quarterly GDP growth rates) are barely unbiased in average but feature pro-cyclical optimism. The latter implies that investors are too optimistic in good times and too pessimistic in bad times and, in turn, they overestimate the persistence of the economic conditions (Baker and Wurgler (2006, 2007)). I use such survey data to parsimoniously characterize the investors’ beliefs about fundamentals.\(^2\) Namely, I consider a closed-form and parsimonious continuous-time general equilibrium model with investors who have unbiased beliefs about economic growth in the long-run, but are optimistic and pessimistic respectively in good and bad times, consistently with the empirical evidence. The analysis bases on a single parameter capturing the strength of the bias in beliefs and shows that the model can capture many features of the asset returns. In particular, the equilibrium leads to a weak or negative risk-return trade off of the stock markets, in line with the empirical data but at odds with standard finance theory, such as the dynamic CAPM of Merton (1980) and more recent leading asset pricing models.

Lettau and Ludvigson (2010) show that the unconditional correlation between expected excess returns and return volatility is negative, whereas general equilibrium models able to produce a time-varying Sharpe ratio –such as the

\(^1\)Recently, Bacchetta, Mertens, and van Wincoop (2009) and Greenwood and Shleifer (2014) have empirically documented the importance, the robustness and the incompatibility with rational expectation models of actual expectations as proxied by survey data.

\(^2\)Recently, Patton and Timmermann (2010) study the evolution of survey expectations about economic growth over the business cycle.
models by Campbell and Cochrane (1999), Barberis, Huang, and Santos (2001) and Bansal and Yaron (2004) predict a strongly positive relationship. The left panel of Figure 1 shows the risk-return trade off through a scatter plot when conditional moments are computed using a small set of instruments as conditioning information.

Insert Figure 1 about here.

In particular, the analysis of Lettau and Ludvigson (2010) documents that the smaller the information set, the more negative the risk-return trade off: intuitively, the negative relationship arises as long as investors overweight simple measures of economic conditions, e.g. the cay, when computing expectations. Such a result is confirmed by survey data. Duke/CFO survey is, to the best of my knowledge, the only data source providing information about the investors’ expectations of both the first and the second moment of market returns. These survey data suggest that investors’ expectations about market returns lead to a nonlinear risk-return trade off, as documented in Graham and Harvey (2001, 2012) and as shown in the right panel of Figure 1.

I document that this nonlinear, namely hump-shaped, dynamics is a result of the pro-cyclical variation of the trade off: market risk and its expected remuneration are positively and negatively correlated respectively in good and bad times. Such a dynamics generates a weak or even negative unconditional trade off and cannot obtain in leading equilibrium models. Such an empirical evidence suggests that the risk-return relation is closely connected with the formation of investors’ beliefs about fundamentals. This paper proposes a general equilibrium model which generates such a connection. Beliefs deviate from the Bayesian assessment generating an alternation of waves of optimism and pessimism. The stronger such a deviation, the more pro-cyclical the relation between market risk and its expected remuneration and, hence, the lower and even negative their unconditional correlation.

To my knowledge, this is the first work providing a rationale for the risk-return trade off and its dynamics in terms of agents behavior and aggregation result. In particular the model offers a theoretical link among the survey expectations of fundamentals and those of market returns. That is the most authentic empirical guidance about respectively the input and the output of an equilibrium asset pricing model. Surprisingly, beyond a huge empirical literature, there is a lack of general equilibrium literature concerning the relationship between the first two moments of market returns.

A notable exception is Whitelaw (2000). His model focuses on the physical dynamics of consumption growth in a

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3 A correlation still negative but close to zero can obtain extending the conditioning information set to a broad battery of financial indicators. At the opposite, if the information set reduces to the only cay as predictive variable (in addition to the lags), the correlation becomes strongly negative, about -0.70. See Lettau and Ludvigson (2010) and references therein.

4 Survey data seem consistent with most recent works using realized returns: Rossi and Timmermann (2015) find strong evidence of non-monotonic hump-shaped risk return relation, Ghosh and Linton (2012) account for non-linearities by verifying structural breaks and Lustig and Verdelhan (2012) document that variations of the first moment of returns are more closely related to business cycle than those of return volatilities.

5 Backus and Gregory (1993) show that the risk-return relation is potentially largely unrestricted from a theoretical point of view.
representative agent model under full information and captures a rich risk-return trade off through multiple regimes with state-dependent transition probabilities. Therefore, the approach that I propose in this paper is both alternative and complementary to the model of Whitelaw (2000). However, my model –which preserves time-separable utility and a simple, namely homoscedastic, specification of fundamentals– also captures many other empirical patterns of asset returns, such as unconditional and conditional moments and long-horizon predictability, which instead fail to obtain in Whitelaw (2000).6

There are three main ingredients which produce the economic implications of the model. First, to model pro-cyclical optimism in the formation of beliefs, I need that agents recognize the state of the economy through a simple state-variable, akin a business cycle indicator. A well known approach in the finance literature is external (linear) habit. I assume that investors feature “catching up with the Joneses” preferences (CuJ, hereafter), in spirit of Abel (1990, 1999): agents have constant relative risk aversion but interpret good and bad states of the economy through the ratio of aggregate consumption over its smoothed past evolution.

Second, to study the risk-return trade off, I need the model generates time-variation in the price of risk. This can obtain exogenously by nonlinear habit as in Campbell and Cochrane (1999) or endogenously by preference heterogeneity as in Chan and Kogan (2002). I follow the latter approach since it provides an economically pertinent micro-foundation as well as additional testable implications, which impose discipline to the calibration of the model. Namely, under preference heterogeneity the cross-sectional distributions of consumption and wealth vary over time and, in turn, lead endogenously to counter-cyclical aggregate risk aversion and price of risk, in line with the real data.

The third main ingredient of the model concerns the formation of investors’ beliefs and represents the main novelty of the paper. I extend the framework by Chan and Kogan (2002) along two dimensions. On the one hand, expected growth is stochastic and mean reverting. On the other hand, information is incomplete: investors do not observe the aggregate consumption drift but can infer information from the consumption realizations and a noisy signal. The main model mechanism is the following. Investors’ beliefs deviate from the Bayesian assessment by a simple term driven by the habit state-variable. Namely, waves of optimism and pessimism alternate over time.7 In turn, beliefs affect investors’ perception about the future evolution of the habit state-variable. Good (bad) states are excessively persistent when investors are optimistic (pessimistic). Such a mutual feedback effect between the habit state-variable and expected growth makes these variables more volatile, more persistent and more positively correlated than in the Bayesian case.

6As suggested by Whitelaw (2000) himself, a more sophisticated modelling of beliefs and preferences could improve many results of his framework, which enhances the complementarity of the two approaches.

7Alternation of optimism and pessimism as a driving source of the business cycle and the financial markets has been studied since the seminal works of Pigou (1920) and Mitchell (1927).
This leads to equilibrium asset prices which help to explain a number of empirical patterns of the stock markets. The risk-return trade-off is as follows. Under Bayesian beliefs, return volatility is always counter-cyclical. This obtains because transient risk—which inherits the counter-cyclical dynamics of the price of risk—dominates the pro-cyclical variation of the forward-looking component of volatility. The latter is given by the elasticity of prices with respect to the current state. Under pro-cyclical optimism, the habit state-variable is excessively persistent and, hence, the forward-looking component of volatility becomes more important. Such an effect is stronger in bad times because risk aversion makes prices more sensitive to the state. Hence, the premium and the return volatility are both counter-cyclical in good times, but the premium is more counter-cyclical than volatility in bad times. As a consequence, the risk-return relation switches from positive to negative in bad times, leading to a pro-cyclical dynamics and to a low or negative unconditional level.

Therefore, the model endogenizes the dynamic risk-return trade-off from survey data.

Although simple and parsimonious the model mechanism is robust to many other aspects of financial markets. Indeed, in addition to i) the weak or negative risk-return relation and its pro-cyclical dynamics, the model helps to explain ii) the equity premium and risk-free rate puzzles, iii) the smooth dynamics of the risk-free rate and the excess volatility of risky returns, iv) the stronger persistence of the dividend yield than that of the risk-free rate, v) the long-horizon predictability of excess returns by dividend yields without predictable dividend growth, vi) the low and upward sloping term structure of interest rates with low and downward sloping volatilities, vii) the counter-cyclical behavior of the price of risk and of the equity premium, viii) the counter-cyclical dynamics of trading volume. Furthermore, the model ix) matches the little dispersion of the cross-sectional distribution of consumption shares and x) features pro-cyclical optimism about economic growth. These ten empirical patterns, on the one hand, provide robustness to the model and, on the other hand, suggest that a stylized representation of survey data about fundamentals is a key ingredient of an equilibrium asset pricing model.

Finally, it is worth emphasizing that, to my knowledge, this is the first paper in which closed form expressions are available for the pricing kernel, the consumption and portfolio strategies, the asset prices and their moments in a multiple-agents economy under incomplete information with a realistic pattern of preference heterogeneity. On the one hand, an in-depth analysis of equilibrium outcomes is feasible. On the other hand, since the equilibrium is stationary, an easy calibration of cash-flows and asset pricing quantities is possible.

The paper is organized as follows. Section II provides further empirical and theoretical motivation to the model and relates the paper to the literature. Section III describes the economy and characterizes the formation of beliefs. Sections IV and V respectively solve for the equilibrium and derive asset prices. Section VI calibrates the model and investigates the asset pricing implications. Section VII highlights potential extensions. Section VIII concludes.
II. Empirical Evidence, Model Intuition and Related Literature

A. Survey Expectations

A.1. Expectations about fundamentals

A large literature studies survey data of investors' forecast: Pesaran and Weale (2006) provide a recent review. La Porta (1996) and, more recently, Bergman and Roychowdhury (2008) study the forecast errors and their findings are consistent with the idea that agents are excessively optimistic and pessimistic respectively in good and bad times and, therefore, they underestimate long-run mean reversion. Moreover, individual beliefs are heterogeneous: Veldkamp (2005), Van Nieuwerburgh and Veldkamp (2006) and Patton and Timmermann (2010) document that forecasts dispersion moves with the business cycle. Investors’ priors, their heterogeneity and their cyclicality jointly determine the “average belief” about economic growth. In particular, aggregation of individual agents seems to lead to a sub-optimal or irrational evolution of the “average belief,” even if individual investors were rational. Therefore, pro-cyclical optimism in aggregate beliefs is not necessarily at odds with individual rational behavior.

The main assumption of the model concerns the pro-cyclical optimism of investors’ beliefs about economic growth. In order to provide empirical support to such an assumption, I consider the following simple model—which is similar to Patton and Timmermann (2010). Let the aggregate output be a random walk: $Y_{t+1} = Y_t + \mu_t + \sigma_y \epsilon_y, t+1$, with unobservable drift $\mu_{t+1} = (1-\kappa)\mu_t + \sigma_\mu \epsilon_\mu, t+1$. Moreover, define a simple business cycle indicator given by:

$$\omega_{t+1} = (1 - \lambda)\omega_t + \Delta Y_{t+1}. \quad (1)$$

Then, denote the “optimal forecast” $\hat{\mu}_{t+1}^{KF}$ as the Kalman filter estimate of expected growth from the observations $\{Y_t\}$. I assume that the average belief obtains by combining the optimal forecast $\hat{\mu}_{t+1}^{KF}$ with a prior $\mu_{t+1}^{Prior}$:

$$\hat{\mu}_{t+1} = \alpha_t \mu_{t+1}^{Prior} + (1 - \alpha_t)\hat{\mu}_{t+1}^{KF}. \quad (2)$$

In line with Patton and Timmermann (2010), I set both the prior and the weight to be potentially state-dependent:

$$\mu_{t+1}^{Prior} = a_0 + a_1 (\omega_t - \bar{\omega}), \quad (3)$$

$$\alpha_t = \frac{E[e_t^2]}{(q_t + E[e_t^2])}, \quad \text{with} \quad \log(q_t) = b_0 + b_1 |\omega_t - \bar{\omega}|. \quad (4)$$
where $\bar{\omega} = \mathbb{E}[\omega_t]$ and $e_t$ is the model error: $\hat{\mu}_{it+1}^\text{Survey} - \hat{\mu}_{it+1}$. I denote with $\hat{\mu}_{it+1}^\text{Survey}$ the average forecast from the Survey of Professional Forecasters. I estimate this model by maximum likelihood using quarterly data on the sample 1968-2005. Table I reports the estimates for several values of the business cycle parameter $\lambda$.\footnote{The term $\hat{\mu}_{it+1}^\text{Prior}$ is not formally a prior since the left hand side of Eq. (2) corresponds to an aggregate quantity. However, it is intended to capture in reduced form the cyclicality due to the aggregation of individual but heterogeneous beliefs (see Patton and Timmermann (2010)).}

Both $a_1$ and $b_1$ are statistically significant and respectively positive and negative. This means that the prior, $\hat{\mu}_{it+1}^\text{Prior}$, is pro-cyclical and that investors put more weight, $\alpha_t$, on the prior when $\omega_t$ is far from its average, i.e., when agents face a scenario they are not experienced with. The resulting forecasts $\{\hat{\mu}_{it+1}\}$ reduce the historical mean square error produced by $\{\hat{\mu}_{it+1}^\text{KF}\}$ for various levels of the parameter $\lambda$. These estimates support the idea that investors feature pro-cyclical optimism.

Notice that asset pricing implications can be considerable even if $\{\hat{\mu}_{it+1}\}$ differs little from $\{\hat{\mu}_{it+1}^\text{KF}\}$. For $a_1 \neq 0, b_1 \neq 0$, beliefs driven by Eq. (2) lead to a first feedback effect from $\omega_t$ to $\hat{\mu}_{it+1}$. However, also a second feedback effect takes place in the mind of the investors. Namely, the investors belief $\hat{\mu}_{it+1}$ affects the perceived evolution of the economy $\Delta \omega_{t+2}$. This can be observed by writing Eq. (1) as $\Delta \omega_{t+2} = \lambda (\hat{\omega}_{t+1} - \omega_{t+1}) + \sigma_y \epsilon_{yt+2}$. The mean-reversion threshold $\hat{\omega}_{t+1} = \hat{\mu}_{it+1} / \lambda$ is increasing with the investors belief $\hat{\mu}_{it+1}$. Therefore, for $a_1 > 0, b_1 < 0$, the persistence of the business cycle indicator $\omega_t$ is overestimated because the mean-reversion threshold $\hat{\omega}_t$ is too high and too low respectively in good a bad times. In summary, a stylized representation of survey data (i.e. $a_1 > 0, b_1 < 0$) leads to a mutual feedback between $\omega_t$ and $\hat{\mu}_{it+1}$ and makes these variables more persistent, more volatile and more correlated. Asset prices will reflect investors beliefs about both expected growth (the so-called cash-flows channel) as well as business cycle (the so-called discount rate channel). I will show that this mechanism is at the heart of the risk-return trade off, once embedded in a general equilibrium model. The next scheme highlights the formation of beliefs implied by the model.
A.2. Expectations about market returns

The risk-return trade off of the market is captured by the unconditional correlation between the expected excess returns and the return volatility, that is the unconditional correlation among the two components of the Sharpe ratio. There is some disagreement among studies that seek to determine the empirical relation between these two components.\(^9\) Recently, Lettau and Ludvigson (2010) document that such a disagreement is likely associated to the amount of conditioning information used in empirical studies. They verify that when using a small information set the correlation between the conditional mean and conditional volatility is strongly negative. When the information set is widely extended to an array of financial indicators, the correlation is still negative but not necessarily statistically different from zero. Harvey (2001) and Graham and Harvey (2001) also document a negative correlation.

These works focus on the risk-return trade off, as it is computed by the "econometrician", and provide very puzzling evidence. Indeed, such a weak or negative relation obtains under various specifications of conditioning information and, then, appears as a robust result. However, the true risk-return trade off—that is the relation based on the actual expectations of the agents, in contrast to those computed ex-post by the "econometrician"—has been yet neither investigated in the empirical literature nor explained by equilibrium models.

Investors’ expectations about market returns from Duke/CFO survey data allow to document a number of stylized facts. Table II reports estimates of the regression of survey expectations on a business cycle indicator.

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Panel A} & \textbf{Panel B} & \\
\hline
\end{tabular}
\end{table}

Insert Table II about here.

Namely, expected premia (\(\mu_P^t - r_t\)) and volatility (\(\sigma_P^t\)) are regressed on a state variable, \(\omega_t\), computed as in Eq. (1):

\[
\begin{align*}
\mu_P^t - r_t &= \alpha_0 + \beta_0 \omega_t + \varepsilon_t, \\
\sigma_P^t &= \alpha_0 + \beta_0 \omega_t + \varepsilon_t
\end{align*}
\]

The first column of Panels A and B show that \(\omega_t\) strongly negatively explains variation in the expected mean and volatility of market returns (\(\beta_0 < 0\)). Therefore, survey expectations are counter-cyclical. The second column of Panels A and B includes in the regressions a nonlinear function of \(\omega_t\):

\[
\begin{align*}
\mu_P^t - r_t &= \alpha_0 + \alpha_1 \mathbb{1}_{\omega_t < \bar{\omega}} + \beta_0 \omega_t + \beta_1 \omega_t \mathbb{1}_{\omega_t < \bar{\omega}} + \varepsilon_t, \\
\sigma_P^t &= \alpha_0 + \alpha_1 \mathbb{1}_{\omega_t < \bar{\omega}} + \beta_0 \omega_t + \beta_1 \omega_t \mathbb{1}_{\omega_t < \bar{\omega}} + \varepsilon_t
\end{align*}
\]

This allows to verify the dynamics of survey expectations over the business cycle. We observe that the negative relation between expected return and the state-variable $\omega_t$ does not vary with the level of $\omega_t$ ($\beta_0 < 0$, $\beta_1 \approx 0$). Instead, the correlation between expected volatility and $\omega_t$ switches from strongly negative ($\beta_0 < 0$) to close to zero when economic conditions deteriorates ($\beta_1 \approx -\beta_0 > 0$).

Finally, panel C reports the estimates from the regression of expected return on expected volatility:

$$\mu_t^P - r_t = \alpha_0 + \beta_0 \sigma_t^P + \epsilon_t$$

In the first column we observe that the unconditional risk-return relation is not statistically different from zero ($\beta_0 \approx 0$). The second column shows that, conditioning on the level of $\omega_t$, we can recover a strongly positive trade off in good times ($\beta_0 > 0$) and a strongly negative trade off in bad times ($\beta_1 < \beta_0 < 0$). Such a pro-cyclical dynamics explains the low unconditional risk-return relation and is due to the lack of counter-cyclical dynamics of expected volatility in bad times.

While the data sample is probably too small to assess an exact value for the unconditional trade off, it suggests that survey expectations are at odds with the strongly positive risk-return relation implied by equilibrium models. Instead, the pro-cyclical dynamics of the trade off seems a robust feature of the data: indeed, the sample covers the various phases of the business cycle, including the recent financial crisis.

The three scatter plots of Figure 2 report the fit from the regressions of Table II (both first and second column).

In particular, the right panel of Figure 2 clearly shows how the risk-return trade off features a pronounced positive or negative sign, depending on the level of the state-variable.

I show that, in general equilibrium, a parsimonious characterization of investors’ beliefs about fundamentals, in line with the model of Eq. (2)-(3)-(4), leads to an endogenous offsetting mechanism among the components of the return volatility, which exactly captures a weak or negative risk-return trade off and its pro-cyclical dynamics, as documented by survey data of market returns.

**B. Other Related Literature**

Behavioral finance offers empirical and theoretical arguments that cast doubts on the hypothesis that security markets are fully justified by economically pertinent news, despite market imperfections. Among others, *Saunders (1993), Elster (1998)*,
Hirshleifer and Shumway (2003) and Baker and Wurgler (2007) document that markets are systematically influenced by investors’ psychology and mood. Charness and Levin (2005) stress that the processing of new information involves some form of reinforcement or extrapolation which often produces deviations from the Bayesian updating and makes choices more conform with recent economic conditions. Similar results are confirmed by neuroeconomics research as in Kuhnen and Knutson (2011). In particular, Shefrin (2008) discusses several studies concerning the perception that risk and return are negatively related.

In the finance literature, Cecchetti, Lam, and Mark (2000) and Abel (2002) show that models in which consumers who exhibit pessimism and doubt could help to explain the average stock return and risk-free rate. More recently, Fuster, Laibson, and Mendel (2010) and Fuster, Hebert, and Laibson (2011) introduce the so-called natural expectations: agents form expectations based on a simpler and more intuitive model than the true one, then they adjust towards rational expectations. The appeal of simple models, the cost of processing information and cognitive biases are used to justify an incomplete adjustment. Extrapolative expectations lead to similar results and are investigated by Hirshleifer and Yu (2011) and Barberis, Greenwood, Jin, and Shleifer (2015) in general equilibrium. Dumas, Kurshev, and Uppal (2009) extend Scheinkman and Xiong (2003) assuming that, when estimating the expected growth, some agents overreact to a noisy signal and, in turn, generate a sentiment risk. In particular, Xiong and Yan (2010) and Jouini and Napp (2010) show that investors with irrational beliefs survive and have price impact as long as they are rational on average, in the sense that incorrect beliefs are symmetric and there is not a long-run bias at the aggregate level. Barone-Adesi, Mancini, and Shefrin (2012) empirically verify that excessive optimism and overconfidence lead to the negative risk-return relation, in line with survey data (e.g. Duke/CFO survey). Baker and Wurgler (2006) and Yu and Yuan (2011) empirically document how investors’ sentiment affects respectively the cross-section of equity return and the risk-return trade off. Duarte, Kogan, and Livdan (2012) recover a negative risk-return relation by assuming preference shocks with time-varying volatility correlated with output. This paper complements the above literature about the role of biased beliefs in financial markets. As a peculiarity, this paper uses survey data about fundamentals as an empirical input of an equilibrium model, whose output (i.e. asset prices) can explain some empirical patterns of survey data about market returns.

This paper also relates with the literature of preference heterogeneity. In particular, Chan and Kogan (2002) show that preference heterogeneity can lead to a number of interesting asset pricing implications, such as countercyclical price of risk and equity premium, and therefore can endogenize some of the results of Campbell and Cochrane (1999).

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Nonetheless, Xiouros and Zapatero (2010) find that just a marginal effect on asset price dynamics does obtain when the model is calibrated with a realistic amount of heterogeneity in risk attitudes. This paper shows that, even under a realistic pattern of heterogeneity, the results of Chan and Kogan (2002) continue to hold because of the peculiar formation of beliefs. A non-negligible re-distribution of wealth among agents takes place because, under their subjective probability, investors overestimate risk and persistence of economic conditions. This leads to endogenous fluctuations in asset prices and also to a weak or negative risk-return trade off.

III. The Model

In a continuous-time infinite-horizon pure exchange economy of Lucas (1978) type, agents feature “catching up with the Joneses” (CuJ) preferences in spirit of Abel (1990). I study both the case of homogeneous agents and the case of agents, who differ from each other with respect to their relative risk aversion as in Chan and Kogan (2002).

A. Preferences and Heterogeneity

In the economy, infinitely many investors maximize expected utility of the form

$$\hat{E}_0 \left[ \int_0^\infty e^{-\rho t} U(C_t, X_t; \gamma) dt \right]$$

where

$$U(C_t, X_t; \gamma) = \frac{1}{1-\gamma} C_t^{1-\gamma} X_t^\gamma$$

which implies $$U_{CX}(C_t, X_t; \gamma) = \gamma C_t^{-\gamma} X_t^{\gamma-1} > 0$$. Thus, for all the agents, the standard of living $$X_t$$ and the current consumption $$C_t$$ act as complementary goods. All agents in the economy have the same time discount rate $$\rho$$ but they differ with respect to their preference parameter $$\gamma$$.

Since in general closed form solutions are not feasible under heterogeneous risk aversion, I assume a specific pattern of heterogeneity to obtain explicit solutions for the optimal consumption and portfolios, the state-price density, the risky asset prices and their moments. I assume that agents have relative risk aversion in the interval between zero and an arbitrary upper bound $$\bar{\gamma}$$. Each agent type $$i$$ is characterized by its constant relative risk aversion and social weight:
\[ \gamma_i = \frac{\bar{\gamma}}{i} \quad \forall i \in \mathbb{N}, \quad \text{with} \quad \bar{\gamma} \in \mathbb{N}, \quad \frac{\nu_i}{\nu_j} = p^{\bar{\gamma} - \gamma_j}, \quad p > 0, \quad \forall i, j \in \mathbb{N}. \]  

(7)

Therefore, the distribution of types is defined on a realistic range of values, \((0, \bar{\gamma})\). Moreover, the average value and the variation of the representative agent’s risk aversion will be characterized in closed form. Section VI.B shows that for reasonable parameters the equilibrium distribution of consumption-shares closely approximates individual consumption data. The homogeneous-agents economy obtains by assuming that all social weights but one are equal to zero.

**B. Primitives**

The aggregate dividend, \(Y_t\), serves as a numeraire and follows the stochastic process:

\[
\begin{align*}
\frac{dY_t}{Y_t} &= \mu_t dt + \sigma_y dB_{y,t}, \\
\frac{d\mu_t}{\mu_t} &= \kappa(\bar{\mu} - \mu_t) dt + \sigma_\mu dB_{\mu,t}.
\end{align*}
\]

(8) \hspace{1cm} (9)

Expected growth, \(\mu_t\), is stochastic and follows an Ornstein-Uhlenbeck (OU) process with long run mean \(\bar{\mu}\) and speed of mean reversion \(\kappa\). The volatilities \(\sigma_y\) and \(\sigma_\mu\) are constant and the standard Brownian motions \(B_{y,t}\) and \(B_{\mu,t}\) are mutually independent for the sake of exposition. Eq. (8)-(9) represent the most common setting in the literature on asset pricing under incomplete information (see, e.g., Kim and Omberg (1996), Brennan and Xia (2001), Pastor and Veronesi (2009)).

As usual in habit models, the standard of living process is a weighted geometric average of past realizations of the aggregate dividend:

\[ x_t = x_0 e^{-\lambda t} + \lambda \int_0^t e^{-\lambda(t-u)} y_u du, \quad \text{such that} \quad dx_t = \lambda(y_t - x_t) dt, \]

(10)

where \(y_t = \log Y_t, \quad x_t = \log X_t\) and \(x_0, \lambda > 0\). The habit state variable, \(\omega_t = y_t - x_t\), has the interpretation of a business cycle indicator. Indeed, it characterizes good and bad times in the economy in the mind of agents with CuJ preferences.\(^{12}\)

An application of Itô’s lemma shows that

\[ d\omega_t = -\lambda \omega_t dt + dy_t = \lambda(\bar{\omega} - \omega_t) dt + \sigma_y dB_{y,t}, \]

(11)

where \(\bar{\omega} = (\mu_t - \sigma_y^2/2)/\lambda\). Thus, \(\omega_t\) is a mean reverting process and is conditionally normally distributed.

\(^{11}\) I set \(p = e^{\bar{\omega}}\) with \(\bar{\omega} = \mathbb{E}_0[\log Y_t/X_t]\) to center the sensitivity of the price of risk at the steady-state. See Section V for the details.

\(^{12}\) Section IV shows that individual marginal utilities as well as the representative agent’s marginal utility evaluated at optimal consumption are functions of \(\bar{\omega}\) only.
I assume incomplete information in the sense that investors cannot observe $\mu_t$. However, they have to infer it from observations of the dividend process $Y_t$ and a noisy signal, $s_t$, about expected growth:

$$ds_t = \mu_t dt + \sigma_s dB_{s,t}$$ (12)

where $\sigma_s$ is constant and the standard Brownian motion $B_{s,t}$ is independent of $B_{y,t}$ and $B_{\mu,t}$. Dynamics in Eq. (8) to (12) define the true model.

C. Investors’ Beliefs

Investors make their consumption and investment decisions without observing $\mu_t$. The investors’ estimate is denoted by $\hat{\mu}_t = \hat{E}_t[\mu_t]$, where the expectation is calculated under the investors’ subjective probability measure. Economies with incomplete information have been studied in Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986), among others. The analysis consists of the filtering and of the consumption and investment decision. I consider at first the benchmark case of Bayesian investors and then I introduce pro-cyclical optimism.

C.1. Optimal forecast

Investors know the structure and the parameters of the true model and update their beliefs in a Bayesian way. Priors are normal with mean $\hat{\mu}_0$ and variance $\sigma_{\mu,0}$. Standard filtering theory, as in Liptser and Shiryaev (2001), leads to the following dynamics of the aggregate dividend, the estimated drift and the habit state variable:\footnote{For the sake of exposition and analytic tractability, I assume a long enough horizon of past observations, so that the variance has already converged towards its steady state. This is an usual assumption, as in Dumas, Kurshev, and Uppal (2009) for instance.}

$$dY_t = \hat{\mu}_t Y_t dt + \sigma_y d\hat{B}_{y,t},$$ (13)
$$d\hat{\mu}_t = \kappa (\bar{\mu} - \hat{\mu}_t) dt + \sigma_{\mu,y} d\hat{B}_{y,t} + \sigma_{\mu,\mu} d\hat{B}_{\mu,t} + \sigma_{\mu,s} d\hat{B}_{s,t},$$ (14)
$$d\omega_t = \lambda (\hat{\omega}_t - \omega_t) dt + \sigma_s d\hat{B}_{s,t},$$ (15)

where $\omega_t = (\hat{\mu}_t - \sigma_y^2/2)/\lambda$ and $\hat{B}_{y,t}, \hat{B}_{\mu,t}$ and $\hat{B}_{s,t}$ are independent standard Brownian motions under the investors’ subjective measure. With a perfect signal, i.e. a signal with zero noise, the drift is observable, therefore $\hat{\mu}_t = \mu_t$. It follows that $\sigma_{\mu,\mu} = \sigma_{\mu,y} = \sigma_{\mu,s} = 0$, $\hat{B}_{y,t} = B_{y,t}$ and $\hat{B}_{\mu,t} = B_{\mu,t}$. With an imperfect signal, i.e. a signal with non-zero noise, $\sigma_{\mu,y} = v_\omega \sigma_y^{-1}$, $\sigma_{\mu,s} = v_\omega \sigma_s^{-1}$ and $\sigma_{\mu,\mu} = 0$ with

\footnote{For the sake of exposition and analytic tractability, I assume a long enough horizon of past observations, so that the variance has already converged towards its steady state. This is an usual assumption, as in Dumas, Kurshev, and Uppal (2009) for instance.}
\[ v_{\infty} = \left( \sqrt{\kappa^2 + \sigma_{\mu}^2 \left( \sigma_y^{-2} + \sigma_s^{-2} \right)} - \kappa \right) \left( \sigma_y^{-2} + \sigma_s^{-2} \right)^{-1}. \]  

(16)

Since investors do not observe the true drift, \( \dot{B}_{\mu,t} \) vanishes from Eq. (14). This will be the case considered in the following of the paper. Finally, the case of a signal with infinite noise obtains by taking the limit \( \sigma_s \to \infty \): the signal becomes useless and both the \( \dot{B}_{\mu,t} \) and \( \dot{B}_{s,t} \) vanish from Eq. (14).

### C.2. Pro-cyclical optimism

I consider now the case of investors who deviate from the Bayesian formation of beliefs, consistently with the empirical findings of Section II.A. The aim is to model pro-cyclical optimism through a simple and intuitive mechanism, governed by a single parameter \( \chi \), which resembles Eq. (2)-(3)-(4). Such a parameter will capture the impact of current economic conditions on the formation of beliefs (i.e. \( \omega_t \to d\hat{\mu}_t \)) and, in turn, the impact of the perceived growth on the perceived evolution of economic conditions (i.e. \( \hat{\mu}_t \to d\omega_t \)). Namely, I assume a simple deviation term, \( \Psi(\omega_t) \), as follows:

\[ d\hat{\mu}_t = \kappa(\bar{\mu} - \hat{\mu}_t)dt + \sigma_{\mu}d\dot{B}_{\mu,t} + \sigma_{\mu,s}d\dot{B}_{s,t}. \]  

(17)

Then, the bias process \( \zeta_t = \hat{\mu}_t - \hat{\mu}_t^{\text{Bayesian}} \) satisfies

\[ \zeta_t = \zeta_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)}\Psi(\omega_u)du, \quad \text{such that} \quad d\zeta_t = -\kappa\zeta_t dt + \Psi(\omega_t)dt. \]  

(18)

The deviation term \( \Psi(\omega_t) \) has a simple functional form since it is just a centered version of the habit state variable scaled by the deviation parameter \( \chi \):

\[ \Psi(\omega_t) = \chi(\bar{\omega} - \bar{\omega}), \]  

(19)

where \( \bar{\omega} = (\bar{\mu} - \sigma_y^2/2)/\lambda \). For \( \chi > 0 \), in good times (i.e. high \( \omega_t \)) the investors tend to overestimate expected growth, while in bad times they tend to underestimate it. A positive parameter \( \chi \) parsimoniously captures the effects of both positive \( a_1 \) and negative \( b_1 \) in Eq. (3) and (4). Indeed, \( d\hat{\mu}_t \) pro-cyclically deviates from \( d\hat{\mu}_t^{\text{Bayesian}} \) (i.e. \( a_1 > 0 \)) and such a deviation is larger in magnitude when \( \omega_t \) is far from its average (i.e. \( b_1 < 0 \)). For \( \chi > 0 \), waves of optimism and pessimism obtain and alternate over time.\(^{14}\)

\(^{14}\)The same dynamics for the expected growth in Eq. (17) can obtain by assuming that the agents are still Bayesian but conditionally to the misleading interpretation of the signal. They observe a signal as in Eq. (12), but they interpret it erroneously as

\[ ds_t = (\mu_t + \Psi'(\omega_t))dt + \sigma_s d\dot{B}_{s,t}. \]
Finally, I set $\Psi(\omega_t)$ to have zero expectation to rule out from equilibrium prices the mechanical effects of the preponderance of optimism over pessimism or vice-versa. The key implication of the term $\Psi(\omega_t)$ concerns the perceived dynamics of $\omega_t$ in the mind of the investors. Under their subjective probability measure, the reversion threshold, $\hat{\omega}_t$, is positively related to $\hat{\mu}_t$:

$$
\begin{align*}
d\omega_t &= \lambda(\hat{\omega}_t - \omega_t)dt + \sigma_y d\hat{B}_t, \\
\hat{\omega}_t &= (\hat{\mu}_t - \sigma_y^2/2)/\lambda.
\end{align*}
$$

Consequently, for $\chi > 0$, in good times (i.e. high $\omega_t$) expected growth is overestimated and the threshold $\hat{\omega}_t$ is higher than in the Bayesian case. It follows that the persistence of good states is overestimated too. Similarly, in bad times (i.e. low $\omega_t$) expected growth is underestimated and the threshold $\hat{\omega}_t$ is lower than for a Bayesian investor. Therefore, bad times appear to the investors more persistent than in the Bayesian case. Such a joint formation of habit $(d\omega_t)$ and beliefs $(d\hat{\mu}_t)$ leads to a perceived evolution of the state of the economy which is riskier than under the true probability measure. Then, the proposed approach provides a rationale for the excess variability of the equilibrium pricing kernel and is consistent with the large fluctuations of stock prices beyond the smooth dynamics of fundamentals.

The left and middle panels of Figure 3 show respectively the cumulative distribution function and the autocorrelation function of $\omega_t$ as perceived by the investors for both the Bayesian case ($\chi = 0\%$) and the pro-cyclical optimism ($\chi = 7.5\%$). The right panel reports the scatter plot of $\omega_t$ and $\hat{\mu}_t$ as they evolve under the investors probability measure for both the two levels of the deviation parameter: the larger $\chi$, the more correlated $\omega_t$ and $\hat{\mu}_t$.

Insert Figure 3 about here.

The Bayesian updating of beliefs obtains when $\chi$ goes to zero. Instead, for $\chi < 0$, waves of optimism and pessimism occur respectively in bad and good times: in such a case—in contrast with the empirical evidence— the perceived persistence of $\omega_t$ is weaker than under the true probability measure.

IV. The Equilibrium

including a deviation term $\Psi'(\omega_t) \propto \Psi(\omega_t)$. See the Appendix A for the details. Such an interpretation is similar to Dumas, Kurshev, and Uppal (2009), where some agents are overconfident about the noisy signal.

The asymmetry in the waves of optimism and pessimism can be easily achieved setting $\Psi(\omega_t) = \chi(\omega_t - \bar{\omega} + \epsilon)$, where $\epsilon$ could be eventually calibrated to capture the long-run bias in the forecast errors. See Section VII for further details.

Results are obtained through simulations: parameters are the same used in the empirical analysis. See Section VLA for the details.
This section considers the representative agent problem to derive the optimal consumption policy and the state price density. Under the investors’ subjective measure, uncertainty is described by the Brownian motions $\hat{B}_{yt}$ and $\hat{B}_{st}$. I assume there exist assets such that markets are complete. Namely, the instantaneously risk-free asset and two long-lived securities: the market asset, which pays the aggregate dividend, and a perpetual bond.\footnote{The market can be completed considering at least two long-lived securities. However, since the price of these securities would be determined endogenously, one would have to verify endogenous completeness. This can be done by using the techniques of Hugonnier, Malamud, and Trubowitz (2012) or by verifying that the prices’ diffusion matrix is invertible for almost all states and times. Otherwise, one can just assume a derivative asset, written on $\hat{B}_{st}$ in zero net supply, with endogenous drift but exogenous volatility such that the market is complete.}

Market completeness implies the existence of a representative agent maximizing a linear combination with positive coefficients of the agents’ utility functions. He acts as a social planner in such a way the resulting allocation is Pareto optimal. Given the social weights, $v_i$, the representative agent solves:

$$\sup \left\{ \mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} \sum_i v_i U(C_{ij}, X_t; y_t) dt \right] \right\} \text{subject to } \sum_i C_{ij} \leq Y_t \ \forall t \geq 0.$$  \hspace{1cm} (22)

Since there is no intertemporal transfer of resources, this optimization reduces to a static problem:

$$\sup \left\{ \mathbb{E}_0 \sum_i v_i U(C_{ij}, X_t; y_t) \right\}, \hspace{1cm} (23)$$

subject to the resource constraint. Lemma 1 characterizes the optimal consumption policy.

**Lemma 1** The optimal consumption sharing rule is given by

$$C_{ij} = c_{ij} Y_t, \quad \text{with} \quad c_{ij} = \left( \frac{e^{\omega t} - \bar{\omega}}{1 + e^{\omega t} - \bar{\omega}} \right)^i e^{-(\omega_{ij} - \bar{\omega})}.$$  \hspace{1cm} (24)

Individual consumption, as a fraction of the aggregate endowment, is a stationary function of $\omega_t$. Thus, consumption (and wealth) of all agents grows at the same average rate and no single agent dominates the economy in the long-run.\footnote{This result is due to the CuJ preferences. As in Chan and Kogan (2002), external habit has an equalizing effect on marginal utilities of the heterogeneous agents such that individual consumption- and wealth-shares remain stationary over time. At the opposite, under standard time-separable CRRA preferences, the least risk averse agent will eventually dominate the economy, as in Wang (1996).}

Let $\xi_{st,u}$ denote the state price density in an Arrow-Debreu economy which supports in equilibrium the Pareto optimal allocation of Lemma 1 (see Duffie and Huang (1985)). The price at time $t$ of an arbitrary payoff stream $\{F_u, u \in (t, \infty)\}$ is given by $\mathbb{E}_t[\int_t^\infty \xi_{st,u} F_u du]$. Corollary 1 characterizes the state price density and the aggregate relative risk aversion.
Corollary 1 The equilibrium state price density is given by

$$\xi_{t,u} = e^{-\rho(u-t)} \frac{U^{RA}(Y_t, X_t)}{U^{RA}(Y_u, X_u)} = e^{-\rho(u-t) - \gamma(\omega_t - \omega_u)} \left( 1 + e^{\omega_t - \omega_u} \right) \gamma,$$

where $U^{RA}(Y_t, X_t) = \sum \nu_i U(C_{i,t}, X_t; \gamma_i)$. The aggregate relative risk aversion is given by

$$\bar{\gamma}(\omega_t) = -Y_t \frac{U^{RA}(Y_t, X_t)}{U^{RA}(Y_t, X_t)} = \frac{\gamma}{1 + e^{\omega_t - \omega}}.$$

Corollary 1 states that the aggregate relative risk aversion is stationary and decreasing in the habit state variable. Furthermore, it is bounded above (below) by the coefficient of relative risk aversion of the most (least) risk averse agent. Intuitively, the risk tolerance of the representative agent is a weighted average of the individual risk tolerances $\gamma^{-1}_i$ with weights given by consumption shares.\footnote{Indeed, in the limit good state ($\omega_t \to +\infty$) the least risk averse agent dominates the economy and his consumption share tends to one. Vice-versa, in the limit bad state ($\omega_t \to -\infty$), the most risk averse agent dominates.}

As a result of heterogeneity, the logarithm of the state price density is a nonlinear function of $\omega_t$. Therefore, its conditional variance depends on the habit state-variable and leads to an endogenous time-variation of the price of risk, even if $\omega_t$ is an homoscedastic process. Instead, the equilibrium state price density of an homogeneous-agents economy obtains when the social weights of all agents are equal to zero but one. If such an agent has arbitrary risk aversion $\gamma > 0$, then the state price density is

$$\xi_{t,u} = e^{-\rho(u-t) - \gamma(\omega_t - \omega_u)}.$$

Such a state price density is exponential affine in $\omega_t$ and therefore inherits its homoscedasticity. To compare asset pricing implications in the two economies, I set the relative risk aversion coefficient of the homogeneous-agents economy equal to the steady state aggregate relative risk aversion in the heterogeneous economy: $\gamma = \bar{\gamma}(\bar{\omega}) = \gamma/2$.

Although a mounting evidence suggests that investors are heterogeneous in both beliefs and risk preferences, modelling simultaneously both these forms of heterogeneity is quite difficult. Online Appendix OA.A offers an equilibrium micro-foundation of the pro-cyclically biased but homogeneous beliefs assumed in Eq. (17). Namely, under heterogeneous risk attitudes, equilibrium aggregation of heterogeneous (e.g. symmetric) beliefs can lead to pro-cyclical optimism for the representative agent. Then, the deviation term $\Psi(\omega_t)$ in Eq. (19) can be interpreted as a reduced form approach: it is useful to build a direct link with the survey data (Section II.A) and to provide analytic tractability.
V. Equilibrium Asset Prices and Portfolio Strategies

This section studies the behavior of security prices. The assets of interest are the risk-free asset, the infinitely lived market security, \( P_t \), that pays the aggregate dividend, and the perpetual bond, \( Q_t \):

\[
P_t = \hat{E}_t \left[ \int_t^\infty \xi_{t,u} Y_u du \right] \quad \text{and} \quad Q_t = \hat{E}_t \left[ \int_t^\infty \xi_{t,u} du \right].
\] (28)

I derive analytically the instantaneous rate of interest and the price of risk, the asset prices and their moments, the individual wealth and the portfolio strategies. The analysis characterizes at first the homogeneous-agents economy and then the heterogeneous one: differences arise endogenously and can be understood as a result of the fluctuations in the cross-sectional distribution of wealth.

A. Homogeneous Agents Economy

Agents have equal relative risk aversion and, hence, the pricing kernel in Eq. (27) allows to discount cash-flows.

**Corollary 2** Under homogeneous risk attitudes, the instantaneous rate of interest is given by

\[
r_t = \rho + \gamma \lambda (\hat{\omega}_t - \omega_t) - \frac{1}{2} \gamma^2 \sigma_y^2,
\] (29)

where \( \hat{\omega}_t = (\hat{\mu}_t - \sigma_y^2/2) / \lambda \), and the price of risk is equal to

\[
\theta_t = \gamma \sigma_y.
\] (30)

The price of risk is constant and thus state-independent. Instead, the interest rate decreases with the habit state variable, \( \omega_t \). Moreover, such a variation increases with both reversion, \( \lambda \), and risk aversion, \( \gamma \). The estimated drift, \( \hat{\mu}_t \), affects the risk-free rate through the reversion threshold \( \hat{\omega}_t \). As commented in Section III.C, pro-cyclical optimism (\( \chi > 0 \)) increases the correlation of \( \omega_t \) and \( \hat{\mu}_t \). Then, it reduces the volatility of \( \hat{\omega}_t - \omega_t \) and, in turn, that of the risk-free rate.

**Proposition 1** Under homogeneous risk attitudes, the equilibrium market price-dividend ratio equals

\[
\frac{P_t}{Y_t} = e^{\gamma \sigma_y} F(\omega_t, \hat{\mu}_t) = e^{\gamma \sigma_y} \int_t^\infty e^{A(u-t, \hat{\xi}) + B(u-t, \hat{\xi}) \omega_t + C(u-t, \hat{\xi}) \hat{\mu}_t} du,
\] (31)
where \( \bar{c} = (0, 1, -\gamma, 0) \). \( F(\cdot, \cdot) \) is implicitly defined and \( A(\tau, \phi), B(\tau, \phi) \) and \( C(\tau, \phi) \) are deterministic functions of time derived in Appendix C. The equilibrium volatility of the market asset is given by

\[
\sigma_P^p = \sqrt{(\sigma_y^p)^2 + (\sigma_s^p)^2},
\]

(32)

where

\[
\sigma_y^p = \sigma_y + \theta_t + \sigma_{\mu, \gamma} F_{(\omega_t, \hat{\mu}_t)} + \sigma_{\mu, \gamma} F_{(\omega_t, \hat{\mu}_t)}
\]

(33)

\[
\sigma_s^p = \sigma_{\mu, \gamma} F_{(\omega_t, \hat{\mu}_t)}.
\]

\( F_\omega \) and \( F_{\hat{\mu}} \) denote the partial derivatives of \( F(\omega_t, \hat{\mu}_t) \). The equity premium is given by

\[
\mu_P^p - r_t = \theta_t \sigma_y^p.
\]

(34)

Similarly to the rate of interest, also the price-dividend ratio, the market volatility and equity premium are stationary functions of \( \omega_t \) and \( \hat{\mu}_t \). Since under homogeneous preferences the price of risk is constant, the equity premium and the priced component of the stock volatility \( \sigma_y^p \) are perfectly correlated.

The volatility of the market asset has two components: \( \sigma_y^p \) and \( \sigma_s^p \). Notice that \( \sigma_y^p \) is the priced component of volatility and is given by the sum of four terms. The first is the volatility of fundamentals such that the other three components lead to excess volatility. The second and the third terms are due to the \( \text{CuJ} \) preferences: the former is exactly the price of risk, the latter accounts for the persistence of the state variables. The fourth term arises when the expected dividend growth is stochastic and captures the effects of the incomplete information. Similarly, \( \sigma_s^p \) captures the effects of the incomplete information due to the noisy signal, but it is not priced in equilibrium.

**Proposition 2** Under homogeneous risk attitudes, the equilibrium perpetual bond price equals

\[
Q_t = e^{\gamma \bar{c}_t} F^\epsilon(\omega_t, \hat{\mu}_t) = e^{\gamma \bar{c}_t} \int_t^\infty e^{A(u-t, \bar{c}) + B(u-t, \bar{c}) \omega_t + C(u-t, \bar{c}) \hat{\mu}_t} du,
\]

(35)

where \( \bar{c} = (0, 0, -\gamma, 0) \). \( F^\epsilon(\cdot, \cdot) \) is implicitly defined and \( A(\tau, \phi), B(\tau, \phi) \) and \( C(\tau, \phi) \) are deterministic functions of time derived in Appendix C. The equilibrium volatility of the bond is given by

\[
\sigma_Q^p = \sqrt{(\sigma_y^p)^2 + (\sigma_s^p)^2},
\]

(36)

where
\[ \sigma_y^Q = \theta_t + \sigma_y \frac{F^s_\omega(\omega_t, \hat{\mu}_t)}{F^s(\omega_t, \hat{\mu}_t)} + \sigma_{\mu,y} \frac{F^s_\mu(\omega_t, \hat{\mu}_t)}{F^s(\omega_t, \hat{\mu}_t)} \quad \text{and} \quad \sigma_y^Q = \sigma_{\mu,y} \frac{F^s_\mu(\omega_t, \hat{\mu}_t)}{F^s(\omega_t, \hat{\mu}_t)}. \]  

(37)

\[ F^s_\omega \text{ and } F^s_\mu \text{ denote the partial derivatives of } F^s(\omega_t, \hat{\mu}_t). \]  

The bond premium is given by

\[ \mu_t^Q - r_t = \theta_t \sigma_y^Q. \]  

(38)

The equilibrium yield of a zero-coupon bond with maturity \( \tau \) is given by

\[ \varepsilon(t, \tau) = -\frac{1}{\lambda} \log \mathbb{E}_t [\xi_t, t+\tau] = -\frac{1}{\lambda} (\gamma \omega_t + A(\tau, \hat{c}) + B(\tau, \hat{c}) \omega_t + C(\tau, \hat{c}) \hat{\mu}_t). \]  

(39)

The perpetual bond price is a stationary function of \( \omega_t \) and \( \hat{\mu}_t \). Its volatility and premium have a decomposition similar to that of the market asset. Real yields are affine in \( \omega_t \) and \( \hat{\mu}_t \). In turn, yields volatility is state-independent.

**B. Heterogeneous Agents Economy: Asset Prices**

This section assumes agents have the heterogeneous relative risk aversion as in Eq. (7) and, hence, the pricing kernel in Eq. (25) allows to discount cash-flows.

**Corollary 3** Under heterogeneous risk attitudes, the instantaneous rate of interest is given by

\[ r_t = \rho + \gamma \omega_t \frac{\hat{\mu}_t - \sigma_y^2}{1 + e^{\omega_t - \hat{\omega}}} - \frac{\gamma^2}{2} \frac{\hat{\mu}_t - \sigma_y^2}{(1 + e^{\omega_t - \hat{\omega}})^2}, \]  

(40)

where \( \hat{\omega} = (\hat{\mu}_t - \sigma_y^2)/\lambda \), and the price of risk is equal to

\[ \theta_t = \frac{\gamma \sigma_y}{1 + e^{\omega_t - \hat{\omega}}}. \]  

(41)

The equilibrium price of risk can be written as \( \theta_t = \tilde{\gamma}(\omega_t) \sigma_y \): its variation is due heterogeneity. In particular, the price of risk inherits the countercyclical behavior of \( \tilde{\gamma}(\omega_t) \): \( \partial_{\omega} \theta_t = -\frac{\gamma}{\lambda + e^{\omega_t - \hat{\omega}} (1 + e^{\omega_t - \hat{\omega}})^2} < 0 \), with maximal sensitivity at \( \omega_t = \omega \). The price of risk reduces to that of an homogeneous-agents economy with risk aversion equal to \( \tilde{\gamma} \) or zero when the habit state variable goes respectively to minus or plus infinity. Indeed, in these cases, respectively the most or the least risk averse agent dominates the economy.
Using the aggregate relative risk aversion, the interest rate can be written as

\[ r_t = \rho + (\hat{\mu}_t - \sigma^2/2)\hat{\gamma}(\omega_t) - \lambda \omega_t \hat{\gamma}(\omega_t) - \frac{\sigma^2}{2} \hat{\gamma}(\omega_t)^2 (1 + e^{\omega_t - \bar{\omega}}/\bar{\gamma}). \]  

(42)

Investors’ patience is inversely related to the common discount rate \( \rho \) which positively affects the level of \( r(\omega_t, \hat{\mu}_t) \). The interest rate increases with \( \hat{\mu}_t \). The higher the rate of consumption growth, the lower the expected marginal utility (relative to the present). The second term on the right hand side of Eq. (42) shows two additional effects. On the one hand, expected growth is scaled by aggregate relative risk aversion. In good (bad) states, marginal utility is low (high) relative to the steady state. Consequently, a low (high) \( \hat{\gamma}(\omega_t) \) reduces (increases) the impact of expected growth on the equilibrium interest rate. On the other hand, incomplete information affects \( \hat{\mu}_t \). In particular, for \( \chi > 0 \), \( \omega_t \) and \( \hat{\mu}_t \) are more correlated than in the Bayesian case. Hence, pro-cyclical optimism mitigates the volatility of the sum of the second and the third term in Eq. (42). Habit persistence measures the incentive to postpone consumption and therefore to save more when consumption today is high with respect to habit: \(-\lambda \omega_t \hat{\gamma}(\omega_t)\) captures such an effect. Heterogeneity adds a countercyclical adjustment proportional to the aggregate relative risk aversion. The volatility of consumption growth increases the expected marginal utility and thus induces a precautionary savings component in the equilibrium interest rate. The last term on the right hand side of Eq. (42) describes this effect. While it is constant in the homogeneous-agents economy, under heterogeneity it also measures the incentive to save less when the price of risk in the economy is low. Therefore, habit persistence and precautionary savings terms are inversely proportional to each other. Consequently, heterogeneity generates an endogenous offsetting mechanism which reduces the level of the interest rates and helps to capture the countercyclical variation of risk premia.

**Proposition 3** Under heterogeneous risk attitudes, the equilibrium market price-dividend ratio equals

\[ \frac{P_t}{Y_t} = \left( \frac{e^{\omega_t - \bar{\omega}}}{1 + e^{\omega_t - \bar{\omega}}} \right)^{\frac{\tau}{2}} \int_{u=0}^{\infty} e^{A(u-t, \bar{\gamma})} + B(u-t, \bar{\gamma}) \omega_t + C(u-t, \bar{\gamma}) \hat{\mu}_t \, du, \]  

\[ F(\omega_t, \hat{\mu}_t) = \left( \frac{e^{\omega_t - \bar{\omega}}}{1 + e^{\omega_t - \bar{\omega}}} \right)^{\frac{\tau}{2}} \sum_{j=0}^{\infty} \left( \frac{\tau}{2} \right)^j \int_{u=0}^{\infty} e^{A(u-t, \bar{\gamma}) + B(u-t, \bar{\gamma}) \omega_t + C(u-t, \bar{\gamma}) \hat{\mu}_t} \, du, \]  

(43)

where \( \bar{c} = (-\bar{\omega}, 1, j - \bar{\gamma}, 0) \), \( F(\cdot, \cdot) \) is implicitly defined and \( A(\tau, \phi), B(\tau, \phi) \) and \( C(\tau, \phi) \) are deterministic functions of time derived in Appendix C. The volatility of the market asset is given by

\[ \sigma^p_t = \sqrt{\left( \sigma^p_t \right)^2 + \left( \sigma^p_t \right)^2}, \]  

(44)

where
\[ \sigma_P^y = \sigma_y + \theta_t + \sigma_{\mu,y} \frac{F_{\mu}(\omega_t, \hat{\mu}_t)}{F(\omega_t, \hat{\mu}_t)} \quad \text{and} \quad \sigma_P^s = \sigma_{\mu,s} \frac{F_{\mu}(\omega_t, \hat{\mu}_t)}{F(\omega_t, \hat{\mu}_t)}. \] (45)

\[ F_\omega \text{ and } F_{\hat{\mu}} \text{ denote the partial derivatives of } F(\omega_t, \hat{\mu}_t). \] The equity premium is given by

\[ \mu_P^t - r_t = \theta_t \sigma_P^y. \] (46)

Similarly to the homogeneous case, the price-dividend ratio, the market volatility and equity premium are stationary functions of the \( \omega_t \) and \( \hat{\mu}_t \). Such a stationarity property does not obtain under heterogeneity in absence of Cul preferences. While the above expressions only hold for a specific pattern of heterogeneity, it is worth emphasizing that, to my knowledge, they are the only closed form expressions available for a stationary equilibrium in a heterogeneous multiple-agents economy under incomplete information.

The equilibrium price dividend ratio is given by the product of two terms: the first comes from the current level of the state price density, while the second one, \( F(\omega_t, \hat{\mu}_t) \), captures the expected evolution of the states \( \omega_t \) and \( \hat{\mu}_t \). Under heterogeneous risk attitudes, this term is given by a weighted sum of integrals of exponential affine functions of the states. This weighted sum obtains from the nonlinearity of the log state price density and, therefore, leads to additional endogenous heteroscedasticity in returns.

The volatility of the market asset has two components: \( \sigma_P^y \) and \( \sigma_P^s \). The term \( \sigma_P^y \) is the priced volatility and is given by the sum of four terms. The first term is the volatility of fundamentals. The second term of \( \sigma_P^y \) is the price of risk and, under preference heterogeneity, is decreasing with \( \omega_t \). Therefore, the whole volatility can be eventually counter-cyclical, whereas it is non-decreasing with the habit state variable in the homogeneous-agents economy. Similarly to the homogeneous case, \( \sigma_P^s \) and the last two terms of \( \sigma_P^y \) are forward-looking terms given by the semi-elasticity of prices with respect to the states. In particular, \( \sigma_P^s \) is the unpriced volatility due to the noisy signal.

Since the price of risk is not constant in the heterogeneous-agents economy, the equity premium is not a scaled version of the priced volatility. Consequently a rich risk-return trade off – that is the unconditional correlation between \( \mu_P^t - r \) and \( \sigma_P^P \) – can obtain, as it will be shown in Section VI.

**Proposition 4** Under heterogeneous risk attitudes, the equilibrium perpetual bond price equals

\[ Q_t = \left( \frac{e^{\bar{\theta}_t} - 1}{1 - e^{\bar{\theta}_t}} \right)^{\gamma} F^e(\omega_t, \hat{\mu}_t) = \left( \frac{e^{\bar{\theta}_t} - 1}{1 - e^{\bar{\theta}_t}} \right)^{\gamma} \sum_{j=0}^{\gamma} \binom{\gamma}{j} \int_t^\infty e^{A(u-t, \bar{\epsilon}) + B(u-t, \bar{\epsilon}) \omega_t + C(u-t, \bar{\epsilon}) \hat{\mu}_t} du. \] (47)
where \( \bar{c} = (-\bar{\omega}, 0, j - \bar{\gamma}, 0) \), \( F^\varepsilon(\cdot, \cdot) \) is implicitly defined and \( A(\tau, \phi), B(\tau, \phi) \) and \( C(\tau, \phi) \) are deterministic functions of time derived in Appendix C. The volatility of the bond is given by

\[
\sigma_t^Q = \sqrt{(\sigma_t^Q)^2 + (\sigma_t^Q)^2},
\]

where

\[
\sigma_t^Q = \theta_t \sigma_y + \sigma_y F_t^z(\omega, \mu_t) + \sigma_{\mu,y} F_t^z(\omega, \mu_t) \quad \text{and} \quad \sigma_t^Q = \sigma_{\mu,\tau} F_t^z(\omega, \mu_t).
\]

\( F^\varepsilon \) and \( F^{\varepsilon}_\tau \) denote the partial derivatives of \( F^\varepsilon(\omega, \mu_t) \). The bond premium is given by

\[
\mu_t^Q - r_t = \theta_t \sigma_t^Q.
\]

The equilibrium yield of a zero-coupon bond with maturity \( \tau \) is given by

\[
\varepsilon(t, \tau) = -\frac{1}{\tau} \log \frac{\hat{A}_t}{\hat{A}_{t+t}} = -\frac{1}{\tau} \log \left\{ \left( \frac{e^{c - \bar{\mu}}}{1 + e^{c - \bar{\mu}}} \right)^{\bar{q}} \sum_{j=0}^{\bar{q}} \left( \frac{e^{c - \bar{\mu}}}{1 + e^{c - \bar{\mu}}} \right)^{\bar{q}} e^{A(\tau, \phi) + B(\tau, \phi) \omega_t + C(\tau, \phi) \mu_t} \right\}.
\]

Under heterogeneous risk preferences, the bond price and its moments are still stationary functions of \( \omega_t \) and \( \mu_t \). The endogenously heteroscedastic state price density leads to real yields which are non-affine in the state variables. In turn, yields volatility is state-dependent. A similar term structure of real interest rates obtains in Buraschi and Jiltsov (2007).

C. Heterogeneous Agents Economy: Wealth and Portfolio Strategies

This section characterizes the equilibrium individual wealth and portfolio holdings.

**Proposition 5** Under heterogeneous risk attitudes, the equilibrium wealth of agent \( i \) equals

\[
W_{i,t} = Y_t \left( \frac{e^{c - \bar{\mu}}}{1 + e^{c - \bar{\mu}}} \right)^{\bar{q}} F^i(\omega_t, \mu_t) = Y_t \left( \frac{e^{c - \bar{\mu}}}{1 + e^{c - \bar{\mu}}} \right)^{\bar{q}} \sum_{j=0}^{\bar{q}} Y(i, j) \int_t^{\infty} e^{A(u, \phi) + B(u, \phi) \omega_t + C(u, \phi) \mu_t} du
\]

where \( \bar{c} = (-\bar{\omega}, 1, \ell(i, j), 0) \), \( F^i(\cdot, \cdot) \) is implicitly defined. \( U, Y(i, j), \ell(i, j) \) and the deterministic functions of time \( A(\tau, \phi), B(\tau, \phi) \) and \( C(\tau, \phi) \) are derived in Appendix C.

The proportions of wealth invested in the market asset, \( \pi_t^P \), and in the perpetual bond, \( \pi_t^Q \), satisfy

\[
\begin{pmatrix}
\sigma_{W_t}^P \\
\sigma_{W_t}^Q
\end{pmatrix} =
\begin{pmatrix}
\sigma_P^P & \sigma_P^Q \\
\sigma_Q^P & \sigma_Q^Q
\end{pmatrix}
\begin{pmatrix}
\pi_t^P \\
\pi_t^Q
\end{pmatrix},
\]

\( \bar{c} \)
where $\sigma^P$, $\sigma^Q$, and $\sigma^y$, $\sigma^s$ are from Eq. (45) and (49).

\[
\sigma^W_i = \sigma_y + \theta_i + \sigma_y F_i^{(\omega_t, \hat{\mu}_t)} + \sigma_{\mu_t} F_i^{(\omega_t, \hat{\mu}_t)} \quad \text{and} \quad \sigma^W_s = \sigma_{\mu_s} F_i^{(\omega_t, \hat{\mu}_t)}.
\]

(54)

Individual wealth is proportional to the aggregate dividend, consequently both wealth shares, $W_{it}/P_t$, and wealth-consumption ratios, $W_{it}/C_{it}$, are stationary functions of the states $\omega_t$ and $\hat{\mu}_t$ only. This result comes out from the equalizing effect of the standard of living $X_t$, which makes marginal utilities and, in turn, the state price density stationary as well. Wealth distribution fluctuates over time around a steady-state. Pro-cyclical optimism ($\chi > 0$) reinforces such an endogenous time-variation – which is otherwise quite limited under a realistic degree of preference heterogeneity. Indeed, for $\chi > 0$, investors overestimate the persistence of good and bad times and, hence, the unconditional volatility of aggregate risk.

The proportions of wealth invested in the market asset, $\pi^P_i$, and in the perpetual bond, $\pi^Q_i$, can be recognized since under completeness the diffusion matrix of the risky assets is invertible. Here, endogenous completeness can be verified as long as $\sigma^P_y \sigma^Q_s - \sigma^Q_y \sigma^P_s \neq 0$.

VI. Asset Pricing Results

This section investigates whether the model can capture the main empirical stylized facts of asset returns together with the weak or negative risk-return trade off and its dynamics. The analysis relies on the single parameter $\chi$ which captures pro-cyclical optimism. Such an approach leads to a number of predictions in line with empirical evidence and improves the Bayesian benchmark ($\chi = 0$) over many dimensions. At first, I describe the model calibration and the cross-sectional implications of heterogeneity. Then, the focus turns on the equilibrium risk-return trade off and its dynamics. Finally, for robustness, I analyse the unconditional and conditional moments of returns, their predictability, the term structure of the interest rates and the dynamics of portfolios and trading volume.

A. Model Calibration

I use annual data from 1933 to 2006 from the S&P Composite Index, the Consumer Price Index and the three-month Treasury bill from the Federal Reserve Bank of St. Louis. Statistics for the price-dividend ratio, the risk-free rate and the market return are computed from the data. I also report other data to account for the recent financial crisis.\footnote{For the purpose of comparison, the empirical analysis considers the same data about stock markets as in Xiouros and Zapatero (2010), Marfè (2011), and Curatola and Marfè (2011) which study Cbl preferences under heterogeneity but full information.}
The values of the unconditional mean $\bar{\mu}$ (1.8%) and the instantaneous volatility $\sigma_y$ (3.6%) of the aggregate dividend growth are calibrated to match the implied moments in Mehra and Prescott (1985). The speed of reversion $\kappa$ (.35) and the instantaneous volatility $\sigma_{\mu}$ (1%) of expected growth are set in the range of values considered in the literature.\footnote{These parameters are in line with most of long-run risk literature: see Bansal, Kiku, Shaliastovich, and Yaron (2014) and references therein.}

Preference parameters are the subjective time-discount rate $\rho$, the maximum individual relative risk aversion $\bar{\gamma}$ and the habit persistence $\lambda$. The maximum individual relative risk aversion is set to $\bar{\gamma} = 10$, which is a conservative upper bound in line with Mehra and Prescott (1985). In turn, the steady state aggregate relative risk aversion is $\bar{\gamma}(\bar{\omega}) = 5$, in line with the estimate by Xiouros and Zapatero (2010) from individual consumption data. In the homogeneous-agents economy, I set $\gamma = \bar{\gamma}(\bar{\omega}) = 5$ for the sake of comparison. Habit persistence is somewhat not observable and there is some disagreement in the literature about its value. Since agents have time-separable utility, I set $\lambda = .35$ to capture enough persistence in the price-dividend ratio but without increasing too much the risk-free rate volatility.\footnote{Accordingly, Lynch and Randall (2011) show that cross-sectional returns are consistent with a low persistence of the habit state-variable when expected growth is mean-reverting.} The time-discount rate is set to $\rho = 3.5\%$, which is in the range of values considered in the literature, in order to match the unconditional risk-free rate, given the other parameters.

The formation of beliefs involves two parameters. I set the volatility of the noisy signal $\sigma_s = 5\%$.\footnote{Although the choice of $\sigma_s$ is arbitrary, it seems safe to assume some monotonicity in the outcomes of the model between the cases of infinite noise (useless signal) and of a perfect signal ($\sigma_s \to 0$). See also Branger, Schlag, and Wu (2011). Notice that the analysis focuses instead on the marginal effect on the asset prices of the parameter $\chi$.} Pro-cyclical optimism depends on $\chi$: I consider several specifications and verify how equilibrium outcomes evolve as a function of such a parameter. In particular, I focus on the Bayesian case ($\chi = 0\%$) and a target value $\chi = 7.5\%$, which captures many empirical patterns of asset returns. Table III summarizes the calibration of the model.

Insert Table III about here.

B. Preference Heterogeneity and Equilibrium Distributions

This section investigates the equilibrium distributions when preferences are heterogeneous. The specification of agents types in Eq. (7) allows for analytical solutions. It is important to notice that such a specific pattern of agents’ types is economically irrelevant. Instead, the endogenous consumption and wealth distributions are economically important and meaningful. Indeed, these are equilibrium outcomes and can be compared to their counterpart in the real data.

Xiouros and Zapatero (2010) study individual consumption data to infer the cross-sectional distribution of consumption-shares as a function of risk tolerance and their results confirm the empirical findings of Kimball, Sahm, and Shapiro (2008). Namely, relative risk aversion has an average of about 5 and most of the probability mass is in the range $(0, 10)$. 

\footnote{These parameters are in line with most of long-run risk literature: see Bansal, Kiku, Shaliastovich, and Yaron (2014) and references therein.}
Under the pattern of heterogeneity in Eq. (7), the equilibrium distributions of consumption- and wealth-shares are given by:

\[
F^C_{\gamma^{-1}}(j, \omega_t) = \sum_{i=1}^{j} \left( \frac{e^{\omega_t - \theta}}{1 + e^{\omega_t - \theta}} \right)^i e^{-(\omega_t - \theta)} \quad \text{and} \quad F^W_{\gamma^{-1}}(j, \omega_t, \hat{\mu}_t) = \sum_{i=1}^{j} \left( \frac{W_i}{\tau} \right) \quad \forall j = 1, 2, \ldots \tag{55}
\]

The upper left panel of Figure 4 shows the consumption-share distribution at the steady state when \( \gamma = \{7.5, 10, 15\} \) together with the estimated distribution of Xiouros and Zapatero (2010). For \( \gamma = 10 \), the equilibrium distribution closely resembles the empirical one.\(^{24}\) The upper right panel displays the wealth-share distribution, which is quite similar to the consumption-share distribution but it is more sensitive to the state.

The lower left panel of Figure 4 shows the density function of aggregate relative risk aversion \( \tilde{\gamma}(\omega_t) \) under the investors probability measure. Under pro-cyclical optimism (\( \chi = 7.5\% \)), agents overestimate the persistence of economic conditions and then a larger variance and a higher probability of extreme events than in the Bayesian case (\( \chi = 0\% \)) obtain. The lower right panel of Figure 4 exhibits the undiscounted log pricing kernel as a function of the habit state variable:

\[
\rho(u-t) + \log \xi_{\omega, u} = \gamma(\omega_t - \omega_u) + \gamma[\log(1 + e^{\omega_t - \theta}) - \log(1 + e^{\omega_u - \theta})], \tag{56}
\]

and its linear counterpart \( \tilde{\gamma}(\omega_t)(\omega_t - \omega_u) \) in the homogeneous-agents economy. The degree of convexity that arises under heterogeneity is an endogenous result of the redistribution of wealth among the agents. Consequently, the log state price density features time-varying volatility even if the state \( \omega \) is homoscedastic. In turn, the price of risk and the aggregate relative risk aversion are endogenously time-varying and countercyclical as in Chan and Kogan (2002). Nonetheless, Xiouros and Zapatero (2010) have shown that a realistic degree of heterogeneity is so small that asset prices barely differ from those of a homogeneous-agents economy. However, under pro-cyclical optimism (\( \chi > 0 \)), even a small and realistic amount of heterogeneity is sufficient to produce interesting equilibrium outcomes, since the agents overestimate aggregate risk.

In summary, the assumed pattern of heterogeneity i) captures at equilibrium the empirical cross-sectional distribution of consumption; ii) is parsimonious (it involves only one parameter) and allows for an upper bound to individual relative

\(^{24}\)The model distribution diverges from that of Xiouros and Zapatero (2010) only for very low values of risk tolerance, that is for unrealistically high levels of risk aversion. The latter are ruled out in the model by the upper bound \( \gamma \) whereas they are taken into account by the parametric distribution of Gamma type in Xiouros and Zapatero (2010).
risk aversion, $\gamma$; iii) leads to analytical solutions not only for the pricing kernel but also for prices, returns and portfolios (under any specification of exogenous uncertainty in the jump-diffusion affine class).

C. Risk-Return Trade Off

The market risk-return trade off is captured by the unconditional correlation between the conditional expected excess return and the conditional return volatility, that is the two components of the Sharpe ratio. The trade off, as it is computed by the “econometrician,” provides very puzzling evidence: a weak or negative relation obtains under broad settings of available information (see Section II.A). Moreover, the actual expectations of the agents (i.e. Duke/CFO survey data), document a weak unconditional trade off and a pro-cyclical dynamics. When economic conditions deteriorate, the relation switches from strongly positive to strongly negative.

The model proposes the following explanation of the risk-return trade off, its dynamics and its potentially negative sign. As long as agents feature pro-cyclical optimism (i.e. $\chi > 0$ in the main model or, similarly, $a_1 > 0, b_1 < 0$ in Section II.A), the two components of the Sharpe ratio do not move together in some states of the world. Namely, this happens in bad times, leading to a weak unconditional risk-return relation.

In order to see such a mechanism, consider the priced component of volatility $\sigma^p_y$ in Eq. (45). It has the following decomposition:

$$\sigma^p_y = \sigma_y + \theta_t + \frac{(\sigma_y F_0 + \sigma_{\mu,y} \hat{\mu}_t)}{F}$$

(57)

Namely, the transient risk is exactly the price of risk in the economy and the forward-looking term is the semi-elasticity of discounted cash-flows with respect to the states. While transient risk moves negatively with $\omega_t$, forward-looking risk moves positively with both $\omega_t$ and $\hat{\mu}_t$. Therefore, the cyclicity of priced volatility depends on which of these two terms dominates. Under reasonable parameters and Bayesian beliefs, priced volatility inherits the counter-cyclical behavior of transient risk, whereas forward-looking risk is residual. Instead, for $\chi > 0$, the excessive persistence perceived by the agents enhances the forward-looking term in comparison with the Bayesian case. In particular, this effect is large in bad times because agents are risk averse and prices are more sensitive to the state. As a consequence, forward-looking risk offset transient risk in bad times. Then, pro-cyclical optimism breaks the cyclicality of volatility in bad times. This effect is consistent with the empirical evidence from survey data (see Table II and Figure 2).
Consider now the equity premium: it is simply the product of the price of risk and the priced volatility:

\[ \mu^P - r = \theta_t \times \sigma^P_y \]

 premium transient risk priced volatility.

The above offsetting mechanism, due to excessive persistence, is less pronounced for the equity premium than for the priced volatility. Indeed, such an effect is mitigated by the transient risk: the latter is not directly affected by \( \hat{\mu}_t \) and moves negatively with \( \omega_t \). Consistently with survey data, the equity premium always moves negatively with \( \omega_t \).

Consider now the risk-return relation. In good times, both volatility and premium are counter-cyclical and move together. This leads to a positive conditional trade off. However, in bad times, the premium is still counter-cyclical, whereas the cyclicality of volatility depends on the strength of pro-cyclical optimism. For \( \chi > 0 \) large enough, volatility can be weakly or even positively correlated with \( \omega_t \). Consequently, pro-cyclical optimism can lead to a negative conditional trade-off in bad times.\(^{25}\) In turn, the risk-return relation has a pro-cyclical dynamics and its unconditional level can be weak or even negative. Such a dynamics is consistent with survey data.

In summary, the endogenous offsetting mechanism among the components of the return volatility leads to the following dynamics for the two components of the Sharpe ratio and for the equilibrium risk-return trade off:

\[
\begin{align*}
\text{corr}(\omega, \mu^P - r | \text{low } \omega) & \approx \text{corr}(\omega, \mu^P - r) \approx \text{corr}(\omega, \mu^P - r | \text{high } \omega), \\
\text{corr}(\omega, \sigma^P_y | \text{low } \omega) & > \text{corr}(\omega, \sigma^P_y) > \text{corr}(\omega, \sigma^P_y | \text{high } \omega), \\
\text{corr}(\mu^P - r, \sigma^P_y | \text{low } \omega) & < \text{corr}(\mu^P - r, \sigma^P_y) < \text{corr}(\mu^P - r, \sigma^P_y | \text{high } \omega).
\end{align*}
\]

These inequalities supply, as an equilibrium outcome, a stylized representation of the three scatter plots in Figure 2. Table VI provides further support. We can observe that, for both the survey data and the model, the trade off is positive in good times (\( \omega_t > \bar{\omega} \)) and becomes strongly negative when economic conditions deteriorate (\( \omega_t < \bar{\omega} \)). This obtains because return volatility is not counter-cyclical in bad times. Therefore, the equilibrium trade off behaves in a similar fashion with what we observe from survey data.

Insert Table VI about here.

Under heterogeneous risk attitudes, pro-cyclical optimism can generate a large range of values for the unconditional correlation between the mean and the volatility of returns. In the Bayesian case (\( \chi = 0 \)) such a correlation is positive\(^{25}\)Unpriced volatility \( \sigma^s_P \) can further reduce the risk-return trade off, however at the market level its role it is supposed to be quite limited.
and almost perfect. As long as the agents feature pro-cyclical optimism ($\chi > 0$), the correlation at first decreases below zero and then increases again up to one. The decrease is due to the offsetting mechanism among the components of the return volatility, as commented above. For the target range of $\chi \in (.07, .08)$, which allows to capture many features of asset returns, the model leads to an unconditional correlation from close to zero to strongly negative, which is in line with the empirical evidence (Lettau and Ludvigson (2010)), but at odds with most of equilibrium asset pricing models.\footnote{For unrealistically high values of the deviation parameter $\chi$, forward-looking risk becomes almost insensitive to the states. Therefore, the whole variation in volatility is due to the price of risk. In turn, the volatility and the price of risk move together and the equity premium moves with the same sign. As a result, the unconditional trade-off increases up to one for very large $\chi$ values.} Figure 5 reports the unconditional correlation between the expected excess returns and the conditional return volatility as a function of $\chi$ in both the homogeneous- and the heterogeneous-agents economy.

Providing a theoretical link among survey expectations of fundamentals and those of returns, the model offers a potential general equilibrium solution to the puzzling evidence about a key quantity in asset pricing, that is the relation among perceived risk and its expected remuneration.

**D. Unconditional Moments of Returns**

Statistics of the model, as a function of the parameter $\chi$, are compared with the data for the cases of the homogeneous- and heterogeneous-agents economy respectively in Table IV and V.

In all the considered cases, the average risk-free rate is quite low (about 1.8%) in line with the data (1.7%). Heterogeneity in risk aversion reduces the unconditional level of the risk-free rate because of the state-dependent behavior of the precautionary savings term. However, such an effect is quantitatively quite limited for a realistic degree of heterogeneity.

Instead, the average risk-free rate is not sensitive to the value of the parameter $\chi$, since $\Psi(\omega_t)$ does not affect the perceived unconditional mean of both $\hat{\mu}$ and $\omega$.

In the Bayesian case ($\chi = 0$), the volatility of the risk-free rate (about 6.4%) is significantly lower than that of the stock returns, but it is somewhat higher than in real data (4.4%). Heterogeneity does not produce substantial differences with respect to the homogeneous-agents economy. Instead, pro-cyclical optimism ($\chi > 0$) helps to explain the smooth dynamics of the risk-free rate: the larger $\chi$, the lower the risk-free rate volatility. Such an effect can be understood from Eq. (29) or (40). The larger the deviation parameter, the stronger the correlation between the states $\hat{\mu}_t$ and $\omega_t$ and, in
turn, the less volatile the difference $\hat{\omega}_t - \omega_t$ (recall $\hat{\omega}_t$ is increasing with $\hat{\mu}_t$). Consequently the risk-free rate volatility reduces too. For $\chi = 7.5\%$, the volatility of the risk-free rate (about 4.6%) matches the historical data.

The unconditional equity premium is unaffected by the choice of homogeneous or heterogeneous risk attitudes. In the Bayesian case ($\chi = 0$), the average expected excess return is quite low (about 3.8%); the historical equity premium is almost twice this value (about 7.4%). Instead, pro-cyclical optimism ($\chi > 0$) increases the unconditional equity premium. Waves of optimism and pessimism alternate over time and induce a riskier perception about the economy than in the Bayesian case. In turn, they generate a higher equity premium, all other things being equal. Notice that such an increase in the premium is not the mechanical result of a long-run bias in the estimation of the expected growth. For $\chi = 7.5\%$, a substantial increase in the equity premium (about 4.8%) can be observed, but it remains still below the historical value. A further increase in the parameter $\chi$ allows to fit the real data but at the cost of too much stock return volatility. Notice that including the recent financial crisis the historical equity premium and price of risk reduce respectively to 5.6% and 28.3%, which are significantly closer to the model predictions for $\chi = 7.5\%$.

Because of the time-separable utility and the realistic aggregate relative risk aversion (5 in average), the unconditional price of risk (about 18%) is lower than the historical Sharpe ratio (41% or 28.3%). Furthermore, neither preference heterogeneity nor pro-cyclical optimism ($\chi > 0$) alter the average price of risk. A relatively low price of risk implies that a high unconditional equity premium should be accompanied by a high unconditional return volatility. This is forthright in the homogeneous-agents economy since the price of risk is constant and the equity premium is just a scaled version of the priced volatility. Under preference heterogeneity, the price of risk decreases with $\omega_t$ as in Eq. (41): while the conditional moments of returns are significantly affected (see Section VI), the unconditional premium and volatility are close to their counterparts in the homogeneous-agents economy. Therefore, the model can achieve a high equity premium only at a cost of a high stock return volatility. However, for $\chi = 7.5\%$, a substantial equity premium is accompanied by a return volatility (about 26%) which is somewhat higher than in real data (18%) but quite realistic.

The equilibrium price-dividend ratio shows some interesting implications. First, pro-cyclical optimism ($\chi > 0$) produces additional volatility with respect the Bayesian case ($\chi = 0$), almost in line with the real data. Second, pro-cyclical optimism allows to disentangle the persistence of the price-dividend ratio from that of the risk-free rate, a feature of the data which is difficult to obtain in most of asset pricing models. The first order autocorrelation of the price-dividend ratio and the risk-free rate are equal in the Bayesian case (about 71%). Instead, the larger the parameter

---

$\text{A better quantitative description of the equity premium can obtain by relaxing some of the simple assumptions of the model (made for the purpose of exposition). Section VII highlights some model extensions.}$

$\text{The unconditional price of risk under preference heterogeneity is given by } \mathbb{E}(\theta) = \sigma \int G_{\omega, \hat{\mu}}(1 + e^{x - \bar{\omega}})^{-1} dG_{\omega, \hat{\mu}}(x, z), \text{ where } G_{\omega, \hat{\mu}} \text{ denotes the joint distribution of } \omega \text{ and } \hat{\mu}. \text{ As long as } \Psi'(\omega) \text{ does not alter the unconditional mean or the symmetry of the distribution of } \omega, \text{ the unconditional mean of } \theta \text{ is only barely affected. Indeed, } (1 + e^{x - \bar{\omega}})^{-1} \text{ is almost linear in the range of values where } G_{\omega, \hat{\mu}} \text{ puts most of the probability mass.}$
χ, the stronger the autocorrelation of the price-dividend ratio and the weaker that of the risk-free rate. For χ = 7.5%, a significant divergence obtains (about 74% and 54%) and goes in the direction of the real data (91% and 33%). The autocorrelation of the price-dividend ratio is somewhat below its historical counterpart since λ is set to capture also the unconditional stock volatility and its cyclical behavior (see Section VI).

The bond premium and volatility decrease with χ since the bond offers an hedge opportunity beyond the stock and its cash-flows do not depend directly on expected growth, \( \hat{\mu}_t \).

Even in a simple and parsimonious form, pro-cyclical optimism (\( \chi > 0 \)) helps to explain some of the stylized features of the markets and improves over the benchmark (\( \chi = 0 \)). Figure 6 shows a summary of the analysis: asset pricing quantities are plotted as a function of the parameter χ under heterogeneous risk attitudes (similar results obtain for the homogeneous case).

E. Conditional Moments of Returns

The conditional price-dividend ratio, stock and bond volatility and premia as functions of \( \omega_t \) and \( \hat{\mu}_t \) are plotted in Figure 7 for the case of heterogeneous preferences. I consider both the Bayesian case (\( \chi = 0 \)) and pro-cyclical optimism for the target value of \( \chi = 7.5\% \).

The first line of panels shows that the log price-dividend ratio increases with \( \omega_t \), capturing the procyclical behavior of prices. Two additional effects obtain. First, heterogeneity in risk attitudes makes the price-dividend ratio a little steeper (flatter) in bad (good) times than in the homogeneous case. Second, for \( \chi > 0 \), the price-dividend increases more in good times and decreases more in bad times, as a result of the waves of optimism and pessimism.

The second line of panels shows the conditional stock return volatility. The model captures the excess of volatility over fundamentals. The specification of risk attitudes significantly affects the dynamics of return volatility. In the homogeneous-agents economy, volatility is procyclical, i.e. increasing with \( \omega_t \). Indeed, both the terms in Eq. (33) increase with \( \omega_t \). Under preference heterogeneity, this is not necessarily true: indeed the second term of \( \sigma_y^p \) in Eq. (45) is the price of risk which is decreasing with \( \omega_t \). Consequently, if \( \lambda \) and \( \kappa \) are large enough, the residual terms of \( \sigma_y^p \) and \( \sigma_s^p \) –which are forward-looking terms– do not offset the countercyclical behavior of the price of risk. Then, under heterogeneous preferences return volatility increases in bad states and decreases in good times.
Also pro-cyclical optimism affects the dynamics of return volatility. The effect is quite limited in the homogeneous-agents economy, but an important result obtains under preference heterogeneity (see also Section VI.C). Indeed, for $\chi > 0$, the states $\omega_t$ and $\hat{\mu}_t$ feature a larger unconditional variance and correlation. Therefore, the forward-looking components of volatility are larger in magnitude than in the Bayesian case. In particular, $\sigma_{P}^{\alpha}$ becomes less countercyclical and $\sigma_{P}^{\beta}$ becomes more markedly procyclical. Furthermore, such an effect is asymmetric: an increase in $\chi$ produces a larger effect on volatility in bad times than in good ones. Figure 8 shows $\sigma_{P}^{\alpha}$ and $\sigma_{P}^{\beta}$ as a function of $\omega_t$ for $\chi = 7.5\%$.

The third line of panels in Figure 7 reports the conditional equity premium. Since in the homogeneous-agents economy the price of risk is constant, the equity premium inherits the procyclical behavior of priced volatility. This is not the case under heterogeneous risk attitudes. The equity premium is strongly countercyclical, in line with the empirical evidence. For $\chi > 0$, the equity premium inherits the above offsetting mechanism between the components of $\sigma_{P}^{\alpha}$, but such an effect is mitigated by the countercyclical price of risk. As a result, for $\chi > 0$, the equity premium is more countercyclical than conditional volatility, in line with the real data (see Lettau and Ludvigson (2010)). Such a result is at the heart of the market risk-return trade off.

The last two lines of panels in Figure 7 show the conditional volatility and premium of the perpetual bond. Similarly to the stock, both the first two moments of bond returns are decreasing with $\omega_t$ under preference heterogeneity. However, the effect of $\chi$ is weaker than that on the stock. Indeed, bond returns reflect the excess of persistence in $\omega_t$ but do not directly account for the perceived dynamics of $\hat{\mu}_t$. In turn, the bond premium and volatility are positively correlated.

**F. Predictability and Other Model Predictions**

This section investigates the predictability of the excess returns. For each model, 1000 simulations are run, with each simulation accounting for 100 observations. For each simulation, I run regressions of cumulative excess market returns from one to seven years on the current price-dividend ratio and on the real 10 years term spread. Tables VII and VIII show the results respectively for the homogeneous- and the heterogeneous-agents economies.

In the Bayesian case ($\chi = 0$), none of the model coefficients is statistically significant. These results are in line with Xiouros and Zapatero (2010): predictability is quite modest since a realistic degree of preference heterogeneity is too small to produce enough variation in the equity premium. Pro-cyclical optimism ($\chi > 0$) leads to some interesting results: the negative coefficients increase in size with the horizon as well as their t-statistics and the explanatory power.
For $\chi = 7.5\%$, the regression coefficients are highly significant at all the horizons, in line with the empirical evidence. Intuitively, investors overestimate the persistence of aggregate risk. In turn, the price-dividend ratio is more persistent and captures the additional variation in the excess return. Figure 9 shows the t-statistics and the $R^2$ from the regressions as functions of $\chi$ under heterogeneous risk aversion.

Insert Figure 9 and 10 about here.

Furthermore, the predictability of stock returns does not come with the counter-factual predictability of dividend growth by valuation ratios. The latter obtains for instance in long-run risk models, as documented by Beeler and Campbell (2012). Figure 10 reports a little explanatory power for any value of the deviation parameter $\chi$ and regression coefficients that are not statistically significant. Such a result is due to the incomplete information and pro-cyclical optimism. Although persistent, true dividend growth is not predictable by equilibrium prices.

The term premium –i.e. the spread between the 10 year interest rate and the risk-free rate– predicts future cumulative excess returns. For $\chi = 7.5\%$, the regression coefficients are highly significant at all the horizons but the first one. The explanatory power obtains because the term premium captures the lack of mean-reversion in $\omega_t$ perceived by the agents.

The real yield is an affine in $\omega_t$ and $\hat{\mu}_t$ in the homogeneous-agents economy, while it is a nonlinear function of the states under heterogeneous risk preferences. A similar result obtains in Buraschi and Jiltsov (2007) because of the nonlinear specification of their habit state variable. Here, non-linearity is due to the endogenous redistribution of wealth among the agents.

Insert Figure 11 about here.

The left panel of Figure 11 shows that the real term structure decreases with $\omega_t$ in the heterogeneous-agents economy. This happens because investors’ marginal utility is high (low) when $\omega_t$ is low (high) implying a higher (lower) demand for consumption. Moreover, the yield curve is upward sloping across different maturities in good states, while it is downward sloping in bad states. Intuitively, in bad states $\omega_t$ is expected to move toward its long-run mean leading to lower risk for the investors at longer maturities, while the opposite holds in good times. Similar results arise in the homogeneous-agents economy. The middle and right panels of Figure 11 report the unconditional mean and volatility of the real yield as a function of maturity. Their term structures are respectively increasing and decreasing. Therefore, the model can lead to a positive slope of the nominal curve (as in actual data), which is not due to an inflation risk premium only. When $\chi > 0$, the curve is slightly lower than in the Bayesian case since the perceived aggregate risk is more volatile and, then, the bond is a better hedge against consumption risk. Nonetheless, for large maturities the curves converge since the agents have unbiased beliefs in the long-run.
Although time-separable utility, the model accounts for a number of features of both interest rates and stock returns. Therefore, this paper complements the literature which considers recursive utility, such as Bansal and Shaliastovich (2013).

Under preference heterogeneity, the model has predictions about cross-sectional portfolios and the resulting trading volume dynamics. Figure 12 reports the wealth proportions invested in the stock and in the bond by the agents types with relative risk aversion between $\gamma = 10$ and one. For reasonable parameters these agents own almost all wealth in the economy. Allocations are plotted as a function of $\omega$. Investments in stock and bond are respectively counter-cyclical and pro-cyclical for the low risk averse agent. The opposite holds for the high risk averse agent. In bad times (low $\omega$), wealth shifts towards the high risk averse agents, who are long in the bond and allow the low risk averse agents to fund their stock investment beyond their wealth. In good times (high $\omega$), all agents tend to invest their entire wealth in the stock since the price of risk and, in turn, prices are little sensitive to the state.

Under pro-cyclical optimism ($\chi > 0$), portfolio allocations become more extreme, that is they are more dispersed in bad times and more aligned in good times. Indeed, allocations reflect the excessive persistence of the habit state variable perceived by the investors. Furthermore, allocations endogenously lead to countercyclical trading volume in line with the empirical evidence. Notice that the model is defined in continuous time: then, continuous trading and Brownian innovations generate infinite trading volume in any finite time interval. However, since the economy has a stationary equilibrium, the state dependent dynamics of trading can be studied by assuming discrete trading at a fixed time interval. Namely, portfolio holdings $\rho_P^i(\omega, \hat{\mu}) = \pi^i P_t W_i / P_t$ are functions of the states only and turnover is given by

$$T(\{\omega', \hat{\mu}'\}, \{\omega'', \hat{\mu}''\}) = \frac{1}{2} \sum_{i=1}^{n} |\rho_P^i(\omega'', \hat{\mu}'') - \rho_P^i(\omega', \hat{\mu}')|,$$

for each pair of states: $\{\omega', \hat{\mu}'\}$ and $\{\omega'', \hat{\mu}''\}$. The sum is truncated to the first $n$ agents such that the remainder owns a negligible share of total wealth.

VII. Extensions

Relaxing some assumptions –made for the purpose of exposition– could help to improve the quantitative implications of the model. Two approaches are: a more general source of uncertainty and a richer construction of beliefs.
**Exogenous Uncertainty:** Consumption dynamics can be generalized by accounting for stochastic volatility in spirit of Campbell and Hentschel (1992), whereas the dividend process, representative of the stock market, can be a distinct (in spirit of Campbell (1996)) but co-integrated process with consumption:

\[
\begin{align*}
    dY_t &= \mu_t Y_t dt + \sqrt{\nu_t} Y_t dB_{y,t}, \\
    d\mu_t &= \kappa_\mu (\bar{\mu} - \mu_t) dt + \phi_\mu \sqrt{\nu_t} dB_{\mu,t}, \\
    d\nu_t &= \kappa_\nu (\bar{\nu} - \nu_t) dt + \phi_\nu \sqrt{\nu_t} dB_{\nu,t},
\end{align*}
\]

and \( \log D_t = \log Y_t - z_t \) with \( dz_t = \kappa_z (\bar{z} - z_t) dt + \phi_z \sqrt{z_t} dB_{z,t} \) and \( \bar{z} > 0 \). This setting of uncertainty is similar to the long-run risk literature and can be further generalized to account for disasters (Wachter (2013)). As long as exogenous uncertainty belongs to the jump-diffusion affine class, all closed forms are preserved. A more realistic exogenous uncertainty can help to match the equity premium, to increase the variation of the price of risk and to capture the downward-sloping term-structure of dividend volatility.

**Beliefs formation:** The paper analyses the case of symmetric waves of optimism and pessimism. A simple way to include a long-run bias, which preserves closed form solutions, is a modified deviation term: \( \Psi(\omega_t) = \chi(\omega_t - \bar{\omega} + \varepsilon) \), where \( \varepsilon \) captures the bias in the long-run forecast errors.\(^{29}\) Such a bias affects the reversion threshold \( \hat{\omega}_t \). For \( \varepsilon > 0 \), good states are perceived as more persistent than bad states. Instead, for \( \varepsilon < 0 \), waves of pessimism are more persistent than waves of optimism. Figure 13 shows the unconditional equity premium as a function of the deviation parameter \( \chi \) and the bias parameter \( \varepsilon \). For \( \varepsilon > 0 \), we observe a negative long-run bias in forecast errors about growth and an increase in the equity premium.

An interesting extension concerns expected growth volatility. Assuming that the volatility of \( \hat{\mu}_t \), is a function of the habit state variable can be interpreted as time-varying investors’ attention and/or time-varying quality of available information. Predictability of forecast errors dispersion could provide empirical support (see Patton and Timmermann (2010)). On a technical side, a characterization, which preserves analytical solutions, can be difficult and is left to future research.

\(^{29}\)The model solution is unchanged: an additional term equal to \( \chi \varepsilon C \) appears on the right hand side of the partial differential equation (C4) in the Appendix C and affects the solution of the deterministic coefficient \( A \), while \( B \) and \( C \) remain unchanged.
VIII. Conclusion

This paper proposes a closed form general equilibrium model under incomplete information. Investors feature heterogeneous risk preferences and pro-cyclical optimism, as suggested by the survey data about fundamentals. In turn, investors overestimate the persistence of the economic environment. This provides a rationale for the excess variability of the equilibrium pricing kernel and, hence, for the disconnect between the smooth dynamics of fundamentals and the large fluctuations of asset prices.

The analysis, based on a single parameter governing pro-cyclical optimism, shows that the model explains:

i the weak or negative empirical relation between market risk and its remuneration;

At the same time, many other asset pricing quantities move in line with the data providing further robustness:

ii the equity premium increases, whereas the risk-free rate stays low;

iii the risk-free rate (and bond) volatility decreases, whereas stock volatility increases;

iv the persistence of the price-dividend ratio and of the risk-free rate respectively increases and decreases;

v the predictability of excess returns by dividend yields increases, predictability of dividend growth does not obtain;

vi the term structure of real yields is upward sloping and yields volatilities decrease with the maturity;

vii the price of risk and the expected returns are more counter-cyclical than return volatility in bad times;

viii portfolio strategies are more sensitive and dispersed in bad times, leading to counter-cyclical trading volume.

Finally, the assumptions of the model are consistent with the data:

ix preference heterogeneity provides a micro-foundation to the heteroscedasticity of the kernel and imposes discipline on the model calibration by matching the empirical cross-sectional distribution of consumption;

x pro-cyclical optimism about economic growth provides a stylized representation of survey data from professional forecasters.

These ten empirical patterns support the main model mechanism which generates in equilibrium a connection between survey data about fundamentals and those about market returns.

The analysis of this paper can be extended in a number of directions. The model can easily account for more general specifications of both exogenous uncertainty and beliefs formation schemes, without losing closed form solutions. An interesting extension of the model –that I leave to future research– is to make the formation of beliefs to depend on both exogenous and endogenous quantities in order to study in general equilibrium, as an additional feature, the informational and self-fulfilling role of prices, in spirit of Bacchetta, Tille, and van Wincoop (2012).
Appendix A – Filtering and state variables

In this appendix I consider the filtering problem of the incomplete information setting of Section III. The agents observe the dividend and the signal, whose dynamics is defined in Eq. (8) and (12), and estimate the unobservable drift in Eq. (9). The dynamics for the estimated drift in the Bayesian case follows from an application of the Kalman-Bucy theorem:

\[
d\hat{\mu}_t = \nu_t [1 \ 1] \left[ \begin{array}{cc} \sigma^2_t & 0 \\ 0 & \sigma^2_t \end{array} \right]^{-1} \left[ \begin{array}{c} d\tilde{Y}_t/\nu_t \\ \kappa \tilde{\omega}_t \end{array} \right] + \kappa (\mu_t - \nu_t [1 \ 1] \left[ \begin{array}{cc} \sigma^2_t & 0 \\ 0 & \sigma^2_t \end{array} \right]^{-1} [1]) \tilde{\mu}_t dt + \varepsilon_t, \tag{A1} \]

\[
d\hat{\mu}_t = \kappa (\hat{\mu}_t - \mu_t) dt + \varepsilon_t \left( \frac{\mu_t dt + \sigma_t d\tilde{B}_{t,t}}{\sigma^2_t} + \frac{\tilde{\mu}_t dt + \sigma_t d\tilde{B}_{t,t}}{\sigma^2_t} - \frac{\mu_t}{\sigma^2_t} (\sigma_t^2 - \sigma^2_{t-1}) dt \right), \tag{A2} \]

where \( \nu_t \) satisfies the following Riccati equation:

\[
\nu_t = -2 \kappa \nu_t - \nu_t^2 (\sigma_t^2 + \sigma^2_{t-1}) + \sigma^2_{t-1}, \tag{A3} \]

\[
\nu_0 = \sigma^2_{t-1}. \tag{A4} \]

At the steady state \( \nu_t \) is as in Eq. (16) and therefore the estimated drift has dynamics

\[
d\hat{\mu}_t = \kappa (\hat{\mu}_t - \mu_t) dt + \frac{\nu_t}{\sigma^2_t} d\tilde{B}_{t,t} + \frac{\tilde{\mu}_t}{\sigma^2_t} d\tilde{B}_{t,t}. \tag{A5} \]

which leads to Eq. (14). If the agents interpret the signal as \( d\tilde{Y}_t = (\mu_t + \Psi(\sigma_t)) dt + \sigma_t d\tilde{B}_{t,t} \) (see Section III.C), therefore Eq. (A5) becomes

\[
d\hat{\mu}_t = \kappa (\hat{\mu}_t - \mu_t) dt + \frac{\nu_t}{\sigma_t} d\tilde{B}_{t,t} + \frac{\tilde{\mu}_t}{\sigma_t} d\tilde{B}_{t,t} + \frac{\sigma_t}{\sigma^2_t} \Psi(\sigma_t) dt. \tag{A6} \]

where \( \nu_t \) is unchanged, \( \Psi(\sigma_t) = \chi' (\sigma_t - \hat{\sigma}) \) and the dynamics in Eq. (17) is obtained for \( \chi = \frac{\sigma_t}{\sigma^2_t} \chi' \).

Both \( \hat{\mu}_t \) and \( \omega_t \) are processes in the affine class under the investors subjective probability measure in both the Bayesian and non-Bayesian cases of Section III.C. Therefore, they have conditional joint Laplace transform of exponential affine form (a special case of Eq. (C2)):

\[
M_{t,\mu}(a, b) = \mathbb{E}_t [e^{\theta_0 + b\mu_t}] = e^{A(a-t)+B(a-t)\theta_0 + C(a-t)\hat{\mu}_0} \tag{A7} \]

for some deterministic functions of time A, B and C. Conditional central (co-)moments can be computed differentiating at zero the logarithm of the Laplace transform. In both the Bayesian and non-Bayesian cases, it can be verified that \( \hat{\mu}_t \) and \( \omega_t \) have unconditional expectation equal respectively to \( \hat{\mu} \) and (\( \hat{\mu} - \sigma^2_{t-1}/2 \)). In the non-Bayesian case, covariance stationarity requires the additional condition: \( \lambda + \kappa > \sqrt{(\lambda - \kappa)^2 + 4\lambda} \).

Appendix B – Equilibrium Solution: Proofs

**Proof of Lemma 1:** Pareto optimality requires that optimal consumption solves the problem (22). Taking the first order condition we get \( v_t e^{-\Phi_t} C_{t,s}^{-\gamma} X_t^\gamma = \zeta_{s,t} \zeta_{s,t} \), where \( \zeta \) is the Lagrange multiplier of the resource constraint. Hence optimal consumption is equal to

\[
C_{t,s} = v_t^{1/\gamma} e^{-\rho_t/\gamma} \zeta_{s,t}^{-1/\gamma} \zeta_{s,t}^{-1/\gamma} X_t = (v_t/v_1)^{1/\gamma} (C_{1,s}/X_t)^{1/\gamma} X_t. \tag{B1} \]

Market clearing implies that

\[
Y_t/x_t = X_t^{1-\gamma} \sum_i C_{i,t} = \sum_i (v_t/v_1)^{1/\gamma} (C_{1,s}/X_t)^{1/\gamma}. \tag{B2} \]

Under the assumption about preference heterogeneity of section III.A, we get

\[
Y_t/x_t = \sum_i e^{(1-\gamma)\hat{\theta}_i} C_{1,s}^{1-\gamma} X_t^{-\gamma} = e^{\hat{\theta}_s} C_{1,s}/(e^{\hat{\theta}_s} X_t - C_{1,s}). \tag{B3} \]
Then $C_{1t} = e^{\theta_0} X_t / (1 + e^{\theta_0 - \theta})$ and the optimal consumption $C_{it}$ in Eq. (24) automatically follows. \textit{q.e.d.}

\textbf{Proof of Corollary 1:} Armed of optimal consumption of agent $i$ as in Eq. (24), the representative agent's utility evaluated at the optimal consumption is

$$U^{RA}(Y_t, X_t) = \sum_i v_i U(C_{ij}, X_t; \gamma_i) = \sum_i \frac{v_i}{1 - \gamma_i} ((v_i / v_1)^{\gamma_i / \gamma} (C_{ij} / X_t)^{\gamma_i / \gamma} X_t^{\gamma_i})$$

(B4)

Then differentiating with respect to the aggregate consumption we get

$$U^{RA}_Y(Y_t, X_t) = \sum_i v_i ((v_i / v_1)^{\gamma_i / \gamma} (C_{ij} / X_t)^{\gamma_i / \gamma} X_t^{\gamma_i}) = v_1 (C_{ij} / X_t)^{-\gamma_i}$$

(B5)

Under the assumption about preference heterogeneity of section IIIA, marginal utility reduces to

$$U^{RA}_Y(Y_t, X_t) = v_1 \left( e^{\theta_0} X_t / (\theta + \theta_0 X_t) \right)^{-\gamma_i}$$

(B6)

The equilibrium state price density $\hat{\xi}_{r,t}$ in Eq.(25) automatically follows. Differentiating again with respect to $Y_t$ leads to $U^{RA}_{YY}(Y_t, X_t) = -(\gamma_i - 1) e^{\gamma_i} X_t^{-\gamma_i - 1}$. Therefore, the Arrow-Pratt coefficient, \( \hat{\gamma}(\omega_k) \), is given by

$$\hat{\gamma}(\omega_k) = \frac{-Y_t U^{RA}_{YY}(Y_t, X_t)}{U^{RA}_Y(Y_t, X_t)} = \frac{\gamma_i}{1 + e^{\theta_0 - \theta}}$$

(B7)

as in Eq. (26). It can be easily verified that the aggregate relative risk aversion is also equal to $\hat{\gamma}(\omega_k) = \left( \sum_i (C_{ij} / X_t)^{-1} \right)^{-1}$. For further details about the optimal consumption and the preferences of the representative agent see Curatola and Marfè (2011).

In the case of homogeneous-agents, since consuming the aggregate dividend must be optimal for the representative agent, the marginal rate of substitution identifies the equilibrium state price density which must consequently be of the form in Eq. (27). \textit{q.e.d.}

\textbf{Appendix C – Equilibrium Prices: Proofs}

Consider the following discounted conditional Laplace transform:

$$M_{r,t}(\tilde{c}) = e^{-A(u-t, \tilde{c})} \mathbb{E}[e^{c_1 \log Y_t + c_2 (\omega_k + \omega_0) + c_3 \hat{\psi}_t} | \{\omega_k \}_{k=1}^t]$$

(C1)

where $\tilde{c} = (c_0, c_1, c_2, c_3)$ is a coefficient vector such that the expectation exists, and guess an exponential affine solution of the kind:

$$M_{r,t}(\tilde{c}) = e^{c_1 \log Y_t + A(u-t, \tilde{c}) + B(u-t, \tilde{c}) \omega_k + C(u-t, \tilde{c}) \hat{\psi}_t}$$

(C2)

Feynman-Kac gives that $M$ has to meet the following partial differential equation

$$M_{t} + M_{Y_t} \hat{\mu}_t Y_t + M_{Y_t^2} \frac{1}{2} \sigma^2_{Y_t} Y_t^2 + M_{\omega_k} \lambda (\omega_k - \omega_0) + M_{\omega_k \omega_0} \frac{1}{2} \sigma^2_{\omega_k} + M_{Y_t \omega_k} \sigma_{Y_t \omega_k} Y_t + M_{\hat{\psi}_t} (\kappa (\hat{\mu} - \hat{\mu}_t) + \hat{\psi}(\omega_k))$$

$$+ M_{\hat{\psi}_t} \frac{1}{2} \sigma^2_{\hat{\psi}_t} + C \sigma^2_{\mu, \nu} + M_{Y_{\hat{\psi}} \sigma_{\nu, Y_t}} Y_t + M_{\hat{\psi}_t} \sigma_{\mu, \nu} Y_t = M \rho$$

(C3)

where the arguments have been omitted for ease of notation. Plugging the resulting partial derivatives from the guess solution into the pde and simplifying gives

$$A_t + B_t \omega_k + C_t \hat{\mu}_t + c_1 \hat{\mu}_t + \frac{1}{2} c_1 (c_1 - 1) \sigma^2_{Y_t} + B (\hat{\mu}_t - \lambda \omega_k) + B (c_1 - 1/2) \sigma^2_{\omega_k} + \frac{1}{2} \sigma^2_{\hat{\psi}_t}$$

$$+ (\kappa (\hat{\mu} - \hat{\mu}_t) + \chi (\omega_k - \hat{\mu}_t / \kappa) + c_1 \sigma_{Y_t \omega_k} + \frac{1}{2} \chi \sigma_{\hat{\psi}_t} Y_t) + \frac{1}{2} (\sigma^2_{\mu, \nu} + \sigma^2_{\mu, \nu} + C \sigma_{\mu, \nu} Y_t + \sigma_{Y_t \omega_k} \sigma_{\mu, \nu} Y_t) = \rho$$

(C4)
This equation has to hold for all \( \omega_t \) and \( \hat{\mu}_t \). We thus get three ordinary differential equations for \( A, B \) and \( C \):

\[
A_t = \rho - \frac{1}{2}c_1(c_1 - 1)\sigma_1^2 - \frac{1}{2}c_2^2B^2 - C(\kappa\mu - \chi\hat{\mu}/\lambda + c_1\sigma_y\sigma_{\mu,y} - \frac{1}{2}\chi\sigma_y^2) - \frac{1}{2}(\sigma_{\mu,y}^2 + \sigma_y^2\sigma_{\mu,y})^2 - B(c_1 - 1/2)\sigma_y^2 - \frac{1}{2}\chi\sigma_y^2\]

\[\text{(C5)}\]

\[
B_t = \lambda B - \chi C
\]

\[\text{(C6)}\]

\[
C_t = \kappa C - B - c_1
\]

\[\text{(C7)}\]

with initial conditions \( A(0, \hat{c}) = c_2C_0, \ B(0, \hat{c}) = c_2 \) and \( C(0, \hat{c}) = c_3 \). The solution is

\[
B(\tau, \hat{c}) = \frac{e^{-(\kappa+\lambda+\chi)t/2}}{2(\lambda - \kappa)} \left[ \chi \left( -c_1 z(1 + e^{\tau t}) \right) + c_2(\kappa - \chi)(1 - e^{\tau t}) \right],
\]

\[\text{(C8)}\]

\[
C(\tau, \hat{c}) = \frac{e^{-(\kappa+\lambda+\chi)t/2}}{2(\lambda - \kappa)} \left[ c_1 \left( (\kappa - \lambda) \lambda z \right) + c_2(\kappa - \lambda)(1 - e^{\tau t}) \right],
\]

\[\text{(C9)}\]

where \( z = \sqrt{(\lambda - \kappa)^2 + 4\chi} \) and \( A(\tau, \hat{c}) \) can be computed in closed form but it is too long to be reported. In the limit case \( \chi \to 0 \) (i.e. under Bayesian beliefs), \( B(\tau, \hat{c}) \) and \( C(\tau, \hat{c}) \) reduce to:

\[
B(\tau, \hat{c}) = c_2e^{-\kappa t},
\]

\[\text{(C10)}\]

\[
C(\tau, \hat{c}) = \frac{c_2\kappa e^{-\kappa t} + c_1(\kappa - \lambda)(1 - e^{-\kappa t}) + \kappa(c_2(\kappa - \lambda) - c_2)e^{-\kappa t}}{\kappa(\kappa - \lambda)}.
\]

\[\text{(C11)}\]

**Proof of Corollary 2:** Applying Itô’s lemma to the state price density \( \xi_{0,t} \) as in Eq. (27) and using the dynamics in Eq. (15) and \( d\xi_{0,t} = -\xi_{0,t}r_t dt - \xi_{0,t}\theta_t dB_t \) gives the expressions for \( r_t \) and \( \theta_t \):

\[
r_t = -\frac{\partial \xi_{0,t}}{\partial \mu_t} - \lambda(\omega_t - \omega) \frac{\partial \xi_{0,t}}{\partial \sigma_{\mu,t}} - \frac{\sigma_y^2}{2} \frac{\partial^2 \xi_{0,t}}{\partial \sigma_{\mu,t}^2} \frac{\partial \omega_t}{\partial \sigma_{\mu,t}} \quad \text{and} \quad \theta_t = -\frac{\sigma_y^2}{2} \frac{\partial^2 \xi_{0,t}}{\partial \sigma_{\mu,t}^2} \frac{\partial \omega_t}{\partial \sigma_{\mu,t}}
\]

\[\text{(C12)}\]

which lead to the Eq. (29) and (30). *q.e.d.*

**Proof of Proposition 1:** To compute the price-dividend ratio, recall that

\[
P_t = \mathbb{E}_t \left[ \int_0^\infty \xi_{0,t}Y_t du \right] = e^{\mu_0} \int_0^\infty e^{-p(u-t)} \mathbb{E}_0 \left[ e^{\log Y_t - \gamma s} \right] du,
\]

\[\text{(C13)}\]

where the latter expectation is a conditional Laplace transform of the type in Eq. (C1). It has solution as in Eq. (C2) where the system (C5)-(C6)-(C7) is solved for \( \hat{c} = (0, 1, -\gamma, 0) \). Therefore, the price-dividend ratio, \( P_t/Y_t \), is a function of the habit state variable and the perceived expected growth only as in Eq. (31). The dynamics of the stock price is obtained by applying Itô’s Lemma:

\[
dP_t = \left[ d\hat{c} + \frac{\partial P_t}{\partial \mu_t} \sigma_{\mu,t} dB_{\mu,t} + \frac{\partial P_t}{\partial \sigma_{\mu,t}} \sigma_{\mu,t} dB_{\sigma_{\mu,t}} + \frac{\partial P_t}{\partial \sigma_y} \sigma_y dB_{\sigma_y} + \sigma_{\mu,t} d\hat{B}_{\mu,t} + \sigma_y d\hat{B}_{\sigma_y} \right]
\]

\[\text{(C14)}\]

and therefore the stock return volatility is given by

\[
\sigma'^2_t = P^{-1}_t \left( \frac{\partial P_t}{\partial \sigma_{\mu,t}} \sigma_{\mu,t} + \frac{\partial P_t}{\partial \sigma_y} \sigma_y \right)^2 + \left( \frac{\partial P_t}{\partial \sigma_{\mu,t}} \sigma_{\mu,t} + \frac{\partial P_t}{\partial \sigma_y} \sigma_y \right)^2 + \left( \frac{\partial P_t}{\partial \sigma_{\mu,t}} \sigma_{\mu,t} + \frac{\partial P_t}{\partial \sigma_y} \sigma_y \right)^2
\]

\[\text{(C15)}\]
which leads to Eq. (32), where $\sigma_{\mu,\mu}$ is equal to zero since the expected growth is not observable. The premium is given by $P_t^\mu - r_t = -\frac{1}{\bar{\gamma}} \left( \frac{dP_t}{P_t} - \frac{\bar{\gamma}}{\bar{\gamma}} \right)$, q.e.d.

**Proof of Proposition 2:** To compute the real yield, recall that

$$\mathbb{E}_t^\xi [\xi_{t,s+t}] = e^{-\rho t + \gamma_0 t} \mathbb{E}_t^\xi \left[ e^{-\gamma_0 t} \xi_t \right],$$

where the latter expectation is a conditional Laplace transform of the type in Eq. (C1). It has solution as in Eq. (C2) where the system (C5)-(C6)-(C7) is solved for $\varepsilon = (0, 0, -\gamma_0, 0)$. Therefore, the real yield, $\varepsilon(t, \tau)$, is a function of the maturity, the habit state variable and the perceived growth only as in Eq. (39). The perpetual bond price is given by $Q_t = \int_0^\infty \mathbb{E}_t^\xi [\xi_{t,s}] du$. The dynamics of the perpetual bond price is obtained by applying Itô’s Lemma:

$$dQ_t = [\gamma] dt + \frac{\partial Q_t}{\partial \xi_t} \sigma_t d\tilde{B}_{t,t} + \frac{\partial Q_t}{\partial \gamma_t} \left( \sigma_{\mu,\gamma} d\tilde{B}_{t,t} + \sigma_{\mu,\mu} d\tilde{B}_{t,t} \right)$$

and therefore the bond return volatility is given by

$$\sigma_t^Q = Q_t^{-1} \sqrt{\frac{\partial Q_t}{\partial \xi_t} \sigma_t \frac{\partial Q_t}{\partial \xi_t} + \left( \frac{\partial Q_t}{\partial \gamma_t} \sigma_{\mu,\gamma} \right)^2 + \left( \frac{\partial Q_t}{\partial \gamma_t} \sigma_{\mu,\mu} \right)^2}$$

which leads to Eq. (36), where $\sigma_{\mu,\mu}$ is equal to zero since the expected growth is not observable. The premium is given by $P_t^\mu - r_t = -\frac{1}{\bar{\gamma}} \left( \frac{dP_t}{P_t} - \frac{\bar{\gamma}}{\bar{\gamma}} \right)$, q.e.d.

**Proof of Corollary 3:** Applying Itô’s lemma to the state price density $\xi_{0,t}$ as in Eq. (25) and using the dynamics in Eq. (15) and $d\xi_{0,t} = -\xi_{0,t} r_t dt - \xi_{0,t} \gamma_0 d\tilde{B}_{t,t}$ gives the expressions for $r_t$ and $\theta_t$:

$$r_t = -\frac{\partial \xi_{0,t}}{\partial \gamma_0} - \lambda (\gamma_0 - \gamma_0) \frac{\partial \xi_{0,t}}{\partial \gamma_0} \frac{\partial \xi_{0,t}}{\partial \gamma_0} - \frac{\sigma_t^2}{2} \frac{\partial^2 \xi_{0,t}}{\partial \gamma_0^2} + \sigma_t \frac{\partial \xi_{0,t}}{\partial \gamma_0}$$

and $\theta_t = -\sigma_t \frac{\partial \xi_{0,t}}{\partial \gamma_0}$

which lead to the Eq. (40) and (41). q.e.d.

**Proof of Proposition 3:** To compute the price-dividend ratio, recall that

$$P_t = \mathbb{E}_t^\xi \left[ \int_0^T \bar{\gamma} Y_u d\xi \right] = \left( \frac{\gamma_0 - \bar{\gamma}}{\gamma_0 - \bar{\gamma}} \right)^\bar{\gamma} \int_0^T e^{-\rho (u-t) \bar{\gamma}} \left[ \frac{1}{\gamma_0 - \bar{\gamma}} \right] Y_u d\xi.$$

In order to evaluate the latter expectation, since $\bar{\gamma}$ is a positive integer, it is possible to apply the Binomial formula $(1 + e^{\gamma_0 - \bar{\gamma}})^\bar{\gamma} = \sum_{j=0}^{\bar{\gamma}} \binom{\bar{\gamma}}{j} e^{j(\gamma_0 - \bar{\gamma})}$ and therefore to obtain:

$$e^{-\rho (u-t) \bar{\gamma}} \left[ \frac{1}{\gamma_0 - \bar{\gamma}} \right] Y_u = e^{-\rho (u-t) \bar{\gamma}} \sum_{j=0}^{\bar{\gamma}} \binom{\bar{\gamma}}{j} Y_u e^{j(\gamma_0 - \bar{\gamma})}.$$

The latter expectation is a conditional Laplace transform of the type in Eq. (C1). It has solution as in Eq. (C2) where the system (C5)-(C6)-(C7) is solved for $\varepsilon = (-\bar{\gamma}, 1, j - \bar{\gamma}, 0) \forall j = 0, \ldots, \bar{\gamma}$. Therefore, the price-dividend ratio, $P_t/Y_t$, is a function of the habit state variable and the perceived expected growth only as in Eq. (43). The dynamics of the stock price is obtained by applying Itô’s Lemma:

$$dP_t = [\gamma] dt + \frac{\partial P_t}{\partial \gamma_t} \sigma_t Y_t d\tilde{B}_{t,t} + \frac{\partial P_t}{\partial \gamma_t} \left( \sigma_{\mu,\gamma} d\tilde{B}_{t,t} + \sigma_{\mu,\mu} d\tilde{B}_{t,t} \right)$$

and therefore the stock return volatility is given by

$$\sigma_t^P = P_t^{-1} \sqrt{\left( \frac{\partial P_t}{\partial \gamma_t} \sigma_t Y_t \right)^2 + \left( \frac{\partial P_t}{\partial \gamma_t} \sigma_{\mu,\gamma} \right)^2 + \left( \frac{\partial P_t}{\partial \gamma_t} \sigma_{\mu,\mu} \right)^2}$$

39
which leads to Eq. (44), where $\sigma_{\mu,\ell}$ is equal to zero since the expected growth is not observable. The premium is given by $\mu_\ell - r_i = -\frac{1}{\ell} \left( \frac{dF}{d\xi} \right)_{\xi = \frac{\bar{r}}{\ell}}$, q.e.d.

**Proof of Proposition 4:** To compute the real yield, recall that

$$E_t [\xi, t + \tau] = e^{-\rho \tau} \left( \frac{1 + \bar{\ell} \gamma}{1 + \bar{\ell} \gamma} \right)^{\frac{\gamma}{\ell}} E_0 \left[ \left( \frac{1 + \bar{\ell} \gamma}{1 + \bar{\ell} \gamma} \right)^{\frac{\gamma}{\ell}} \right].$$

(C24)

In order to evaluate the latter expectation, since $\gamma$ is a positive integer, it is possible to apply the Binomial formula $(1 + e^{\delta \gamma})^{-\gamma} = \sum_{j=0}^{\gamma} \binom{\gamma}{j} e^{j(\delta \gamma)}$ and therefore to obtain:

$$e^{-\rho (u-\ell)} E_0 \left[ \left( \frac{1 + \bar{\ell} \gamma}{1 + \bar{\ell} \gamma} \right)^{\frac{\gamma}{\ell}} \right] = e^{-\rho (u-\ell)} \sum_{j=0}^{\gamma} \binom{\gamma}{j} E_0 \left[ e^{j(\delta \gamma)} \right].$$

(C25)

The latter expectation is a conditional Laplace transform of the type in Eq. (C3). It has solution as in Eq. (C2) where the system (C5)-(C6)-(C7) is solved for $\bar{c} = (-\bar{c}, 0, \ldots, 0)$, $\gamma = 0, \ldots, \gamma$. Therefore, the real yield, $\varepsilon(t, \tau)$, is a function of the maturity, the habit state variable and the perceived expected aggregate dividend growth only as in Eq. (5). The perpetual bond price is given by $Q_t = \int_0^\infty \xi_t E_0 [\xi, u] du$. The dynamics of the perpetual bond price is obtained by applying Itô’s Lemma:

$$dQ_t = \left[ \int_0^\infty \xi_t d\tilde{B}_t + \int_0^\infty \left( \sigma_{\mu,\ell} d\tilde{B}_t + \sigma_{\mu,\bar{r}} d\tilde{B}_{\bar{r},t} + \sigma_{\mu,\bar{r}} d\tilde{B}_{\bar{r},t} \right) \right] dt$$

(C26)

and therefore the bond return volatility is given by

$$\sigma_t^2 = \frac{Q_t^{-1}}{2} \left( \frac{\partial Q_t}{\partial \xi} \sigma_{\mu,\ell} + \frac{\partial Q_t}{\partial \xi} \sigma_{\mu,\bar{r}} \right)^2 + \frac{\partial Q_t}{\partial \xi} \sigma_{\mu,\bar{r}}^2 + \frac{\partial Q_t}{\partial \xi} \sigma_{\mu,\bar{r}}^2 + \frac{\partial Q_t}{\partial \xi} \sigma_{\mu,\bar{r}}^2$$

(C27)

which leads to Eq. (48), where $\sigma_{\mu,\ell}$ is equal to zero since the expected growth is not observable. The premium is given by $\mu_\ell - r_i = -\frac{1}{\ell} \left( \frac{dQ}{d\xi} \right)_{\xi = \frac{\bar{r}}{\ell}}$, q.e.d.

**Proof of Proposition 5:** To compute the individual wealth, recall that $\forall i \in N$

$$W_{i,t} = \xi_0 \int_0^\infty E_t \left[ C_{i,u} d\tilde{u} \right] = \left( \frac{1 + \bar{\ell} \gamma}{1 + \bar{\ell} \gamma} \right) \xi_0 \int_0^\infty e^{-\rho (u-\ell)} E_0 \left[ \left( \frac{1 + \bar{\ell} \gamma}{1 + \bar{\ell} \gamma} \right)^{\frac{\gamma}{\ell}} \right] e^{-(\delta \gamma)Y_t} d\tilde{u}.$$  

(C28)

In order to evaluate the latter expectation, since $\gamma$ is a positive integer, it is possible to apply the Binomial formula $(1 + e^{\delta \gamma})^{-\gamma} = \sum_{j=0}^{\gamma} \binom{\gamma}{j} e^{j(\delta \gamma)}$ where

$$U = \begin{cases} q - i & \gamma_i = 1 \\ \infty & \gamma_i < 1 \end{cases}$$

and

$$Y_t(i, j) = \begin{cases} (\bar{c}_j) & \gamma_i = 1 \\ (\ell(i,j))(-1)^j & \gamma_i < 1 \end{cases}$$

(C29)

with $\ell(i, j) = j - (\gamma - i + 1)$ and therefore to obtain:

$$e^{-\rho (u-\ell)} E_0 \left[ \left( \frac{1 + \bar{\ell} \gamma}{1 + \bar{\ell} \gamma} \right)^{\frac{\gamma}{\ell}} \right] e^{-(\delta \gamma)Y_t} = e^{-\rho (u-\ell)} \sum_{j=0}^{U} Y_t(i, j) \xi_0 \left[ e^{\log Y_t + \gamma(i, j)(\delta \gamma)Y_t} \right].$$

(C30)

The latter expectation is a conditional Laplace transform of the type in Eq. (C3). It has solution as in Eq. (C2) where the system (C5)-(C6)-(C7) is solved for $\bar{c} = (-\bar{c}, 1, \ell(i, j), 0)$, $\gamma = 0, \ldots, \gamma$. Therefore, the ratio $W_{i,t}/Y_t$ is a function of the habit state variable and the perceived expected growth only as in Eq. (52). The dynamics of individual wealth is obtained by applying Itô’s Lemma:

$$dW_{i,t} = W_{i,t} \left[ (r + \pi^t \mu^P - r) + \pi^t \sigma^W \right] dt + \sigma^W d\tilde{B}_{\bar{r},t}$$

(C31)
where

\[
\sigma^W_i = \pi^p_i \sigma^p_i + \pi^Q_i \sigma^Q_i = W_{i,j}^{-1} \left( \frac{\partial W_{i,j}}{\partial \sigma_j} \sigma_j + \frac{\partial W_{i,j}}{\partial \sigma_{\mu,y}} \sigma_{\mu,y} \right),
\]

(C32)

\[
\sigma^W_i = \pi^p_i \sigma^p_i + \pi^Q_i \sigma^Q_i = W_{i,j}^{-1} \left( \frac{\partial W_{i,j}}{\partial \sigma_{\mu,y}} \sigma_{\mu,y} \right),
\]

(C33)

as in Eq. (54). Wealth proportions invested in the risky assets, \( \pi^p_i \) and \( \pi^Q_i \), satisfy Eq. (53) as long as the market asset and the perpetual bond lead to endogenous completeness, that is their diffusion matrix is invertible (i.e., \( \sigma^p_i \sigma^Q_i \neq \sigma^p_i \sigma^Q_i \) for almost all states and times). q.e.d.

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Tran, Ngoc-Khanh, and Richard J. Zeckhauser, 2012, The Behavior of Savings and Asset Prices When Preferences and Beliefs are Heterogeneous, Unpublished manuscript. [61]
Figures & Tables

Figure 1. Empirical risk-return trade off

Left panel. Scatter plot of the conditional mean and conditional volatility of excess returns on the CRSP value weighted stock index. Source: Lettau and Ludvigson (2001). The conditioning information set is given by the cay and two lags of volatility. The sample is quarterly and spans the period 1953:Q4 to 2000:Q4. Right panel. Scatter plot of the conditional mean and conditional volatility of returns from the Duke/CFO survey (see Graham and Harvey (2012)). The sample is quarterly and spans the period 2000:Q3-2012:Q2. The solid line denotes the nonlinear (quadratic) fit (panel C, third column in Table II).

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Figure 2. Survey risk-return trade off vs economic conditions

The left and middle panels show the scatter plot of state variable $\omega$ (built as in Eq. (1)) and respectively the conditional mean and the conditional volatility of returns from the Duke/CFO survey. The right panel shows the scatter plot the survey conditional mean and volatility of returns. The sample is quarterly and spans the period 2000:Q3-2012:Q2. Blue and red colors denote observations associated to values of $\omega$ respectively above and below its long-run average $\bar{\omega}$. Lines denote predicted values from regressions in panels A, B and C of Table II. Black lines correspond to the case where the only predictive variable is the variable on the horizontal axis (first column, Table II); coloured lines correspond to the case where also a dummy is considered (second column, Table II).

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Joint distribution of $\xi$ouros and Zapatero attitudes (red line) as a function of $\chi$. Upper right panel: Equilibrium cumulative distribution of consumption- and wealth-shares and for $\bar{\gamma}$. Habit state variable: cumulative distribution $= \Omega$. Figure 3. Pro-cyclical optimism.

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Figure 4. Preference Heterogeneity and Equilibrium Distributions

Upper left panel: Equilibrium cumulative distribution of consumption-shares at the steady state $\theta_k = \bar{\theta}$. Colours green, red and blue denote respectively the model distribution for $\bar{\gamma} = \{7.5, 10, 15\}$. Black line denotes the Gamma distribution estimated by Xiouros and Zapatero (2018) from individual consumption data. Upper right panel: Equilibrium cumulative distribution of consumption- and wealth-shares and for $\bar{\gamma} = 10$. Black line denotes wealth-shares at the steady state $\theta_k = \bar{\theta}$, red and blue areas denote respectively the range of variation of consumption- and wealth-shares. Lower left panel: Density function of aggregate relative risk aversion under Bayesian beliefs ($\chi = 0\%$, blue line) and pro-cyclical optimism ($\chi = 7.5\%$, red line). Lower right panel: Undiscounted log state price density under homogeneous (blue line) and heterogeneous risk attitudes (red line) as a function of $\theta_k$. Model calibration is from Table III.

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Figure 5. Equilibrium risk-return trade off

Equilibrium unconditional correlation between expected excess return and stock return volatility as a function of the parameter $\chi$. Colours blue and red denote respectively the homogeneous- and the heterogeneous-agents economy. Model calibration is from Table III.

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Figure 6. Unconditional moments and pro-cyclical optimism

Unconditional moments as a function of the parameter $\chi$. First panel: expected stock (solid) and bond (dashed) returns and risk-free rate (dotted line). Second panel: risk-free rate volatility. Third panel: stock (solid) and bond (dashed line) return volatility. Fourth panel: first order autocorrelation of price-dividend ratio (solid), bond (dashed) and risk-free rate (dotted line). Model calibration is from Table III.

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Log price-dividend ratio (first line), stock return volatility (second line), equity premium (third line), bond return volatility (fourth line) and bond premium (fifth line) are plotted as a function of $\omega_t$ and $\hat{\mu}_t$. Left and right panels consider respectively Bayesian beliefs ($\chi = 0\%$) and pro-cyclical optimism ($\chi = 7.5\%$). Model calibration is from Table III.

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Figure 8. Stock return volatility and pro-cyclical optimism

Priced (left) and unpriced (right) components of stock return volatility are plotted as a function of the habit state variable. Solid and marked lines denote the cases: $\hat{\mu}_t = \bar{\mu} \pm 3\%$. Colours blue and red denote respectively Bayesian beliefs ($\chi = 0\%$) and pro-cyclical optimism ($\chi = 7.5\%$). Model calibration is from Table III.

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Figure 9. Long-horizon stock return predictability and pro-cyclical optimism

Regressions of cumulative excess stock returns on the log price-dividend ratio (upper panels) and on the 10 years term premium (lower panels). Colours blue, green, red, cyan and orange denote the horizon of respectively 1, 2, 3, 5 and 7 years. $t$-statistics (left panels) and $R^2$ (right panels) are plotted as a function of the parameter $\chi$. Model calibration is from Table III.

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Figure 10. Long-horizon dividend growth predictability and pro-cyclical optimism

Regressions of cumulative dividend growth on the log price-dividend ratio (upper panels) and on the 10 years term premium (lower panels). Colours blue, green, red, cyan and orange denote the horizon of respectively 1, 2, 3, 5 and 7 years. t-statistics (left panels) and $R^2$ (right panels) are plotted as a function of the parameter $\chi$. Model calibration is from Table III.

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Figure 11. Term structure of interest rates and pro-cyclical optimism

Left panel: Real yield as a function of the habit state variable and the maturity (with $\hat{\mu}_t = \bar{\mu}$) under pro-cyclical optimism ($\chi = 7.5\%$) and heterogeneous preferences. Middle and right panels: Unconditional mean (middle) and volatility (right) of the real yield as a function of the maturity under Bayesian beliefs ($\chi = 0\%$, blue line) and pro-cyclical optimism ($\chi = 7.5\%$, red line). Model calibration is from Table III.

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Figure 12. Portfolio allocations and pro-cyclical optimism

Wealth proportions invested in the market asset (upper panels) and in the perpetual bond (lower panels) as a function of $\omega_t$ (with $\hat{\mu}_t = \bar{\mu}$) respectively under Bayesian beliefs ($\chi = 0\%$, left panels) and pro-cyclical optimism ($\chi = 7.5\%$, right panels). Model calibration is from Table III.

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Figure 13. Long-run bias in forecast errors and equity premium

Unconditional equity premium as a function of the deviation parameter $\chi$ and the bias parameter $\varepsilon$ (model extension of Section VII). Model calibration is from Table III.

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Table I
Investors’ Beliefs and Forecast Errors

Maximum Likelihood Estimation

<table>
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<tr>
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<th>$\lambda_{\text{quarterly}} = .15$</th>
<th>$\lambda_{\text{quarterly}} = .20$</th>
<th>$\lambda_{\text{quarterly}} = .25$</th>
<th>Kalman filter</th>
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<td>$a_0$</td>
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<td>.006</td>
<td>.006</td>
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<tr>
<td>$t$-stat</td>
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<td>6.22</td>
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<td>$a_1$</td>
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<td>.313</td>
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<tr>
<td>$\sigma_e$</td>
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<td>.003</td>
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<tr>
<td>$t$-stat</td>
<td>17.15</td>
<td>17.15</td>
<td>17.15</td>
<td>17.15</td>
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</table>

$1000 \times \text{RMSE}$ | 3.01 | 2.99 | 2.97 | 3.32 |

Maximum likelihood estimation of

$$\hat{\mu}_{t+1} = \hat{\mu}_{t+1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_e),$$

where $\hat{\mu}_{t+1}^{\text{Survey}}$ is the one-quarter ahead average forecast from the Survey of Professional Forecasters (1968-2005) and $\hat{\mu}_{t+1}$ is the model forecast given by

$$\hat{\mu}_{t+1} = \alpha_t \hat{\mu}_{t+1}^{\text{Prior}} + (1 - \alpha_t) \hat{\mu}_{t+1}^{\text{KF}},$$

with

$$\hat{\mu}_{t+1}^{\text{Prior}} = \hat{a}_0 + \hat{a}_1 (\omega_t - \bar{\omega}),$$

$$\alpha_t = \frac{\mathbb{E}[(\epsilon_t^2)]}{\mathbb{E}[(\epsilon_t^2) + \mathbb{E}[(\epsilon_t^2)])}, \quad \text{and} \quad \log(q_t) = b_0 + b_1 (\omega_t - \bar{\omega}),$$

where $\omega_t$ is the habit state variable computed as in Eq. (1) (with $\lambda_{\text{quarterly}}$ adjusted to quarterly frequency and $\omega_0 = \bar{\omega}$) and $\hat{\mu}_{t+1}^{\text{KF}}$ is the Kalman filter estimator of expected log GDP growth from the observation of realized log GDP $Y_t$. RMSE denotes the root mean square error.

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### Table II

**Survey Risk-Return Trade Off**

#### Panel A – expected returns vs economic conditions:

\[
\mu_P^t - r_t = \alpha_0 + \alpha_1 1_{\omega < \bar{\omega}} + \beta_0 \omega_t \ + \beta_1 \omega_t 1_{\omega < \bar{\omega}} + \epsilon_t
\]

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<th>Parameter</th>
<th>Equation</th>
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<th>(2)</th>
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<td>-1.56**</td>
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<tr>
<td>( \beta_1 )</td>
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<td>( R^2 )</td>
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#### Panel B – expected volatility vs economic conditions:

\[
\sigma_P^t = \alpha_0 + \alpha_1 1_{\omega < \bar{\omega}} + \beta_0 \omega_t \ + \beta_1 \omega_t 1_{\omega < \bar{\omega}} + \epsilon_t
\]

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<th>Parameter</th>
<th>Equation</th>
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<td>( R^2 )</td>
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<td>.49</td>
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#### Panel C – expected returns vs expected volatility:

\[
\mu_P^t - r_t = \alpha_0 + \alpha_1 1_{\omega < \bar{\omega}} + \beta_0 \sigma_P^t \ + \beta_1 \sigma_P^t 1_{\omega < \bar{\omega}} + \beta_2 (\sigma_P^t)^2 + \epsilon_t
\]

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<td>1.17**</td>
<td>8.28***</td>
<td>5.71***</td>
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<td>( \beta_1 )</td>
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<td>.02</td>
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Regressions are run with data about expected returns (\( \mu_P^t - r_t \)) and expected return volatility (\( \sigma_P^t \)) from the Duke/CFO survey 2000:Q3-2012:Q2. The state variable \( \omega_t \) is built as in Eq. \( (1) \) using GDP growth data (yearly parameter \( \lambda = .35 \) is adjusted to quarterly frequency: \( \lambda^{\text{quarterly}} = 1 - (1 - \lambda)^{1/4} \) and \( \omega_0 \) starts at the steady state \( \bar{\omega} \)). Symbols *, ** and *** denote 10%, 5% and 1% significance levels.

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## Table III
### Model Calibration

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<td>Long-run growth.</td>
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<td>$\sigma_y$</td>
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<td>Instantaneous volatility.</td>
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<td>Speed of reversion of long-run growth.</td>
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<tr>
<td>$\sigma_{\mu}$</td>
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<td><strong>Implied Parameters</strong></td>
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<td>$\sigma_{\mu,y}$</td>
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<td>Volatility of perceived long-run growth due to dividend observations.</td>
</tr>
<tr>
<td>$\sigma_{\mu,s}$</td>
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<td>Volatility of perceived long-run growth due to signal observations.</td>
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<td>$\bar{\omega}$</td>
<td>4.96%</td>
<td>Steady-state level of the habit state variable.</td>
</tr>
<tr>
<td>$\bar{\gamma}(\bar{\omega})$</td>
<td>5</td>
<td>Steady-state aggregate relative risk aversion.</td>
</tr>
</tbody>
</table>

→ Back to the text.
Table IV
Unconditional Moments – Homogeneous-Agents Economy

<table>
<thead>
<tr>
<th></th>
<th>Data (a)</th>
<th>Data (b)</th>
<th>$\chi = 0%$</th>
<th>$\chi = 5%$</th>
<th>$\chi = 7.5%$</th>
<th>$\chi = 10%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{E}(r)%$</td>
<td>1.70</td>
<td>0.60</td>
<td>1.93</td>
<td>1.92</td>
<td>1.90</td>
<td>1.90</td>
</tr>
<tr>
<td>$\sigma_r%$</td>
<td>4.38</td>
<td>3.00</td>
<td>6.40</td>
<td>5.17</td>
<td>4.67</td>
<td>4.32</td>
</tr>
<tr>
<td>$\mathbb{E}(\mu - r)%$</td>
<td>7.38</td>
<td>5.60</td>
<td>3.86</td>
<td>4.26</td>
<td>4.78</td>
<td>7.37</td>
</tr>
<tr>
<td>$\sigma\mu%$</td>
<td>17.88</td>
<td>19.80</td>
<td>20.80</td>
<td>23.10</td>
<td>26.10</td>
<td>41.10</td>
</tr>
<tr>
<td>$\mathbb{E}(\theta)%$</td>
<td>41.27</td>
<td>28.28</td>
<td>17.90</td>
<td>17.90</td>
<td>17.90</td>
<td>17.90</td>
</tr>
<tr>
<td>$\mathbb{E}(\log P/Y)$</td>
<td>3.36</td>
<td>3.38</td>
<td>4.09</td>
<td>4.17</td>
<td>4.33</td>
<td>6.12</td>
</tr>
<tr>
<td>$\sigma(\log P/Y)$</td>
<td>.44</td>
<td>.45</td>
<td>.24</td>
<td>.28</td>
<td>.33</td>
<td>.58</td>
</tr>
<tr>
<td>$AC(\log P/Y)$</td>
<td>.91</td>
<td>.88</td>
<td>.71</td>
<td>.72</td>
<td>.74</td>
<td>.77</td>
</tr>
<tr>
<td>$AC(r)$</td>
<td>.33</td>
<td>.67</td>
<td>.72</td>
<td>.60</td>
<td>.52</td>
<td>.45</td>
</tr>
</tbody>
</table>

Unconditional statistics of the model, in the homogeneous-agents economy and for various levels of $\chi$, are compared with the data sample estimates. In (a) I use annual data between 1933 and 2006: price-dividend ratio and stock returns refer to the S&P Composite Index and the risk-free rate is equal to rate of return on the three-month Treasury bill. In (b) data are from 1931 to 2009 as in Constantinides and Ghosh (2011). Model calibration is from Table III.

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Table V  
Unconditional Moments – Heterogeneous-Agents Economy

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
<th>( \chi = 0% )</th>
<th>( \chi = 5% )</th>
<th>( \chi = 7.5% )</th>
<th>( \chi = 10% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{E}(r) ) ( (%) )</td>
<td>1.70</td>
<td>0.60</td>
<td>1.93</td>
<td>1.87</td>
<td>1.84</td>
<td>1.82</td>
</tr>
<tr>
<td>( \sigma_r ) ( (%) )</td>
<td>4.38</td>
<td>3.00</td>
<td>6.04</td>
<td>5.10</td>
<td>4.60</td>
<td>4.20</td>
</tr>
<tr>
<td>( \mathbb{E}(\mu^P - r) ) ( (%) )</td>
<td>7.38</td>
<td>5.60</td>
<td>3.86</td>
<td>4.26</td>
<td>4.79</td>
<td>7.37</td>
</tr>
<tr>
<td>( \sigma^P ) ( (%) )</td>
<td>17.88</td>
<td>19.80</td>
<td>21.46</td>
<td>23.10</td>
<td>26.10</td>
<td>41.10</td>
</tr>
<tr>
<td>( \mathbb{E}(\theta) ) ( (%) )</td>
<td>41.27</td>
<td>28.28</td>
<td>17.90</td>
<td>17.90</td>
<td>17.90</td>
<td>17.90</td>
</tr>
<tr>
<td>( \mathbb{E}(\log P/Y) )</td>
<td>3.36</td>
<td>3.38</td>
<td>4.09</td>
<td>4.17</td>
<td>4.33</td>
<td>6.12</td>
</tr>
<tr>
<td>( \sigma(\log P/Y) )</td>
<td>.44</td>
<td>.45</td>
<td>.24</td>
<td>.28</td>
<td>.33</td>
<td>.57</td>
</tr>
<tr>
<td>( AC(\log P/Y) )</td>
<td>.91</td>
<td>.88</td>
<td>.71</td>
<td>.72</td>
<td>.74</td>
<td>.77</td>
</tr>
<tr>
<td>( AC(r) )</td>
<td>.33</td>
<td>.67</td>
<td>.72</td>
<td>.60</td>
<td>.51</td>
<td>.44</td>
</tr>
</tbody>
</table>

Unconditional statistics of the model, in the heterogeneous-agents economy and for various levels of \( \chi \), are compared with the data sample estimates. In (a) I use annual data between 1933 and 2006: price-dividend ratio and stock returns refer to the S&P Composite Index and the risk-free rate is equal to rate of return on the three-month Treasury bill. In (b) data are from 1931 to 2009 as in Constantinides and Ghosh (2011). Model calibration is from Table III.

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### Table VI
Equilibrium Risk-Return Trade Off

<table>
<thead>
<tr>
<th></th>
<th>Survey data</th>
<th>Model</th>
<th>Model</th>
<th>Model</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\chi = 0%$</td>
<td>$\chi = 5%$</td>
<td>$\chi = 7.5%$</td>
<td>$\chi = 10%$</td>
</tr>
<tr>
<td>$\forall \omega$</td>
<td>$\text{corr}(\omega, \mu^P - r)$</td>
<td>-0.49***</td>
<td>-0.99</td>
<td>-0.99</td>
<td>-0.98</td>
</tr>
<tr>
<td></td>
<td>$\text{corr}(\omega, \sigma^P)$</td>
<td>-0.61***</td>
<td>-0.99</td>
<td>-0.95</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>$\text{corr}(\mu^P - r, \sigma^P)$</td>
<td>0.15</td>
<td>0.99</td>
<td>0.97</td>
<td>-0.32</td>
</tr>
<tr>
<td>$\omega &gt; \bar{\omega}$</td>
<td>$\text{corr}(\omega, \mu^P - r)$</td>
<td>-0.43*</td>
<td>-0.99</td>
<td>-0.99</td>
<td>-0.99</td>
</tr>
<tr>
<td></td>
<td>$\text{corr}(\omega, \sigma^P)$</td>
<td>-0.76***</td>
<td>-0.99</td>
<td>-0.97</td>
<td>-0.42</td>
</tr>
<tr>
<td></td>
<td>$\text{corr}(\mu^P - r, \sigma^P)$</td>
<td>0.42**</td>
<td>0.99</td>
<td>0.99</td>
<td>0.54</td>
</tr>
<tr>
<td>$\omega &lt; \bar{\omega}$</td>
<td>$\text{corr}(\omega, \mu^P - r)$</td>
<td>-0.43**</td>
<td>-0.99</td>
<td>-0.99</td>
<td>-0.95</td>
</tr>
<tr>
<td></td>
<td>$\text{corr}(\omega, \sigma^P)$</td>
<td>-0.12</td>
<td>-0.99</td>
<td>-0.83</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td>$\text{corr}(\mu^P - r, \sigma^P)$</td>
<td>-0.39**</td>
<td>0.99</td>
<td>0.83</td>
<td>-0.52</td>
</tr>
</tbody>
</table>

Data about expected returns ($\mu^P - r$) and expected return volatility ($\sigma^P$) are from the Duke/CFO survey 2000:Q3-2012:Q2. The state variable $\omega$ is built as in Eq. (1) using GDP growth data ($\lambda$ is adjusted to quarterly frequency). Symbols *, ** and *** denote 10%, 5% and 1% significance levels. Numbers in parentheses denote p-values. Model calibration is from Table III. Model correlations are computed from simulations.

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## Table VII
Long-Horizon Regressions – Homogeneous-Agents Economy

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Log price-dividend ratio</th>
<th>Data</th>
<th>( \chi = 0% )</th>
<th>( \chi = 5% )</th>
<th>( \chi = 7.5% )</th>
<th>( \chi = 10% )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coeff.</td>
<td>t-stat</td>
<td>( R^2 )</td>
<td>Coeff.</td>
<td>t-stat</td>
<td>( R^2 )</td>
</tr>
<tr>
<td>1</td>
<td>-.11</td>
<td>-2.70</td>
<td>.072</td>
<td>-.09</td>
<td>-1.78</td>
<td>.015</td>
</tr>
<tr>
<td>2</td>
<td>-.19</td>
<td>-2.80</td>
<td>.112</td>
<td>-.16</td>
<td>-1.04</td>
<td>.026</td>
</tr>
<tr>
<td>3</td>
<td>-.22</td>
<td>-2.68</td>
<td>.121</td>
<td>-.22</td>
<td>-1.19</td>
<td>.035</td>
</tr>
<tr>
<td>5</td>
<td>-.33</td>
<td>-3.19</td>
<td>.164</td>
<td>-.30</td>
<td>-1.38</td>
<td>.047</td>
</tr>
<tr>
<td>7</td>
<td>-.46</td>
<td>-3.57</td>
<td>.223</td>
<td>-.37</td>
<td>-1.54</td>
<td>.056</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Horizon</th>
<th>10 years term premium</th>
<th>Data</th>
<th>( \chi = 0% )</th>
<th>( \chi = 5% )</th>
<th>( \chi = 7.5% )</th>
<th>( \chi = 10% )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coeff.</td>
<td>t-stat</td>
<td>( R^2 )</td>
<td>Coeff.</td>
<td>t-stat</td>
<td>( R^2 )</td>
</tr>
<tr>
<td>1</td>
<td>-.51</td>
<td>-1.30</td>
<td>.015</td>
<td>-1.30</td>
<td>.024</td>
<td>-1.20</td>
</tr>
<tr>
<td>2</td>
<td>-.91</td>
<td>-1.84</td>
<td>.044</td>
<td>-1.92</td>
<td>.046</td>
<td>-3.19</td>
</tr>
<tr>
<td>3</td>
<td>-1.21</td>
<td>-2.19</td>
<td>.060</td>
<td>-3.36</td>
<td>-.65</td>
<td>-4.74</td>
</tr>
<tr>
<td>5</td>
<td>-1.71</td>
<td>-2.65</td>
<td>.084</td>
<td>-4.89</td>
<td>-2.91</td>
<td>.093</td>
</tr>
<tr>
<td>7</td>
<td>-2.09</td>
<td>-2.87</td>
<td>.098</td>
<td>-5.86</td>
<td>-3.19</td>
<td>.111</td>
</tr>
</tbody>
</table>

The \( i \)-years cumulative excess log returns are regressed on the log price-dividend ratio (upper panel) and the 10 years term premium (lower panel): \( r_{t+1} + \ldots + r_{t+i} = \alpha + \beta x_t + \epsilon \). Regression estimates are averages of ten thousand simulations with each simulation having 100 observations.
The \( i \)-years cumulative excess log returns are regressed on the log price-dividend ratio (upper panel) and the 10 years term premium (lower panel): \( r_{t+1}^e + \ldots + r_{t+i}^e = \alpha + \beta x_t + \epsilon \). Regression estimates are averages of ten thousand simulations with each simulation having 100 observations.
Online Appendix of “Survey Expectations and the Equilibrium Risk-Return Trade Off”

Online Appendix A – Equilibrium Homogenization of Beliefs

This section provides an explanation of the cyclical bias \( \Psi(\omega_t) \) in Eq. (17) as an aggregation result of investors featuring heterogeneity in both beliefs and risk preferences. Indeed, both the two sources of heterogeneity can be embedded in a model with heterogeneity only in risk aversion and an endogenously adjusted dynamics for the aggregate dividend. Such an endogenous adjustment can be proxied, in reduced form, by the deviation term \( \Psi(\omega_t) \). To gain intuition the framework of this section is simplified: the economy is populated by only two agents who do not learn about expected growth and traded assets are such that agents face complete markets.

The economy consists of two agents \( i = \{A, B\} \), who differ both in their risk preferences as well as in their beliefs. Namely, they maximize utility of the form

\[
E_i \left[ \int_0^\infty e^{-\rho t} U^i(C_i(t))dt \right] = E_i \left[ \int_0^\infty e^{-\rho t} \frac{1}{\gamma_i} C_i(t) \eta^i X_i(t) dt \right], \quad i = \{A, B\},
\]

where \( E_i^0 \) is the expectation operator under the probability measure of the investor \( i \), \( C_i(t) \) is individual consumption and \( X_i(t) \) is the exogenous standard of living as in Section IIIA. The agents differ in their beliefs about the growth rates of the aggregate dividend and their probability measures are characterized as follows:

\[
q_t^i = \left. \frac{dP_t^i}{dP_t^{\gamma_i}} \right|_{t} = \exp \left( -\frac{1}{2} \int_0^t (\eta^i_u)^2 du - \int_0^t \eta^i_u dB_{y,u} \right), \quad i = \{A, B\},
\]

where the function \( \eta^i \) represents the deviation of agent \( i \)'s beliefs on the growth rate \( \mu^i_t \) from its true value \( \mu_t \): when \( \eta^i \) is negative, the agent \( i \) is optimistic with respect to the physical measure and vice-versa. Indeed, the agents observe the realizations of the aggregate dividend but interpret the Brownian shocks under their subjective information set:

\[
\mu^A_t dt + \sigma_A dB^A_{y,t} = \frac{dY_t}{Y_t} = \mu^B_t dt + \sigma_B dB^B_{y,t},
\]

where

\[
dB^A_{y,t} = dB_{y,t} + \eta^A_t dt, \quad \eta^A_t = \frac{\mu_t - \mu^A_t}{\sigma_A},
\]

\[
dB^B_{y,t} = dB_{y,t} + \eta^B_t dt, \quad \eta^B_t = \frac{\mu_t - \mu^B_t}{\sigma_B}.
\]

The two agents assign different but equivalent measures to the future uncertain dividend process and still can trade in financial assets such that the market is complete. Therefore, a representative agent optimization problem can be constructed to account for heterogeneous beliefs:

\[
\sup_{C_{A,t} + C_{B,t} = Y_t} \left. \nu^A E_0^A \int_0^\infty e^{-\rho u} U^A(C_{A,u})du + \nu^B E_0^B \int_0^\infty e^{-\rho u} U^B(C_{B,u})du \right|_{t}.
\]
The optimization problem can be written under the physical measure (see Detemple and Murthy (1994) and Basak (2005)) as

$$\sup_{C_{A,t}+C_{B,t}=Y_t} E_0 \left[ \int_0^\infty p^A e^{-\rho^A u} U^A (C_{A,u}) du + \int_0^\infty p^B e^{-\rho^B u} U^B (C_{B,u}) du \right].$$

Then, the economy is characterized by the following first-order conditions:

$$\nu^A p^A e^{-\rho^A u} (C_{A,t}/X_t)^{-\gamma_A} = -\xi_{A,t} = \nu^B p^B e^{-\rho^B u} (C_{B,t}/X_t)^{-\gamma_B},$$

and the market clearing condition: \(C_{A,t} + C_{B,t} = Y_t\) or, equivalently, \(C_{A,t}/X_t + C_{B,t}/X_t = e^{\omega_t}\). Let denote with \(\Theta_t\) the following process:

$$\Theta_t = \exp \left( \int_0^t \beta_t du + \int_0^t \delta_t dB_t \right),$$

where

$$\beta_t = \frac{(\eta^A_t)^2/2 - (\eta^B_t)^2/2}{\gamma_A - \gamma_B}$$

and

$$\delta_t = \frac{\eta^A_t - \eta^B_t}{\gamma_A - \gamma_B}.$$

It is easy to verify that \(\Theta_t^{-\gamma_A}/p^A_t = \Theta_t^{-\gamma_B}/p^B_t\). Consider now the following simple transformation:

$$\begin{align*}
C_{A,t} &\rightarrow \tilde{C}_{A,t} = \Theta_t C_{A,t}, \\
C_{B,t} &\rightarrow \tilde{C}_{B,t} = \Theta_t C_{B,t}, \\
Y_t &\rightarrow \tilde{Y}_t = \Theta_t Y_t,
\end{align*}$$

with \(\Theta_t \equiv \log \tilde{Y}_t - \log X_t\). Under such a transformation the dynamics from the first order conditions simplifies to

$$\nu^A e^{-\rho^A u} (\tilde{C}_{A,t}/X_t)^{-\gamma_A} = \xi_{A,t} = \nu^B e^{-\rho^B u} (\tilde{C}_{B,t}/X_t)^{-\gamma_B},$$

where \(\xi_{A,t} = \xi_{A,t}/\Theta_t^{-\gamma_A}/p^A_t\), market clearing requires \(\tilde{C}_{A,t} + \tilde{C}_{B,t} = \tilde{Y}_t\) and \(\omega_t\) fully characterizes the equilibrium state-price density. The latter equation represents the first order conditions of a two-agent Cuj economy whose agents differ only in their relative risk aversions whereas they have homogeneous beliefs (similarly to the problem in Eq. (22)). Heterogeneous beliefs have been homogenized at equilibrium by a change in the (perceived) dynamics of the aggregate dividend. Such a result can be generalized to an arbitrary number of agents, to a wide class of biases in beliefs and to heterogeneous time preferences: see Tran and Zeckhauser (2012) for further details.

In particular, assume the case where individual beliefs feature a cyclical bias which evolves with the habit state variable, \(\omega_t\):

$$\mu^i_t = a_{0,i} + a_{1,i}(\omega_t - \bar{\omega}_t), \quad i = \{A, B\}.$$  

Consequently, \(\eta^A_t, \eta^B_t, \beta_t\) and \(\delta_t\) are functions of \(\omega_t\) and the adjusted dynamics of the aggregate dividend becomes:

$$d\tilde{Y}_t = \left( \mu_t + \tilde{\beta}_t + \frac{\delta_t^2}{2} + \sigma_t \tilde{\delta}_t \right) \tilde{Y}_t dt + \left( \sigma_t + \tilde{\delta}_t \right) \tilde{Y}_t dB_{3,t},$$

$$= \tilde{\mu}(\mu, \omega_t) \tilde{Y}_t dt + \tilde{\sigma}(\omega_t) \tilde{Y}_t dB_{3,t}.$$

State-dependent biases in individual beliefs translate, as an endogenous result, into an adjusted dynamics for the aggregate dividend of an economy where risk aversion is the only form of heterogeneity. Consider the simple case \(a_{0,A} = a_{0,B}\) and \(a_{1,A} = -a_{1,B} > 0\) but \(\gamma_A > \gamma_B\). The degree of optimism in good times (and pessimism in bad times) of agent \(A\) exactly offset the degree of pessimism in good times (and optimism in bad times) of agent \(B\). However, the former agent is more risk averse than the latter. Under such simple conditions the adjusted-drift \(\tilde{\mu}(\mu, \omega_t)\) can be increasing in \(\omega_t\) in most of the states, whereas the aggregate relative risk aversion is decreasing in \(\omega_t\), as shown in Figure OAI. This is an endogenous result of heterogeneity and does not need asymmetry in individual beliefs. Pro-cyclical optimism about expected growth and counter-cyclical aggregate risk aversion are the two key ingredients of the model of this paper and drive all asset pricing implications. They can be understood as an aggregation result and
Figure OA.1. Equilibrium homogenization of beliefs

Left panel: Adjusted expected growth $\hat{\mu}$ (red line) and long-run growth $\bar{\mu}$ (black line) are plotted as a function of the habit state variable. Right panel: Aggregate relative risk aversion $\gamma(\hat{\omega}_t) = -\partial_{\hat{\omega}_t} \xi_t / \xi_t$ is plotted as a function of the (adjusted) habit state variable. Parameters: $\bar{\mu} = a_{0,A} = a_{0,B} = .02, \sigma_y = .035, a_{1,A} = -a_{1,B} = .05, \gamma_A = 10, \gamma_B = 2$.

are not the mechanical implication of either a long-run bias or asymmetry in individual beliefs. Therefore, the homogeneous belief driven by $\Psi(\hat{\omega}_t)$ in Eq. (17) can be interpreted as a reduced form of the equilibrium homogenization of heterogeneous beliefs and is adopted for the sake of exposition and analytic tractability throughout the paper.