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Abstract

The traditional multivariate Lévy process constructed by subordinating a Brownian motion through a univariate subordinator presents a number of drawbacks, including the lack of independence and a limited range of dependence. In order to face these, we investigate multivariate subordination, with a common and an idiosyncratic component. We introduce generalizations of some well known univariate Lévy processes for financial applications: the multivariate compound Poisson, NIG, Variance Gamma and CGMY. In all these cases the extension is parsimonious, in that one additional parameter only is needed.

First we characterize the subordinator, then the time changed processes via their Lévy measure and characteristic exponent. Finally we study the subordinator association, as well as the subordinated processes' linear and non linear dependence. We show that the processes generated with the proposed time change can include independence and that they span the whole range of linear dependence. We provide some examples of simulated trajectories, scatter plots and both linear and non linear dependence measures. The input data for these simulations are calibrated values of major stock indices.

Journal of Economic Literature Classification: G12, G13

Keywords: Lévy processes, multivariate subordinators, dependence (association, correlation), multivariate asset modelling, multivariate time changed processes..

Introduction

The technique of time change is a well established way to introduce Lévy processes at the univariate level: it has proven to be theoretically helpful for financial applications, thanks to Monroe's theorem. Geman, Madan, Yor [11] do report that " price processes, being semi-martingales [..] are time changed Brownian motions. [..] As time changes are increasing random processes, they are for practical purposes purely discontinuous, if they are not locally deterministic" This remark led them and a number of prominent Authors to consider purely discontinuous models for univariate asset prices, generated as time changed diffusions.

At the multivariate level, however, time changing has been studied much less. Multivariate Lévy processes have been generally constructed by subordinating a Brownian motion by means of a univariate subordinator. Such processes present a number of drawbacks, including the lack of independence and a limited range of dependence (see [9]).

In order to face these problems, we investigate multivariate subordination. ¿From the intuitive point of view, the main feature of such a multivariate subordination is that it allows to incorporate both a common time transform, which can be interpreted in finance applications as a measure of the overall market activity, and an idiosyncratic time shift, linked to the asset specific trade and information update. In particular, we introduce generalizations of the multivariate compound Poisson, Normal Inverse Gaussian (NIG), Variance Gamma (VG) and Carr Geman Madan Yor (CGMY) processes. In all of them, first we characterize the subordinator, then the time changed processes via their Lévy Measure and characteristic exponent. Finally we study the subordinator association, as well as the single processes linear and non linear dependence. We show that the aforementioned processes can include independence and that they span the whole range of linear dependence. We provide some examples of simulated trajectories, scatter plots and both linear and non linear dependence measures. The input data for the simulation are calibrated values of major stock indicies.

The paper proceeds as follows: section 1 presents the class of multivariate subordinators which we are going to adopt and recalls some features of stable subordinators. Section 2 considers the general properties (Lévy nature, characteristic function, Lévy triplet and measure) of the corresponding subordinated processes, using Barndorff-Nielsen et alii [2]. The results are then specified in relation to the compound Poisson, NIG, VG and CGMY cases. Section 3 studies the association of the subordinator and the linear and non linear dependence of the subordinated processes. Section 4 concentrates on linear dependence. Section 5 provides simulations of the Compound Poisson, VG and NIG cases.

1 A class of multivariate subordinators

In this section we follow the construction in Semeraro [20] in order to introduce a class of multivariate time changes: each one is a sum of an idiosyncratic and a common component.

We define n subordinators as follows: let X_i, $i = 1, ..., n$ and Z be independent and infinitely divisible random variables, with characteristic functions respectively ψ_i, $i = 1, ..., n$ and ψ_Z. Define the random vector $\boldsymbol{W}$ as the sum

$$\boldsymbol{W} = (W_1, W_2, ..., W_n)^T = (X_1 + \alpha_1 Z, X_2 + \alpha_2 Z, ..., X_n + \alpha_n Z)^T, \tag{1.1}$$

where α_j, $j = 1, ..., n$ are real parameters (the same method to construct multivariate infinitely divisible distribution is adopted in Barnorff-Nielsen et al. [3]).

The vector $\boldsymbol{W}$ is jointly infinitely divisible and, due to independency, its characteristic function, $\psi_{\boldsymbol{W}}$, is:

$$\psi_{\boldsymbol{W}}(u_1, u_2, ...u_n) = \prod_{j=1}^{n} \psi_j(u_j)\psi_Z(\sum_{j=1}^{n} \alpha_j u_j), \tag{1.2}$$

Define $\boldsymbol{G} = \{\boldsymbol{G}(t), t \geq 0\}$ as the Lévy process which has the law $\mathcal{L}$ of $\boldsymbol{W}$ at time one:

$$\mathcal{L}(\boldsymbol{G}(1)) = \mathcal{L}(\boldsymbol{W}). \tag{1.3}$$

Semeraro [20] characterized the process $\boldsymbol{G}$ in terms of its Lévy triplet and of its characteristic exponent as follows.

Let $\tilde{X}_j = \{X_j(t), t \geq 0\}$, $j = 1, ..., n$ and $\tilde{Z} = \{Z(t), t \geq 0\}$ be the Lévy processes defined by:

$$\mathcal{L}(X_j(1)) = \mathcal{L}(X_j), \quad \mathcal{L}(Z(1)) = \mathcal{L}(Z). \tag{1.4}$$

Let ν_j, ν_Z be respectively the Lévy measures of the processes $\tilde{X}_j, j = 1, ..., n$ and $\tilde{Z}$, then the Lévy measure $\nu_{\boldsymbol{G}}$ of $\boldsymbol{G}$ satisfies

$$\nu_{\boldsymbol{G}}(E) = \sum_{j=1}^{n} \nu_j(E_j) + \nu_Z(E_\Delta), \tag{1.5}$$

where $E \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$, $E_j = E \cap A_j$ and $A_j = \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0, k \neq j, k = 1, ..., n\}$. If $\tilde{X}_j$, $j = 1, ..., n$, and $\tilde{Z}$ are real subordinators, then the process $\boldsymbol{G}$ is a multivariate subordinator. Moreover if

$$\begin{array}{c} \Psi_j(w) = \int_{\mathbb{R}_+} (e^{iwz} - 1)\nu_j(dz) + il_j w, \quad j = 1, ..., n \\ \Psi_Z(w) = \int_{\mathbb{R}_+} (e^{iwz} - 1)\nu_Z(dz) + il_z w. \end{array} \tag{1.6}$$

are respectively the exponents of the processes $\tilde{X}_j$, $j = 1, ..., n$ and $\tilde{Z}$, then the characteristic exponent $\Psi_{\boldsymbol{G}}$ of $\boldsymbol{G}$ satisfies:

$$\begin{aligned}\Psi_{\boldsymbol{G}}(\boldsymbol{w}) &= \sum_{j=1}^{n} \Psi_j(w_j) + \Psi_Z(\sum_{j=1}^{n} \alpha_j w_j) \\ &= \sum_{j=1}^{n} \int_{\mathbb{R}^+} (e^{iw_j z_j} - 1)\nu_j(dz_j) + i(l_j w_j) \\ &+ \int_{\mathbb{R}^+} (e^{i(\sum_{j=1}^{n} \alpha_j w_j)s} - 1)\nu_Z(ds) + i(l_z(\sum_{j=1}^{n} \alpha_j w_j)),\end{aligned} \tag{1.7}$$

Observe that if $\tilde{X}_j$, $j =, ..., n$ and $\tilde{Z}$ have zero drift, so does $\boldsymbol{G}$. Throughout the paper the subordinator we are going to consider for $\tilde{X}_j$, $j =, ..., n$ and $\tilde{Z}$ will have zero drift.

1.1 Stable subordinators

In what follows, we will be concerned mainly with stable and tempered stable subordinators (for a complete treatment see Samorodnitsky and Taqqu [18]). We will therefore spend some words about their general properties, before focusing on the ones of main interest for asset price modelling.

A random variable X has stable distribution with parameters $0 < \alpha \leq 2$, $\sigma > 0$, $-1 < \beta < 1$ and $\gamma \in \mathbb{R}$, shortly $X \sim S_\alpha(\sigma, \beta, \gamma)$, if its characteristic function has the form:

$$\psi_X(z) = \begin{cases} \exp\{-\sigma^\alpha |z|^\alpha (1 - \beta(signz) tan\frac{\pi\alpha}{2}) + i\gamma z\}, & \alpha \neq 1 \\ \exp\{-\sigma|z|(1 + \beta(signz)\beta\frac{2}{\pi} ln|z|) + i\gamma z\} & \alpha = 1, \end{cases} \tag{1.8}$$

Since γ affects only location, we assume $\gamma = 0$ for simplicity.

An α-stable real subordinator G is given by a stable random variable X with support $[0, \infty)$, $X \sim S_\alpha(\sigma, 1, 0)$ with $0 < \alpha < 1$.

The Lévy measure of a stable subordinator has the following expression

$$\nu_G(dx) = \frac{c_G}{x^{\alpha+1}} 1_{x>0}, \tag{1.9}$$

where $c_G = c(\alpha)\sigma^\alpha$, $c(\alpha) > 0$.

If the subordinators $\tilde{X}_j$ and Z defined in (1.4) are α-stable then $\boldsymbol{G}$ has α-stable margins. Let $X_j \sim S_\alpha(\sigma_j, 1, 0)$ and $Z \sim S_\alpha(\sigma_z, 1, 0)$, so that $\alpha_j Z \sim S_\alpha(\sigma_z \alpha_j, 1, 0)$. By Propositions 1.2.1 and 1.2.3 in Samorodnitsky and Taqqu [18], if independency holds, $X_j + \alpha_j Z$ is α-stable and its law is

$$\mathcal{L}(X_j + \alpha_j Z) = S_\alpha(\sigma_{G_j}, 1, 0), \tag{1.10}$$

where $\sigma_{G_j} = (\sigma_j^\alpha + (\sigma_z \alpha_j)^\alpha)^{1/\alpha}$.

We will also consider tempered stable subordinators, first introduced by Tweedie [24]. They are characterized by the following Lévy measure:

$$\nu(x) = \frac{ce^{-\lambda x}}{x^{\alpha+1}} 1_{x>0}. \tag{1.11}$$

Let us denote the corresponding infinitely divisible distribution by $X \sim TS_\alpha(c, \lambda)$, where $0 < \alpha < 1$, $\lambda > 0$ and $c > 0$. The distribution of the sum of two tempered stable processes, analogously to the non-tempered case, can be characterized as follows: if $X \sim TS_\alpha(c_X, \lambda)$ and $Y \sim TS_\alpha(c_Y, \lambda)$ their sum is $TS_\alpha(c_X + c_Y, \lambda)$ and $\alpha_i X \sim TS_\alpha(c_X \alpha_i^\alpha, \frac{\lambda}{\alpha_i})$. Therefore if $X_i \sim TS_\alpha(c_i, \frac{\lambda}{\alpha_i})$ for $i = 1, ..., n$ and $Z \sim TS_\alpha(c_z, \gamma_z, \lambda)$, then $\boldsymbol{G}$ has margins $TS_\alpha(c_i + c_z \alpha_i^\alpha, \frac{\lambda}{\alpha_i})$.

We end this section with a more general remark. Consider a stable subordinator G_B with Lévy measure given by (1.9). A subordinator G_A is absolutely continuous with respect to G_B (see Madan and Yor [13] and Sato [19] for a more general definition), if

$$\nu_A(dx) = f(x)\nu_B(dx) = f(x)\frac{c_G}{x^{1+\alpha}}dx. \tag{1.12}$$

and

$$\int_0^\infty \nu_B(dt)(\sqrt{f(t)} - 1)^2 < \infty. \tag{1.13}$$

Obviously if $\tilde{X}_j$ and $\tilde{Z}$ are α-stable continuous with the same density, their sum is.

All the previous classes of subordinators are characterized by the fact that the difference between the Lévy measures of $\tilde{X}_j$, $\tilde{Z}$ and G_j is a constant.

2 Time-changed Brownian motions

We are ready to use the multivariate subordinators above in order to time change independent Brownian motions. This construction was first introduced in Semeraro [20] for the variance gamma model.

The general construction relies on the characterization of Barndorff-Nielsen et al. [2] (Theorem 3.3). We need a preliminary notion.

Consider n independent Lévy processes $X_1(t), ..., X_n(t)$. The stacked process $\boldsymbol{X}(t) = (X_1(t), ..., X_n(t))^T$, where the superscript T denotes the transpose, is then a Lévy process on $\mathbb{R}^n$. Consider the multi-parameter $\boldsymbol{s} = (s_1, ..., s_n)^T \in \mathbb{R}^n_+$ and the partial order on $\mathbb{R}^n_+$:

$$\boldsymbol{s}^1 \preceq \boldsymbol{s}^2 \;\; \Leftrightarrow \;\; s^1_j \leq s^2_j,\ j = 1, ...n.$$

The multi-parameter process $\{\boldsymbol{X}(\boldsymbol{s}),\ \boldsymbol{s} \in \mathbb{R}^n_+\}$ is defined by

$$\boldsymbol{X}(\boldsymbol{s}) = (X_1(s_1), ..., X_n(s_n))^T.$$

We consider as particular multi-parameter processes the ones obtained from independent Brownian motions.

Let $B_j = \{B_j(t), t \geq 0\}$ $j = 1, ..., n$ be independent standard Brownian motions. Consider the process $\boldsymbol{B} = \{\boldsymbol{B}(t), t > 0\}$

$$\boldsymbol{B}(t) = (\mu_1 t + \sigma_1 B_1(t), ..., \mu_n t + \sigma_n B_n(t))^T, \tag{2.1}$$

the Lévy triplet of $\boldsymbol{B}$ is obviously $(\boldsymbol{\mu}, \Sigma, 0)$, where

$$\Sigma = diag(\sigma_1, ..., \sigma_n) := \begin{pmatrix} \sigma_1 & 0... & 0 \\ 0 & \sigma_2.... & 0 \\ 0 & 0... & \sigma_n \\ & & \end{pmatrix}$$

The time changed processes at time t will be collected in the vector $\boldsymbol{Y}(t)$ and interpreted as log returns or log prices: $\boldsymbol{Y}(t) = log\boldsymbol{S}(t)$ where $\boldsymbol{S}(t)$ collects the time t prices of n assets.

The $\mathbb{R}^n$ valued log price process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t > 0\}$ is defined as:

$$\boldsymbol{Y}(t) = \begin{pmatrix} Y_1(t) \\ ... \\ Y_n(t) \end{pmatrix} = \begin{pmatrix} \mu_1 G_1(t) + \sigma_1 B_1(G_1(t)) \\ .... \\ \mu_n G_n(t) + \sigma_n B_n(G_n(t)) \end{pmatrix}, \tag{2.2}$$

where $\mathbf{B}$ is given by (2.1) and $\boldsymbol{G}$ is a multivariate subordinator defined by (1.3), independent from $\boldsymbol{B}$.

The process $\boldsymbol{Y}$, as given by (2.2), is a Lévy process with characteristic function

$$E[e^{i\langle \boldsymbol{z}, \boldsymbol{Y}(t)\rangle}] = \exp(t\Psi_{\boldsymbol{G}}(\log \psi_{\boldsymbol{B}}(\boldsymbol{z}))), \ \boldsymbol{z} \in \mathbb{R}^n_+,$$

where $\psi_{\boldsymbol{B}}$ is the characteristic function of the Brownian motion $\boldsymbol{B}$ and for any $\boldsymbol{w} = (w_1, ..., w_n)^T \in \mathbb{C}^n$ with $Re(w_j) \leq 0, \ j = 1, ..., n$,

$$\Psi_{\boldsymbol{G}}(\boldsymbol{w}) = \langle \boldsymbol{m} \cdot \boldsymbol{w}\rangle + \int_{\mathbb{R}^n} (e^{\langle \boldsymbol{w}, \boldsymbol{x}\rangle} - 1)\nu(d\boldsymbol{x}).$$

is the characteristic exponent of G. The subordinator involved in the construction of $\boldsymbol{G}$ will have zero drift, i.e. $\boldsymbol{m} = \mathbf{0}$. Therefore we characterize the process $\boldsymbol{Y}$ under this condition and we refer to Barndorff-Nielsen et al. [2] (Theorem 3.3) for the general case.

The characteristic triplet $(\boldsymbol{\gamma}_{\boldsymbol{Y}}, \Sigma_{\boldsymbol{Y}}, \nu_{\hat{\boldsymbol{Y}}})$ of $\boldsymbol{Y}$ is as follows

$$\begin{aligned} \boldsymbol{\gamma}_{\boldsymbol{Y}} &= \textstyle\int_{\mathbb{R}^n_+} \nu_{\boldsymbol{G}}(d\boldsymbol{s}) \int_{|\boldsymbol{x}|\leq 1} \boldsymbol{x}\rho_{\boldsymbol{s}}(d\boldsymbol{x}), \\ \Sigma_{\boldsymbol{Y}} &= \mathbf{0}, \\ \nu_{\boldsymbol{Y}}(B) &= \textstyle\int_{\mathbb{R}^n_+} \rho_{\boldsymbol{s}}(B)\nu_{\boldsymbol{G}}(d\boldsymbol{s}), \end{aligned} \tag{2.3}$$

where $\rho_s = \mathcal{L}(\tilde{\boldsymbol{B}}(\boldsymbol{s}))$, $s \in \mathbb{R}^n_+$, $\boldsymbol{x} = (x_1, ..., x_n)^T$ and $B \in \mathbb{R}^n \setminus \{0\}$. Observe that the process $\boldsymbol{Y}$ is a pure jump. Starting from the previous theorem we can also discuss the

regularity of the trajectories of the process $\boldsymbol{Y}$, namely its finite/infinite activity and its bounded/unbounded variation.

As concerns the activity, an immediate consequence of

$$\nu_{\boldsymbol{Y}}(\mathbb{R}^d) = \int_{\mathbb{R}^n_+} \rho_{\boldsymbol{s}}(\mathbb{R}^d)\nu_{\boldsymbol{G}}(d\boldsymbol{s}) = \int_{\mathbb{R}^n_+} \nu_{\boldsymbol{G}}(d\boldsymbol{s}) = \nu_{\boldsymbol{G}}(\mathbb{R}^n), \tag{2.4}$$

is that $\boldsymbol{Y}$ has finite activity ($\nu_{\boldsymbol{Y}}(\mathbb{R}^d) < \infty$) if and only if $\boldsymbol{G}$ does ($\nu_{\boldsymbol{G}}(\mathbb{R}^d) < \infty$). Sufficient conditions on the subordinator's Lévy measure for $\boldsymbol{Y}$ to have finite variations are also provided by Theorem 3.3 in Barndorff-Nielsen et al. [2]. These conditions are not always satisfied by the processes examined here. However, since their margins are completely characterized, we can infer the path regularity of the process as a whole from its margin properties, as follows. Since the marginal Lévy measures are defined by (see Cont and Tankov [9])

$$\nu_j(A) = \nu_{\boldsymbol{Y}}(\mathbb{R} \times A_j ... \times \mathbb{R}), A_j \in \mathcal{B}(\mathbb{R}), j = 1, ..., n, \tag{2.5}$$

$\nu_j(\mathbb{R}) < \infty$ for all $j = 1, .., n$ iff $\nu(\mathbb{R}^n) < \infty$.

As concerns the variations, $\boldsymbol{Y}$ has finite variations if and only if the margins do. The paths of $\boldsymbol{Y}$ are vectorial functions whose components are the paths of its marginal processes. Therefore the previous statement is a consequence of the fact that a vectorial function has bounded variation (has finite length) if and only if its components have bounded variations.

We now discuss different specifications of the $\boldsymbol{Y}$ process and we characterize them completely. They are multivariate generalizations of log prices models widely studied in Finance. The main properties of the corresponding univariate versions are recalled in the Appendix.

2.1 Compound Poisson margins

Geman, Madan, Yor [11] proved that the Poisson model with reflected normal jumps' intensity can be constructed by Poisson time-changing a univariate Brownian Motion. We further demonstrate that it can be extended to the multivariate case using our construction.

Consider the univariate compound Poisson process:

$$\hat{Y}(t) = \sum_{j=1}^{N(t)} M_j, \tag{2.6}$$

where $N(t)$ is a Poisson process with rate λt, $\lambda > 0$, and the random variables M_j are i.i.d, independent from the process N, with reflected normal density

$$f(x) = \frac{\sqrt{2}exp(-\frac{x^2}{2\sigma^2})}{\sigma\sqrt{\pi}}, \quad x > 0. \tag{2.7}$$

Geman, et al. [11] considered the log price process defined as

$$Y(t) = \hat{Y}_1 - \hat{Y}_2, \tag{2.8}$$

where $\hat{Y}_1, \hat{Y}_2$ are independent copies of $\hat{Y}$. They proved that $\boldsymbol{Y}$ can be defined as a time changed Brownian motion through the following construction:

$$Y(t) = \sigma B(N_1(t) + N_2(t)), \tag{2.9}$$

where B is a standard Brownian motion and N_1 and N_2 are two independent Poisson processes with the same arrival rate λt.

Since the Poisson distribution is closed under convolution, $N_1 + N_2$ is a Poisson process with rate 2λ ($N_1 + N_2 \sim Poisson(2\lambda)$). In order to extend the compound Poisson construction to multivariate subordination, we now specify the subordinator $\boldsymbol{G}$ defined by (1.1), so that the resulting multivariate log price model has compound Poisson margins, as in [11]. Let $X_i \sim Poisson(2\lambda_i - a)$, $i = 1, ..., n$ and $Z \sim Poisson(a)$, where $0 < a < 2\lambda_j$, $j = 1, ..., n$. It follows that $X_i + Z \sim Poisson(2\lambda_i)$. Define $\boldsymbol{W}$ as in (1.1), and choose unit weighting parameters $\alpha_i = 1$, $i = 1, ..., n$. Let $\boldsymbol{G}$ be as in Section 1. In this way the marginal process G_j is compound Poisson with parameter $2\lambda_j t$:

$$\mathcal{L}(G_j(t)) = Poisson(2\lambda_j t), \quad j = 1, ..., n.$$

Using (1.7), the characteristic function of $\boldsymbol{G}(1)$ is

$$\psi_{\boldsymbol{G}}(\boldsymbol{u}) = \exp(\sum_{j=1}^{n}((2\lambda_i - a)(\exp(iu_j) - 1)) + (a(\exp(i\sum_{j=1}^{n}\alpha_j uj) - 1)). \tag{2.10}$$

The Lévy measure of $\boldsymbol{G}$ can be derived applying (1.5).

The log-price process $\boldsymbol{Y}$ defined as in Section 2, imposing $\mu_j = 0$, $j = 1, ..., n$, namely

$$\boldsymbol{Y}(t) = \begin{pmatrix} \sigma_1 B_1(G_1(t)) \\ ..... \\ \sigma_n B_2(G_2(t)) \end{pmatrix}, \tag{2.11}$$

has therefore compound Poisson margins. Because the subordinator has zero drift, the results of Section 1 hold and the process $\boldsymbol{Y}$ defined in (2.11) is a pure jump. We are able to provide its Lévy triplet, as explained in Section 1. Moreover its characteristic function at time one is the following:

$$\psi_{\boldsymbol{Y}}(\boldsymbol{u}) = \exp(\sum_{j=1}^{n}((2\lambda_i - a)(\exp(-i\frac{1}{2}\sigma_j^2 u_j^2) - 1)) + (a(\exp(-i\sum_{j=1}^{n}\alpha_j\frac{1}{2}\sigma_j^2 u_j^2) - 1))$$

The process has finite activity, because its margins do.

2.2 Normal inverse gaussian (NIG) margins

Following Barndorff-Nielsen [4] (see also Schoutens [21]) we can define an inverse gaussian process by subordination of a Brownian motion using an inverse gaussian subordinator G. The subordinator used to obtain the NIG process as a time changed Brownian motion belongs to the tempered stable family.

First we recall the univariate definition of the NIG process as a time changed Brownian motion. Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and $\{G(t), t \geq 0\}$ an IG process with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$, such that $\alpha > 0$, $-\alpha < \beta < \alpha$, $\delta > 0$. It can be proven that the process

$$Y(t) = \beta\delta^2 G(t) + \delta B(t), \tag{2.12}$$

is a NIG process with parameters (α, β, δ).

We construct a multivariate Lévy process with NIG margins assuming that the subordinator $\boldsymbol{G}$ defined by (1.3) has IG margins: define $X_i \sim IG(1 - a\gamma_i, \frac{b}{\gamma_i})$, $i = 1, ..., n$ and $Z \sim IG(a, b)$. Since the IG distribution is tempered stable it follows that $\gamma_i^2 Z \sim IG(a\gamma_i, \frac{b}{\gamma_i})$. In order for the marginal distributions to have non negative parameters, the following constraints must be satisfied:

$$b > 0, 0 < a < \frac{1}{\gamma_i}, \;\; i = 1, ..., n. \tag{2.13}$$

¿From stability it follows that $X_i + \gamma_i^2 Z$ is IG; from independence it follows that its characteristic function is

$$\begin{aligned}\psi_{X_i+\gamma_i^2 Z} &= \exp(-\gamma_i a(\sqrt{-2iu + (\frac{b}{\gamma_i})^2} - \frac{b}{\gamma_i}))\exp(-(1 - a\gamma_i)(\sqrt{-2iu + (\frac{b}{\gamma_i})^2} - \frac{b}{\gamma_i})) \\ &= \exp(-(\sqrt{-2iu + (\frac{b}{\gamma_i})^2} - \frac{b}{\gamma_i})),\end{aligned} \tag{2.14}$$

Therefore: $X_i + \gamma_i^2 Z \sim IG(1, \frac{b}{\gamma_i})$. Let $\boldsymbol{W}$ be as in (1.1) and choose as weighting parameters $\alpha_i = \gamma_i^2$, $i = 1, ..., n$. Let $\boldsymbol{G}$ be as in section 1. In this way the marginal process G_j is IG with parameters t and $\frac{b}{\gamma_j}$

$$\mathcal{L}(G_j(t)) = IG(t, \frac{b}{\gamma_j}), \;\; j = 1, ..., n,$$

The characteristic function of $\boldsymbol{G}(1)$ is $\psi_{\boldsymbol{G}}(\boldsymbol{u}) = \psi_{X_i+\gamma_i^2 Z}$. ¿From equation (1.5) we can derive also the Lévy measure of $\boldsymbol{G}$.

We now impose some constraints on the parameters which lead the subordinated process to have NIG margins. Let $\alpha_j, \beta_j, \delta_j$ be such that $\alpha_j > 0$, $-\alpha_j < \beta < \alpha_j$, $\delta > 0$. In order to get NIG margins we choose the parameter of the subordinator so

that $b_j = \frac{b}{\gamma_j} = \delta_j\sqrt{\alpha_j^2 - \beta_j^2}$. Furthermore, we define the independent Brownian motions $B_j(t) = \beta_j\delta_j^2 t + \delta_j B_j(t)$, $j = 1, ..., n$, according to (2.12).

In accordance to the general construction of the previous section, we form the process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t > 0\}$ by time changing the previous Brownian motions:

$$\boldsymbol{Y}(t) = \begin{pmatrix} \beta_1\delta_1^2 G_1(t) + \delta_1 B_1(G_1(t)) \\ ..... \\ \beta_n\delta_n^2 G_2(t) + \delta_n B_2(G_2(t)) \end{pmatrix}. \tag{2.15}$$

The process $\boldsymbol{Y}$ defined in (2.15) is a Lévy process with NIG margins. Its Lévy triplet $(\boldsymbol{\gamma_Y}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ can be derived from (2.3). Its characteristic function at time one is the following:

$$\begin{aligned} \psi_{\boldsymbol{Y}}(\boldsymbol{u}) = \exp[-\sum_{j=1}^{n}(1 - a\gamma_j)(\sqrt{-2i(i\beta_j\delta_j^2 u_j - \frac{1}{2}\delta_j^2 u_j^2) + \frac{b^2}{\gamma_j^2}} - \frac{b}{\gamma_j}) \\ - a\gamma_j(\sqrt{-2i\sum_{j=1}^{n}\gamma_j(i\beta_j\delta_j^2 u_j - \frac{1}{2}\delta_j^2 u_j^2) + \frac{b^2}{\gamma_j^2}} - \frac{b}{\gamma_j})] \end{aligned} \tag{2.16}$$

Since the subordinator has zero drift it is a pure jump process. It has unbounded variation, since the marginal processes do.

2.3 Variance gamma (VG) margins

Another example of multivariate subordinator with the features of section 1 above is the $\boldsymbol{\alpha}$-gamma process introduced in Semeraro [20], that leads to a log price model with variance gamma (VG) margins. Its subordinator, which is a gamma process, can be considered as a tempered stable process, if one includes in that class $\alpha = 0$ (see for example Cont and Tankov [9]).

The $\boldsymbol{\alpha}$-gamma process is generalization of the multivariate VG process introduced for the symmetric case in Madan and Seneta [15] and calibrated in Luciano and Schoutens [12]. The latter process was constructed by subordination of a multivariate Brownian motion $\boldsymbol{B}$ using a common gamma subordinator. The model we are going to discuss contains the VG as the limit subcase that leads to the maximal correlation for the subordinator[1].

The starting point is the univariate VG model, which is constructed as follows: let $\{B(t), t \geq 0\}$ be a standard Brownian motion, $\{G(t), t \geq 0\}$ be a gamma process with parameters $(\frac{1}{\nu}, \frac{1}{\nu})$ and $\sigma > 0$, μ be real parameters, then the real process X_{VG} is defined as

$$X_{VG}(t) = \mu G(t) + \sigma B(G(t)).$$

[1]This aspect will be investigated in the section devoted to dependence.

The multivariate VG is obtained by extending the previous construction considering n independent Brownian motions subordinated by a common gamma process.

The $\boldsymbol{\alpha}$-gamma process instead is constructed as follows: consider a, b, α_j, $j = 1, ..., n$ real parameters that satisfy the constraints

$$0 < \alpha_j < \tfrac{b}{a} \quad j = 1, ..., n. \tag{2.17}$$

Let $\mathcal{L}(X_j) = \Gamma(\frac{b}{\alpha_j} - a, \frac{b}{\alpha_j})$ and $\mathcal{L}(Z) = \Gamma(a, b)$; assume that $X_j, j = 1, ..., n$ and Z are independent random variables; the random vector $\boldsymbol{W}$ defined in (1.1) satisfies $\mathcal{L}(W_j) = \Gamma(\frac{b}{\alpha_j}, \frac{b}{\alpha_j})$, $j = 1, ..., n$ (the proof is in Semeraro [20]).

Let $\boldsymbol{G} = \{\boldsymbol{G}(t), t \geq 0\}$ be the Lévy process associated to the distribution of $\boldsymbol{W}$, then

$$\mathcal{L}(G_j(t)) = \Gamma(\tfrac{tb}{\alpha_j}, \tfrac{b}{\alpha_j}), \;\; j = 1, ..., n,$$

The process $\boldsymbol{G}$ is a subordinator, following Semeraro [20] its Lévy measure can be shown to be:

$$\begin{aligned}\nu_{\boldsymbol{G}}(B) &= \sum_{j=1}^{n} \nu_j(B_j) + \nu_Z(B_\Delta) \\ &= \sum_{j=1}^{n} \int_{B_j} (\frac{b}{\alpha_j} - a) exp(-\frac{b}{\alpha_j} x) x^{-1} 1_{(0,+\infty)}(x) dx + \int_{B_\Delta} a exp(-bx) x^{-1} 1_{(0,+\infty)}(x) dx,\end{aligned}$$

where $B \in \boldsymbol{B}(\mathbb{R}^2)$ and B_j, B_Δ are defined as in Section 1.

The process $\boldsymbol{Y}$ defined from $\boldsymbol{G}$ as in (2.2) is a pure jump Lévy process. Its Lévy triplet $(\boldsymbol{\gamma_Y}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ is given by (2.3). Its characteristic function is

$$\psi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = \prod_{j=1}^{n} (1 - \frac{\alpha_j(i\mu_j u_j - \frac{1}{2}\sigma_j^2 u_j^2)}{b})^{-t(\frac{b}{\alpha_j} - a)} (1 - \frac{\sum_{j=1}^{n} \alpha_n(i\mu_j u_j - \frac{1}{2}\sigma_j^2 u_j^2)}{b})^{-ta}. \tag{2.18}$$

The $\boldsymbol{\alpha}$-VG process has infinite activity and bounded variation, as we can derive from the properties of its components.

2.4 CGMY margins

Madan and Yor [13] proved that the CGMY process, first introduced in Carr et al [8], can be constructed as a time changed Brownian motion.

Let Y be a $CGMY(C, G, M, Y)$ process, with parameters $C, G, M > 0$ and $Y < 2$. Let us consider the stable subordinator $G' \sim S_{\frac{Y}{2}}(K, \gamma)$, with Lévy measure

$$\nu'(dx) = \frac{K}{x^{1+\frac{Y}{2}}} dx. \tag{2.19}$$

Define as Γ_k the gamma random variable with law $\Gamma(k,1)$, and $\Gamma(k)$ the gamma function at k.

Madan and Yor [13] assume that G is a subordinator absolutely continuous with respect to G', with density

$$f(y) = e^{-\frac{(B^2-A^2)y}{2}} E[\exp\{-\frac{B^2 y}{2}\frac{\Gamma_{Y/2}}{\Gamma_{1/2}}\}], \tag{2.20}$$

where

$$A = \frac{G-M}{2}, \quad B = \frac{G+M}{2}, \quad K = \frac{C\Gamma(Y/4)\Gamma(1-Y/4)}{2\Gamma(1+Y/2)}. \tag{2.21}$$

They then define the process Y by the following

$$Y(t) = \frac{G-M}{2}G(t) + B(G(t)). \tag{2.22}$$

We now construct a multivariate subordinator of the type introduced in Section 1 as to obtain a multivariate Lévy model with $CGMY(C_j, G_j, M_j, Y)$ margins, where $C_j, G_j, M_j > 0$, $Y < 2$. We denote the subordinator of Madan and Yor as $Su(CGMY)$.

Let $\tilde{Z} \sim Su(C', G, M, Y)$, where $C', G, M > 0$ and $Y < 2$ and $\tilde{X}_j$ is $Su(C_j'' G_j M_j Y)$, where $C_j'' > 0$; then $\boldsymbol{G}$ has marginal processes $G_j \sim Su(C_j G_j M_j Y)$, with $C_j = C_j' + C_j''$ and $C_j' = C'\alpha_j^{Y/2}$. In fact if $\tilde{Z} \sim Su(C', G, M, Y)$ then $\alpha_j \tilde{Z} \sim Su(C_j', G_j, M_j, Y)$ where $G = \frac{G'}{\sqrt{\alpha_j}}$, $M = \frac{M'}{\sqrt{\alpha_j}}$ and $C_j' = C'\alpha_j^{Y/2}$. Assume now that $\tilde{X}_j$ is $Su(C_j'' G_j M_j Y)$, where $C_j'' > 0$; then the assert stems from the properties of continuous stable subordinators.

The Lévy measure of $\boldsymbol{G}$ follows by equation (1.5). In accordance to the general construction of the previous section, define the process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t > 0\}$ by time changing n independent Brownian motions:

$$\boldsymbol{Y}(t) = \begin{pmatrix} \frac{G_1-M_1}{2}G_1(t) + B_1(G_1(t)) \\ ..... \\ \frac{G_n-M_n}{2}G_n(t) + B_n(G_n(t)) \end{pmatrix}. \tag{2.23}$$

The process $\boldsymbol{Y}$ is a Lévy process with CGMY margins with parameters C_j, M_j, G_j, Y. Since the subordinator has zero drift its Lévy triplet $(\boldsymbol{\gamma_Y}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ can be derived from (2.3). The variations of $\boldsymbol{Y}$, as for the marginal processes, depend on the parameter Y. If $Y < 1$ the path have bounded variation, if $Y \in [1, 2)$ they have unbounded variation. Moreover if $Y < 0$ the process has also finite activity. In fact the marginal Y_j are CGMY processes and they have finite activity if $Y < 0$. Since the Lévy measures of G_j and $\tilde{X}_j$ only differ for constant terms, also the Lévy measures of the subordinated processes $Y_j = B_j(G_j)$ and $B_j(\tilde{X}_j)$ only differ for constant terms. Thus, if $Y < 0$ the margins Y_j have finite activity then $B_j(\tilde{X}_j)$ have finite activity that implies (see Appendix B) $\boldsymbol{Y}$ has finite activity.

3 Dependence

This section is devoted to discussing the dependence structure of the above models.

The subordinator $\boldsymbol{G}$ is always positively associated, see Semeraro [20].

As concerns the single subordinated models presented above, only the dependence features of the Compound Poisson case are an easy consequence of known closure properties of dependence with respect to mixture. Indeed, the following proposition holds:

The process $\boldsymbol{Y}$ defined in (2.11) is PA.

To prove the previous assert observe that the compound Poisson process $\hat{\boldsymbol{Y}}$ can be written as follows: $\hat{\boldsymbol{Y}}(t) = \sum_{j=1}^{\boldsymbol{N}(t)} \boldsymbol{X}_j$, where $\boldsymbol{X}$ are i.i.d. and positive. Since $[\hat{\boldsymbol{Y}}(t)|N(t) = \boldsymbol{n}]$ is stochastically increasing in $\boldsymbol{n}$, and $\boldsymbol{N}(t)$ is PA, then by Proposition 2.1 in [5] so is $\hat{\boldsymbol{Y}}(t)$.

Consider now $-\hat{\boldsymbol{Y}}$. Since the same argument holds if $[\hat{\boldsymbol{Y}}(t)|N(t) = \boldsymbol{n}]$ is stochastically decreasing, we derive that $-\hat{\boldsymbol{Y}}$ is PA. The thesis follows because PA is closed under convolution.

The subordinated Lévy model $\boldsymbol{Y}$ has non linear dependence. To prove this, we observe that the process has dependent margins also in the symmetric case ($\rho = 0$): indeed the Lévy measure of $\boldsymbol{Y}$ is given by

$$\nu_{\boldsymbol{Y}}(B) = \int_{\mathbb{R}_+^n} \rho_{\boldsymbol{s}}(B)\nu_{\boldsymbol{G}}(d\boldsymbol{s}). \tag{3.1}$$

¿From the expression of $\nu_{\boldsymbol{G}}$ it follows that the components of $\boldsymbol{Y}$ may jump together. Thus the processes $\sigma_j B_j(G_j(t))$ have non-linear dependence, unless the random variable Z is degenerate.

Remark 1. *A detailed study of the subordinated process dependence - apart from the Compound Poisson case - would require the use of copulas or Lévy copulas. In principle, these can be obtained through Sklar's theorem. In our case, since we can only provide integral expressions for the marginal and joint distributions of $\boldsymbol{Y}(t)$, even when the subordinators $\tilde{X}_j$, $j = 1, ..., n$ and $\tilde{Z}$ have known distributions at each time t, we do not have a closed formula for the copula at time t. Anyway the copula function can be studied numerically, by providing both scatter plots and contour levels of it. The Lévy copula function should also be studied numerically, since we have only integral expressions for the tail integrals of the marginal and of the joint Lévy measures.*

4 Linear dependence

We now turn to linear dependence, which can be useful in order to calibrate the previous models. In order to show that the processes constructed so far are actually a generalization of the existing ones, we study their linear correlation and show that it can span the whole range [-1,1], including independence. We will spend some words about linear

correlation for the multivariate time changed class as a whole. Then we will specify it for the models considered. We focus mainly on the α-VG model, a similar discussion also holds for the other models.
We start from the correlation matrix $\rho_{\boldsymbol{G}(t)} = (\rho_{\boldsymbol{G}(t)}(l,j))$ of the subordinator. Since

$$Cov(G_l(t), G_j(t)) = \alpha_l \alpha_j V(Z(t)) \text{ and } V(G_j) = V(X_j(t)) + \alpha^2 V(Z(t)), \tag{4.1}$$

we have

$$\rho_{\boldsymbol{G}(t)}(l,j) = \frac{\alpha_l \alpha_j V(Z(t))}{\sqrt{[V(X_l(t)) + \alpha_l^2 V(Z(t))][V(X_j(t)) + \alpha_j^2 V(Z(t))]}}$$

Concerning the subordinated process $\boldsymbol{Y}$, the variance of $Y_j(t)$ is:

$$V[Y_j(t)] = E[V[Y_j(t)|G_j(t)]] + V[E[Y_j(t)|G_j(t)]] = \sigma_j^2 E[G_j(t)] + \mu_j^2 V[G_j(t)]. \tag{4.2}$$

The lj covariance of the process at time t is:

$$cov[Y_l(t), Y_j(t)] = \mu_l \mu_j cov[G_1(t), G_2(t)] = \mu_l \mu_j \alpha_l \alpha_j V(Z(t)).$$

Therefore the linear correlation coefficients are

$$\rho_{\boldsymbol{Y}(t)}(l,j) = \frac{\mu_l \mu_j \alpha_l \alpha_j V(Z(t))}{\sqrt{V(Y_l(t)) V(Y_j(t))}}$$

Since all the processes involved are Lévy ones, by infinite divisibility $V(Z(t)) = V(Z)t$, $V(Y_j(t)) = V(Y_j(1))t$, $j = 1, ..., n$ and $\rho_{\boldsymbol{Y}(t)}(l,j)$ is independent from t. In addition[2], $\rho_{\boldsymbol{Y}(t)}(l,j) \leq \rho_{\boldsymbol{G}(t)}(l,j)$.

Under the conditions $\mu_j > 0$ and $\alpha_j > 0$, $j = 1, ..., n$, $\rho_{\boldsymbol{Y(t)}}(l,j) = 0$ iff $V[Z(t)] = 0$, that is $\tilde{Z}$ is degenerate iff the margins are independent. This is the case which

[2]Indeed

$$\mu_1 \mu_2 \alpha_1 \alpha_2 = \sqrt{\mu_1^2 \alpha_1 \mu_2^2 \alpha_2} \sqrt{\alpha_1 \alpha_2} \leq \sqrt{(\sigma_1^2 + \mu_1^2 \alpha_1)(\sigma_2^2 + \mu_2^2 \alpha_2)} \sqrt{\alpha_1 \alpha_2}$$

implies

$$\mu_1 \mu_2 \sqrt{\alpha_1 \alpha_2} \leq \sqrt{(\sigma_1^2 + \mu_1^2 \alpha_1)(\sigma_2^2 + \mu_2^2 \alpha_2)} \tag{4.3}$$

and

$$\mu_1 \mu_2 \alpha_1 \alpha_2 a = \mu_1 \mu_2 \sqrt{\alpha_1 \alpha_2} \sqrt{\alpha_1 \alpha_2} a \leq \sqrt{(\sigma_1^2 + \mu_1^2 \alpha_1)(\sigma_2^2 + \mu_2^2 \alpha_2)} \sqrt{\alpha_1 \alpha_2} a, \tag{4.4}$$

from this it follows that

$$\rho_{\boldsymbol{Y}(t)}(l,j) = \frac{\mu_l \mu_j \alpha_l \alpha_j a}{\sqrt{(\sigma_l^2 + \mu_l^2 \alpha_l)(\sigma_j^2 + \mu_j^2 \alpha_j)}} \leq \sqrt{\alpha_j \alpha_j} a = \rho_{\boldsymbol{G}(t)(l,j)}. \tag{4.5}$$

cannot be captured by the standard multivariate time changed models with a univariate subordinator.

Observe that linear correlation ρ depends on the variance of the subordinator's common factor. By construction we are able to move the variance of the common factor leaving fixed margins. Since $\alpha_j^2 V(Z(t)) = V(G_j) - V(X_j(t))$, if we fix the margins of $\boldsymbol{G}$ (so that $V(G_j(t))$ is constant), to increase the variance of Z we have to decrease the variance of X_j. The maximal correlation is then reached when[3] $V(X_j(t)) \to 0$. This implies that $\rho_{\boldsymbol{G(t)}}(l,j) \to 1$ and $G_j(t) = \alpha_j Z(t)$. Togheter with the constraints $E[G_j(t)] = t$ this makes $\alpha_i = \alpha_j$ and $\boldsymbol{G}$ a.s. a univariate subordinator.

Before analyzing case by case the way linear correlation changes with parameters, we would like to observe that in the symmetric case the correlation is zero. The same happened in the case with one subordinator. In order to have linear correlation in the symmetric models, the only way out is to consider a more general construction with correlated Brownian motions. Anyway also in the symmetric case we are able to model non linear and tail dependence.

4.1 Compound Poisson

Consider the general model (2.1). If the subordinator is the Poisson one introduced in Section 2.1 the linear correlation coefficients of the process at time t are:

$$\rho_{\boldsymbol{Y}(t)}(l,j) = \frac{\mu_l \mu_j a}{2\sqrt{\lambda_l(\sigma_l^2 + \mu_l^2)\lambda_j(\sigma_j^2 + \mu_j^2)}},$$

If we focus on the Poisson compound log price of Geman, Madan and Yor [11], in which $\mu_j = 0$, $j = 1, ..., n$, the linear correlation of $\boldsymbol{Y}$ is zero, while we can capture non linear dependence. Indeed if $a \neq 0$, then $V[Z(t)] = at \neq 0$, the correlation of the subordinator is different from zero and the margins of $\boldsymbol{Y}$ are positively associated. Moreover we have independence if $a \to 0$ and maximal dependence, that corresponds to maximal correlation for the subordinator, if $a \to 2\lambda_j$ for each $j = 1, ..., n$; in the last case $\boldsymbol{G}$ is a.s. a univariate subordinator.

4.2 Normal inverse gaussian

We now consider the NIG log-price model. The linear correlations of the subordinator are

$$\rho_{\boldsymbol{G(t)}}(l,j) = \frac{\gamma_l^2 \gamma_j^2 \frac{a}{b^3}}{\sqrt{[\frac{(1-a\gamma_l)\gamma_l^3}{b^3} + \gamma_l^2 \frac{a}{b^3}][\frac{(1-a\gamma_j)\gamma_j^3}{b^3} + \gamma_j^2 \frac{a}{b^3}]}}$$

[3]The limit value $V(X_j(t)) = 0$ requires some constraints on the marginal distributions, which we will discuss in details for the $\boldsymbol{\alpha}$-variance gamma case.

Observe that $\rho_{\boldsymbol{G(t)}}(l,j) = 1$ if $a = \frac{1}{\gamma_j} = \frac{1}{\gamma}$, $j = 1, ...n$ (this way $\rho_{\boldsymbol{G(t)}(l,j)} = \gamma$) and $\gamma = 1$. By so doing we obtain the subcase with one subordinator Z. Its law becomes $IG(1, b)$.

The linear correlation coefficients of the subordinated process at time t are:

$$\rho_{\boldsymbol{Y}(t)}(i,j) = \frac{\beta_i \delta_i^2 \beta_j \delta_j^2 \gamma_i^2 \gamma_j^2 \frac{a}{b^2}}{\sqrt{(\delta_i^2 \gamma_i + \frac{\beta_i^2 \delta_i^4 \gamma_i^3}{b^2})(\delta_j^2 \gamma_j + \frac{\beta_j^2 \delta_j^4 \gamma_j^3}{b^2})}}$$

For given marginal distributions, i.e. fixed $\delta_i, \gamma_i, \beta_i$, the correlation is a function of both a and b, the parameters of the common factor. Therefore changing them we can move the correlation matrix of the process leaving fixed margins. The only way to capture independence is to let a go to zero. In order to capture the maximal dependence, we have to impose to the parameters the constraints that lead to $\rho_{\boldsymbol{G(t)}}(l,j) = 1$. This reduces our model to one subordinator with parameters $(1, b)$, as observed above. Notice that $\rho_{\boldsymbol{Y}(t)}(i,j)$ can be written as

$$\rho_{\boldsymbol{Y}(t)}(i,j) = \frac{\beta_i \delta_i^2 \beta_j \delta_j^2 \frac{\gamma_i^2}{b^2} \frac{\gamma_j^2}{b^2} ab}{\sqrt{(\delta_i^2 \frac{\gamma_i}{b} + \beta_i^2 \delta_i^4 \frac{\gamma_i^3}{b^3})(\delta_j^2 \frac{\gamma_j}{b} + \beta_j^2 \delta_j^4 \frac{\gamma_j^3}{b^3})}}.$$

¿From this rapresentation it is clear that in order to study the correlation the assumption $b = 1$ is not restrictive.

4.3 α-variance gamma

The linear correlations of the $\boldsymbol{\alpha}$-gamma subordinator do not depend on t and are increasing in α_j:

$$\rho_{\boldsymbol{G}(t)}(l,j) = \frac{a}{b}\sqrt{\alpha_l \alpha_j}.$$

The linear correlation coefficients of the process at time t are:

$$\rho_{\boldsymbol{Y}(t)}(l,j) = \frac{\mu_l \mu_j \alpha_l \alpha_j \frac{a}{b^2}}{\sqrt{(\sigma_l^2 + \mu_l^2 \frac{\alpha_l}{b})(\sigma_j^2 + \mu_j^2 \frac{\alpha_j}{b})}} = \frac{\mu_l \mu_j \alpha_l \alpha_j a}{b\sqrt{(b\sigma_l^2 + \mu_l^2 \alpha_l)(b\sigma_j^2 + \mu_j^2 \alpha_j)}},$$

The correlations of the process involve all the parameters, and for any couple of fixed marginal distributions the linear correlation is a function of a only. This is the main contribution of the $\boldsymbol{\alpha}$-VG generalization with respect to VG correlation, since changing a we can modify the correlation of the process, without modifying the marginal distributions of the process. On the contrary, in the Variance Gamma process with a common gamma subordinator used in the previous literature ($\rho_{\boldsymbol{G}(t)} = 1$), for fixed parameters of the lj marginal processes, the correlation coefficient is uniquely determined.

Moreover, the correlation coefficient depends on α_l, α_j, b only through the ratios $\frac{\alpha_l}{b}$ and $\frac{\alpha_j}{b}$: this means that in order to study the correlation the assumption $b = 1$ is not restrictive. Therefore we fix $b = 1$. Let us examine the extreme correlation cases.

1. $|\rho_{\boldsymbol{Y}(t)}(l,j)| = 1$ iff $\sigma_l = \sigma_j = 0$ and $a = \frac{1}{\alpha_l} = \frac{1}{\alpha_j}$ holds.

2. the lj-th marginal processes are independent iff $a = 0$

3. the lj margins are dependent and uncorrelated ($\rho_{\boldsymbol{Y}(t)}(l,j) = 0$) processes iff $\mu_1 = \mu_2 = 0$ and $a \neq 0$.

The previous asserts are an evident consequence of the construction and the expression of $\rho_{\boldsymbol{Y}(t)}(l,j)$. We now discuss them case by case.

1. $|\rho_{\boldsymbol{Y}(t)}(l,j)| = 1$ can be reached if and only if the model becomes the traditional multivariate VG. Indeed the condition $a = \frac{1}{\alpha_l} = \frac{1}{\alpha_j}$ corresponds to the limit case with only one subordinator ($\rho_{\boldsymbol{G}(t)}(l,j) = 1$, see Semeraro [20]). With a unique subordinator a necessary condition for $|\rho_{\boldsymbol{Y}(t)}(l,j)| = 1$ is that $\sigma_j = \sigma_l = 0$. Anyway we can reach high correlation also with the general model, thus with different marginal processes, assuming that $\sigma_l \ll |\mu_l|$ and $\sigma_j \ll |\mu_j|$.

2. The limit case $a = 0$ leads to a subordinator with independent components. The process $\boldsymbol{Y}$ is a mixture of independent processes and has independent margins. We also capture low correlation with $a \neq 0$, **as the calibrations will show**.

3. If $\mu_l = \mu_j = 0$ the correlation is 0, even if the margins are correlated ($a \neq 0$). This means that the linear correlation is not sufficient to capture the dependence structure of the model. In order to model linear dependence we should use correlated Brownian motions. To introduce correlation in the Brownian motions we could come back to the bivariate variance gamma with one subordinator, i.e. $a = \frac{1}{\alpha_l} = \frac{1}{\alpha_j}$. In any case we observe that also in the symmetric case we can describe independence by choosing $a = 0$.

For completeness we observe that in the dependence cases ($a \neq 0$) $\rho_{\boldsymbol{Y}(t)}(l,j) > 0$ if and only if $\mu_l \mu_j > 0$.

5 Simulation and dependence

In this section we simulate the subordinators and subordinated processes introduced so far, in order to discuss their behaviour. The simulation tecnique is described in appendix B. For each given type of process (compound Poisson, NIG, $\alpha-$VG) we will

1. fix the marginal parameters,

2. choose the values of a corresponding to independence and to maximal dependence;

3. compute the corresponding maximal linear correlation;

4. construct the trajectories of both the subordinators and the corresponding subordinated processes;

5. build the scatter plots of both, for $t = 1$.

By so doing, we aim at

- pointing out the flexibility features of multivariate subordinators, as opposed to the standard univariate subordinators. The range of dependence captured by the former indeed spans from independence to maximal dependence, while the latter correspond to the perfect linear correlation between the subordinators,

- compare the flexibility features of the different specifications (compound Poisson, NIG..) introduced above.

In all but the compound Poisson case we will use the estimates obtained for seven stock indices, using the Bloomberg quotes of the corresponding options with three months to expiry. For each index, six strikes (the closest to the initial price) were selected, and the corresponding option prices were monitored over a one hundred days window, from 7/14/06 to 11/30/06. In correspondence to the alpha-VG marginal model we estimated the marginal parameters as follows: using the quotes of the first day only, we obtained the parameter values which minimized the mean square error between theoretical and observed prices, the theoretical ones being obtained by FRFT. We used the results as guess values for the second day, the second day results as guess values for the third day, and so on. The marginal parameters used here are the average ones. The previous procedure is intended to provide marginal parameters which are actually "representative" of the corresponding stock index price, and are not dependent on an initial arbitrary guess. The marginal values for the VG processes are reported in the following table:

Asset	μ	σ	α	b
S&P	-0.6490	0.0224	0.1021	1.0000
Nasdaq	-0.6730	0.1062	0.1317	1.0000
CAC 40	-0.4674	0.1031	0.1109	1.0000
FTCE	-0.5865	0.0450	0.0313	1.0000
Nikkei	-0.3386	0.1595	0.1042	1.0000
DAX	-0.2700	0.1334	0.1410	1.0000
Hang Seng	-1.6790	0.0788	0.0279	1.0000

Please notice that, without loss of generality, b has been fixed to the value 1 (see section 4.3).

For the NIG, we computed the marginal parameters by moment matching. More precisely, we fixed them by matching the first four moments of the VG and NIG cases. The relationships between the moments and the process parameters are in Appendix A. The values so obtained are in the following table:

α	β	δ	b	γ
1.0910	-0.2170	3.1740	1.0000	0.294668349
1.1690	-0.2920	2.5850	1.0000	0.341754741
1.0670	-0.2560	2.3440	1.0000	0.411862303
1.2540	-0.3930	0.8180	1.0000	1.026593053
1.2780	-0.2600	1.6120	1.0000	0.495773039
0.7390	-0.1220	9.9440	1.0000	0.137973197
1.0280	-0.1480	4.0250	1.0000	0.244224438

Please notice that, without loss of generality, b has been fixed to the value 1 (see section 4.2).

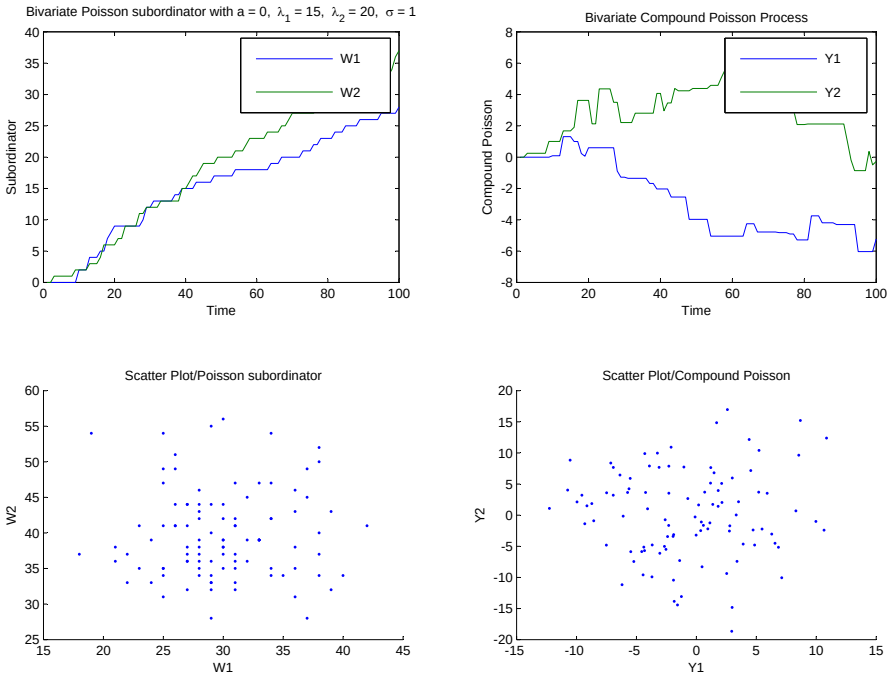
As for the compound Poisson, with respect to the theoretical model presented above we provide an example adding the drift. In the following table we list the marginal parameters chosen.

Asset	λ	μ	σ
1	15.0000	1.0000	1.0000
2	20.0000	1.0000	1.0000

5.1 Compound Poisson

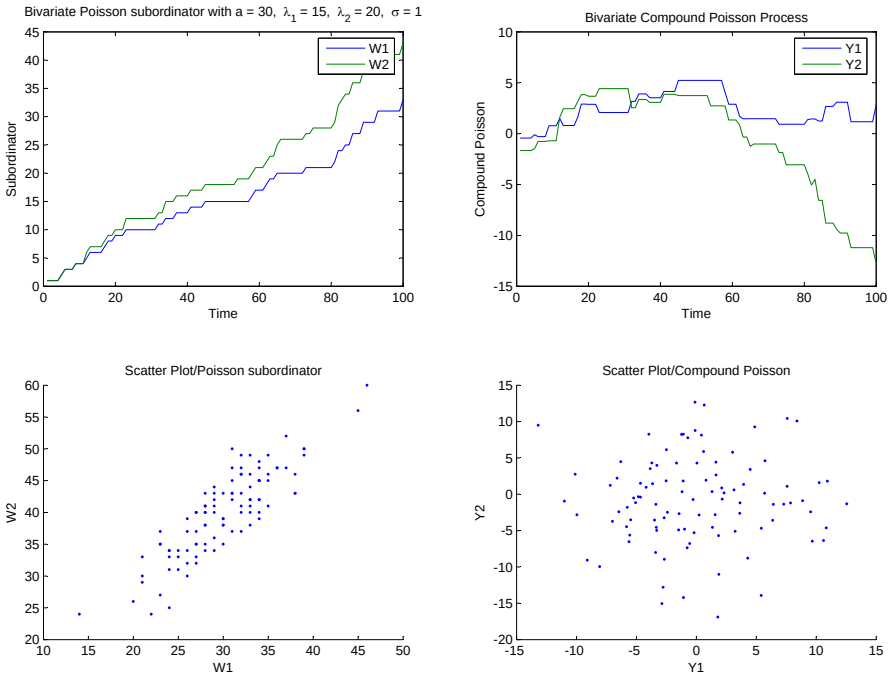
5.1.1 $a = 0$

This value of a corresponds to independence. The picture below shows the scatter plot and the simulated trajectories.



5.1.2 $a = min\{2\lambda_1, 2\lambda_2\}$

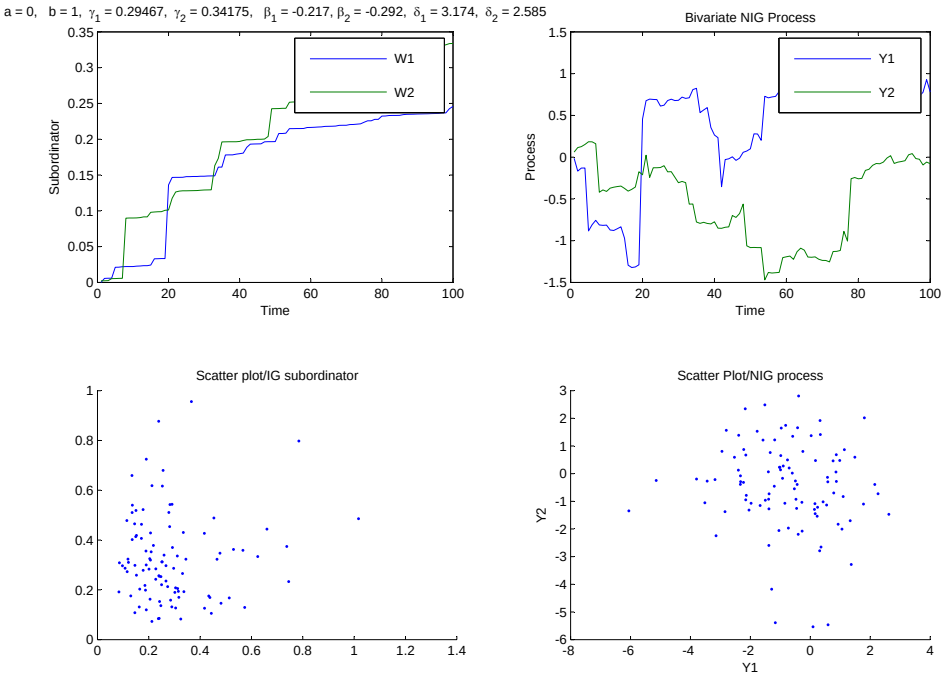
This value of a, namely 30, corresponds to $\rho = 0.433$ and rapresents the maximal correlation captured by the model. The picture below shows the scatter plot and the simulated trajectories.



5.2 Normal inverse gaussian

5.2.1 $a = 0$

This value of a corresponds to independent components. The picture below shows the scatter plot and the simulated trajectories for the pair S&P and Nasdaq.

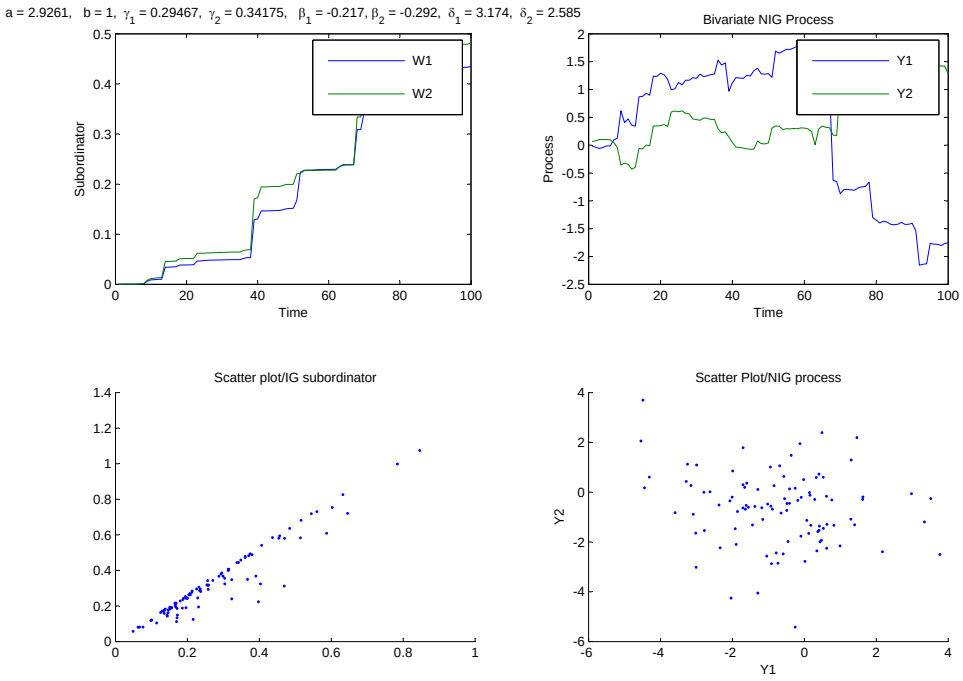


5.2.2 $a = \min\{\frac{1}{\gamma_i}, \frac{1}{\gamma_j}\}$

For each pair of assets the following table gives the values of ρ (upper entry) and a (lower one) corresponding to the maximal correlation:

	S&P	Nasdaq	CAC 40	FTSE	Nikkei	Dax
Nasdaq	0.047					
	2.926					
CAC 40	0.041	0.056				
	2.428	2.428				
FTSE	0.034	0.047	0.049			
	0.974	0.974	0.974			
Nikkei	0.032	0.044	0.046	0.047		
	2.017	2.017	2.017	0.974		
Dax	0.023	0.027	0.024	0.020	0.018	
	3.394	2.926	2.428	0.974	2.017	
Hang Seng	0.027	0.031	0.027	0.023	0.021	0.018
	3.394	2.926	2.428	0.974	2.017	4.095

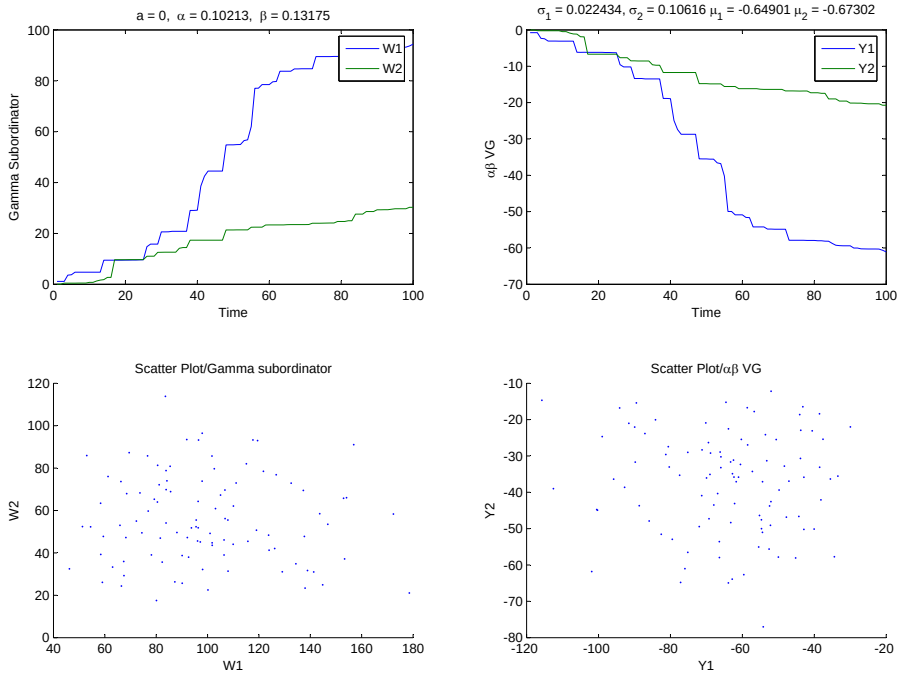
The picture below shows the scatter plot and the simulated trajectories for the pair S&P and Nasdaq.



5.3 α-variance gamma

5.3.1 $a = 0$

This value of a corresponds to independent components. The picture below shows the scatter plot and the simulated trajectories for the pair S&P and Nasdaq.

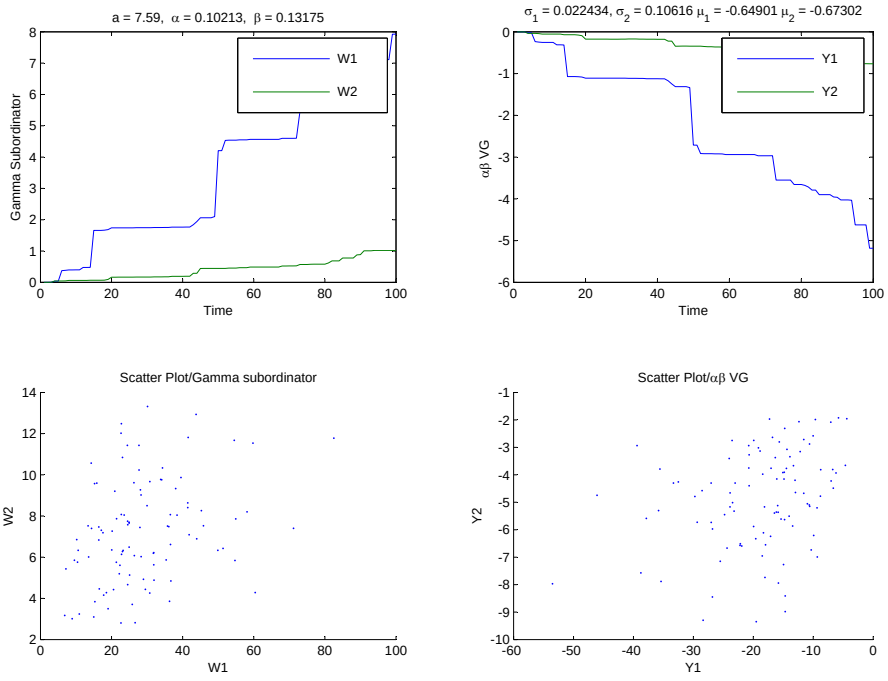


5.3.2 $a = \min\{\frac{b}{\alpha_i}, \frac{b}{\alpha_j}\}$

For each pair of assets the following table gives the values of ρ (upper entry) and a (lower one) corresponding to the maximal correlation:

	S&P	Nasdaq	CAC 40	FTCE	Nikkei	Dax
Nasdaq	0.803					
	7.590					
CAC 40	0.795	0.701				
	9.020	7.590				
FTCE	0.505	0.410	0.406			
	9.791	7.590	9.020			
Nikkei	0.556	0.461	0.457	0.284		
	9.593	7.590	9.020	9.593		
Dax	0.512	0.536	0.447	0.261	0.294	
	7.092	7.092	7.092	7.092	7.092	
Hang Seng	0.500	0.406	0.403	0.834	0.282	0.259
	9.791	7.590	9.020	31.976	9.593	7.092

The picture below shows the scatter plot and the simulated trajectories for the pair S&P and Nasdaq.



6 Appendix A

Here we recall the definitions of the real processes which are the basis of our multivariate generalization.

6.1 Normal inverse gaussian

An *inverse gaussian* (IG) process with parameters (a, b) is a Lévy process with the following characteristic function:

$$\psi_{IG}(z) = \exp t(-a(\sqrt{-2iu + b^2} - b). \tag{6.1}$$

The Lévy measure of the IG process is 1/2-stable, in fact

$$\nu_G(x) = (2\pi)^{1/2} a x^{-3/2} \exp(-1/2b^2 x)\mathbf{1}_{(0,+\infty)}(x)dx. \tag{6.2}$$

A *normal inverse gaussian* (NIG) process with parameters $\alpha > 0$, $-\alpha < \beta < \alpha$, $\delta > 0$ is a Lévy process $X_{NIG} = \{X_{NIG}(t), t \geq 0\}$ with characteristic function

$$\psi_{NIG}(z) = \exp t(-\delta(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}). \tag{6.3}$$

A NIG process has no gaussian component; for its characterization see Schoutens [21].

The process is of infinite variation.

We end with the moments of the distribution: the mean m, the variance v, the sweakness s and the curtosis k.

$$m = \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} \tag{6.4}$$

$$v = \alpha^2\delta(\alpha^2 - \beta^2)^{-\frac{3}{2}} \tag{6.5}$$

$$s = 3\beta\alpha^{-1}\delta^{-\frac{1}{2}}(\alpha^2 - \beta^2)^{-\frac{1}{4}} \tag{6.6}$$

$$k = 3(1 + \frac{\alpha^2 + 4\beta^2}{\delta\alpha^2\sqrt{\alpha^2 - \beta^2}}) \tag{6.7}$$

6.2 Variance gamma

A *variance gamma* process is a real Lévy process $X_{VG} = \{X_{VG}(t), t \geq 0\}$ which can be obtained as a Brownian motion with drift time-changed by a gamma process.

A gamma process $\{G(t), t \geq 0\}$ with parameters (a, b) is a Lévy process so that the defining distribution of $X(1)$ is gamma with parameters (a, b) (shortly $\mathcal{L}(X(1)) = \Gamma(a, b)$). It is a finite variation Lévy process. Its Lévy triplet is

$$\gamma = \tfrac{a(1-\exp(-b))}{b},$$
$$A = 0$$
$$\nu(dx) = a\exp(-bx)x^{-1}\mathbf{1}_{(0,+\infty)}(x)dx.$$

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion, $\{G(t), t \geq 0)$ be a gamma process with parameters $(\frac{1}{\nu}, \frac{1}{\nu})$ and $\sigma > 0$, θ be real parameters; then the process X_{VG} is defined as

$$X_{VG}(t) = \theta G(t) + \sigma B(G(t)).$$

The characteristic function of X_{VG} is the following

$$\psi_{X_{VG}(t)}(u) = [\psi_{X_{VG}(1)}(u)]^t = (1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2)^{-\frac{t}{\nu}}.$$

The paths of the VG process are of infinite activity and finite variation. We end with the moments of the distribution: the mean m, the variance v, the sweakness s and the curtosis k.

$$m = \theta \tag{6.8}$$

$$v = \sigma^2 + \nu\theta^2 \tag{6.9}$$

$$s = \frac{\theta\nu(3\sigma^2 + 2\nu\theta^2)}{(\sigma^2 + \nu\theta^2}^{3/2} \tag{6.10}$$

$$k = 3(1 + 2\nu - 4\nu\sigma^4(\sigma^2 + \nu\theta^2)^{-2}) \tag{6.11}$$

6.3 CGMY

The CGMY process is a Lévy process $X = \{X(t), t \geq 0\}$ whose characteristic function is

$$\psi_{X(t)}(u) = \exp(Ct\Gamma(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y)), \tag{6.12}$$

where $C, G, M > 0$ and $Y < 2$. Carr et al. [8] introduced this distribution. The path regularity changes for different values of the parameter Y: if $Y < 0$ the paths have finite activity; if $Y \in [0, 1)$ they have infinite activity and finite variation; if $Y \in [1, 2)$ they have infinite variation.

6.4 The return processes

The price process is the exponential of the process $\boldsymbol{Y}$. The i-th component of S is

$$S_i(t) = S_i(0)\exp(Y_i(t)), \quad t \geq 0.$$

According to Cont and Tankov [9] (Section 9.5) our model is arbitrage free, since it has both positive and negative jumps. Since this is an exponential-Lévy model which is arbitrage-free, there exists an equivalent martingale measure Q. The model however belongs to the class of incomplete models: the equivalent martingale measure is not unique.

Among the possible equivalent martingale measures, we select the mean-correcting one. Under this measure, the risk neutral return process $\tilde{\mathbf{Y}}(t)$ has the same linear correlation matrix as the process $\boldsymbol{Y}(t)$ at any point in time. This happen since $\tilde{\mathbf{Y}}(t)$ is a translation $\boldsymbol{Y}(t)$. We call $\boldsymbol{\rho}$ their common correlation matrix.

7 Appendix B: simulation and dependence assessment techniques

For each one of the models studied above, the scatter plots of the processes $\boldsymbol{Y}$ at time $t = 1$ can be easily obtained. We illustrate the technique in the bivariate case.

- Simulate N realizations from the independent laws $\mathcal{L}(X_1)$, $\mathcal{L}(X_2)$, $\mathcal{L}(Z)$; let them be respectively x_1^n, x_2^n, z^n for $n = 1, ..., N$;
- obtain N realizations (w_1^n, w_2^n) of $\boldsymbol{W}$ through the relations $W_1 = X_1 + Z$ and $W_2 = X_2 + Z$;
- generate N independent random draws from each of the independent random variables M_1 and M_2 with laws $N(0, W_1)$ and $N(0, W_2)$. The draws for M_1 are in turn obtained from N normal distributions with zero mean and variance w_1^n, namely

  $$M_1(n) = N(0, w_1^n)$$

  The draws for M_2 are from normal distributions with zero mean and variance w_2^n, namely

  $$M_2(n) = N(0, w_2^n)$$

- obtain N realizations (y_1^n, y_2^n) of $\boldsymbol{Y}(1)$ by means of the relations

  $$\begin{aligned} y_1^n &= \mu_1 w_1^n + \sigma_1 M_1(n) \\ y_2^n &= \mu_2 w_2^n + \sigma_2 M_2(n) \end{aligned}$$

  where the parameters μ_j and σ_j, $j = 1, 2$ depend on the specific model under consideration.

The realizations (y_1^n, y_2^n) give the scatter plots of

$$\begin{aligned} Y_1(1) &= \mu_1 W_1 + \sigma_1 M_1 \\ Y_2(1) &= \mu_2 W_2 + \sigma_2 M_2 \end{aligned}$$

.

For completeness one can also simulate the value of the process $\boldsymbol{Y}$ at time points $\{n\Delta t, n = 0, ...N\}$ as follows

- simulate N realizations $\{(x_1(n), x_2(n), z(n)), n = 1, ...N\}$ from the independent random variables $\mathcal{L}(X_1(\Delta t))$, $\mathcal{L}(X_2(\Delta t))$, $\mathcal{L}(Z(\Delta t))$, where $X_i(\Delta t) := X_i(s+\Delta t) - X_i(s)$ is independent from s. The same notation for $Z(\Delta t)$;

- obtain N realizations of the increments of $\boldsymbol{W}$, $\{(w_1(n), w_2(n)), n = 1, ..., N\}$ through the relations $W_1 = X_1 + Z$ and $W_2 = X_2 + Z$;

- generate N independent random numbers $\{M_1(n) : n = 1, ...N\}$ extracted from a variable M_1 with $\mathcal{L}(M_1) = N(0, 1)$

- generate N independent random numbers $\{M_2(n) : n = 1, ...N\}$ extracted from a variable M_2 with $\mathcal{L}(M_2) = N(0, 1)$, independent from M_1;

The N independent simulated increments of the process $\boldsymbol{Y}$ are then

$$\begin{aligned} y_1(n) &= \mu_1 \cdot w_1(n) + \sigma_1 M_1(n)\sqrt{w_1(n)} \\ y_2(n) &= \mu_2 \cdot w_2(n) + \sigma_2 M_2(n)\sqrt{w_2(n)} \end{aligned}$$

For $j = 1, 2$ the simulated trajectories are:

$$\begin{aligned} &Y_j(0) = 0, \\ &Y_j(n\Delta t) = Y_j((n-1)\Delta t) + y_j(n). \end{aligned}$$

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