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Are information and portfolio diversification substitutes or complements?*

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Abstract

Whenever a new financial product is offered by the financial industry, a rational investor faces a trade off between diversification benefits and costs of "getting to know" the newly introduced asset. In this paper the investor who can diversify can also decide either to pay a fee and separate the information on different risks affecting his asset value, or to remain uninformed and receive a non-separating signal. The uninformed investor optimally filters his pooled signal. The paper provides conditions under which diversification benefits are exploited, with or without information acquisition. We discuss lack of diversification and under-diversification and provide conditions under which each of them applies.

Keywords: Information costs, Optimal filtering, Portfolio diversification.

JEL Classification: G11, G14.

In recent years, both as a result of natural market developments and of the fears left by the Great Recession, the quest for financial products able to increase the diversification of investors' portfolios rose. New financial products were and are supplied by the financial industry. A rational investor is sometimes reluctant to buy them, since he faces costs to acquire information about them. This paper studies diversification benefits in the presence of information costs, and the optimal behavior of a rational investor facing them. We assume that the investor maximizes the long run rate of growth of wealth. The investor can decide to stick to a financial product depending on a single mean-reverting risk factor. As an alternative, he can invest also in financial product depending on two mean-reverting risk factors, after having decided whether to get perfect information on its return dynamics, or to get only partial information about it. Information disclosure will be described as in Guasoni (2006): an uninformed investor observes only the "combined" return from each risk factor, while an informed investor receives separately the two risk factors. A priori, information

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is imperfect, and the investor adopts optimal filtering to process it. However information can be made perfect (each shock entering into it can be distinguished) by paying a fee. To anticipate on our results, we show that, when the additional risk factor is less persistent than the initial one, diversification benefits are exploited. Information and diversification are complements for low information costs and substitutes otherwise. When the additional risk factor is more persistent investors do not diversify. We provide a market calibrated example of the last occurrence. We discuss lack of diversification and under-diversification and provide conditions under which each of them applies. Our results on under-diversification are more nuanced than in the literature (Nieuwerburgh and Veldkamp (2010)), because both the noise of the signal and information costs play a role in determining it.

The outline of the paper is as follows: Section 1 gives a brief overview on the previous literature. In Section 2 we describe the portfolio choice of an investor with one-factor assets only. In Section 3 we extend the model to two-factor assets, and characterize the optimal filtering of information in that case. In Section 3.1 we determine the maximum cost or fee that an investor is willing to pay to acquire financial information. At that point, each investor can choose between different scenarios: he can buy a one-factor asset or he can buy two-factor assets, being informed or not. In Section 4 we study under which conditions on information fees the investor selects one specific scenario. Section 5 studies under-diversification and lack of diversification. To show the robustness of our results in Sections 6 and 7 we introduce an additional risk factor. In Section 8 we summarize and conclude.

1. Literature Review

Two literatures are behind this paper: incomplete information in continuous time and diversification in the presence of information update. We use a model of the first type to answer a question of the second type.

Incomplete information models in continuous-time stochastic economies were pioneered by Detemple (1986), where investors do not observe the realizations of the state variables that drive the economy, Dothan and Feldman (1986) and Genotte (1986), where the parameter investors do not observe is the drift of production or dividend processes. In all of them, investors are homogeneous (they have the same prior and same information). They use Bayesian filtering in order to incorporate new information and update their beliefs. A similar construction, including unobservable drift and optimal Bayesian filtering, is proposed in the models of Detemple and Murthy (1997), Zapatero (1998) Basak (2005) and in the general equilibrium of Dumas et al. (2009). In the latter papers, each individual faces incomplete information but agents are heterogeneous in their priors or information updates. By construction, agents do not act strategically and update information on a stand-alone basis. Both the homogeneous and heterogeneous agents' models considered so far take the information update as given and concentrate on portfolio allocation.

In the sequel, we will consider a-priori-homogeneous investors which may become heterogeneous by paying a fee (information cost) and maintain optimal filtering. So, our learning mechanism is standard. What is less standard is the fact that investors can choose at the same time whether to collect information and how allocate the portfolio. They can choose a market where there is a single source of risk and no unobservability and a world where the risky asset is subject to two risks, but there is incomplete information about them, unless a fee is paid. We interpret the possibility of switching from the former to the latter market as a possibility of diversifying. And we wonder how the interaction between diversification and information acquisition – at a cost – works. Then we extend to two versus three risks.

In terms of questions, the paper closest to our is Nieuwerburgh and Veldkamp (2010). There the set-up is quite different from our: while there is still an investor maximizing the utility of returns on his wealth, there is a maximum amount of information that can be gathered. Two possible learning technologies can produce that amount. The model is static, because information is released once and only for expositional convenience has three dates: at time 1 information is acquired, at time 2 portfolio decisions are taken, at time 3 a static return realizes. Information improves the accuracy of the estimates of the return parameters. When restricted to log (or CRRA) case, the result Nieuwerburgh and Veldkamp (2010) obtain is that investors are always under-diversified. In our model investors can acquire separate information on two risk sources - instead of observing only their "combination", at a cost. And we distinguish the cases of the second risk factor being less or more persistent than the first. As explained in Section 5, in our model under-diversification applies when the added risk is less persistent and information fees are high. The under-diversification result is reversed in all other cases, namely when the added risk is still less persistent, but separate information on it is not very costly to acquire, or when it is more persistent than the initial one. This reversion does not occur in Nieuwerburgh and Veldkamp (2010). This shows that the dynamic aspect - how much information is valuable over time, as a result of the interaction of its cost and persistence - is an important feature to capture. In a static model, when the total amount of information is given, incentives to diversify are reduced. In a dynamic model, as expected, even though information may be not very valuable at a single point in time, it may become valuable over time, especially if the process on which information is acquired does not mean revert too quickly, or if it does, but acquiring information on it is not very costly. More details will be given in Section 5.

2. Investor's preferences and one-factor assets

The representative investor maximizes the long run rate of growth of wealth. He can invest in a riskless asset and participate either in a market with a risky asset subject to one risk source or factor, or in a market where a risky asset subject to two risk sources is traded. For the sake of simplicity we normalize the riskless rate to zero, so that the riskless asset is worth $B_0 \in \mathcal{R}^+$ at all times.

In the first market the risk factor follows an Ornstein-Uhlenbeck process,

$$dJ_1(t) = -\lambda_1 J_1(t)dt + dW_1(t) \tag{1}$$

where $\lambda_1 > 0$ and $W_1(t)$ is a Brownian motion. Call $\mathcal{F}(t)$ the augmented filtration generated by $W_1(t)$. On $(\Omega, \mathcal{F}, P, \mathcal{F}(t))$ the risky asset has the price dynamics

$$\frac{dM(t)}{M(t)} = (\mu - \sigma\lambda_1 J_1(t))\, dt + \sigma dW_1(t). \tag{2}$$

with $\mu > 0$, $\sigma > 0$. The idea behind the above assumption on the asset dynamics consists of having a shock dJ_1 with a mean-reverting component. The higher is the mean-reverting parameter λ_1, the less persistent will be the shock.[1]

We solve the logarithmic utility maximization problem for the investor. The investor seeks for

$$\sup_{z(t)} \left(\lim_{T\to\infty} \frac{ln(W(T))}{T} \right) \tag{3}$$

where, in the current market, $W(T) = X(T) + B_0$, $X(0) = x$, $x > 0$ and $z(t) = \pi(t)$ is the fraction of wealth invested in M at t. The budget constraint or self-financing condition is:

$$\frac{dX(t)}{X(t)} = \pi(t)\frac{dM(t)}{M(t)} \tag{4}$$

with $\dfrac{dM(t)}{M(t)}$ given by (2). The total amount invested in M at time t is

$$X(t) = x \exp\left[\int_0^t \left[\pi(s)\mu - \pi(s)\sigma\lambda_1 J_1(s) - \frac{1}{2}\pi(s)^2\sigma^2 \right] ds + \int_0^t \pi(s)\sigma dW_1(s) \right]. \tag{5}$$

It is easy to prove that:

Theorem 2.1. *The optimal strategy $\pi^*(t)$, and the asymptotic log utility $u(x)$, which solve problem* (3) *subject to* (4) *are*

$$\pi^*(t) = \frac{\mu - \sigma\lambda_1 J_1(t)}{\sigma^2}, \tag{6}$$

$$u(x) = \frac{\mu^2}{2\sigma^2} + \frac{\lambda_1}{4}. \tag{7}$$

See Appendix A for the proof.

[1] As in Summers (1986) and Poterba and Summers (1988) the temporary shock, or "fad", may represent the mispricing of the risky asset, i.e. the difference between its price and its fundamental value.

To make more concrete the above intuition, let us consider the mean reverting factor, J_1, and its long term variance, $\frac{\sigma^2}{2\lambda_1}$. It is evident that the higher is the parameter of the mean reversion the less noise will be accumulated over time and the lower will be the long term variance.

3. Two-factor assets

Suppose now that, instead of investing in the asset M, the investor can enter a new market, where a risky asset N, whose returns depend on J_1 and on another risk source J_2, is traded. Also J_2 follows a mean-reverting Ornstein-Uhlenbeck process

$$dJ_2(t) = -\lambda_2 J_2(t)dt + dW_2(t) \tag{8}$$

with $W_1(t), W_2(t)$ independent Brownian motions,

Let us define as $Y(t)$ the weighted sum of the two processes J_1 and J_2

$$Y(t) = p_1 J_1(t) + p_2 J_2(t) \tag{9}$$

It follows that N satisfies the following dynamics

$$\frac{dN(t)}{N(t)} = \mu dt + \sigma dY(t) \tag{10}$$

Assumption 3.1. *Let us assume that* $\lambda_i > 0$, $p_i > 0$, $\lambda_1 \neq \lambda_2 > 0$ *and* $\sum_{i=1}^{2} p_i^2 = 1$, $i = 1, 2$.

The investor has incomplete information because, while he knows μ and σ and the weights p_i, he does not observe separately the realizations of the processes J_i. If he decides not to acquire specific information about the dynamics of the single factors, he just observes the "pooled signal" or risk factor Y. As an alternative, the investor can decide to get full information on both risk factors and observe separately J_1 and J_2. It follows from (10) that - all others equal - the additional source of risk J_2 has a stronger effect on the uninformed investor the higher is the mean reversion parameter λ_2 associated to it. The greater is the difference between λ_2 and λ_1, with $\lambda_2 > \lambda_1$, the greater is the "gap" in terms of information between an *informed* and an *uninformed* investor. If the investor decides to stay uninformed, his filtration is $(\mathcal{F}_U(t))_{t\in[0,+\infty)}$. The filtration for the informed investor is $(\mathcal{F}_I(t))_{t\in[0,+\infty)}$. $\mathcal{F}_U(t)$ is the augmented filtration generated by $Y(t)$ alone while $\mathcal{F}_I(t)$ denotes the augmented filtration generated by $W_1(t)$ and $W_2(t)$. Further, $\mathcal{F}_U(t) \subset \mathcal{F}_I(t)$.

It is proved in Guasoni and Tolomeo (2016) that the dynamics for the *uninformed* investor are

$$\frac{dN_U(t)}{N_U(t)} = \left[\mu + \sigma\left(\tilde{J}_1(t) + \tilde{J}_2(t)\right)\right]dt + \sigma d\tilde{W}(t) \tag{11}$$

where $\tilde{J}_1(t)$, $\tilde{J}_2(t)$ are

$$\tilde{J}_1(t) = \int_0^t \hat{j}_1^N e^{-\lambda_1(t-s)} d\tilde{W}(s) \tag{12}$$

$$\tilde{J}_2(t) = \int_0^t \hat{j}_2^N e^{-\lambda_2(t-s)} d\tilde{W}(s) \tag{13}$$

where

$$\hat{j}_1^N = \frac{-\lambda_1^2 + \lambda_1\sqrt{p_1^2\lambda_2^2 + p_2^2\lambda_1^2}}{(\lambda_1 - \lambda_2)}, \qquad \hat{j}_2^N = \frac{\lambda_2^2 - \lambda_2\sqrt{p_1^2\lambda_2^2 + p_2^2\lambda_1^2}}{(\lambda_1 - \lambda_2)}$$

and $\tilde{W}$ represents the innovation process obtained from the filtering procedure.

For the *informed* investor the dynamics can be written substituting (9) in (10) as follows

$$\frac{dN_I(t)}{N_I(t)} = (\mu - p_1\sigma\lambda_1 J_1(t) - p_2\sigma\lambda_2 J_2(t))\, dt + \sigma dW_I(t) \tag{14}$$

where $W_I(t) = p_1 W_1(t) + p_2 W_2(t)$ is an $\mathcal{F}_I(t)$-Brownian motion.

3.1. Utility Maximization

As in intermediate step towards our final goal, we spell out the conditions under which the investor participates in the N-market as an *uninformed* (U) or as an *informed* (I) investor. For this reason we solve the logarithmic utility maximization problem in the two cases, namely (3) with $W = X_\kappa + B_0$, $\kappa = U, I$, and $z = \pi_\kappa$, $\kappa = U, I$. The self-financing condition for the *uninformed* investor is

$$\frac{dX_U(t)}{X_U(t)} = \pi_U(t)\frac{dN_U(t)}{N_U(t)} \tag{15}$$

The condition in (15) is different for the *informed* investor because we introduce for him an *information fee* ϕ. It stands for a cost per unit of time t that the *informed* investor pays to access to the full information on risk factors.[2] The self-financing condition for the *informed* investor is

$$\frac{dX_I(t)}{X_I(t)} = \pi_I(t)\frac{dN_I(t)}{N_I(t)} - \phi dt. \tag{16}$$

In Appendix A we prove that

Theorem 3.1. *The optimal strategies π_κ^*, $\kappa = U, I$ and the asymptotic log utilities u_i, $i = U, I$, which solve problem (3) subject to (15) for the uninformed investor, (3) subject to (16) for the*

[2] The concept of the fee that we consider in this paper refers to the "total shareholder costs" (TSCs) as in Khorana et al. (2008). Funds fees can be measured in different ways and the TSC is a more complete measure rather than a simple management fee. The TSC is equal to the "expense ratio" plus annualized loads. While a management fee includes the charges levied each year by fund for management services, the expense ratio also covers other expenses, such as investment management, administration, servicing, transfer agency, audit, legal.

informed one, exist. They are

$$\pi_U^*(t) = \frac{\mu + \sigma\left(\tilde{J}_1(t) + \tilde{J}_2(t)\right)}{\sigma^2} \tag{17}$$

$$u_U(x) = \frac{\mu^2}{2\sigma^2} + \frac{\lambda_1(1+p_2^2) + \lambda_2(1+p_1^2) - 2\sqrt{\lambda_2^2 p_1^2 + \lambda_1^2 p_2^2}}{4} \tag{18}$$

$$\pi_I^*(t) = \frac{\mu - \lambda_1 \sigma p_1 J_1(t) - \lambda_2 \sigma p_2 J_2(t)}{\sigma^2} \tag{19}$$

$$u_I(x,\phi) = \frac{\mu^2}{2\sigma^2} + \frac{p_1^2\lambda_1 + p_2^2\lambda_2}{4} - \phi. \tag{20}$$

The previous results allow us to consider under which conditions an investor participates in the market as informed or uninformed. The level of indifference fees which match $u_U(x)$ in (18) and $u_I(x,\phi)$ in (20) is

$$\phi_N^* = \frac{1}{2}\left(\sqrt{\lambda_2^2 p_1^2 + \lambda_1^2 p_2^2} - \lambda_2 p_1^2 - \lambda_1 p_2^2\right). \tag{21}$$

The level of indifference fees ϕ_N^* is also equal to $u_I(x,0) - u_U(x)$ and Lemma A.1 in Appendix A shows that it is positive. Note that ϕ_N^* does not depend on μ and σ, which are common knowledge, but only on the weights p_j and the mean reversion parameters λ_1, λ_2, which enter the risk factors J_1, J_2 and their weighted sum Y, and therefore determine the value of information. If the level of ϕ remains below the value ϕ_N^* it is better to be informed, because

$$\phi < \phi_N^* \Rightarrow u_I(x,\phi) > u_U(x). \tag{22}$$

If the level of ϕ is greater than the value ϕ_N^* it is better to be uninformed

$$\phi > \phi_N^* \Rightarrow u_I(x,\phi) < u_U(x). \tag{23}$$

4. Cost-of-information threshold effect

This Section starts by assessing which scenario the investor chooses, when he can access either the one-factor market or the two-factor one, (Theorem 4.1). It studies two specific cases, first when $\lambda_2 > \lambda_1$ then when the opposite inequality holds. Section 4.3 focuses on ϕ_N^*, interpreted as the value of information. Section 4.4 refers to a calibrated example.

The following Theorem holds:

Theorem 4.1. *Given Assumption 3.1*

a) *If $\lambda_2 > \lambda_1$, information and diversification are first complements then substitutes, because*

$$u(x) < u_U(x) < u_I(x,\phi) \qquad \textit{for } \phi < \phi_N^*, \tag{24}$$

$$u_I(x,\phi) < u_U(x) \qquad \textit{for } \phi > \phi_N^* \tag{25}$$
$$u(x) < u_U(x) \tag{26}$$

b) *If $\lambda_2 < \lambda_1$, information and diversification are neither substitutes nor complements, because*

$$u_U(x) < u_I(x,\phi) < u(x) \qquad \textit{for } \phi < \phi_N^*, \tag{27}$$

$$u_I(x,\phi) < u_U(x) < u(x) \qquad \textit{for } \phi > \phi_N^*. \tag{28}$$

The proof follows is in Appendix A; we provide an example below, separately for the cases $a)$ $\lambda_2 > \lambda_1$, $b)$ $\lambda_2 < \lambda_1$.

4.1. Case a)

Assume that $p_1 = p_2$ and the mean reversion parameters are increasing, $\lambda_2 > \lambda_1$. We show below that the standard diversification effect holds, and it is better to invest in the two-factor product than to stick to a one-factor asset.

If $\lambda_2 > \lambda_1$ from the optimal log utilities in (7), (18), (20) and the fact that the difference $u_I(x,0) - u_U(x)$ is positive at $\phi = 0$, decreasing in ϕ and null at ϕ_N^*, while the difference $u_U(x) - u(x)$ is positive for $\lambda_2 > \lambda_1$, as in Theorem 4.1, it follows that the scenarios open to the investor depend on the actual cost-of-information fees.

If the actual level of fees is lower than the level of indifference fees, there is an advantage for the investor to be informed and participate in the market that includes the second risk factor. Indeed, it is evident from (24) that the greater utility when $\phi < \phi_N^*$ is $u_I(x,\phi)$.

If the actual level of fees exceeds the threshold ϕ_N^*, the optimal choice is to buy two-factor assets and being uninformed. In this scenario the cost-of-information is too high. Consequently, the optimal utility is obtained by selecting the market with greater diversification. The investor utility is greater than the utility that he could get otherwise.

So, under optimal filtering, log utility and the assumptions on weights and mean reversion parameters introduced so far, the investor always exploits diversification (buys the two-factor asset) and at most decides not to be informed. Diversification and information are complements if $\phi < \phi_N^*$, substitutes if $\phi > \phi_N^*$, because the investor diversifies and acquires information in the first case, diversifies but does not acquire information in the second. Suppose for instance that the mean reversion parameters are $\lambda_1 = 1$ and $\lambda_2 = 2$ and the weights are equal. Given that $\sum_{i=1}^{2} p_i^2 = 1$, this means $p_1 = p_2 = \dfrac{1}{\sqrt{2}}$. We obtain the values of derived utilities and switching fees shown in

Table 1 and represented in Figure 1.[3]

[Table 1 about here.]

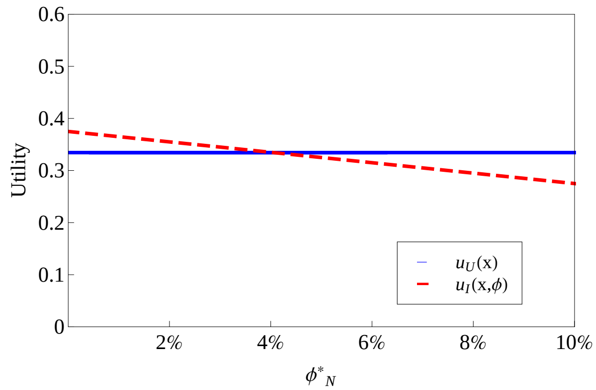

Figure 1. Utility level and level of indifference fees ϕ_N^*, given $\lambda_1 = 1$, $\lambda_2 = 2$ and equal weights p_i.

4.2. Case b)

Consider now the case in which $p_1 = p_2$ and $\lambda_2 < \lambda_1$, so that the mean reversion parameters are decreasing, from the first to the second factor, so that the second shock is more persistent than the first. As a consequence of Theorem 4.1, the optimal choice of the investor is to stay undiversified in the market with one risk factor, independently of the cost of information. Buying information on an asset which diversifies but is more persistent than the less diversified asset is of no value. Information and diversification are neither substitutes nor complements, since no one of them takes places. For instance, given the mean reversion parameters equal to $\lambda_1 = 2$, $\lambda_2 = 1$ and equal weights $p_i = \dfrac{1}{\sqrt{2}}$, the maximized utility values for this case are in Table 2 and shown in Figure 2.

[Table 2 about here.]

[3] Hereafter we will not consider in the computation the term $\dfrac{\mu^2}{2\sigma^2}$, since it is common to all utilities and does not affect the level of indifference fees.

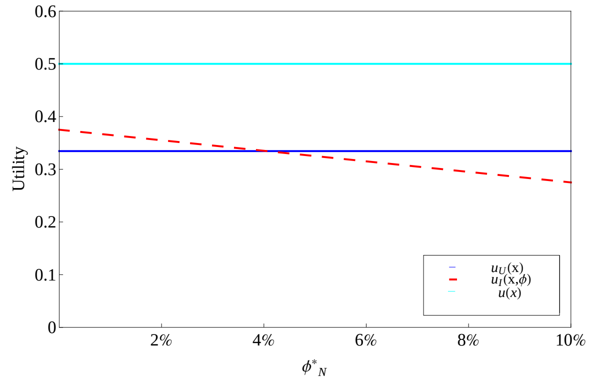

Figure 2. Utility level and level of indifference fees ϕ_N^*, given $\lambda_1 = 2$, $\lambda_2 = 1$ and equal weights p_i.

4.3. The value of information

A simple numerical analysis shows that when $\lambda_2 > \lambda_1$ namely $\lambda_i = i\lambda, i = 1, 2$, ϕ_N^*, which represents the value of information, is concave in p, with

$$\lim_{\substack{p\to 0\\ p\to 1}} \phi_N^* = 0 \tag{29}$$

and has a maximum around $p = 0.7$ ($p^2 \simeq 1/2$). Figure 3 illustrates the behavior of ϕ_N^* as a function of p and λ and shows that the maximum of ϕ_N^* is increasing in λ. An higher level of mean reversion implies that both informed and uninformed utilities, (18) and (20), increase, however the informed utility rises more than the uninformed one. This is because an higher mean reversion provides more information and less noise, that allows an investor who has access to all the information to have an advantage and benefit from any rise in mean reversion, while an uninformed investor can only benefit from the aggregate level of information. The overall effect translates in a greater level of indifference fees.[4]

[4]If we consider the opposite case, with decreasing lambdas, the result will not change since the level of information fees ϕ_N^* is symmetric in parameters.

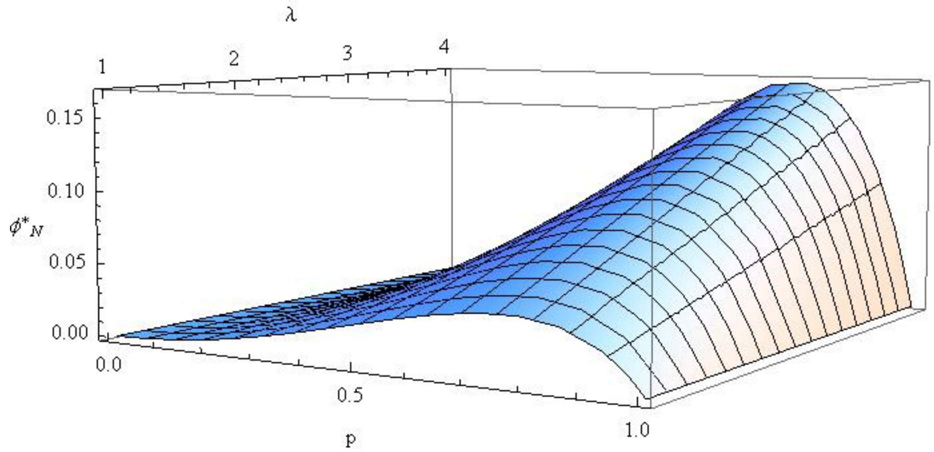

Figure 3. Level of indifference fees ϕ_N^* as a function of p and λ.

4.4. A calibrated example

As a calibrated example of the theory built so far, Luciano and Tolomeo (2016) provide a rationale for investors not to buy longevity bonds: indeed, in the presence of a standard interest-rate bond, which pays a fixed rate μ and the EURIBOR (J_1), and a longevity bond which pays, on top of that, a spread linked to the longevity of a population (J_2), if the information that investors can acquire is of the type described so far, they would better stick to the first product. This holds if the EURIBOR and the longevity index are uncorrelated. The Authors show that this is the case if the EURIBOR is calibrated to the term structure of March 2016, using an O-U process as dictated above, and the longevity is the Italian males' one, as of December 2015, as provided by the Italian National Statistics. The value of optimal fees obtained from the calibration of the model is equal to 1.4% and is consistent with the value-weighted average levels of fees as in Khorana et al. (2008).

5. Lack of diversification and under-diversification

This section introduces the concepts of *lack of diversification* and *under-diversification* and compares with the literature.

Given a benchmark case, we say that there is *lack of diversification* if in the benchmark case there is diversification while in the model set up above there is not. We say that there is *under-diversification* if the instantaneous variance of the investor's wealth - which coincides with that of the amount invested in the risky asset - is smaller in the benchmark case than in the model, when the expected return is the same.

We have two benchmark cases. The first benchmark is the case in which investors would be informed at no cost, so that they get $u_I(x, 0)$ and invest $\pi_I^*(t)$ in the risky asset. The second benchmark occurs when investors are obliged to stay uninformed but have the possibility of diversifying, so that they choose max $(u_U(x), u(x))$ and invest either $\pi_U^*(t)$ or $\pi^*(t)$. Appendix B proves the following Theorems. Based on Assumption 3.1

Theorem 5.1. *In comparison to the first benchmark*

i) *If $\lambda_2 > \lambda_1$, no lack of diversification occurs.*
ii) *If $\lambda_2 < \lambda_1$, lack of diversification occurs.*

Theorem 5.2. *In comparison to the first benchmark*

i) *If $\lambda_2 > \lambda_1$, no under-diversification occurs when $\phi < \phi_N^*$, under-diversification occurs when $\phi > \phi_N^*$.*
ii) *If $\lambda_2 < \lambda_1$, no under-diversification occurs.*

Theorem 5.3. *In comparison to the second benchmark*

i) *no lack of diversification occurs.*
ii) *no under-diversification occurs.*

So, *lack of diversification* and *under-diversification* only occur when we compare with the first benchmark. *Lack of diversification* occurs when $\lambda_2 < \lambda_1$, because the new signal is more persistent and the investor refrains from introducing it into its portfolio. *Under-diversification* occurs if the relationship between the mean reversion parameters is reversed ($\lambda_2 > \lambda_1$), so that the investor has an incentive to introduce the new risk source, and the fees are high. In that case the optimal choice at no cost (the benchmark) is $\pi_I^*(t)$ while the model portfolio choice is $\pi_U^*(t)$. It is proved in Appendix B that in this circumstance the variance of the model wealth is greater than the variance of the benchmark, while the return is the same. *Under-diversification* arises, and the model achieves an inefficient portfolio choice, because high information costs cause a lack of information. If, all others equal ($\lambda_2 > \lambda_1$), fees are low, the model portfolio strategy is efficient and no *under-diversification* occurs.

Theorem 5.3 tells us that neither *lack nor under-diversification* take place, when we compare with the second benchmark, and this happens independently of the relative magnitude of the signals' mean reversions. This is a result of the fact that, with $\lambda_2 > \lambda_1$ and low fees, the latter cause the investor's portfolio choice to be efficient. With $\lambda_2 > \lambda_1$ and high fees, as well as with $\lambda_2 < \lambda_1$, we have the same strategy in both the benchmark and the model case.

Our *under-diversification* can be compared with the "under-diversification" of Nieuwerburgh and Veldkamp (2010). Their benchmark case is the one prescribed by standard portfolio theory, which assumes that the investor knows the true distribution of returns. So their benchmark is comparable with our first benchmark. Both in that paper and in our case, we solve jointly for information and investment choices. In both cases asset returns are Gaussian and signals are independent signals.[5] To properly compare both results we need to take into account the outcome of Nieuwerburgh and Veldkamp (2010) in the presence of the logarithmic utility function. It is proved in Nieuwerburgh and Veldkamp (2010) that a model with log utility has the same properties of the CRRA model

[5] Note that in Nieuwerburgh and Veldkamp (2010) the model consists of n assets, while in our setting there is only one asset affected by one or more risk factors.

with risk aversion greater than one. It means that the investor will specialize in learning about only one asset and hold a less diversified portfolio than in the benchmark case because he devotes part of his wealth to a single asset, which has the highest Sharpe ratio. So, in Nieuwerburgh and Veldkamp (2010) a log investor is always under-diversified, because she systematically deviates from holding a diversified portfolio. We have just seen that, when we compare to the first benchmark, *under-diversification* in our model occurs only when the new risk source is less persistent and the costs of getting informed are high ($\lambda_2 > \lambda_1$, $\phi > \phi_N^*$). The intuition is that, while in Nieuwerburgh and Veldkamp (2010) the presence of a limited supply of information, together with Gaussian returns and myopic preferences, results in under-diversification, the two extra parameters in our model, namely the noise of the signal, as represented by λ, and the cost ϕ, permit to achieve a more nuanced result. *Under-diversification* occurs if one has to pay an high cost for a relatively less persistent signal, because in that case the investor stays optimally under-diversified, while we would diversify if there were no costs to afford (in the benchmark).

6. Robustness: three-factor assets

In this section we investigate the two versus three factors case. This serves as a robustness check of previous analysis. Theorem 7.1 in Section 7 will indeed state that, also for three factors, with increasing lambdas information and diversification are first complements then substitutes while, with decreasing lambdas, information and diversification are neither complements nor substitutes.

Suppose that also the third risk factor follows an Ornstein-Uhlenbeck process

$$dJ_3(t) = -\lambda_3 J_3(t)dt + dW_3(t) \tag{30}$$

where $W_3(t)$ is a Brownian motion and it is independent from $W_1(t), W_2(t)$ associated to risk factors J_1 and J_2.

We consider the process $Z(t)$ defined as:

$$Z(t) = q_1 J_1(t) + q_2 J_2(t) + q_3 J_3(t) \tag{31}$$

The risky asset dynamics evolve according to

$$\frac{dQ(t)}{Q(t)} = \mu dt + \sigma dZ(t) \tag{32}$$

with μ and σ common knowledge to informed and uninformed investors.

Assumption 6.1. *Let us assume $\lambda_j > 0$, $q_j > 0$, $\lambda_j \neq \lambda_k$ and $\sum_{j=1}^{3} q_j^2 = 1$, $j = 1, 2, 3$.*

The asset dynamics for the *uninformed* investor can be derived, in closed-form solution, following the Kalman-Bucy filter approach in Lipster and Shiryaev (2001) and Guasoni and Tolomeo (2016):

$$\frac{dQ_U(t)}{Q_U(t)} = \left[\mu + \sigma\left(\tilde{J}_1(t) + \tilde{J}_2(t) + \tilde{J}_3(t)\right)\right]dt + \sigma d\tilde{W}(t). \tag{33}$$

where the dynamics of $\tilde{J}_1(t)$, $\tilde{J}_2(t)$, $\tilde{J}_3(t)$ evolve as follows

$$\tilde{J}_1(t) = \int_0^t \frac{-\lambda_1^3 + \lambda_1^2\sqrt{\nu} - \lambda_1\sqrt{\psi}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} e^{-\lambda_1(t-s)} d\tilde{W}(s) \tag{34}$$

$$\tilde{J}_2(t) = \int_0^t \frac{-\lambda_2^3 + \lambda_2^2\sqrt{\nu} - \lambda_2\sqrt{\psi}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} e^{-\lambda_2(t-s)} d\tilde{W}(s) \tag{35}$$

$$\tilde{J}_3(t) = \int_0^t \frac{-\lambda_3^3 + \lambda_3^2\sqrt{\nu} - \lambda_3\sqrt{\psi}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} e^{-\lambda_3(t-s)} d\tilde{W}(s) \tag{36}$$

where

$$\nu = \lambda_1^2(q_2^2 + q_3^2) + \lambda_2^2(q_1^2 + q_3^2) + \lambda_3^2(q_1^2 + q_2^2) + 2\sqrt{\psi} \tag{37}$$

$$\psi = q_1^2\lambda_2^2\lambda_3^2 + q_2^2\lambda_1^2\lambda_3^2 + q_3^2\lambda_1^2\lambda_2^2. \tag{38}$$

The process $\tilde{W} = (\tilde{W}(t), \mathcal{F}_U(t))$, $0 < t < T$, is a Wiener process with respect to the investor's filtration $\mathcal{F}_U(t)$.

The asset dynamics of the informed investor can be derived substituting (31) in (32)

$$\frac{dQ_I(t)}{Q_I(t)} = [\mu - \sigma(q_1\lambda_1 J_1(t) + q_2\lambda_2 J_2(t) + q_3\lambda_3 J_3(t))]dt + \sigma dW_I(t) \tag{39}$$

where $dW_I(t) = [q_1 dW_1(t) + q_2 dW_2(t)) + q_3 dW_3(t)]$ is an $\mathcal{F}_I(t)$-Brownian motion.

6.1. Utility Maximization

In this Section we solve the logarithmic utility maximization problem for both *informed* and *uninformed* investors. Given $W = X_{\kappa D} + B_0$, $\kappa = U, I$, and $z = \pi^*_{\kappa D}$, $\kappa = U, I$, the self-financing condition for the *uninformed* investor becomes

$$\frac{dX_{UD}(t)}{X_{UD}(t)} = \pi_{UD}(t)\frac{dQ_U(t)}{Q_U(t)} \tag{40}$$

As in Section 3 we introduce for the *informed* investor an *information fee*. Therefore, his self-financing condition is

$$\frac{dX_{ID}(t)}{X_{ID}(t)} = \pi_{ID}(t)\frac{dQ_I(t)}{Q_I(t)} - \phi dt. \tag{41}$$

Theorem 6.1. *The optimal strategies $\pi_{\kappa D}$, $\kappa = U, I$ and the asymptotic log utilities, which solve* *problem* (3) *subject to* (40) *for the uninformed investor,* (3) *subject to* (41) *for the informed one exist. They are*

$$\pi^*_{UD}(t) = \frac{\mu + \sigma\left(\tilde{J}_1(t) + \tilde{J}_2(t) + \tilde{J}_3(t)\right)}{\sigma^2} \tag{42}$$

$$u_{UD}(x) = \frac{\mu^2}{2\sigma^2} + \frac{1}{4}\left[\left(1 + q_2^2 + q_3^2\right)\lambda_1 + \left(1 + q_1^2 + q_3^2\right)\lambda_2 + \left(1 + q_1^2 + q_2^2\right)\lambda_3 - 2\sqrt{\nu}\right] \tag{43}$$

$$\pi^*_{ID}(t) = \frac{\mu - \sigma(q_1\lambda_1 J_1(t) + q_2\lambda_2 J_2(t) + q_3\lambda_3 J_3(t))}{\sigma^2} \tag{44}$$

$$u_{ID}(x, \phi) = \frac{\mu^2}{2\sigma^2} + \frac{q_1^2\lambda_1 + q_2^2\lambda_2 + q_3^2\lambda_3}{4} - \phi. \tag{45}$$

The level of indifference fees which matches $u_{UD}(x)$ in (43) and $u_{ID}(x)$ in (45) is

$$\begin{aligned}\phi^*_Q = \frac{1}{2}\Bigg[&\sqrt{q_3^2\left(\lambda_1^2 + \lambda_2^2\right) + q_2^2\left(\lambda_1^2 + \lambda_3^2\right) + q_1^2\left(\lambda_2^2 + \lambda_3^2\right) + 2\sqrt{q_3^2\lambda_1^2\lambda_2^2 + q_2^2\lambda_1^2\lambda_3^2 + q_1^2\lambda_2^2\lambda_3^2}} \\ &- \left(q_2^2 + q_3^2\right)\lambda_1 - \left(q_1^2 + q_3^2\right)\lambda_2 - \left(q_1^2 + q_2^2\right)\lambda_3\Bigg].\end{aligned} \tag{46}$$

Note that ϕ^*_Q does not depend on μ and σ, but only on weights p_j and the mean reversion parameters λ_1, λ_2, λ_3.

If the level of ϕ remains below the value ϕ^*_Q it is better to be informed, because

$$\phi < \phi^*_Q \Rightarrow u_{ID}(x, \phi) > u_{UD}(x). \tag{47}$$

If the level of ϕ is greater than the value ϕ^* it is better to be uninformed.

$$\phi > \phi^*_Q \Rightarrow u_{ID}(x, \phi) < u_{UD}(x). \tag{48}$$

7. Cost-of-information threshold effect with three factors

The following Theorem holds:

Theorem 7.1. *Given Assumption 6.1*

a) *If $\lambda_j = j\lambda$, information and diversification are first complements then substitutes, because*

$$u_U(x) < u_I(x,\phi) < u_{UD}(x) < u_{ID}(x,\phi) \qquad for\ \phi < \phi^*_N, \tag{49}$$

$$u_I(x,\phi) < u_U(x) < u_{UD}(x) < u_{ID}(x,\phi) \qquad for\ \phi^*_N < \phi < \phi^*_Q, \tag{50}$$

$$u_{ID}(x,\phi) < u_{UD}(x) \qquad for\ \phi > \phi^*_Q \tag{51}$$

$$u_I(x,\phi) < u_U(x) < u_{UD}(x) \qquad for\ \phi > \phi^*_Q. \tag{52}$$

b) *If $\lambda_j = \lambda/j$, information and diversification are neither complements nor substitutes, because*

$$u_{UD}(x) < u_{ID}(x,\phi) < u_U(x) < u_I(x,\phi) \qquad for\ \phi < \phi^*_N, \tag{53}$$

$$u_{UD}(x) < u_{ID}(x,\phi) < u_I(x,\phi) < u_U(x) \qquad for\ \phi^*_N < \phi < \phi^*_Q, \tag{54}$$

$$u_I(x,\phi) < u_U(x) \qquad for\ \phi > \phi^*_Q \tag{55}$$

$$u_{ID}(x,\phi) < u_{UD}(x) < u_U(x) \qquad for\ \phi > \phi^*_Q. \tag{56}$$

The proof is in Appendix A; we provide an example below, separately for the cases *a*) $\lambda_j = j\lambda$, *b*) $\lambda_j = \lambda/j$.

7.1. Case a)

Assume that the mean reversion parameters are in the following arithmetic sequence $\lambda_j = j\lambda$, $j = 1, 2, 3$.

The costs-of-information ϕ^*_N in (21), when the model consists of two factors only, and ϕ^*_Q in (46), when the third factor is added, satisfy the following inequality

$$\phi^*_N < \phi^*_Q. \tag{57}$$

It follows from Theorem 7.1 that the scenarios open to the investor depend on the actual cost-of-information fees:

1. If the actual level of fees is lower than the level of indifference fees obtained from the model with two risk factors, there is an advantage for the investor to be informed and participates in the market that includes the third risk factor. By so doing he will get an optimal log utility, net of information costs, greater than all other alternatives. Indeed, it is evident from (49) that the greater utility is $u_{ID}(x,\phi)$.
2. When the actual level of fees stays between the two thresholds, ϕ^*_N and ϕ^*_Q, the optimal decision for each investor is to pay such commission and invest in the market with three-

factor assets since (50). This choice will still return to the investor an optimal log utility that will be greater than all other ones.

3. If the actual level of fees exceeds the threshold ϕ_Q^*, the optimal choice is to buy three-factor assets, being uninformed. In this scenario the cost-of-information is too high in both markets, with or without the third risk factor. Consequently, the optimal utility is obtained by selecting the market with greater diversification (without paying any cost of information). The investor's utility is greater than the utility that he could get otherwise, see (52).

So, in case a), the investor always exploits diversification (buys the three-factor asset) and at most decides not to be informed. Information and diversification are complements if $\phi < \phi_Q^*$ and substitutes if $\phi > \phi_Q^*$.

If the weights q_j are the same across factors (be them two or three), we obtain the values of derived utilities and switching fees shown in Table 3.

[Table 3 about here.]

7.2. Case b)

We now examine the impact of having decreasing mean reversion parameters on utilities, among the two and three-factor assets. Suppose that $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = .5$. From Theorem 7.1 it follows that the scenarios we are going to explore are the following:

1. If the actual level of ϕ is less than the threshold ϕ_N^*, the optimal choice for the investor is to acquire information, remaining undiversified in the two-factor market, $u_I(x, \phi)$, since (53).
2. If the actual level of fees is greater than the threshold ϕ_N^*, the cost-of-information is too high and it is better to buy the two-factor asset, being uninformed, $u_U(x)$, since (54).
   Table 4 summarizes the previous results.

[Table 4 about here.]

So, also case b) is robust in comparison to his one - to - two counterpart, since no diversification is optimal in both. When starting with two risk sources there is still room for information, while a non-diversified investor with one source of risk, by definition, did not need information, by definition.

8. Summary and Conclusions

In this paper we investigate diversification benefits in a market with information costs, and the optimal behavior of a rational investor facing them and choosing at the same time how much information to acquire and how to invest. The investor maximizes the long run rate of growth of wealth. When information is imperfect the investor adopts optimal filtering to process it. To get perfect information (each shock entering into asset returns can be distinguished) the investor has to pay a fee. Two cases are relevant: the first occurs when the shocks on the additional factor are less persistent than the initial ones, the second otherwise. In the first case diversification benefits are exploited. The investor acquires information when fees are low and remains uninformed when information fees are too high, so that information and diversification are first complements and then substitutes. No diversification occurs in the second case. The extension to three risk factors shows that the results are robust.

Lack of diversification occurs only in the second case. If the benchmark is being informed at no cost, *under-diversification* occurs only in the first case above, when the model implements an inefficient portfolio choice, because of high information costs on a relatively low-persistent risk. If the benchmark is the best portfolio strategy when information on the second factor is not available, no *under-diversification* occurs.

Appendix A

The following proof of Theorem 3.1 is standard, is reported also in Scolozzi and Tolomeo (2015), and included only for the sake of completeness. We can use the same method in order to prove that investors with one factor (6), informed investors with two factors (19), informed and uninformed investors with three factors, (44) and (42), are the unique optimal strategies for the logarithmic maximization problem.

Proof of Theorem 3.1. To prove that $\pi_U^*(t) = \dfrac{\mu + \sigma\left(\tilde{J}_1(t) + \tilde{J}_2(t)\right)}{\sigma^2}$ is the unique optimal strategy for the logarithmic utility maximization problem in the case of uninformed investors, we consider two portfolios $X^\pi(t)$ and $X^\rho(t)$. Their dynamics are

$$\begin{aligned}\frac{dX^\pi(t)}{X^\pi(t)} &= \pi(t)\left(\mu + \sigma\left(\tilde{J}_1(t) + \tilde{J}_2(t)\right)\right)dt + \sigma\pi(t)d\tilde{W}(t),\\ \frac{dX^\rho(t)}{X^\rho(t)} &= \rho(t)\left(\mu + \sigma\left(\tilde{J}_1(t) + \tilde{J}_2(t)\right)\right)dt + \sigma\rho(t)d\tilde{W}(t).\end{aligned}$$

Then, we observe first that

1. $\forall\, \pi\ \exists\, \bar{\rho} = \dfrac{\mu + \sigma\left(\tilde{J}_1(t) + \tilde{J}_2(t)\right)}{\sigma^2} : \dfrac{X^\pi(t)}{X^{\bar{\rho}(t)}} = Z^\pi(t)$ is a supermartingale.

This follows from

- *Itô's* formula on the dynamics $\dfrac{dZ^\pi(t)}{Z^\pi(t)}$. Setting the dt term equal to 0 yields that $\dfrac{X(t)^\pi}{X(t)^\rho}$ is a local martingale $\forall\, \pi$, iff $\rho(t)$ is equal to:

$$\rho(t) = \frac{\mu + \sigma\left(\tilde{J}_1(t) + \tilde{J}_2(t)\right)}{\sigma^2}. \tag{58}$$

- The fact that $\dfrac{X(t)^\pi}{X(t)^{\bar{\rho}}}$ is also positive.
- The fact that A positive local martingale is a supermartingale (Karatzas and Shreve, 1991, 1, 5.19, p. 36).

Because $Z^\pi(t)$ is a supermartingale, by applying Fatou's Lemma and Jensen's inequality, we can state that

$$E\left[\frac{X^\pi(t)}{X^{\bar{\rho}}(t)}\right] \leq E\left[\frac{X^\pi(0)}{X^{\bar{\rho}}(0)}\right] = 1 \tag{59}$$

Consequently we obtain

$$E\log X(t)^\pi \leq E\log X^{\bar{\rho}}(t) \qquad \forall\, \pi. \tag{60}$$

This inequality is consistent with the statement that $\rho(t)$ is optimal for log-utility. $\square$

The following Lemmas are needed in the proof of Theorem 4.1

Lemma A.1. *Given Assumption 3.1 then* $u_I(x,\phi) - u_U(x) > 0 \ \forall \phi < \phi_N^*$.

Proof.

$$u_I(x,0) - u_U(x) = p_1^2 p_2^2 (\lambda_2 - \lambda_1)^2 > 0.$$

Then

$$u_I(x,\phi) - u_U(x) > 0 \qquad\qquad \forall \phi < \phi_N^*.$$

□

Lemma A.2. *Given Assumption 3.1 then* $u_U(x) > u(x) \iff \lambda_2 > \lambda_1$.

Proof. The statement follows from $u(x)$ in (7) and $u_U(x)$ in (18), by which $u_U(x) - u(x) = (\lambda_2 - \lambda_1)(3\lambda_1 + \lambda_1 p_1^2 + \lambda_2 p_2^2)p_2^2$. □

Lemma A.3. *Given Assumption 3.1 then* $u_I(x,0) > u(x) \iff \lambda_2 > \lambda_1$.

Proof. The statement follows from $u(x)$ in (7) and $u_I(x,0)$ in (20), by which $u_I(x,0) - u(x) = p_2^2(\lambda_2 - \lambda_1)$. □

Proof of Theorem 4.1. Let us consider all inequalities in Theorem 4.1

- The first inequality in (24) follows from Lemma A.2 while the second inequality follows from Lemma A.1 since the actual fees are lower than ϕ_N^*.
- The inequality in (25) follows from Lemma A.1 since the actual level of fees is higher than ϕ_N^* and then $u_I(x,\phi) > u_U(x)$, while the inequality in (26) follows from Lemma A.2 because $\lambda_2 > \lambda_1$.
- The first inequality in (27) follows from Lemma A.1 since the actual level of fees is lower than ϕ_N^*. Lemma A.3 proves that $u(x) > u_I(x,0) > u_I(x,\phi)$ therefore the second inequality in (27) holds.
- The first inequality in (28) follows from Lemma A.1 because the actual level of fees is greater than ϕ_N^*. The second one follows from Lemma A.2.

□

The following Lemmas are needed in the proof of Theorem 7.1.

Lemma A.4. *If* $\lambda_j = j\lambda$ *or* $\lambda_j = \lambda/j$, $\lambda > 0$, $q_j = \frac{1}{\sqrt{3}}$, *then* $u_{ID}(x,\phi) - u_{UD}(x) > 0 \ \forall \phi < \phi_Q^*$.

Proof. The statement follows from $u_{ID}(x,\phi)$ in (45) and $u_{UD}(x)$ in (43), since $u_{ID}(x,0) - u_{UD}(x) = \phi_Q^* > 0$, $u_{ID}(x,\phi) - u_{UD}(x) > 0 \ \forall \phi < \phi_Q^*$ by the definition of ϕ_Q^*.

□

For the next four Lemmas we assume equal weights within each case. Since $\sum_{i=1}^{2} p_i^2 = 1$ and $\sum_{j=1}^{3} q_j^2 = 1$ then $p_i = \frac{1}{\sqrt{2}}$ and $q_j = \frac{1}{\sqrt{3}}$.

Lemma A.5. *If the actual level of fees is below ϕ_N^**

- *If $\lambda_j = j\lambda$, $\lambda > 0$ then*

$$u_{ID}(x, \phi) - u_I(x, \phi) = \frac{\lambda}{2} > 0 \qquad \forall\lambda > 0. \tag{61}$$

- *If $\lambda_j = \lambda/j$, $\lambda > 0$ then*

$$u_{ID}(x, \phi) - u_I(x, \phi) = -\frac{5}{36}\lambda < 0 \qquad \forall\lambda > 0. \tag{62}$$

Proof. The statements follow from $u_{ID}(x, \phi)$ in (45) and $u_I(x, \phi)$ in (20). □

Lemma A.6. *If the actual level of fees is $\phi_N^* < \phi < \phi_Q^*$*

- *If $\lambda_j = j\lambda$, $\lambda > 0$ then*

$$u_{ID}(x, \phi) - u_U(x) > 0 \qquad \forall\lambda > 0. \tag{63}$$

- *If $\lambda_j = \lambda/j$, $\lambda > 0$ then*

$$u_{ID}(x, \phi) - u_U(x) < 0 \qquad \forall\lambda > 0. \tag{64}$$

Proof. The statements follow from $u_{ID}(x, \phi)$ in (45) and $u_U(x)$ in (18). To obtain (63) we have that

$$u_{ID}(x, \phi) - u_U(x) = \frac{1}{8}\left((-5 + 2\sqrt{10})\lambda - 8\phi\right). \tag{65}$$

Since the previous difference is decreasing in the level of information fees we substitute in (65) the upper bound of the range of ϕ, ϕ_Q^*, and get 0.138λ. Because the difference is positive for $\phi = \phi_Q^*$, it is positive for any $\phi_N^* < \phi < \phi_Q^*$.

To obtain (64) we have that

$$u_{ID}(x, \phi) - u_U(x) = \lambda\left(-\frac{59}{144} + \frac{1}{4}\sqrt{\frac{5}{2}}\right) - \phi. \tag{66}$$

Since the previous difference is decreasing in the level of information fees we substitute in (66) the lower bound of the range of ϕ, ϕ_N^*, and get -0.055λ. Because the difference is negative for $\phi = \phi_N^*$, it is negative for any $\phi_N^* < \phi < \phi_Q^*$.

□

Lemma A.7.

- *If* $\lambda_j = j\lambda$, $\lambda > 0$ *then*

$$u_{UD}(x) - u_U(x) = \lambda \left(11/2 + \sqrt{10} - 2\sqrt{14/3}\sqrt{2+\sqrt{3}} \right) > 0 \qquad \forall \lambda > 0. \tag{67}$$

- *If* $\lambda_j = \lambda/j$, $\lambda > 0$ *then*

$$u_{UD}(x) - u_U(x) = \lambda \left(29 + 18\sqrt{10} - 4\sqrt{6}\sqrt{49 + 6\sqrt{42}} \right) < 0 \qquad \forall \lambda > 0. \tag{68}$$

Proof. The statements follow from $u_{UD}(x)$ in (43) and $u_U(x)$ in (18). □

Lemma A.8. *If* $\lambda_j = j\lambda$, $\lambda > 0$, *then*

$$u_I(x,\phi) - u_{UD}(x) = -1.43\lambda - \phi < 0 \qquad \forall \lambda > 0. \tag{69}$$

Proof. The statement follows from $u_I(x,\phi)$ in (20) and $u_{UD}(x)$ in (43). In (69) we have

$$u_I(x,\phi) - u_{UD}(x) = \frac{83\lambda}{12} - 2\lambda\sqrt{\frac{14}{3}}\sqrt{2+\sqrt{3}} - \phi < 0 \qquad \forall \lambda > 0. \tag{70}$$

Since the previous equation is decreasing in the level of fees, it is negative for all positive level of fees. Therefore (69) is proved. □

Proof of Theorem 7.1. Let us consider all inequalities in Theorem 7.1, case a)

- The first inequality in (49) follows from Lemma A.1 since the actual fee is lower than ϕ^*_N and therefore $u_U(x) < u_I(x,\phi)$. The second inequality follows from Lemma A.8 since the difference $u_I(x,\phi) - u_{UD}(x)$ is always negative for all values of fees and $\lambda_j = j\lambda$. Last inequality follows from Lemma A.4 because $u_{UD}(x) < u_{ID}(x,\phi)$ $\forall \phi < \phi^*_Q$. Since it can be easily shown that $\phi^*_N < \phi^*_Q$ then the inequality in (49) is proved.
- The first inequality in (50) follows from Lemma A.1. Lemma A.7 proves that the second inequality holds if $\lambda_j = j\lambda$. Last inequality follows from Lemma A.4.
- The inequality in (51) follows from Lemma A.4. The first inequality in (52) follows from Lemma A.4 and from the fact that it can be easily shown that $\phi^*_N < \phi^*_Q$. Last inequality in (52) follows from Lemma A.7.

Let us consider now case b)

- With $\lambda_j = \lambda/j$and fees below ϕ^*_N, the first inequality and the third one in (53) follows from Lemmas A.4 and A.1. Lemma A.6 proves that the second inequality holds because the result in (66) is negative also for $\phi = 0$.

- The first inequality in (54) follows from Lemmas A.4. The second one follows from the second point in Lemma A.5 and Lemma A.1 proves that the third inequality holds when the level of fees is above ϕ_N^*.
- The inequality in (55) follows from Lemmas A.1. The first inequality in (56) follows from Lemma A.4 and the last form Lemma A.7.

□

Appendix B

The instantaneous expected return and variance of $\pi^*(t)$ will be done as $\mathbb{E}(\pi^*(t))$ and $var(\pi^*(t))$

Lemma B.1. *Given Assumption 3.1*

$$\lim_{t\to+\infty} \mathbb{E}(\pi^*(t)) = \mathbb{E}(\pi_I(t)) = \mathbb{E}(\pi_U(t)) = \frac{\mu}{\sigma^2}, \tag{71}$$

$$\lim_{t\to+\infty} var(\pi^*(t)) = \frac{\lambda_1}{2\sigma^2}, \tag{72}$$

$$\lim_{t\to+\infty} var(\pi_I^*(t)) = \frac{\lambda_1+\lambda_2}{2\sigma^2}, \tag{73}$$

$$\lim_{t\to+\infty} var(\pi_U^*(t)) = \frac{(\hat{j}_1^N)^2}{\sigma^2}\frac{1}{2\lambda_1} + \frac{(\hat{j}_2^N)^2}{\sigma^2}\frac{1}{2\lambda_2} + 2\hat{j}_1^N\hat{j}_2^N\frac{1}{(\lambda_1+\lambda_2)}. \tag{74}$$

Proof. Given the optimal strategies $\pi^*(t) = \dfrac{\mu - \sigma\lambda_1 J_1(t)}{\sigma^2}$, $\pi_I^*(t) = \dfrac{\mu - \lambda_1\sigma p_1 J_1(t) - \lambda_2\sigma p_2 J_2(t)}{\sigma^2}$ and $\pi_U^*(t) = \dfrac{\mu + \sigma\left(\tilde{J}_1(t) + \tilde{J}_2(t)\right)}{\sigma^2}$, the expected values are always equal to $\dfrac{\mu}{\sigma^2}$, while the variance is

$$var(\pi^*(t)) = \mathbb{E}\left[\int_0^t -\frac{\lambda_1}{\sigma^2}e^{-\lambda_1(t-s)}dW_1(s)\right]^2, \tag{75}$$

$$var(\pi_I^*(t)) = \mathbb{E}\left[\int_0^t -\frac{\lambda_1}{\sigma^2}e^{-\lambda_1(t-s)}dW_1(s)\right]^2 + \mathbb{E}\left[\int_0^t -\frac{\lambda_2}{\sigma^2}e^{-\lambda_2(t-s)}dW_2(s)\right]^2, \tag{76}$$

$$\begin{aligned} var(\pi_U^*(t)) &= \mathbb{E}\left[\int_0^t \hat{j}_1^N e^{-\lambda_1(t-s)}d\tilde{W}(s)\right]^2 + \mathbb{E}\left[\int_0^t \hat{j}_2^N e^{-\lambda_2(t-s)}d\tilde{W}(s)\right]^2 \\ &\quad + 2\mathbb{E}\left[\int_0^t \hat{j}_1^N e^{-\lambda_1(t-s)}d\tilde{W}(s)\int_0^t \hat{j}_2^N e^{-\lambda_2(t-s)}d\tilde{W}(s)\right] \end{aligned} \tag{77}$$

where $\hat{j}_1^N = \frac{-\lambda_1^2+\lambda_1\sqrt{p_1^2\lambda_2^2+p_2^2\lambda_1^2}}{(\lambda_1-\lambda_2)}$, $\hat{j}_2^N = \frac{\lambda_2^2-\lambda_2\sqrt{p_1^2\lambda_2^2+p_2^2\lambda_1^2}}{(\lambda_1-\lambda_2)}$ and its limits are as in (72), (73) and (74). □

Proof of Theorem 5.1. Based on Theorems 3.1 and 4.1, in case i) of Theorem 5.1, in the model the optimal strategy is $\pi_I^*(t)$ if $\phi < \phi_N^*$, $\pi_U^*(t)$ if $\phi > \phi_N^*$ while in the benchmark the optimal strategy is $\pi_I^*(t)$. No *lack of diversification* occurs because both $\pi_I^*(t)$ and $\pi_U^*(t)$ entail investing in N, the two-factor asset. In in case ii) of Theorem 5.1, the optimal strategy in the model is $\pi^*(t)$ and in the benchmark $\pi_I^*(t)$ and *lack of diversification* occurs since in the benchmark there is diversification while in the model there is not. □

Proof of Theorem 5.2. Based on Theorems 3.1 and 4.1, in case i), if $\phi < \phi_N^*$, the investor chooses $\pi_I^*(t)$ in both the benchmark and the model. If $\phi < \phi_N^*$ the investor chooses $\pi_I^*(t)$ in the benchmark and $\pi_U^*(t)$ in the model. In case ii), the investor chooses $\pi_I^*(t)$ in the benchmark and $\pi^*(t)$ in the model. It follows from Lemma B.1 that the statements of the Theorem follow. □

Proof of Theorem 5.3. Based on Theorems 3.1 and 4.1, when $\lambda_2 > \lambda_1$, in the model the optimal strategy is $\pi_I^*(t)$ if $\phi < \phi_N^*$, $\pi_U^*(t)$ if $\phi > \phi_N^*$ while in the benchmark the optimal strategy is $\pi_U^*(t)$. No *lack of diversification* occurs because both $\pi_I^*(t)$ and $\pi_U^*(t)$ entail investing in N, the two-factor asset. When $\lambda_2 < \lambda_1$ no *lack of diversification* applies because in both the benchmark and the model the optimal strategy is $\pi^*(t)$.

The statement in case ii) of the Theorem follows from Lemma B.1. □

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Table 1: Case a) - Comparing Log Utility

	Asymptotic Log Utility			
$\mathbf{p_i}$ **equal**	$\mathbf{u(x)}$	$\mathbf{u_U(x)}$	$\mathbf{u_I(x,0)}$	$\phi^*_{\mathbf{N}}$
	.25	.334	.375	.041

Table 2: Case b) - Comparing Log Utility

	Asymptotic Log Utility			
$\mathbf{p_i}$ **equal**	$\mathbf{u(x)}$	$\mathbf{u_U(x)}$	$\mathbf{u_I(x,0)}$	$\phi^*_{\mathbf{N}}$
	.5	.334	.375	.041

Table 3: Case a) - Comparing Log Utility

	Asymptotic Log Utility			
$\mathbf{q_j}$ **equal**	$\mathbf{u_U(x)}$	$\mathbf{u_I(x,0)}$	$\mathbf{u_{UD}(x)}$	$\mathbf{u_{ID}(x,0)}$
	.334	.375	.413	.50
	$\mathbf{u_I(x,\phi) - u_U(x)}$ $\phi^*_{\mathbf{N}}$	$\mathbf{u_{ID}(x,\phi) - u_{UD}(x)}$ $\phi^*_{\mathbf{Q}}$		
	.041	.087		

Table 4: Case b) - Comparing Log Utility (two and three factor assets)

	Asymptotic Log Utility			
$\mathbf{q_j}$ **equal**	$\mathbf{u_U(x)}$	$\mathbf{u_I(x,0)}$	$\mathbf{u_{UD}(x)}$	$\mathbf{u_{ID}(x,0)}$
	.334	.375	.219	.292
	$\mathbf{u_I(x,\phi) - u_U(x)}$ $\phi^*_{\mathbf{N}}$	$\mathbf{u_{ID}(x,\phi) - u_{UD}(x)}$ $\phi^*_{\mathbf{Q}}$		
	.041	.073		