THE CARLO ALBERTO NOTEBOOKS

Fondazione Collegio CarloAlberto

# On Concavity and Supermodularity

Massimo Marinacci
Luigi Montrucchio

No.5, May 2006
www.collegiocarloalberto.it

# On Concavity and Supermodularity

Massimo Marinacci and Luigi Montrucchio[1]

Dipartimento di Statistica e Matematica Applicata and

Fondazione Collegio Carlo Alberto

Università di Torino

Draft of April 2006

[1]We thank an ananymous referees for helpful discussions and suggestions. We also gratefully acknowledge the financial support of the Ministero dell'Istruzione, dell'Università e della Ricerca.

**Abstract**

Concavity and supermodularity are in general independent properties. A class of functionals defined on a lattice cone of a Riesz space has the Choquet property when it is the case that its members are concave whenever they are supermodular. We show that for some important Riesz spaces both the class of positively homogeneous functionals and the class of translation invariant functionals have the Choquet property. We extend in this way the results of Choquet [2] and Konig [5].

*JEL classification:* C60

*Keywords:* Concavity, Supermodularity

# 1 Introduction

Let $E_+ = \{x \in E : x \geq 0\}$ be the positive convex cone of a Riesz space $E$. In this paper we consider functionals $I : E_+ \to \mathbb{R}$ defined on $E_+$ and we study the relations among two classic properties they may have, that is, concavity and supermodularity.

In general, these two properties are altogether independent, there are concave functionals that are not supermodular, as well as supermodular functionals that are not concave. However, in a classic article Choquet [2, Thm 54.1] claimed that supermodularity implies concavity for the important class of positively homogeneous functionals defined on ordered vector spaces in which only $E_+$ is required to be a lattice in its order. Unfortunately, his proof of this remarkable claim considered only the special case $E = \mathbb{R}^n$ with coordinate-wise order, and even for this special case his argument was incomplete and hence his claim remained open.

It turned out that Choquet's claim is true in the special case $E = \mathbb{R}^n$ with coordinate-wise order, but beyond that it need not be true even for finite dimensional Riesz spaces. In fact, after a half century König [5] on the one hand disproved the assertion by the example $E = \mathbb{R}^2$ with lexicographic order, thus in the present context by the simplest non-Archimedean Riesz space. On the other hand, he proved rigorously the Choquet's assertion in the case $E = \mathbb{R}^n$ with coordinate-wise order (under a certain moderate additional assumption, which now with our present Theorem 3 will be shown to be superfluous).[1]

Our purpose in this paper is to study to what extent Choquet's claim holds in general Riesz spaces. Our first main result, Theorem 8, fully characterizes the Riesz spaces for which Choquet's assertion holds, when no other additional assumptions on the functionals is made besides positive homogeneity. It turns out that this is the well know class of Riesz spaces that have Archimedean quotient spaces, often called hyper-Archimedean spaces.

Though hyper-Archimedean spaces are relatively few, fortunately they are dense in many other Riesz spaces. Hence, by imposing a continuity condition on the functionals, in Section 5 we show how Choquet's claim holds in a large number of Riesz spaces. In Section 6 we actually show that for some important classes of Riesz spaces Choquet's claim holds more generally for upper semicontinuous functionals.

Besides studying the validity of Choquet's assertion in general Riesz spaces, in Section 7 we show that supermodularity implies concavity also for the important class of translation invariant functionals, that is, functionals $I : E \to \mathbb{R}$ such that $I(x + \alpha e) = I(x) + \alpha I(e)$ for all $x \in E$ and $\alpha \in \mathbb{R}$, where $e$ is an order unit of $E$. In this way we provide a new important class of functionals that have the remarkable property that Choquet envisaged for positively homogeneous functionals.

Interestingly, positive homogeneity and translation invariance are the two main

[1] The further contents of [5] go in a direction different from ours, motivated by his specific goals.

properties enjoyed by Choquet integrals, the class of functionals in which Choquet [2] was mostly interested in.[2] As a result, Choquet integrals turn out to be only a quite special class of functionals for which the property postulated by Choquet holds.

## 2 Preliminaries

We follow [8] for notation and terminology on Riesz spaces. Given a Riesz space $E$ (i.e., a vector lattice), we denote by $E_+$ its positive cone $\{x \in E : x \geq 0\}$. A vector subspace $L$ of $E$ is a *Riesz subspace* if $u, v \in L$ implies $u \wedge v \in L$; $E[u, v]$ denotes the Riesz subspace generated by two elements $u, v \in E$. Two elements $u, v \in E$ are disjoint, written $u \perp v$, if $|u| \wedge |v| = 0$. Given a subset $M \subseteq E$, $M^{\perp}$ denotes the set $\{u \in E : u \perp x \text{ for all } x \in M\}$.

A vector subspace $J$ is called an *ideal* if $|u| \leq v$ and $v \in J_+$ implies $u \in J$. The symbol $J_u$ denotes the ideal generated by $u$. An ideal $J$ is a *principal ideal* if $J = J_u$ for some $u$. An element $e \in E_+$ is said to be an *order unit* if $J_e = E$. An ideal $P$ is *prime* if $u \wedge v = 0$ implies that either $u$ or $v$ belongs to $P$.

A Riesz space is *Archimedean* if $nu \leq v$ for $u \geq 0$ and all the integers $n$ implies $u = 0$. Given an ideal $J$ of $E$, the vector quotient space $E/J$ has a natural structure of Riesz space. Observe that, in general, $E/J$ may fail to be Archimedean, even if $E$ is Archimedean.

A *band* $B$ is an ideal such that $u \in B$, provided $0 \leq u_\alpha \uparrow u$ and $\{u_\alpha\} \subseteq B$. A band $B$ is a *principal band* if there exists $u \in B$ such that $B$ is the smallest band containing $u$. In this case, we write $B_u$. A band $B$ is a *projection band* if there exists a linear projection $P : E \to B$ such that $0 \leq Px \leq x$ for all $x \in E_+$. Equivalently, a band $B$ is a projection band if $E = B \oplus B^{\perp}$. A Riesz space $E$ is said to have *the principal projection property* if any principal band is a projection band (see [8, Ch. 4]).

A linear map $T : E \to F$ between the two Riesz spaces $E$ and $F$ is a *Riesz homomorphism* if it preserves the lattice operations. When it is one-to-one, $T$ is a *Riesz isomorphism* and the two spaces are called Riesz isomorphic.

A linear topology $\tau$ on a Riesz space is compatible if the lattice operations are continuous with respect to $\tau$ (for comprehensive study of the so-called Riesz locally solid topologies we refer to [1]). A *Riesz normed space* (or a normed lattice) is a Riesz space equipped with a norm $\|.\|$ such that $|u| \leq |v|$ implies $\|u\| \leq \|v\|$. When the space is norm complete, it is called a *Banach lattice*.

A Riesz normed space is an $M$ *space* if $\|x \vee y\| = \|x\| \vee \|y\|$ for all $x, y \in E_+$, while it is an $L$ *space* if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in E_+$. When $E$ is a Banach lattice, they are called $AM$ and $AL$ spaces, respectively.

[2] See, e.g., [10] for a detailed study of the properties of Choquet integrals.

Let $C$ be either $E_+$ or $E$. A functional $I : C \to \mathbb{R}$ is

1. *concave* if $I(tx + (1-t)y) \geq tI(x) + (1-t)I(y)$ for all $t \in [0,1]$ and all $x, y \in C$,

2. *supermodular* if $I(x \vee y) + I(x \wedge y) \geq I(x) + I(y)$ for all $x, y \in C$,

3. *positively homogeneous* if $I(\alpha x) = \alpha I(x)$ for all $\alpha \geq 0$ and all $x \in C$,

4. *superadditive* if $I(x + y) \geq I(x) + I(y)$ for all $x, y \in C$,

5. *translation invariant* (or additively homogeneous) if $I(x + \alpha e) = I(x) + \alpha I(e)$ for all $\alpha \geq 0$ and all $x \in C$, where $e$ is an order unit of $E$.

Observe that a functional $I : E \to \mathbb{R}$ is translation invariant if and only if $I(x + \alpha e) = I(x) + \alpha I(e)$ for all $\alpha \in \mathbb{R}$ and all $x \in E$. For, given $\alpha < 0$,

$$\begin{aligned} I(x) + \alpha I(e) &= I(x + \alpha e - \alpha e) + \alpha I(e) \\ &= I(x + \alpha e) - \alpha I(e) + \alpha I(e) = I(x + \alpha e). \end{aligned}$$

The next lemma, whose routine proof is omitted, gives another simple property of translation invariant functionals.

**Lemma 1** *Every translation invariant functional $I : E_+ \to \mathbb{R}$ has a unique translation invariant extension on the entire space $E$. Moreover, If $I$ is supermodular, then the extension is supermodular, and if $I$ is concave, then the extension is concave.*

Next we give a key definition for our purposes.

**Definition 2** *A class of functionals $I : C \to \mathbb{R}$ has the Choquet property if its members are concave whenever they are supermodular.*

In the paper we will consider the class of positively homogeneous functionals and the class of translation invariant functionals, and for them we will study the validity of the Choquet property. For brevity, we will say that positively homogeneous (or translation invariant) functionals have the Choquet property instead of saying that the class of such functionals has the Choquet property.

Observe that for positively homogeneous functionals concavity and superadditivity are equivalent properties, and so for this case Definition 2 can be equivalently stated in terms of supermodularity and superadditivity.

# 3 The $\mathbb{R}^n$ Case

The starting point of our study is the following theorem for the $\mathbb{R}^n$ case,[3] a slight improvement of König's [5] main result that will turn out to be very useful for our purposes.

**Theorem 3** *The positively homogeneous functionals $I : \mathbb{R}^n_+ \to \mathbb{R}$ have the Choquet property.*

In other words, a positively homogeneous functional $I : \mathbb{R}^n_+ \to \mathbb{R}$ is superadditive whenever it is supermodular. To complete König's theorem, we rely on the following Lemma, which is a version of a property of supermodular functions established in [9, Lm 6].

**Lemma 4** *Let $E$ be a Riesz space and $(a_i)_{i=1}^n \subseteq E_+$ be mutually disjoint elements. If $I : E_+ \to \mathbb{R}$ is supermodular and $I(0) = 0$, then it is superadditive over $(a_i)_{i=1}^n$, i.e.,*

$$I\left(\sum_{i=1}^n a_i\right) \geq \sum_{i=1}^n I(a_i). \tag{1}$$

**Proof.** As $a_i \wedge a_j = 0$, we have that $\vee_{i=1}^n a_i = \sum_{i=1}^n a_i$. We prove the result by induction. For $n = 1$, (1) is trivially true. Suppose that it is true for $n \geq 1$. We have

$$\begin{aligned} I\left(\vee_{i=1}^{n+1} a_i\right) &= I\left(\left(\vee_{i=1}^n a_i\right) \vee a_{n+1}\right) = I\left(\left(\vee_{i=1}^n a_i\right) \vee a_{n+1}\right) + I\left(\left(\vee_{i=1}^n a_i\right) \wedge a_{n+1}\right) \\ &\geq I\left(\vee_{i=1}^n a_i\right) + I(a_{n+1}) = I\left(\sum_{i=1}^n a_i\right) + I(a_{n+1}) \geq \sum_{i=1}^{n+1} I(a_i), \end{aligned}$$

as desired. ■

**Proof of Theorem 3.** Let $(e_i)_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. The elements of this basis are mutually disjoint. By (1), we have

$$I(x) = I\left(\sum_{i=1}^n x_i e_i\right) \geq \sum_{i=1}^n x_i I(e_i) \tag{2}$$

for all $x \in \mathbb{R}^n_+$. If we consider the scalar function $t \to I(tx + (1-t)y)$, for all $x, y \in \mathbb{R}^n_+$, by (2) we have

$$I(tx + (1-t)y) \geq -\sum_{i=1}^n (x_i \vee y_i) |I(e_i)|.$$

Therefore the function is bounded from below $[0, 1]$. By König's theorem [5, Thm 2.10], $I$ is then superadditive. ■

The converse of Theorem 3 holds in $\mathbb{R}^2$, something not surprising in view of the key role that $\mathbb{R}^2$ plays in König's proof.

[3] Unless otherwise stated, throughout the paper $\mathbb{R}^n$ is endowed with its component-wise order.

**Proposition 5** *A positively homogeneous functional $I : \mathbb{R}^2_+ \to \mathbb{R}$ is superadditive if and only if it is supermodular.*

**Proof.** The proof is based on the following simple property of $\mathbb{R}^2$ (see Lemma 16 in the Appendix): given any $u, v \in \mathbb{R}^2_+$, there exist $\alpha, \sigma \in [0,1]$ such that $x \wedge y = \sigma (\alpha x + \overline{\alpha} y)$, where $\overline{\alpha} = 1 - \alpha$.

As $x \wedge y + x \vee y = x + y$, it follows that $x \vee y = \sigma_1 \left(\beta x + \overline{\beta} y\right)$, where $\sigma_1 = 2 - \sigma$ and $\beta = (1 - \alpha\sigma)(2 - \sigma)^{-1}$. Assume that $I$ is concave. We then obtain

$$\begin{aligned} I(x \wedge y) &= \sigma I(\alpha x + \overline{\alpha} y) \geq \sigma \alpha I(x) + \sigma \overline{\alpha} I(y), \\ I(x \vee y) &= \sigma_1 I\left(\beta x + \overline{\beta} y\right) \geq \sigma_1 \beta I(x) + \sigma_1 \overline{\beta} I(y), \end{aligned}$$

and so

$$\begin{aligned} I(x \wedge y) + I(x \vee y) &\geq (\sigma\alpha + \sigma_1\beta) I(x) + \left(\sigma\overline{\alpha} + \sigma_1\overline{\beta}\right) I(y) \\ &= I(x) + I(y), \end{aligned}$$

as desired. ∎

The example on p. 288 of Choquet [2] shows that Proposition 5 does not hold in general in $\mathbb{R}^n$ when $n > 2$. His example can be generalized as follows. Consider an auxiliary function $\phi : \mathbb{R}^2_+ \to \mathbb{R}$ that is positively homogeneous and concave (or supermodular, by Proposition 5). Assume that $\phi$ satisfies the following mild strict concavity property:

$$\phi\left(1, 2^{-1}(a+b)\right) > 2^{-1}\phi(1,a) + 2^{-1}\phi(1,b) \tag{3}$$

for some $a, b \in \mathbb{R}_+$. Under these conditions, the superadditive and positively homogeneous functional $I(x_1, ..., x_n) = \phi(x_1, x_2 + ... + x_n)$ with $n \geq 3$ is not supermodular. Suppose *per contra* that $I$ is supermodular. Take the two points $x$ and $y$ of $\mathbb{R}^n$ given by $x = (2, a, b, 0, ...., 0)$ and $y = (2, b, a, 0, ...., 0)$. It would hold

$$\begin{aligned} I(x \vee y) + I(x \wedge y) &\geq I(x) + I(y), \\ \phi(2, 2a) + \phi(2, 2b) &\geq 2\phi(2, a+b). \end{aligned}$$

Dividing by 4, we get

$$2^{-1}\phi(1,a) + 2^{-1}\phi(1,b) \geq \phi\left(1, 2^{-1}(a+b)\right)$$

which contradicts (3). Simple specifications of this general construction are for instance $I = x_1^{\alpha}(x_2 + ... + x_n)^{1-\alpha}$ with $\alpha \in (0,1)$ and $I = x_1 \wedge (x_2 + ... + x_n)$.

Despite of this argument, there are special classes of functionals for which the converse of Theorem 3 holds. For instance, this is the case for Choquet integrals (see [2], [5], and [10]).

# 4 The General Case

Consider the following class of Riesz spaces, which has been extensively studied in literature.

**Definition 6** *A Riesz space $E$ is said to be hyper-Archimedean if all quotient spaces $E/J$, with $J$ ideal in $E$, are Archimedean.*

Several alternative characterizations of hyper-Archimedean spaces are known (see [7], [8, Thms 37.6, 61.1, and 61.2] and [14]). For later use, we collect some of them in the following lemma. Here

$$Q(u) = \{v \in E_+ : v \wedge (u - v) = 0\}$$

is the set of all *quasi units* with respect to $u \in E_+$ ([11, p. 20]).

**Lemma 7** *A Riesz space $E$ is hyper-Archimedean if and only if any of the following equivalent conditions holds:*

*(i) every principal ideal in $E$ is a projection band,*

*(ii) every ideal in $E$ is uniformly closed,*

*(iii) every proper prime ideal is a maximal ideal,*

*(iv) $\operatorname{span} Q(u) = J_u$ for all $u \in E_+$.*

We can now state and prove our first main result. It shows that hyper-Archimedean Riesz spaces are the class of Riesz spaces $E$ in which the Choquet property holds for positively homogeneous functionals $I : E_+ \to \mathbb{R}$. We thus provide a further characterization of hyper-Archimedean Riesz spaces.

**Theorem 8** *A Riesz space $E$ is hyper-Archimedean if and only if the positively homogeneous functionals $I : E_+ \to \mathbb{R}$ have the Choquet property.*

**Proof.** Assume every positively homogeneous and superadditive functional $I : E_+ \to \mathbb{R}$ has the Choquet property. Suppose, *per contra*, that $E$ is not hyper-Archimedean. By Lemma 7-(iii), there exists a prime ideal $P$ which is not maximal. Consider the quotient space $E/P$ and the quotient map $\pi : E \to E/P$. The map $\pi$ is a lattice homomorphism between $E$ and $E/P$. As $P$ is prime, the quotient space $E/P$ is linearly ordered (see [8, Thm 33.2]). On the other hand, $E/P$ is linearly isomorphic to $\mathbb{R}$ if and only if $P$ is maximal (see [8, Thm 27.3]). Therefore, $E/P$ is not isomorphic to $\mathbb{R}$. Moreover, $E/P$ is then not Archimedean, since $\mathbb{R}$ is the unique linearly ordered Archimedean space. Pick any two points $[u], [v] \in E/P$

that are linearly independent and positive. By using an Hamel basis, construct a linear functional $L : E/P \to \mathbb{R}$ such that $L([u]) = 1$ and $L([v]) = -1$. The functional $|L(x)|$ is positively homogeneous and trivially supermodular, as $E/P$ is totally ordered. Consequently, the functional $I(x) = |L(\pi(x))|$, defined over $E_+$, is convex, positively homogeneous and supermodular. On the other hand, $I(u) = I(v) = 1$, while $I(u+v) = 0$, and thus $I$ is strictly subadditive, a contradiction.

To prove the converse implication, suppose that $E$ is hyper-Archimedean. We first show that, for any $u, v \in E_+$, the Riesz subspace $E[u, v]$ is finite-dimensional. Assume first that $E$ has a order unit $e \in E_+$. By [8, Thm 37.7], $E$ is Riesz isomorphic to a space $B_0(\Sigma)$ for some algebra $\Sigma$ of subsets of some space $X$[4]. By using this identification, if $u = \sum_i \lambda_i 1_{A_i}$ and $v = \sum_j \mu_j 1_{B_j}$, we can find a common finite partition $\{C_k\} \subseteq \Sigma$ of $X$ such that $u = \sum_k \lambda'_k 1_{C_k}$ and $v = \sum_k \lambda''_k 1_{C_k}$. Hence, $E[u, v] \subseteq Span\{1_{C_k}\}$ and $E[u, v]$ is finite-dimensional.

Assume now that $E$ has no order unit. By Lemma 7-(ii), every ideal $J$ of $E$ is in turn hyper-Archimedean. On the other hand, for any $u, v \in E_+$, we have that $E[u, v] \subseteq J_{u+v}$, where $J_{u+v}$ is the principal ideal generated by $u + v$. The desired property then follows from the previous result, as $u + v$ is a order unit in $J_{u+v}$. We conclude that, for any $u, v \in E_+$, the Riesz subspace $E[u, v]$ is finite-dimensional.

By the Judin Theorem (see [8, Thm 26.11]), $E[u, v]$ is then Riesz isomorphic to some $\mathbb{R}^n$ with the coordinate-wise ordering. Let $I : E_+ \to \mathbb{R}$ be a functional which is positively homogeneous and supermodular. Fix any two points $u$, $v \in E_+$ and consider the restriction of $I$ to $E[u, v]$. In view of what has been proved, by Theorem 3, it has the Choquet property on $E[u, v]$. In particular, $I(u + v) \geq I(u) + I(v)$ and the proof is complete. ■

**Remark.** In the proof of Theorem 8 we have shown that in each non hyper-Archimedean Riesz space $E$ we can construct a functional which is strictly convex, positively homogeneous and supermodular. Though it is likely to be highly irregular, all its one-dimensional restrictions $t \to I(tu + (1-t)v)$ are continuous, as it is convex. Therefore, this type of regularity does not suffice to rule out these pathological examples and stronger continuity conditions are needed.

We now illustrate our result with few examples.

- Given a set $X$, let $\mathcal{F}_{00}(X)$ be the Riesz space of all the function $f : X \to \mathbb{R}$ having a finite support (namely, such that the set $\{f \neq 0\}$ has finite cardinality). The Riesz space $\mathcal{F}_{00}(X)$ is hyper-Archimedean.
- Given an algebra $\Sigma$ of subsets of a space $X$, consider the Riesz space $B_0(\Sigma)$ of all simple $\Sigma$-measurable functions $f$. The space $B_0(\Sigma)$ is hyper-Archimedean.

[4] $B_0(\Sigma)$ denotes the space of all $\Sigma$-measurable simple functions; i.e., $B_0(\Sigma) = span\{1_A : A \in \Sigma\}$.

If $\mu : \Sigma \to \mathbb{R}$ is a measure, the set $\mathcal{M}(\Sigma, \mu)$ of all $\mu$-a.e. $\Sigma$-measurable simple functions is also hyper-Archimedean.

- The spaces $C(K)$, with $K$ compact and Hausdorff, are an important example of Riesz spaces that are not hyper-Archimedean, unless $K$ is finite. In fact, when $K$ is infinite, $C(K)$ has more prime ideals than maximal ideals ([8, Thm 34.3]), and so by Lemma 7-(iii) it fails to be hyper-Archimedean. As a result, the Kakutani Theorem ([11, Thm 2.1.3]) implies that in all infinite dimensional $AM$ spaces with order unit there are functionals violating the Choquet property.

# 5 Topological Riesz Spaces

Turn now to Riesz spaces having compatible linear topologies. In this setting it is natural to consider the Choquet property for continuous functionals. The next fact, an immediate consequence of Theorem 8, already shows that the continuous and positively homogeneous functionals of a large family of Riesz spaces have the Choquet property.

**Lemma 9** *Suppose the Riesz space $E$ contains an hyper-Archimedean Riesz subspace that is dense in $E$ for some lattice compatible linear topology $\tau$. Then, the $\tau$-continuous and positively homogeneous functionals $I : E_+ \to \mathbb{R}$ have the Choquet property.*

In view of this lemma, the following Riesz spaces are examples where the Choquet property holds for continuous and positively homogeneous functionals.

- The space $\mathcal{F}_0(X)$, the supnorm completion of $\mathcal{F}_{00}(X)$.
- The space $B(\Sigma)$, the supnorm completion of $B_0(\Sigma)$. When $\Sigma$ is a $\sigma$-algebra, $B(\Sigma)$ is the space of all bounded $\Sigma$-measurable functions.
- For all $p > 0$, let $\ell_p(X)$ be the space all functions $f : X \to \mathbb{R}$ such that

$$\sup \left\{ \sum_{x \in D} |f(x)|^p : D \subseteq X \text{ finite} \right\} < +\infty.$$

  It is a Banach lattice for $p \geq 1$, and a metrizable and complete metric space for $0 < p < 1$. Observe that $\mathcal{F}_{00}(X)$ is dense in $\ell_p(X)$ with respect to the strong topology.

- The spaces $L_p(\Omega, \Sigma, \mu)$, with $0 < p \leq \infty$. In fact, in all these spaces $\mathcal{M}_0(\Omega, \Sigma, \mu)$ is dense in the strong topology. By the Kakutani Representation Theorem, the Choquet property then holds for continuous and positively homogeneous functionals defined on $AL$ spaces and on abstract $L_p$ spaces.

The next simple lemma shows how to find new Riesz spaces on which continuous and positively homogeneous functionals satisfy the Choquet property.

**Lemma 10** *Let $\pi : E \to F$ be a continuous and surjective Riesz homomorphism between two normed Riesz spaces $E$ and $F$. If the continuous and positively homogeneous functionals on $E_+$ have the Choquet property, then the same is true for the continuous and positively homogeneous functionals on $F_+$.*

**Proof.** Assume *per contra* that the Choquet property does not hold in $F$ for some continuous functional $I : F_+ \to \mathbb{R}$ that is positively homogeneous and supermodular, but non superadditive. Namely, there exist $f_1, f_2 \in F_+$ such that $I(f_1 + f_2) < I(f_1) + I(f_2)$. Consider the continuous functional $\widetilde{I} = I \circ \pi$ over $E$. Clearly, it is positively homogeneous and supermodular. By hypothesis, $\widetilde{I}$ is then superadditive. As $\pi$ is onto, there are two elements $x_1, x_2 \in E_+$ such that $\pi(x_1) = f_1$ and $\pi(x_2) = f_2$. We have

$$\begin{aligned}
\widetilde{I}(x_1 + x_2) &\geq \widetilde{I}(x_1) + \widetilde{I}(x_2), \\
I(\pi(x_1) + \pi(x_2)) &\geq I(\pi(x_1)) + I(\pi(x_2)), \\
I(f_1 + f_2) &\geq I(f_1) + I(f_2),
\end{aligned}$$

a contradiction. ■

We now state our key lemma.

**Lemma 11** *Suppose $X$ is a zero-dimensional normal space. Then, the supnorm continuous and positively homogeneous functionals $I : C_b^+(X) \to \mathbb{R}$ have the Choquet property. If, in addition, $X$ is compact, then the Choquet property also holds for the continuous and positively homogeneous functionals $I : J_+ \to \mathbb{R}$, where $J$ is a closed ideal of $C(X)$.*

**Proof.** If $X$ is a zero-dimensional normal space, then, its inductive dimension is null as well, namely $Ind(X) = 0$ (see [12, p. 45]). Therefore, given any two disjoint closed sets $F_1$ and $F_2$, there exists a clopen set $G$ such that $F_1 \subseteq G \subseteq F_2^c$. Let $\Sigma$ be the algebra of the clopen sets of $X$. It is easy to check that $C_b(X) = B(\Sigma)$, i.e., $B_0(\Sigma)$ is supnorm dense in $C_b(X)$ (see, e.g., the proof of [11, Prop. 2.1.19]). As $B_0(\Sigma)$ is hyper-Archimedean, we conclude that any supnorm continuous functional $I : C_b(X) \to \mathbb{R}$ has the Choquet property.

Let us prove the last statement. Let $J \subset C(X)$ be a closed ideal. We know that $J$ is an algebraic ideal as well. Namely, there is a compact set $X_0 \subseteq X$, such that $f \in J \iff f(X_0) = 0$ (see for instance [11, Prop. 2.1.9]).

Consider again the simple functions $\sum_i \lambda_i 1_{A_i}$, where $A_i$ are clopen sets and $\{A_i\}$ is a partition of the space $X$. Restrict this family to those having the property that if $A_i \cap X_0 \neq \emptyset \implies \lambda_i = 0$. Clearly, this family lies in $J$. Moreover, they are an hyper-Archimedean space. Our objective is to show that such a family is dense in $J$.

Fix a function $f \in J$ and a scalar $\varepsilon > 0$. Consider the closed set $X_\varepsilon = \{x \in X : |f(x)| \geq \varepsilon\}$. Clearly $X_\varepsilon \cap X_0 = \emptyset$. As before, there is a clopen set $G$ such that $X_\varepsilon \subseteq G \subseteq X_0^c$. Moreover, there is a simple function $\sum_i \lambda_i 1_{A_i}$ such that $\|f - \sum_i \lambda_i 1_{A_i}\| < \varepsilon$ and $A_i$ are clopen sets. If we define the new simple function $\sum_i \lambda_i 1_{A_i \cap G}$, we have $\|f - \sum_i \lambda_i 1_{A_i \cap G}\| < \varepsilon$ as well and $\sum_i \lambda_i 1_{A_i \cap G}$ is a simple function of the above type. This concludes the proof. ■

The following result is the main consequence of our key lemma.

**Theorem 12** *If the Riesz normed space $E$ has the principal projection property, then the norm continuous and positively homogeneous functionals $I : E_+ \to \mathbb{R}$ have the Choquet property.*

**Remark.** The principal projection property is implied by the $\sigma$-Dedekind completeness, but the converse implication does not hold. The interrelationships between the principal projection property and the other classes of Archimedean Riesz spaces is the subject of the so-called "Main Inclusion Theorem" (see [8, Ch. 4]). Recall that spaces satisfying the principal projection property include $AL$ spaces and $L_\infty(\mu)$ spaces (and $B(\Sigma)$).

**Proof.** Suppose first that $E$ has an order unit $e$. Let $\|\cdot\|$ be the lattice norm of $E$ and $\rho_e$ the order norm induced by $e$. Consider the isomorphism $T : (E, \rho_e) \to (E, \|\cdot\|)$ given by $T(x) = x$ for each $x \in E$. Since $\|x\| \leq \rho(x)\|e\|$ for all $x \in E$, we have $T(x_n) \overset{\|\cdot\|}{\to} T(x)$ if $x_n \overset{\rho_e}{\to} x$. By Lemma 10, to prove the result it is then enough to show that all $\rho_e$-continuous functionals $I : E_+ \to \mathbb{R}$ have the Choquet property.

The lattice $(E, \rho_e)$ is an $M$-space. By the Kakutani Theorem ([6, p. 164]), there is an isometric lattice isomorphism $T$ from $(E, \rho_e)$ into $(C(X), \|\cdot\|_s)$, where $X$ is a suitable compact Hausdorff space and $\|\cdot\|_s$ is the supnorm. Moreover, $T(e) = 1_X$ and $T(E)$ is dense in $C(X)$.

Since $E$ has the principal projection property, also $T(E)$ does. By [3, Thm 2.9], $X$ is totally disconnected. Hence, $X$ is zero-dimensional ([12, p. 46]) and so, by Lemma 11, all continuous functionals $I : C_+(X) \to \mathbb{R}$ have the Choquet property. Hence, any $\rho_e$-continuous functional $I : E_+ \to \mathbb{R}$ has the Choquet property, as desired.

Suppose now that $E$ does not have a unit. For any $u, v \in E_+$, consider the principal ideal $J_{u+v}$ generated by $u + v$ and the restriction $I : J_{u+v} \to \mathbb{R}$ of our functional to the ideal $J_{u+v}$. As the principal projection property is inherited by ideals [8, Thm 25.2] and $u + v$ is an order unit in $J_{u+v}$, from what we just proved before, $I : J_{u+v} \to \mathbb{R}$ is superadditive, provided $I$ is supermodular and linearly homogeneous. In particular, as $u, v \in J_{u+v}$, we have $I(u+v) \geq I(u) + I(v)$. ■

Spaces $C(K)$, with $K$ compact, having the principal property are those for which $K$ is $\sigma$-Stonian ([11, Prop. 2.1.5]). Therefore, Theorem 12 covers few $AM$ spaces, and it has eluded us whether the Choquet property is valid for continuous functionals defined over general $AM$ spaces.

# 6 The Semicontinuous Case

In the previous section we have investigated the Choquet property for continuous functionals. The next theorem considers this property for functionals that are only semicontinuous.

**Theorem 13** *If $E$ is an $AL$ space, then the upper semicontinuous and positively homogeneous functionals $I : E_+ \to \mathbb{R}$ have the Choquet property. The same property holds for Banach lattices having a $p$ additive norm, with $p > 1$, and for $L_\infty(\mu)$ spaces with $\mu$ finite.*

**Proof.** Observe that the upper semicontinuity of $I$ at 0 and the property $I(\alpha u) = \alpha I(u)$ imply that $I(u) \leq L\|u\|$ for all $u \in E_+$ for some $L \geq 0$. Moreover, by the Kakutani Representation Theorem [11, Thm 2.7.1] $E$ is isometrically isomorphic to some $L_1(\mu)$ space of functions.

*Step 1.* The norm $\|\cdot\|$ is "modular" over $E$, namely, $\|x \wedge y\| + \|x \vee y\| = \|x\| + \|y\|$ holds for all $x, y \in E$. Actually, from the obvious identities

$$\begin{aligned} (x \wedge y)^+ &= x^+ \wedge y^+, \ (x \wedge y)^- = x^- \vee y^- \\ (x \vee y)^+ &= x^+ \vee y^+, \ (x \vee y)^- = x^- \wedge y^-, \end{aligned}$$

we obtain

$$|x \wedge y| = x^+ \wedge y^+ + x^- \vee y^-, \quad |x \vee y| = x^+ \vee y^+ + x^- \wedge y^-.$$

Hence,

$$\begin{aligned} &\|x \wedge y\| + \|x \vee y\| = \left\|x^+ \wedge y^+\right\| + \left\|x^- \vee y^-\right\| \\ &+ \left\|x^+ \vee y^+\right\| + \left\|x^- \wedge y^-\right\| = \||x| + |y|\| = \|x\| + \|y\| \end{aligned}$$

where the property of additivity over $E_+$ for the norm is repeatedly used.

*Step 2.* The norm $\|\cdot\|$ is *ultramodular* over $E$ (see [9]). Namely,

$$\|x + h\| - \|x\| \leq \|y + h\| - \|y\| \tag{4}$$

holds for all $x \leq y$ in $E$ and all $h \in E_+$. For, this ultramodularity property holds for the function $t \rightarrow |t|$, as it is convex. Hence, by representing the elements of $E$ by functions, we have that

$$|x(t) + h(t)| - |x(t)| \leq |y(t) + h(t)| - |y(t)|$$

for $x \leq y$ and $h \geq 0$. By integration,

$$\begin{aligned} & \int |x(t) + h(t)| \mu(dt) - \int |x(t)| \mu(dt) \\ \leq & \int |y(t) + h(t)| \mu(dt) - \int |y(t)| \mu(dt), \end{aligned}$$

which yields (4).

*Step 3.* We now show that the function $(x, y) \rightarrow \|x - y\|$ from $E \times E \rightarrow \mathbb{R}$ is submodular. Actually, by Step 1, the maps $x \rightarrow \|x - y\|$ and $y \rightarrow \|x - y\|$ are modular. Hence, by [13, Thm 2.6.2] it suffices to check that $\|x - y\|$ has decreasing differences. That is, the function $x \rightarrow \|x - y_2\| - \|x - y_1\|$ decreases for all $y_2 \geq y_1$. Namely,

$$\|x + h - y_2\| - \|x + h - y_1\| - \|x - y_2\| + \|x - y_1\| \leq 0$$

for $h \geq 0$ and $y_2 \geq y_1$. By setting $x' = x - y_2$ and $y' = x - y_1$, this inequality follows from (4).

*Step 4.* Define the sequence of functionals

$$I_n(x) = \sup_{y \in E_+} [I(y) - n\|x - y\|] \tag{5}$$

over $E_+$. By virtue of $I(u) \leq L\|u\|$, $I_n$ are finitely-valued for $n \geq L$. Clearly, $I_n$ are Lipschitz continuous and positively homogeneous. Moreover, $I_n \geq I$ and the sequence decreases. Let us prove that $I_n(x) \downarrow I(x)$. Fix $x$ and $\varepsilon > 0$. Then, for all $n \geq L$, there is a sequence $y_n \in E_+$ such that

$$\begin{aligned} n\|x - y_n\| & \leq I(y_n) - I_n(x) + \varepsilon \leq L\|y_n\| - I_n(x) + \varepsilon \\ & \leq L\|y_n\| - I(x) + \varepsilon. \end{aligned}$$

As $\|y_n\| \leq \|y_n - x\| + \|x\|$, we have $(n - L)\|x - y_n\| \leq L\|x\| - I(x) + \varepsilon$. Hence, $\|x - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Now, from

$$I(y_n) \geq I(y_n) - n\|x - y_n\| \geq I_n(x) - \varepsilon,$$

by the upper semicontinuity,

$$I(x) \geq \limsup_n I(y_n) \geq \lim_n I_n(x) - \varepsilon$$

and we conclude that $I_n(x) \downarrow I(x)$ for all $x \in E_+$.

*Step 5.* To conclude the proof, we observe that $I_n$ are supermodular. For, given that the map $(x,y) \to I(y) - n\|x-y\|$ is supermodular by Step 3, the sup is supermodular by [13, Thm 2.7.6]. We infer that each $I_n$ is superadditive, by Theorem 12. From $I_n(a+b) \geq I_n(a) + I_n(b)$, by taking the limit we have $I(a+b) \geq I(a) + I(b)$.

*Step 6.* If $E$ is a Banach lattice with a $p$-additive norm, then $E$ is isometrically isomorphic to $L_p(X,\Sigma,\mu)$ (see [1, Th. 3.34]). As $L_p(X,\Sigma,\mu) \subset L_1(X,\Sigma,\mu)$ is a projection band in $L_1(X,\Sigma,\mu)$, the band projection $P : L_1(X,\Sigma,\mu) \to L_p(X,\Sigma,\mu)$ is an onto homomorphism. $P$ is continuous, as $\|Pf\| \leq \|f\|$. Hence, the result follows by Lemma 10. The same argument holds for $L_\infty(\mu)$, which is a projection band in $L_1(\mu)$, provided $\mu$ is finite. ■

# 7 Translation Invariant Functionals

In this last section we consider the class of translation invariant functionals. For these functionals the relations between supermodularity and concavity turn out to be similar to the ones that we have established in the previous sections for positively homogeneous functions. For brevity, we do not detail all such properties, but we limit ourselves to state and prove the counterparts of Theorems 3 and 8, leaving to the interested reader the counterparts of the other results proved in Sections 5 and 6.

We begin with the counterpart of Theorem 3. Here we consider both functionals defined on the positive cone $\mathbb{R}^n_+$ and functionals defined on the entire space $\mathbb{R}^n$.

**Theorem 14** *The translation invariant functionals $I : \mathbb{R}^n \to \mathbb{R}$ have the Choquet property, as well as the translation invariant functionals $I : \mathbb{R}^n_+ \to \mathbb{R}$.*

In other words, both a translation invariant functional $I : \mathbb{R}^n \to \mathbb{R}$ and a translation invariant functional $I : \mathbb{R}^n_+ \to \mathbb{R}$ is concave whenever it is supermodular. Observe that if in the definition of translation invariance we do not require $e$ to be an order unit, then Theorem 14 fails. In fact, consider $I(x,y) = x + \phi(y)$ over $\mathbb{R}^2$, where $\phi$ is not concave. The function $I$ is both translation invariant, with $e = (1,0)$, and supermodular, but it is not concave.

**Proof.** Begin with $I : \mathbb{R}^n \to \mathbb{R}$. As it is translation invariant, there is $u \in \mathbb{R}^n_{++} = \left\{x \in \mathbb{R}^n_+ : x_i > 0 \ \forall i = 1,...,n\right\}$ such that $I(x + \alpha u) = I(x) + \alpha I(u)$ for all $x \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$. Let $e = (1,1,...,1)$, the new function $\widetilde{I}(x) = I(ux)$, where $ux = (u_i x_i)_{i=1}^n$, satisfies $\widetilde{I}(x + \alpha e) = \widetilde{I}(x) + \alpha \widetilde{I}(e)$. As $u_i > 0$ for all $i$, we can assume $u = e$, w.l.o.g. Moreover, by normalizing the function, we can always set $I(e) = 1, -1, 0$. Our proof goes through in the similar way in all these three cases. We shall set $I(e) = 1$, namely, $I(x + \alpha e) = I(x) + \alpha$.

The proof proceeds by induction. As it is trivially true for $n = 1$, we show that it holds in $\mathbb{R}^{n+1}$ provided it is true in $\mathbb{R}^n$. In the sequel, we shall adopt the following notation. Vectors in $\mathbb{R}^{n+1}$ are denoted by $\overline{x}$ and the following decompositions are used: $\overline{x} \equiv (x_0, x) \equiv (x_0, x_1, x')$, with $x \in \mathbb{R}^n$ and $x' \in \mathbb{R}^{n-1}$. Note further that $(x_0, x_1, x')$ is understood as $(x_0, x_1)$, when $n = 2$.

If $I(x_0, x_1, x')$ is a function over $\mathbb{R}^{n+1}$, and $c \in \mathbb{R}$, $I_c : \mathbb{R}^n \to \mathbb{R}$ denotes the function $I_c(x_0, x') = I(x_0, x_0 + c, x')$. Clearly, $I_c$ is translation invariant and supermodular whenever $I$ is.

Since $I(x_0, x) = I(0, x - x_0 e) + x_0$, where $e = (1, 1, ..., 1) \in \mathbb{R}^n$, to prove the theorem it suffices to show that $I(0, x)$ is concave.

Take any two points $(0, u) \equiv (0, u_1, u')$ and $(0, v) \equiv (0, v_1, v')$ of $\mathbb{R}^{n+1}$. By Lemma 16-(iii), there are $\sigma_1, \sigma_2, \lambda, \mu$ such that

$$\begin{aligned}\frac{1}{2}(0, u_1) + \frac{1}{2}(0, v_1) + \sigma_1(1, 1) &= [(0, u_1) + \lambda(1, 1)] \wedge [(0, v_1) + \mu(1, 1)] \\ \frac{1}{2}(0, u_1) + \frac{1}{2}(0, v_1) + \sigma_2(1, 1) &= [(0, u_1) + \lambda(1, 1)] \vee [(0, v_1) + \mu(1, 1)]\end{aligned}$$

with $\sigma_1 + \sigma_2 = \lambda + \mu$. Hence, by considering the two points $\overline{a} = (a_0, a_1, a')$ and $\overline{b} = (b_0, b_1, b')$ in $\mathbb{R}^{n+1}$, defined by,

$$\begin{aligned}\overline{a} &= [(0, u) + \lambda e] \wedge [(0, v) + \mu e], \\ \overline{b} &= [(0, u) + \lambda e] \vee [(0, v) + \mu e],\end{aligned}$$

where $\lambda$ and $\mu$ are as above, we obtain

$$\begin{aligned}a_0 &= \sigma_1, \quad a_1 = \sigma_1 + 2^{-1}(u_1 + v_1) \\ b_0 &= \sigma_2, \quad b_1 = \sigma_2 + 2^{-1}(u_1 + v_1).\end{aligned} \tag{6}$$

If we set $c = 2^{-1}(u_1 + v_1)$, (6) implies that $I(\overline{a}) = I_c(\sigma_1, a')$ and $I(\overline{b}) = I_c(\sigma_2, b')$. As the function $I_c$ is concave by assumption, we have

$$\begin{aligned}I_c\left(\frac{1}{2}(\sigma_1 + \sigma_2), \frac{1}{2}(a' + b')\right) &\geq \frac{1}{2}I_c(\sigma_1, a') + \frac{1}{2}I_c(\sigma_2, b') = \\ \frac{1}{2}I(\overline{a}) + \frac{1}{2}I(\overline{b}) &\geq \frac{1}{2}I((0, u) + \lambda e) + \frac{1}{2}I((0, v) + \mu e) \\ &= \frac{1}{2}I(0, u) + \frac{1}{2}I(0, v) + \frac{1}{2}(\lambda + \mu),\end{aligned} \tag{7}$$

where in the second line it is used the fact that $I$ is supermodular.

On the other hand, the first term of (7) equals

$$\begin{aligned}&I\left(\frac{1}{2}(\sigma_1 + \sigma_2), \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(u_1 + v_1), \frac{1}{2}(u' + v') + \frac{1}{2}(\lambda + \mu)e'\right) \\ = \; &I\left(0, \frac{1}{2}(u + v)\right) + \frac{1}{2}(\lambda + \mu),\end{aligned}$$

as $\sigma_1 + \sigma_2 = \lambda + \mu$. Consequently,

$$I\left(0, 2^{-1}(u+v)\right) \geq 2^{-1} I(0,u) + 2^{-1} I(0,v),$$

and so the function $I(0,x)$ is mid-concave. By [4, Thm 111], $I(0,x)$ is concave since $I(0,x)$ is bounded from below by Lemma 4. This proves the Theorem for the case $I: \mathbb{R}^n \to \mathbb{R}$.

Consider now a translation invariant and supermodular functional $I: \mathbb{R}^n_+ \to \mathbb{R}$. By Lemma 1, there exists a translation invariant and supermodular extension $\widetilde{I}: \mathbb{R}^n \to \mathbb{R}$. By what it has been just proved, $\widetilde{I}$ is concave, and so $I$ is. ■

Clearly the analogous property established in Proposition 5 holds: any translation invariant and concave function on $\mathbb{R}^2$ is supermodular. We omit the simple proof based on the property that for any two vectors $u, v \in \mathbb{R}^2$, we have $u \wedge v + \sigma e = \alpha u + \overline{\alpha} v$ for some $\sigma \geq 0$ and $\alpha \in [0,1]$.

We close with the counterpart of Theorem 8.

**Theorem 15** *For a Riesz space $E$ with order unit, the following conditions are equivalent:*

*(i) is hyper-Archimedean,*

*(ii) the translation invariant functionals $I: E_+ \to \mathbb{R}$ have the Choquet property,*

*(iii) the translation invariant functionals $I: E \to \mathbb{R}$ have the Choquet property.*

**Proof.** The equivalence of (ii) and (iii) follows from Lemma 1. The proof that (i) and (iii) are equivalent is rather similar to that of Theorem 8, and so we only mention the points at which they differ. In the first implication we assume *per contra* that $E$ is not hyper-Archimedean. The proof then goes on in constructing a functional that is not concave, though translation invariant and supermodular. This is obtained by of the same quotient map $\pi: E \to E/P$ of Theorem 8. Note that if $e$ is an order unit of $E$, then $[e]$ is an order unit of the quotient space $E/P$. Pick a point $[u] \in E/P$ linearly independent of $[e]$, and construct two linear functionals $L_1$ and $L_2$ over $E/P$ such that $L_1([u]) = -1$, $L_1([e]) = 1$, $L_2([u]) = 1$ and $L_2([e]) = 1$. The functional $(L_1 \vee L_2)(x)$ is translation invariant with respect $[e]$ and trivially supermodular. Note that $(L_1 \vee L_2)(-[u]) = 1$, $(L_1 \vee L_2)([u]) = 1$ and $(L_1 \vee L_2)(2^{-1}[u] - 2^{-1}[u]) = 0$. Therefore, $L_1 \vee L_2$ is not concave.

As to converse, it suffices to prove here that the Riesz subspace $E[u,v,e]$ is finite-dimensional, where $e$ is the order unit. ■

# 8 Appendix: The Space $\mathbb{R}^2$

The space $\mathbb{R}^2$ plays a fundamental role in view of the geometrical properties described below. Property (ii) below is closely related to König's construction, while (iii) is its translation invariant counterpart.

**Lemma 16** *(i) For all $u, v \in \mathbb{R}^2_+$ there is some $\sigma \in [0,1]$ and $\alpha \in [0,1]$, such that*

$$u \wedge v = \sigma(\alpha u + \overline{\alpha} v). \tag{8}$$

*If $u \wedge v \neq 0$ and $u$, $v$ are linearly independent, $\sigma$ and $\alpha$ are uniquely determined.*
*(ii) For all $u, v \in \mathbb{R}^2_+$ there is a unique $\alpha \in [0,1]$ and $\sigma \in [0,1]$ such that*

$$\alpha u \wedge \overline{\alpha} v = \sigma(\alpha u \vee \overline{\alpha} v). \tag{9}$$

*If in addition $u, v \in \mathbb{R}^2_{++}$, then $\alpha \in (0,1)$. More precisely,*

$$\begin{aligned} \alpha &= \frac{\sqrt{v_1 v_2}}{\sqrt{u_1 u_2} + \sqrt{v_1 v_2}} \\ \sigma &= \frac{\sqrt{u_1 v_2} \wedge \sqrt{v_1 u_2}}{\sqrt{u_1 v_2} \vee \sqrt{v_1 u_2}}. \end{aligned}$$

*(iii) For all $u, v \in \mathbb{R}^2$, there are $\sigma_1, \sigma_2, \lambda, \mu \in \mathbb{R}$, with $\sigma_2 \geq \sigma_1$ and $\sigma_1 + \sigma_2 = \lambda + \mu$, such that*

$$\begin{aligned} \frac{1}{2}(u+v) + \sigma_1 e &= (u + \lambda e) \wedge (v + \mu e) \qquad (10) \\ \frac{1}{2}(u+v) + \sigma_2 e &= (u + \lambda e) \vee (v + \mu e) \end{aligned}$$

*where $e = (1,1)$.*

**Proof.** (i) If $u \wedge v = 0$, set $\sigma = 0$. If $u$ and $v$ are comparable, set $\sigma = 1$ and $\alpha \in \{0,1\}$. Hence, it remains to check it when $u = (u_1, u_2)$ and $v = (v_1, v_2)$, with $u_1 < v_1$, $v_2 < u_2$ and $u_1$, $v_2$ not both equal to 0. Clearly $\sigma \neq 0$, in this case. Suppose first that $u_1, v_2 > 0$. For (8) holds, it must be

$$\frac{1}{\sigma} = \frac{\alpha u_1 + \overline{\alpha} v_1}{u_1} = \frac{\alpha u_2 + \overline{\alpha} v_2}{v_2}. \tag{11}$$

The function $\varphi(\alpha) = (\alpha u_1 + \overline{\alpha} v_1) u_1^{-1}$ decreases, as $\varphi(0) = v_1 u_1^{-1} > 1$ and $\varphi(1) = 1$. While the function $\psi(\alpha) = (\alpha u_2 + \overline{\alpha} v_2) v_2^{-1}$ increases, as $\psi(0) = 1$ and $\psi(1) = u_2 v_2^{-1} > 1$. Hence, a unique $\alpha \in (0,1)$ exists such that $\psi(\alpha) = \varphi(\alpha)$. This $\alpha$, along with $\sigma = \psi(\alpha)^{-1}$, solves (8). By taking the inverse of (11), we can deal with the case in which either $u_1$ or $v_2$ vanishes. The uniqueness, when $u$ and $v$ are linearly independent, is obvious. Otherwise, $u \wedge v = \sigma(\alpha u + \overline{\alpha} v) = \sigma_1(\alpha u + \overline{\alpha} v)$ which implies $\sigma = \sigma_1$.

(ii) This property has been proved by König [5]. It suffices to check that

$$\begin{aligned}\sqrt{v_1v_2}u \wedge \sqrt{u_1u_2}v &= \left(\sqrt{u_1v_2} \wedge \sqrt{v_1u_2}\right)\sqrt{uv} \\ \sqrt{v_1v_2}u \vee \sqrt{u_1u_2}v &= \left(\sqrt{u_1v_2} \vee \sqrt{v_1u_2}\right)\sqrt{uv},\end{aligned}$$

where $\sqrt{uv} = \left(\sqrt{u_1v_1}, \sqrt{u_2v_2}\right)$.

(iii) It suffices to check that (10) is true by setting

$$\begin{aligned}\lambda &= -\mu = \frac{1}{4}(v_1 - u_1) + \frac{1}{4}(v_2 - u_2) \\ \sigma_2 &= -\sigma_1 = \frac{1}{4}\left|(v_2 - u_2) - (v_1 - u_1)\right|.\end{aligned}$$

■

# References


[1] C. D. Aliprantis, O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics*, American Mathematical Society, Providence, Rhode Island, 2003.

[2] G. Choquet, Theory of Capacity, *Annales Institute Fourier*, 5, 131-295, 1953-4.

[3] W. Filter, Representations of Archimedean Riesz spaces, *Rocky Mountain Journal of Mathematics*, 24, 771–851, 1994.

[4] G. Hardy, J.E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[5] H. König, The (Sub/Super) Additivity Assertion of Choquet, *Studia Mathematica,* 157, 171-197, 2003.

[6] G. Jameson, *Ordered Linear Spaces*, Springer-Verlag, Berlin, 1970.

[7] W. A. J. Luxemburg and L. C. Moore, Archimedean Quotient Riesz Spaces, *Duke Mathematical Journal*, 34, 725–739, 1967.

[8] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces*, vol. 1, North-Holland, Amsterdam, 1971.

[9] M. Marinacci and L. Montrucchio, Ultramodular Functions*, Mathematics of Operations Research*, 30, 311-332, 2005.

[10] M. Marinacci and L. Montrucchio, Introduction to the Mathematics of Ambiguity, in *Uncertainty in Economic Theory,* (I. Gilboa, ed.), Routledge, New York, 2004.

[11] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, 1991.

[12] K. Nagami, *Dimension Theory*, Academic Press, New York, 1970.

[13] D. M. Topkis, *Supermodularity and Complementarity*, Princeton University Press, Princeton, 1998.

[14] A. C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, 1983.