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Disputed Lands

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Abstract

In this paper we consider the classical problem of dividing a land among many agents so that everybody is satisfied with the parcel she receives. In the literature, it is usually assumed that all the agents are endowed with cardinally comparable, additive, and monotone utility functions. In many economic and political situations violations of these assumptions may arise. We show how a family of cardinally comparable utility functions can be obtained starting directly from the agents' preferences, and how a fair division of the land is feasible, without additivity or monotonicity requirements. Moreover, if the land to be divided can be modelled as a finite dimensional simplex, it is possible to obtain envy-free (and *a fortiori* fair) divisions of it into subsimplexes.

The main tool is an extension of a representation theorem of Gilboa and Schmeidler (1989).

JEL classification: D01; D74

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1 Introduction

In this paper we consider the classical problem of dividing a land among many agents so that everybody is satisfied with the parcel she receives. See Brams and Taylor (1996) and Robertson and Webb (1998) for complete references on this subject, which was pioneered by Steinhaus (1948) and Dubins and Spanier (1961).

In the fair division literature, agents' preferences are usually described by cardinally comparable, additive, and monotone utility functions on parcels of land.[1] Whether this setup (utility functions on parcels of land) and these assumptions (additivity and monotonicity) are the most natural ones, or not, critically depends on the land division problem at hand.

Relative to the setup, land properties frequently consist of several shares of different parcels rather than referring to entire parcels. For example, an agent may own 50% of her flat (the remaining part belonging to her husband) and 20% of her family house (the remaining parts belonging to her brethren). Analogously, two different ethnic groups may be sharing different regions of the same territory with different population densities, and for each region one might consider the percentage of inhabitants belonging to a group and living in the region as the group's property share of it. For this reason, we will consider preferences on land property shares rather than utility functions on land parcels.

This ordinal perspective also has the advantage of referring to observable choice behavior, and it clarifies the economic underpinning of the standing mathematical assumption of existence of cardinally comparable utility functions.

Relative to the behavioral assumptions, the additivity of the agents' utility functions:

$$\upsilon(A \cup H) - \upsilon(A) = \upsilon(H)$$

for all disjoint parcels A and H, means literally that the agent's utility is marginally constant. This assumption is not more innocuous in land division than it is in the rest of economics. Clearly, the value added by the annexation of a coastal region may greatly differ according to whether or not the annexing region already has a coast line or not. Analogously, increasing maintenance costs might lead to decreasing marginal utilities, while scale or combination effects might lead to increasing marginal ones.

Last, but not least, monotonicity of preferences is violated in any situation in which there are some undesirable regions in the land. An agent might not be willing to expand her estate because this might make it adjacent to the one of a neighbor she dislikes, or worse still, the addition of a contaminated field might cause serious troubles for a whole farm.

Motivated by these considerations, in this paper we show that simple assumptions on the agents' preferences on land property shares guarantee the existence of cardinally comparable utility functions (neither additive nor monotone, in general), and guarantee the existence of a solution to the fair division problem.

Surprisingly enough, in the important special case of a segment to be divided among agents with equal initial shares, such fair solution can be constructed by the classical Steinhaus-Banach-Knaster technique,[2] and a division into subsegments is obtained.

More in general, wherever land is concerned, it is important that the parcels into which it is divided are nicely shaped (think of owning a one kilometer-square park consisting of one million disconnected plots measuring a square meter each). We show that, if the land to be divided can

[1] Important exceptions are discussed below.

[2] See next section.

be modelled as a simplex, our preference assumptions guarantee the existence of a division into subsimplexes which is not only fair but also envy free, thus extending the results obtained by Ichiishi and Izdik (1999) in the classical setup.

The literature of fair division has already dealt, at least singly, with some departures from the classical settings. The cited Ichiishi and Izdik (1999) model agents' preferences as signed measures with finite positive total mass, thus dropping monotonicity (while retaining additivity). In another direction, Berliant, Dunz and Thomson (1992) tackle the case of decreasing marginal (but monotone) utilities with a notion of concavity based on set-dependent density functions. Maccheroni and Marinacci (2003) extend the results of Dubins and Spanier (1961) to the class of nonatomic concave capacities.

A different approach is the one taken by Berliant and Dunz (2004) who – under suitable continuity, "separation by hyperplanes," and "nonwasteful partitions" assumptions on preferences over land parcels – establish welfare theorems and the existence of the core for a land trading economy. These solution notions rely on the agents' willingness to trade and cooperate. This might not be the case for bitterly disputing agents who, for example, rely on an external referee to achieve a satisfactory division.

When the division regards chores exclusively, the original assumptions regarding the additivity and monotonicity of the agents' utilities are usually maintained with the proviso that the agents aim at partitions which "reward" them as little as possible. An example of this approach is given by Peterson and Su (2002).

Finally, for the representation of binary relations over the subsets of a given set by means of probabilities, the reader is referred to the classical works of de Finetti (1931), Koopman (1940), Savage (1954), and the recent Barbanel and Taylor (1995). While the – closer to our setting – problem of non-additive representation of preferences over random variables was pioneered by Gilboa and Schmeidler (1989).

The paper is organized as follows. Section 2 briefly reviews some classic results of Dubins and Spanier (1961) and clarifies a "hidden" assumption of classic fair division models. Section 3 introduces our setup and behavioral assumptions. In Section 4, we present our main results: preference representation (Theorem 2) and land division (Theorem 3). In the final Section 5 (Proposition 1), we consider the special case in which the land to be divided can be represented as a simplex. Proofs and related material are collected in the Appendix.

2 Dubins-Spanier Theorem

A finite set of n agents owns a piece of land S. The set of all parcels into which S can be divided is modelled as a σ-algebra Σ. For all $i \in I$, α_i denotes the property share of S owned by agent i, and the nonatomic probability measure $\mu_i : \Sigma \to [0, 1]$ represents agent's i utility function on the various parcels of S.

Theorem 1 (Dubins-Spanier) *Let $\mu_1, \mu_2, ..., \mu_n$ be nonatomic probability measures on Σ. Given any $\alpha_1, \alpha_2, ..., \alpha_n \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, there exists a partition $\{A_1, A_2, ..., A_n\}$ of S in Σ such that*

$$\mu_i(A_i) \geq \alpha_i \tag{1}$$

for all $i = 1, ..., n$. Moreover, if $\mu_j \neq \mu_h$ for some $j \neq h$ and $\alpha_1, \alpha_2, ..., \alpha_n > 0$, then $A_1, A_2, ..., A_n$ can be chosen to satisfy $\mu_i(A_i) > \alpha_i$ for all $i = 1, ..., n$.

As anticipated in the introduction, Theorem 1 guarantees the existence of a fair division of the land, provided the agents' preferences on land parcels can be represented by cardinally comparable utility functions which are marginally constant, monotone, nonatomic; under the implicit assumption that the comparison $\mu_i(A_i) \geq \alpha_i$ of a *subjective utility* and an *objective property share* can be interpreted as: "the agent is satisfied that the parcel she receives is worth at least as much as the share of the whole land which she was due (before the division took place)."

However, from an economic perspective, it is important to notice that in the Dubins and Spanier model (as well as in all the derived literature) preference comparisons between land parcels A and property shares α of S are not explicitly modelled, but implicitly related to the comparison between $\mu_i(A)$ and $\alpha = \mu_i(\alpha\chi_S)$. Here χ_S is the indicator function of S, and μ_i is naturally extended to the space of all simple measurable functions $X : S \to \mathbb{R}$ by $\mu_i(X) = \int_S X d\mu_i$.

Dubins and Spanier (1961) also rephrase the Steinhaus-Banach-Knaster technique to obtain division (1) in the case in which $\alpha_1 = \alpha_2 = ... = \alpha_n = 1/n$:

> *"... A knife is slowly moved over the cake. The first person to indicate satisfaction with the slice then determined by the position of the knife receives that slice... The process is repeated with the other $n-1$ participants and with that remains of the cake..."*

3 Setup and Axioms

While we maintain the description of the object to be divided as a measurable space (S, Σ), we model the agents' preferences explicitly by a family $\{\succsim_i\}_{i=1,...,n}$ of binary relations on *land property shares*. The simple measurable function

$$X = \beta_1\chi_{B_1} + \beta_2\chi_{B_2} + ... + \beta_m\chi_{B_m}$$

represents the property of share β_1 of parcel B_1, β_2 of parcel B_2, ... , β_m of parcel B_m. Therefore, the set $\mathcal{X}$ of all land property shares consists of all Σ-measurable simple functions on S taking values in $[0, 1]$, denoted by X, Y, Z.... In this perspective, it is natural to identify parcel A with a 100% property share of parcel A itself, that is with the indicator function χ_A. Under this identification, $\succsim_i$ induces a binary relation on Σ, still denoted by $\succsim_i$.

A parcel N in Σ is *null* for $\succsim_i$ if $A \cup N \sim_i A$ for all A in Σ. An *atom* for $\succsim_i$ is a non-null part A such that for every $B \subseteq A$, either B or $A \setminus B$ is null.

Next are listed some properties of the preferences that will be used in the sequel. As usual, $\succ_i$ and $\sim_i$ denote the asymmetric and symmetric parts of $\succsim_i$, respectively.

Axiom 1 (Weak Order) *(a) For all X, Y, either $X \succsim_i Y$ or $Y \succsim_i X$. (b) If $X \succsim_i Y$ and $Y \succsim_i Z$, then $X \succsim_i Z$.*

Axiom 2 (Continuity) *If $X_k \to X$ pointwise and $X_k \succsim_i Y$ (resp. $\precsim_i$) for all $k \in \mathbb{N}$, then $X \succsim_i Y$ (resp. $\precsim_i$).*

Axiom 3 (Desirability) *$S \succ_i \emptyset$.*

These first three axioms are adapted versions of the standard rationality, continuity, and non-degeneracy assumptions for preferences. Axioms 1 and 2 are commonly used and well discussed in the literature. Axiom 3, a non-triviality assumption, also means that the entire land is valuable to agent i, that is, he is willing to take part in the division of it.

Axiom 4 (Preference for Concentration) *If $X \sim_i Y$, then $X \succsim_i \frac{1}{2}X + \frac{1}{2}Y$.*

This axiom means that the agent prefers concentrated property. Consider, for example, two non-overlapping parcels: the axiom says that if the agent (e.g., an ethnic group) considers them indifferent, then he prefers to be in full control of one of them rather than sharing both of them 50%-50% with other agents (e.g., because of coexistence problems). See also Berliant and Dunz (2004, p. 597) for further discussion of this kind behavior in land division problems.

Axiom 5 (Constant Independence) *If* $0 \leq \alpha, \beta \leq 1$ *and* $\alpha \neq 0$*, then* $X \succsim_i Y$ *if and only if* $\alpha X + (1-\alpha)\beta\chi_S \succsim_i \alpha Y + (1-\alpha)\beta\chi_S$*.*

Constant Independence requires that preference is not reverted by proportional changes in the owned shares. For example, if A is preferred to B, then a 50% property share of A is preferred to the same share of B, and this preference is not affected by adding to both shares a constant share, say 10%, of the whole land S.

The final nonatomicity Axiom 6 is easily seen to be crucial for division. For example, no division is possible in the extreme case in which all the agents only care about a single point s in S.

Axiom 6 (Atomlessness) *There are no atoms for* $\succsim_i$*.*

It is important to notice that we will not assume additivity or monotonicity of the preference, that is:

Independence *If* $0 \leq \alpha \leq 1$ *and* $Z \in \mathcal{X}$*, then* $X \succsim_i Y$ *implies* $\alpha X + (1-\alpha) Z \succsim_i \alpha Y + (1-\alpha) Z$*.*

Monotonicity *If* $X \geq Y$*, then* $X \succsim_i Y$*.*

4 Main Results

Denote by $\mathcal{M}$ the set of all atomless signed measures on (S, Σ) that assign value 1 to S.

Theorem 2 *A binary relation* $\succsim_i$ *on* $\mathcal{X}$ *satisfies Axioms 1 - 6 if and only if there exist a product compact and convex set* $\mathcal{C}_i \subseteq \mathcal{M}$ *such that*

$$X \succsim_i Y \Leftrightarrow \max_{\mu \in \mathcal{C}_i} \mu(X) \geq \max_{\mu \in \mathcal{C}_i} \mu(Y). \tag{2}$$

The set $\mathcal{C}_i$ *is unique.*

A possible interpretation is the following: There is a set $\mathcal{C}_i$ of alternative development policies on the land agent i knows that he is able to apply ($\mu(A)$ being the payoff produced by plot A under policy μ), then he evaluates each property share with the highest possible payoff it potentially guarantees.

Technically, the result is an extension of Gilboa and Schmeidler (1989, Theorem 1) even if the economic setup is quite different. The extension is not trivial since Monotonicity plays a crucial role in their proof, but, as discussed in the introduction, it is not always sensible from a land division perspective.

Finally, it is easy to check that the preferences discussed in Theorem 2 satisfy Independence if and only if the set $\mathcal{C}_i$ is a singleton, while they satisfy Monotonicity if and only if $\mathcal{C}_i$ consists of probability measures.

We are now ready to state our main fair division result:

Theorem 3 *Let the binary relations* $\succsim_1, \succsim_2, ..., \succsim_n$ *on* $\mathcal{X}$ *satisfy Axioms 1 - 6. Given any* $\alpha_1, \alpha_2, ..., \alpha_n \geq 0$ *with* $\sum_{i=1}^n \alpha_i = 1$*, there exists a partition* $\{A_1, A_2, ..., A_n\}$ *of* S *in* Σ *such that*

$$A_i \succsim_i \alpha_i \chi_S$$

for each $i = 1, 2, ..., n$. *Moreover, if* $\succsim_j \neq \succsim_k$ *for some* $j \neq k$ *and* $\alpha_1, \alpha_2, ..., \alpha_n > 0$, *the partition* $\{A_1, A_2, ..., A_n\}$ *can be chosen to satisfy*

$$A_i \succ_i \alpha_i \chi_S$$

for each $i = 1, 2, ..., n$.

An important corollary of Theorem 3 is the extension of the results of Maccheroni and Marinacci (2003) for decreasing marginal utilities to the non-monotonic case. Before stating it, we need to define some properties of utility functions $\upsilon_i : \Sigma \to \mathbb{R}$. First, analogously to what happens for preferences, a parcel N in Σ is *null* for υ_i if $\upsilon_i (A \cup N) = \upsilon_i (A)$ for all A in Σ; an *atom* of υ_i is a non-null part A such that for every $B \subseteq A$, either B or $A \setminus B$ is null. As observed by Marinacci and Montrucchio (2004), when υ is a measure these notions coincide with the standard ones. Moreover, a set function $\upsilon_i : \Sigma \to \mathbb{R}$ is:

(a) *bounded* if $\sup_{A \in \Sigma} |\upsilon_i (A)| < \infty$,

(b) *concave* if $\upsilon_i (A \cup B) + \upsilon_i (A \cap B) \leq \upsilon_i (A) + \upsilon_i (B)$ for all A, B in Σ,[3]

(c) *continuous* if $\lim_k \upsilon_i (A_k) = 0 = 1 - \lim_k \upsilon (A_k^c)$ for all $A_k \downarrow \emptyset$,[4]

(d) *atomless* if it has no atoms.

Corollary 1 *Let* $\upsilon_1, \upsilon_2, ..., \upsilon_n : \Sigma \to \mathbb{R}$ *be bounded, concave, continuous, and atomless. Given any* $\alpha_1, \alpha_2, ..., \alpha_n \geq 0$ *with* $\sum_{i=1}^n \alpha_i = 1$, *there exists a partition* $\{A_1, A_2, ..., A_n\}$ *of* S *in* Σ *such that*

$$\upsilon_i(A_i) \geq \alpha_i$$

for each $i = 1, 2, ..., n$. *Moreover, if* $\upsilon_j \neq \upsilon_k$ *for some* $j \neq k$ *and* $\alpha_1, \alpha_2, ..., \alpha_n > 0$, *the partition* $\{A_1, A_2, ..., A_n\}$ *can be chosen to satisfy*

$$\upsilon_i(A_i) > \alpha_i$$

for each $i = 1, 2, ..., n$.

5 Envy freeness and nicely shaped parcels

A partition $\{A_1, A_2, \ldots, A_n\}$ of S in Σ is *envy free* if

$$A_i \succsim_i A_j \text{ for all } i, j = 1, \ldots, n.$$

This means that everybody prefers the parcel A_i that she received to all the parcels the others received.

If all agents have the same initial endowment (i.e. $\alpha_1 = \alpha_2 = ... = \alpha_n = 1/n$) and their preferences can be represented by convex functions ν_i,[5] then envy free partitions are a fortiori *fair*, that is

$$A_i \succsim_i \frac{1}{n} \chi_S \text{ for all } i = 1, 2, ..., n.$$

[3] As observed in Maccheroni and Marinacci (2003), this is equivalent to $\upsilon_i (B \cup C \cup A) - \upsilon_i (B \cup C) \leq \upsilon_i (B \cup A) - \upsilon_i (B)$ for all disjoint A, B, C in Σ, thus capturing decreasing marginal utility.

[4] In particular, $\upsilon (\emptyset) = 0 = 1 - \upsilon (S)$.

[5] A function $\nu_i : \mathcal{X} \to \mathbb{R}$ is *convex* if for all $X, Y \in \mathcal{X}$ and all $\beta \in [0, 1]$

$$\nu_i (\beta X + (1 - \beta) Y) \leq \beta \nu_i (X) + (1 - \beta) \nu_i (Y) .$$

Clearly this is the case for the functions $\nu_i (X) = \max_{\mu \in \mathcal{C}_i} \mu (X)$ that represent the preferences we considered in the previous section.

In fact, envy freeness implies that $\nu_i(\chi_{A_i}) \geq \nu_i(\chi_{A_j})$ for all $i, j = 1, \ldots, n$, then

$$\nu_i(\chi_{A_i}) \geq \frac{1}{n}\sum_{j=1}^{n}\nu_i(\chi_{A_j}) \geq \nu_i\left(\sum_{j=1}^{n}\frac{1}{n}\chi_{A_j}\right) = \nu_i\left(\frac{1}{n}\chi_S\right).$$

For historical background on these concepts, as well as further reference, we refer to Brams and Taylor (1996).

As pointed out by Ichiishi and Izdik (1999):

> *"... While much of the literature concerns divisions of a good into merely measurable subsets... it is desirable from a practical point of view to have divisions into geometrically simple subsets..."*

Important results in this direction were obtained by the same authors, who proved the existence of envy free partitions for linear preferences defined on (the simplex) $[0,1]$ or on the $n-1$-dimensional simplex Δ^n, endowed with their Lebesgue σ-fields. Next we generalize their results to our nonlinear preferences setup.

Consider the following axiom:

Axiom 7 (Null cuts) *If S is a simplex, then (the intersection of S with) any finite union of hyperplanes in* aff$S$ *is $\succsim_i$-null.*

We call *partition into polytopes* of a simplex S a collection $\{A_1, A_2, ..., A_n\}$ of polytopes (convex hulls of finite subsets of S) with disjoint interiors (relative to aff$S$) and such that $S = \bigcup_{i=1}^{n} A_i$. Notice that, since any subset of a $\succsim_i$-null set is $\succsim_i$-null, if each preference satisfies Axiom 7, then the union of all boundaries (relative to aff$S$) of the elements of the collection is null for every agent. The definitions of fairness and envy freeness hold unchanged as well as their meaning.

Proposition 1 *Let S be either $[0,1]$ or Δ^n endowed with its Lebesgue σ-field, and $\succsim_1, \succsim_2, ..., \succsim_n$ be binary relations on $\mathcal{X}$ satisfying Axioms 1 - 5 and 7. There exists a partition $\{A_1, A_2, ..., A_n\}$ of S into polytopes such that*

$$A_i \succsim_i A_j \quad \text{for all } i, j = 1, \ldots, n, i \neq j,$$

and this partition is also fair.

The case $S = [0,1]$ is especially important since it naturally represents a transportation route (such as a highway), a river, or the time interval in which a resource is managed.

In all cases, the management of a given segment clearly involves costs and benefits which are influenced by scale factors. For example, when a highway is considered, costs are given by maintenance, while benefits derive mainly from pay-tolls. For a river, pollution-cleaning is clearly a cost, while revenues derive from fishing, irrigation, energy production, and again transportation activities. Finally, in the case of a time span, it is clear how costs and benefits depend on the period of exploitation (think for example of the multi-property of a mountain cottage).

Moreover, for $S = [0,1]$, it is easily checked that the Steinhaus-Banach-Knaster technique quoted in Section 2 delivers a fair division for the preferences considered here. This partition, however, may fail to be envy free. The technique of Su (1999) can be used instead to obtain an approximately envy free division.

A Proofs and related material

In this section we denote by $B_0(I)$ the set of all real valued measurable simple functions on (S,Σ) taking value in the interval I (notice that $B_0([0,1]) = \mathcal{X}$). For any interval I, $B_0(I)$ is a convex subset of the normed space $B_0(\mathbb{R})$ (endowed with the supnorm). A functional $\nu : B_0(I) \to \mathbb{R}$ is *constant affine* if $\nu(\alpha X + (1-\alpha)\beta) = \alpha\nu(X) + (1-\alpha)\beta$ for all $X \in B_0(I)$, all $\alpha \in [0,1]$, and all $\beta \in I$. It is easy to see that a constant affine functional $\nu : B_0(\mathbb{R}) \to \mathbb{R}$ is *constant linear*, that is, $\nu(aX+b) = a\nu(X) + b$ for all $X \in B_0(\mathbb{R})$, $a, b \in \mathbb{R}$, $a \geq 0$. Moreover, any constant affine functional $\nu : B_0(I) \to \mathbb{R}$ admits a unique constant linear extension to $B_0(\mathbb{R})$, provided I contains at least two points.

The vector space of all bounded and additive set functions – *charges* – on (S,Σ) is denoted by $ba(\Sigma)$. The subspaces of $ba(\Sigma)$ consisting of signed measures and atomless signed measures are denoted by $ca(\Sigma)$ and $na(\Sigma)$, respectively. The cone of non-negative charges is $ba^+(\Sigma)$, while $ba_1(\Sigma) = \{\mu \in ba(\Sigma) : \mu(S) = 1\}$; $ca^+(\Sigma)$, $na^+(\Sigma)$, $ca_1(\Sigma)$, and $na_1(\Sigma)$ are analogously defined (notice that $na_1(\Sigma) = \mathcal{M}$).

When endowed with the total variation norm, $ba(\Sigma)$ is isometrically isomorphic to the norm dual of $B_0(\mathbb{R})$, the duality being

$$\mu(X) = \int X d\mu$$

for all $X \in B_0(\mathbb{R})$ and all $\mu \in ba(\Sigma)$. The weak* topology $\sigma(ba(\Sigma), B_0(\mathbb{R}))$ induced by this duality on $ba(\Sigma)$ coincides with the product topology (when $ba(\Sigma)$ is regarded as a subset of $\mathbb{R}^\Sigma$) and it is denoted by τ_Σ. For the properties of such classical spaces and topologies, we refer to Dunford and Schwartz (1958) and Gänssler (1971); here we just recall some definitions. Let $\mathcal{C}$ be a subset of $ca(\Sigma)$:

- $|\mathcal{C}| = \{|\mu| : \mu \in \mathcal{C}\}$, where $|\mu|$ denotes the total variation of μ.[6]
- $\mathcal{C}^\#$ is the set of all $\lambda \in ca^+(\Sigma)$ of the form $\lambda = \sum_{n\in\mathbb{N}} 2^{-n} \dfrac{|\mu_n|}{1 + |\mu_n|(S)}$ with $\mu_n \in \mathcal{C}$ for every $n \in \mathbb{N}$.
- $\mathcal{C}$ is *dominated* by $\lambda \in ca^+(\Sigma)$, denoted by $\mathcal{C} \ll \lambda$, if every element of $\mathcal{C}$ is absolutely continuous with respect to λ.
- $\mathcal{C}$ is *uniformly dominated* by $\lambda \in ca^+(\Sigma)$, denoted by $\mathcal{C} \lll \lambda$, if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\lambda(A) < \delta$ implies $\sup_{\mu\in\mathcal{C}} |\mu(A)| < \varepsilon$.

Lemma 1 *Let $\{\mu_l\}$ be a sequence in $ca(\Sigma)$ and $\lambda \in ca_+(\Sigma)$ such that $\{\mu_l\} \ll \lambda$. If $\mu_l \overset{\tau_\Sigma}{\to} \mu_0$, and $\{X_l\}$ is a norm bounded sequence in $L^\infty(S,\Sigma,\lambda)$ converging to X_0 in measure (w.r.t. λ), then*

$$\int X_l d\mu_l \to \int X_0 d\mu_0.$$

Proof. Notice that:

- $\lambda(A) = 0$ implies $\mu_l(A) = 0$ for every $l \in \mathbb{N}$, and $\mu_0(A) = \lim_l \mu_l(A) = 0$, that is $\mu_0 \ll \lambda$.
- By the Vitali-Hahn-Saks Theorem (see Gänssler, 1971, 1.9), $\{\mu_l\} \lll \lambda$ and by the Nikodym Theorem (ibid. 1.12) $\sup_{l\in\mathbb{N}} \|\mu_l\| < \infty$.
- $L^\infty(S,\Sigma,\mu_l) \subseteq L^\infty(S,\Sigma,\lambda)$ for all $l \in \mathbb{N} \cup \{0\}$.
- $X_0 \in L^\infty(S,\Sigma,\lambda)$ since there exists a subsequence $\{X_{l_j}\}$ converging to X_0 λ-almost everywhere.

[6] For all $A \in \Sigma$ the *total variation* of μ on A is defined as $|\mu|(A) = \sup \sum_{i=1}^n |\mu(A_i)|$ where the supremum is taken over all finite partitions $\{A_1, ..., A_n\}$ of A in Σ.

For all $l \in \mathbb{N}$,

$$\begin{aligned}\left|\int X_0 d\mu_0 - \int X_l d\mu_l\right| &\leq \left|\int X_0 d\mu_0 - \int X_0 d\mu_l\right| + \left|\int X_0 d\mu_l - \int X_l d\mu_l\right| \\ &\leq \left|\int X_0 d\mu_0 - \int X_0 d\mu_l\right| + \left|\int (X_0 - X_l)\, d\mu_l\right| \\ &\leq \left|\int X_0 d\mu_0 - \int X_0 d\mu_l\right| + \left|\int_{\mathbb{R}^+} \mu_l \{(X_0 - X_l) \geq t\}\, dt - \int_{\mathbb{R}^-} \mu_l \{(X_0 - X_l) \leq t\}\, dt\right| \\ &\leq \left|\int X_0 d\mu_0 - \int X_0 d\mu_l\right| + \int_{(0,\infty)} |\mu_l \{(X_0 - X_l) \geq t\}|\, dt + \int_{(-\infty,0)} |\mu_l \{(X_0 - X_l) \leq t\}|\, dt,\end{aligned}$$

and $\left|\int X_0 d\mu_0 - \int X_0 d\mu_l\right| \to 0$ as $l \to \infty$, since μ_l weak converges to μ_0 (ibid. 2.15).

Next we show that

$$\int_{(0,\infty)} |\mu_l \{(X_0 - X_l) \geq t\}|\, dt \to 0 \text{ as } l \to \infty.$$

- Given $l \in \mathbb{N}$, for all $k \in \mathbb{N}$ the function $f_{l,k} : (0,\infty) \to \mathbb{R}^+$ defined by $f_{l,k}(t) = |\mu_k \{(X_0 - X_l) \geq t\}|$ is Lebesgue measurable, then $g_l(t) = \sup_{k\in\mathbb{N}} |\mu_k \{(X_0 - X_l) \geq t\}|$ is Lebesgue measurable too, and $g_l(t) \geq |\mu_l \{(X_0 - X_l) \geq t\}|$ for all $t > 0$. So that

  $$\int_{(0,\infty)} |\mu_l \{(X_0 - X_l) \geq t\}|\, dt \leq \int_{(0,\infty)} g_l(t)\, dt$$

  for all $l \in \mathbb{N}$.

- By norm boundedness of $\{X_l\}$, there exists $T > 0$ such that for all $l \in \mathbb{N}$ and all $t > T$, we have $\lambda \{s \in S : (X_0 - X_l)(s) \geq t\} = 0$, that is $g_l(t) = 0$ ($\mu_k \ll \lambda$ for all $k \in \mathbb{N}$). Whence $\int_{(0,\infty)} g_l(t)\, dt = \int_{(0,T)} g_l(t)\, dt$ for all $l \in \mathbb{N}$.

- For all $l \in \mathbb{N}$ and all $t \in (0,T)$,

  $$g_l(t) = \sup_{k\in\mathbb{N}} |\mu_k \{(X_0 - X_l) \geq t\}| \leq \sup_{k\in\mathbb{N}} \|\mu_k\| < \infty.$$

- If $t > 0$, convergence in measure implies

  $$0 \leq \lambda \{(X_0 - X_l) \geq t\} \leq \lambda \{|X_0 - X_l| \geq t\} \to 0 \text{ as } l \to \infty.$$

  Then for every $\delta > 0$ there is $l(\delta) \geq 0$ such that $\lambda \{(X_0 - X_l) \geq t\} < \delta$ if $l \geq l(\delta)$. Let $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\lambda(A) < \delta(\varepsilon)$ implies $\sup_{k\in\mathbb{N}} |\mu_k(A)| < \varepsilon$; therefore for $l \geq l(\delta(\varepsilon))$, $\lambda \{(X_0 - X_l) \geq t\} < \delta(\varepsilon)$ and $\sup_{k\in\mathbb{N}} |\mu_k \{(X_0 - X_l) \geq t\}| < \varepsilon$. This allows us to conclude that

  $$0 \leq g_l(t) = \sup_{k\in\mathbb{N}} |\mu_k \{(X_0 - X_l) \geq t\}| \to 0 \text{ as } l \to \infty.$$

  Since this is true for all $t \in (0,T)$, the Dominated Convergence Theorem yields $\int_{(0,T)} g_l(t)\, dt \to 0$ and $\int_{(0,\infty)} |\mu_l \{(X_0 - X_l) \geq t\}|\, dt \to 0$ as $l \to \infty$.

Next we show that

$$\int_{(-\infty,0)} |\mu_l \{(X_0 - X_l) \leq t\}|\, dt \to 0 \text{ as } l \to \infty.$$

- Given $l \in \mathbb{N}$, for all $k \in \mathbb{N}$ the function $f_{l,k} : (-\infty,0) \to \mathbb{R}^+$ defined by $f_{l,k}(t) = |\mu_k \{(X_0 - X_l) \leq t\}|$ is Lebesgue measurable, then $h_l(t) = \sup_{k\in\mathbb{N}} |\mu_k \{(X_0 - X_l) \leq t\}|$ is Lebesgue measurable too, and $h_l(t) \geq |\mu_l \{(X_0 - X_l) \leq t\}|$ for all $t < 0$. So that

  $$\int_{(-\infty,0)} |\mu_l \{(X_0 - X_l) \leq t\}|\, dt \leq \int_{(-\infty,0)} h_l(t)\, dt$$

  for all $l \in \mathbb{N}$.

- By norm boundedness of $\{X_l\}$, there exists $T<0$ such that for all $l\in\mathbb{N}$ and all $t<T$, we have $\lambda\{s\in S:(X_0-X_l)(s)\le t\}=0$, that is $h_l(t)=0$ ($\mu_k\ll\lambda$ for all $k\in\mathbb{N}$). Whence $\int_{(-\infty,0)}h_l(t)\,dt=\int_{(T,0)}h_l(t)\,dt$ for all $l\in\mathbb{N}$.

- For all $l\in\mathbb{N}$ and all $t\in(T,0)$,

$$h_l(t)=\sup_{k\in\mathbb{N}}|\mu_k\{(X_0-X_l)\le t\}|\le\sup_{k\in\mathbb{N}}\|\mu_k\|<\infty.$$

- If $t<0$, convergence in measure implies

$$0\le\lambda\{(X_0-X_l)\le t\}\le\lambda\{|X_0-X_l|\ge -t\}\to 0\text{ as }l\to\infty.$$

  Then for every $\delta>0$ there is $l(\delta)\ge 0$ such that $\lambda\{(X_0-X_l)\le t\}<\delta$ if $l\ge l(\delta)$. Let $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that $\lambda(A)<\delta(\varepsilon)$ implies $\sup_{k\in\mathbb{N}}|\mu_k(A)|<\varepsilon$; therefore for $l\ge l(\delta(\varepsilon))$, $\lambda\{(X_0-X_l)\le t\}<\delta(\varepsilon)$ and $\sup_{k\in\mathbb{N}}|\mu_k\{(X_0-X_l)\le t\}|<\varepsilon$. This allows us to conclude that

$$0\le h_l(t)=\sup_{k\in\mathbb{N}}|\mu_k\{(X_0-X_l)\le t\}|\to 0\text{ as }l\to\infty.$$

  Since this is true for all $t\in(T,0)$, the Dominated Convergence Theorem yields $\int_{(T,0)}h_l(t)\,dt\to 0$ and $\int_{(-\infty,0)}|\mu_l\{(X_0-X_l)\le t\}|\,dt\to 0$ as $l\to\infty$. ■

Lemma 2 *Let $\mathcal{C}$ be a τ_Σ conditionally compact subset of $ca(\Sigma)$ and $\lambda\in ca_+(\Sigma)$ such that $\mathcal{C}\ll\lambda$. If $\{X_l\}$ is a norm bounded sequence in $L^\infty(S,\Sigma,\lambda)$ converging to X_0 in measure (w.r.t. λ), then*

$$\lim_n\left(\sup_{\mu\in\mathcal{C}}\int X_n d\mu\right)=\sup_{\mu\in\mathcal{C}}\int X_0 d\mu.$$

Proof. Notice that $\sup_{\mu\in\mathcal{C}}\|\mu\|<\infty$ (ibid. 2.14), hence $\sup_{\mu\in\mathcal{C}}\int Y d\mu<\infty$ for all $Y\in L^\infty(S,\Sigma,\lambda)$. Therefore, for all $n\in\mathbb{N}$, there exists $\mu_n\in\mathcal{C}$ and $c_n\in(0,1/n)$ such that $\sup_{\mu\in\mathcal{C}}\int X_n d\mu=\int X_n d\mu_n+c_n$.

First we show that $\sup_{\mu\in\mathcal{C}}\int X_0 d\mu\le\underline{\lim}_n\left(\sup_{\mu\in\mathcal{C}}\int X_n d\mu\right)$. Let $\{X_{n_j}\}$ be a subsequence of X_n such that

$$\sup_{\mu\in\mathcal{C}}\int X_{n_j}d\mu\to\underline{\lim}_n\left(\sup_{\mu\in\mathcal{C}}\int X_n d\mu\right)=\underline{\ell}$$

as $j\to\infty$, then $\int X_{n_j}d\mu_{n_j}+c_{n_j}\to\underline{\ell}$ as $j\to\infty$. But $X_{n_j}\xrightarrow{\lambda}X_0$, then there exists a subsequence $\left\{X_{n_{j_k}}\right\}$ converging to X_0 λ-almost everywhere. For all $\mu\in\mathcal{C}\ll\lambda$, $\left\{X_{n_{j_k}}\right\}$ converges to X_0 μ-almost everywhere and $\int X_{n_{j_k}}d\mu_{n_{j_k}}+c_{n_{j_k}}\ge\int X_{n_{j_k}}d\mu$, passing to the limits as $j\to\infty$ (by the Dominated Convergence Theorem) we obtain $\underline{\ell}\ge\int X_0 d\mu$; therefore $\underline{\ell}\ge\sup_{\mu\in\mathcal{C}}\int X_0 d\mu$.

Then we show that $\overline{\lim}_n\left(\sup_{\mu\in\mathcal{C}}\int X_n d\mu\right)\le\sup_{\mu\in\mathcal{C}}\int X_0 d\mu$. Let X_{n_j} be a subsequence of X_n such that

$$\sup_{\mu\in\mathcal{C}}\int X_{n_j}d\mu\to\overline{\lim}_n\left(\sup_{\mu\in\mathcal{C}}\int X_n d\mu\right)=\overline{\ell}$$

as $j\to\infty$, then $\int X_{n_j}d\mu_{n_j}=\sup_{\mu\in\mathcal{C}}\int X_{n_j}d\mu-c_{n_j}\to\overline{\ell}$ as $j\to\infty$. Since $\mathcal{C}$ is τ_Σ conditionally compact, then $\mathcal{C}$ is τ_Σ sequentially conditionally compact (ibid. 2.6). In particular, there exists a subsequence $\mu_{n_{j_k}}$ of μ_{n_j} that τ_Σ converges to $\mu_0\in ca(\Sigma)$. By Lemma 1, $\lim_k\int X_{n_{j_k}}d\mu_{n_{j_k}}=\int X_0 d\mu_0$, but, since $\mu_{n_{j_k}}$ weak converges to μ_0 (ibid. 2.15), then $\int X_0 d\mu_0=\lim_k\int X_0 d\mu_{n_{j_k}}\le\sup_{\mu\in\mathcal{C}}\int X_0 d\mu$, summing up

$$\overline{\ell}=\lim_k\int X_{n_{j_k}}d\mu_{n_{j_k}}=\int X_0 d\mu_0=\lim_k\int X_0 d\mu_{n_{j_k}}\le\sup_{\mu\in\mathcal{C}}\int X_0 d\mu.$$

Finally, $\sup_{\mu\in\mathcal{C}}\int X_0 d\mu\le\underline{\ell}\le\overline{\ell}\le\sup_{\mu\in\mathcal{C}}\int X_0 d\mu$, as wanted. ■

Lemma 3 *Let $\mathcal{C}$ be a τ_Σ compact subset of $ba(\Sigma)$. The following statements are equivalent:*

(a) $\mathcal{C} \subseteq ca(\Sigma)$.

(b) $\lim_n \left(\max_{\mu\in\mathcal{C}} \int X_n d\mu\right) = \max_{\mu\in\mathcal{C}} \int (\lim_n X_n)\, d\mu$ *for every uniformly bounded pointwise convergent sequence* $\{X_n\}$ *in* $B_0(\mathbb{R})$.

(c) $\lim_n \left(\max_{\mu\in\mathcal{C}} \int X_n d\mu\right) = \max_{\mu\in\mathcal{C}} \int (\lim_n X_n)\, d\mu$ *for every monotonely convergent sequence* $\{X_n\}$ *in* $B_0(\mathbb{R})$.

Moreover, if $\mu_1(S) = \mu_2(S)$ *for all* $\mu_1, \mu_2 \in \mathcal{C}$, *then (a) is equivalent to:*

(d) $\max_{\mu\in\mathcal{C}} \mu(A_n) \to 0$ *and* $\max_{\mu\in\mathcal{C}} \mu(A_n^c) \to \max_{\mu\in\mathcal{C}} \mu(S)$ *for every sequence* $\{A_n\}$ *in* Σ *such that* $A_n \downarrow \emptyset$.

Finally, if all the elements of $\mathcal{C}$ *are positive, then (a) is also equivalent to:*

(e) $\max_{\mu\in\mathcal{C}} \mu(A_n) \to 0$ *for every sequence* $\{A_n\}$ *in* Σ *such that* $A_n \downarrow \emptyset$.

Proof. (a) $\Rightarrow$ (b) Since $\mathcal{C}$ is τ_Σ compact subset in $ca(\Sigma)$, there exists $\lambda \in ca^+(\Sigma)$ such that $\mathcal{C} \ll \lambda$ (ibid. 2.6). Moreover, every uniformly bounded pointwise convergent sequence $X_n \to X_0$ in $B_0(\mathbb{R})$ converges to X_0 in measure (w.r.t. λ). Apply Lemma 2.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a) Let $E_n \downarrow \emptyset$ and $\bar{\mu} \in \mathcal{C}$. Then, by (b), $\overline{\lim}_n \bar{\mu}(E_n) \leq \lim_n (\max_{\mu\in\mathcal{C}} \mu(E_n)) = 0$. Moreover, $-1_{E_n} \uparrow 0$, hence

$$0 = \lim_n \left(\max_{\mu\in\mathcal{C}} \mu(-1_{E_n})\right) = \lim_n \left(-\min_{\mu\in\mathcal{C}} \mu(E_n)\right) = -\lim_n \left(\min_{\mu\in\mathcal{C}} \mu(E_n)\right)$$

and $\underline{\lim}_n \bar{\mu}(E_n) \geq \lim_n (\min_{\mu\in\mathcal{C}} \mu(E_n)) = 0$. As wanted.

Since (a) implies (c), a fortiori it implies (d) and (e).

Let $\mu_1(S) = \mu_2(S) = m$ for all $\mu_1, \mu_2 \in \mathcal{C}$.

(d) $\Rightarrow$ (a) Let $E_n \downarrow \emptyset$ and $\bar{\mu} \in \mathcal{C}$. Then $\overline{\lim}_n \bar{\mu}(E_n) \leq \lim_n (\max_{\mu\in\mathcal{C}} \mu(E_n)) = 0$ and $\overline{\lim}_n \bar{\mu}(E_n^c) \leq \lim_n (\max_{\mu\in\mathcal{C}} \mu(E_n^c)) = \max_{\mu\in\mathcal{C}} \mu(S) = m$. Hence $-\underline{\lim}_n (-m + \bar{\mu}(E_n)) = \overline{\lim}_n \bar{\mu}(E_n^c) \leq m$, $\underline{\lim}_n (-m + \bar{\mu}(E_n)) \geq -m$, $\underline{\lim}_n \bar{\mu}(E_n) \geq 0$. As wanted.

Let $\mathcal{C}$ consist of positive set functions.

(e) $\Rightarrow$ (a) Let $E_n \downarrow \emptyset$ and $\bar{\mu} \in \mathcal{C}$. Then $0 \leq \lim_n \bar{\mu}(E_n) \leq \lim_n (\max_{\mu\in\mathcal{C}} \mu(E_n)) = 0$. As wanted. ■

Marinacci and Montrucchio (2004, p. 55-57) show that if $\mu \in ba(\Sigma)$, $E \in \Sigma$ is null for μ iff $|\mu|(E) = 0$ (that is, E is null for $|\mu|$, or equivalently, $\mu(F) = 0$ for all $F \in \Sigma \cap E$), and they prove the following:

Lemma 4 *Let $\mathcal{C}$ be a τ_Σ compact subset of $ca_1(\Sigma)$ and $\upsilon(A) = \max_{\mu\in\mathcal{C}} \mu(A)$ for all $A \in \Sigma$.*

1. *$E \in \Sigma$ is υ-null iff it is μ-null for every $\mu \in \mathcal{C}$.*

2. *If $\mathcal{C} \ll \lambda \in ca^+(\Sigma)$, every λ-null set is υ-null.*

3. *If $\lambda \in \mathcal{C}^\#$ is such that $\mathcal{C} \lll \lambda$,[7] then $E \in \Sigma$ is υ-null iff it is λ-null.*

4. *υ is atomless iff $\mathcal{C} \subseteq na(\Sigma)$.*

[7] Such a λ always exists (see Gänssler, 1971, 2.6).

For the sake of completeness we provide a short proof.

Proof. 1. Let E be υ-null, $\mu \in \mathcal{C}$, and $F \in \Sigma \cap E$. Then

$$\mu(F) \leq \upsilon(F) = 0$$

and

$$\mu(F) = 1 - \mu(F^c) \geq 1 - \upsilon(F^c) = 1 - \upsilon(F^c \cup F) = 0.$$

Hence E is μ null. Conversely, if E is μ-null for every $\mu \in \mathcal{C}$, then

$$\upsilon(A \cup E) = \max_{\mu \in \mathcal{C}} \mu(A \cup E) = \max_{\mu \in \mathcal{C}} (\mu(A) + \mu(E) - \mu(A \cap E)) = \max_{\mu \in \mathcal{C}} \mu(A) = \upsilon(A).$$

2. If $\lambda(E) = 0$, then $\lambda(F) = 0$ for all $F \in \Sigma \cap E$, then $\mu(F) = 0$ for all $F \in \Sigma \cap E$ and all $\mu \in \mathcal{C}$. Apply point 1.

3. If E is υ-null, by point 1., it is μ-null for every $\mu \in \mathcal{C}$, hence it is $|\mu|$-null for every $\mu \in \mathcal{C}$, and it is λ-null. The converse follows from point 2.

4. Let $\lambda \in \mathcal{C}^{\#}$ be such that $\mathcal{C} \lll \lambda$. Notice that: (i) A is a υ-atom iff A is not υ-null and for every $B \in \Sigma \cap A$, either B or $A \setminus B$ is υ-null iff, by point 3., A is a λ-atom; (ii) λ belongs to the weak closure of the span of $|\mathcal{C}|$ in $ba(\Sigma)$, which coincides with the norm closure of the span of $|\mathcal{C}|$ in $ba(\Sigma)$.

Assume $\mathcal{C} \subseteq na(\Sigma)$. Then $|\mathcal{C}| \subseteq na(\Sigma)$, and, since $na(\Sigma)$ is a norm closed subspace of $ba(\Sigma)$, the norm closure of the span of $|\mathcal{C}|$ in $ba(\Sigma)$ is contained in $na(\Sigma)$. Conclude that λ has no atoms and hence υ has no atoms either.

Conversely if υ has no atoms, then λ has no atoms and $\mathcal{C} \ll \lambda$ implies $\mathcal{C} \subseteq na(\Sigma)$. ■

A.1 Proof of Theorem 1

Since we focus on the preferences of a single agent, we drop the subscript i. By definition, $\mathcal{X} = B_0([0,1])$. Under the identification of the constant elements of $B_0([0,1])$ with the elements of the interval $[0,1]$, Axioms 1, 2, 5, and the Mixture Space Theorem of Herstein and Milnor (1953) guarantee that there exists an affine function u representing $\succsim$ on $[0,1]$, u is unique up to a positive affine transformation, and Axiom 3 guarantees $u(1) > u(0)$. Hence the identity is the unique affine function representing $\succsim$ on $[0,1]$ and such that $u(0) = 0$, $u(1) = 1$.

In particular, $1 \succ \dfrac{1}{2} \succ 0$ and Axiom 2 guarantee that there exists $\varepsilon > 0$ (and $< \frac{1}{4}$) such that $1 \succ W \succ 0$ for all $W \in B_0\left(\left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)\right)$ (Axiom 2 implies continuity in the supnorm). For all $X \in B_0([0,1])$, $\varepsilon X + (1-\varepsilon)\frac{1}{2} \in B_0\left(\left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)\right)$.[8] Therefore, $1 \succ \varepsilon X + (1-\varepsilon)\frac{1}{2} \succ 0$ for all $X \in B_0([0,1])$.

Next we show that: For all X there exists a unique $\gamma = \gamma_X \in (0,1)$ such that

$$\varepsilon X + (1-\varepsilon)\frac{1}{2} \sim \gamma.$$

Set $\gamma = \sup\left\{\delta : \delta \prec \varepsilon X + (1-\varepsilon)\frac{1}{2}\right\}$, by Axiom 2, $\gamma \precsim \varepsilon X + (1-\varepsilon)\frac{1}{2}$ (hence $\gamma < 1$). If $\gamma \prec \varepsilon X + (1-\varepsilon)\frac{1}{2}$ there would exist a neighborhood of γ such that $W \prec \varepsilon X + (1-\varepsilon)\frac{1}{2}$ for all W in the neighborhood, hence $\tau \prec \varepsilon X + (1-\varepsilon)\frac{1}{2}$ for some $\tau > \gamma$, which is absurd. Clearly, it cannot be that $\gamma \neq \delta$ and $\delta \sim \varepsilon X + (1-\varepsilon)\frac{1}{2} \sim \gamma$.

Set $\nu(X) = \dfrac{\gamma_X - \frac{(1-\varepsilon)}{2}}{\varepsilon}$ for all $X \in B_0([0,1])$. If $\varepsilon X + (1-\varepsilon)\frac{1}{2} \sim \gamma_X$ and $\varepsilon Y + (1-\varepsilon)\frac{1}{2} \sim \gamma_Y$, Axiom 5 guarantees that

$$X \succsim Y \Leftrightarrow \varepsilon X + (1-\varepsilon)\frac{1}{2} \succsim \varepsilon Y + (1-\varepsilon)\frac{1}{2} \Leftrightarrow \gamma_X \succsim \gamma_Y \Leftrightarrow \gamma_X \geq \gamma_Y \Leftrightarrow \nu(X) \geq \nu(Y).$$

[8] $0 \leq X(s) \leq 1 \Rightarrow -1 < X(s) - \frac{1}{2} < 1 \Rightarrow -\varepsilon < \varepsilon\left(X(s) - \frac{1}{2}\right) < \varepsilon$, that is $\frac{1}{2} - \varepsilon < \varepsilon\left(X(s) - \frac{1}{2}\right) + \frac{1}{2} < \frac{1}{2} + \varepsilon$.

Notice that for all $X \in B_0([0,1])$ and $\alpha, \beta \in [0,1]$, Axiom 5 and $\varepsilon X + (1-\varepsilon)\frac{1}{2} \sim \gamma_X$ imply

$$\alpha\left(\varepsilon X + (1-\varepsilon)\frac{1}{2}\right) + (1-\alpha)\left(\varepsilon\beta + (1-\varepsilon)\frac{1}{2}\right) \sim \alpha\gamma_X + (1-\alpha)\left(\varepsilon\beta + (1-\varepsilon)\frac{1}{2}\right)$$

but $\alpha\left(\varepsilon X + (1-\varepsilon)\frac{1}{2}\right) + (1-\alpha)\left(\varepsilon\beta + (1-\varepsilon)\frac{1}{2}\right) = \varepsilon(\alpha X + (1-\alpha)\beta) + (1-\varepsilon)\frac{1}{2}$, whence

$$\varepsilon(\alpha X + (1-\alpha)\beta) + (1-\varepsilon)\frac{1}{2} \sim \alpha\gamma_X + (1-\alpha)\left(\varepsilon\beta + (1-\varepsilon)\frac{1}{2}\right) \in (0,1)$$

therefore

$$\begin{aligned}\nu(\alpha X + (1-\alpha)\beta) &= \frac{\alpha\gamma_X + (1-\alpha)\left(\varepsilon\beta + (1-\varepsilon)\frac{1}{2}\right) - \frac{(1-\varepsilon)}{2}}{\varepsilon} \\ &= \alpha\frac{\gamma_X - \frac{(1-\varepsilon)}{2}}{\varepsilon} + (1-\alpha)\beta = \alpha\nu(X) + (1-\alpha)\beta.\end{aligned}$$

We can conclude that ν is constant affine on $B_0([0,1])$, and denote by $\hat{\nu}$ its unique constant linear extension to $B_0(\mathbb{R})$. In particular, $\hat{\nu}$ is positively homogeneous.

Next we show that supnorm continuity of ν, and of $\hat{\nu}$, descend from Axiom 2, and constant linearity. Consider the subbase of the Euclidean topology of $\mathbb{R}$ consisting of all the open half-lines. Let $a \in \mathbb{R}$. If $a \in \nu(B_0([0,1]))$, say $a = \nu(X)$, the sets

$$\{Y \in B_0([0,1]) : \nu(Y) < a\} = \{Y \in B_0([0,1]) : Y \prec X\}$$

and

$$\{Y \in B_0([0,1]) : \nu(Y) > a\} = \{Y \in B_0([0,1]) : Y \succ X\}$$

are open (by Axiom 2). If $a \notin \nu(B_0([0,1]))$, since $\nu(B_0([0,1]))$ is an interval containing 0, then

- either $\nu(Y) > a$ for all $Y \in B_0([0,1])$, hence $\{\nu < a\} = \emptyset$ and $\{\nu > a\} = B_0([0,1])$;
- or $\nu(Y) < a$ for all $Y \in B_0([0,1])$, hence $\{\nu < a\} = B_0([0,1])$ and $\{\nu > a\} = \emptyset$.

Therefore ν is continuous on $B_0([0,1])$ and so is $\hat{\nu}$. If $\{X_n\} \subseteq B_0(\mathbb{R})$ and $X_n \to X$ in the supnorm, then $\{X_n\}$ is norm bounded and there exist $a > 0$ and $b \in \mathbb{R}$ such that $\{aX_n + b\} \subseteq B_0([0,1])$ and $aX_n + b \to aX + b$, this implies $\hat{\nu}(aX_n + b) \to \hat{\nu}(aX + b)$, and, by constant linearity $\hat{\nu}(X_n) \to \hat{\nu}(X)$, that is, $\hat{\nu}$ is continuous.

Next we show that Axiom 4 guarantees that $\hat{\nu}$ is subadditive. If $\hat{\nu}(X) = \hat{\nu}(Y)$, take $a > 0$ and $b \in \mathbb{R}$ such that $aX + b, aY + b \in B_0([0,1])$, hence $\nu(aX + b) = \nu(aY + b)$, it follows that

$$\begin{aligned}\hat{\nu}\left(\frac{1}{2}(aX+b) + \frac{1}{2}(aY+b)\right) &= \nu\left(\frac{1}{2}(aX+b) + \frac{1}{2}(aY+b)\right) \leq \nu(aX+b) \\ &= \frac{1}{2}\nu(aX+b) + \frac{1}{2}\nu(aY+b) = \frac{1}{2}\hat{\nu}(aX+b) + \frac{1}{2}\hat{\nu}(aY+b)\end{aligned}$$

whence $\hat{\nu}(X+Y) \leq \hat{\nu}(X) + \hat{\nu}(Y)$. If $\hat{\nu}(X) \neq \hat{\nu}(Y)$ there exists $b \in \mathbb{R}$ such that $\hat{\nu}(X) = \hat{\nu}(Y) + b = \hat{\nu}(Y+b)$, so $\hat{\nu}(X+Y+b) \leq \hat{\nu}(X) + \hat{\nu}(Y+b)$, and again $\hat{\nu}(X+Y) \leq \hat{\nu}(X) + \hat{\nu}(Y)$.

In particular, we have shown that $\hat{\nu} : B_0(\mathbb{R}) \to \mathbb{R}$ is positively homogeneous, subadditive, and supnorm continuous, therefore

$$\hat{\nu}(X) = \max_{\mu \in \mathcal{C}} \mu(X) \tag{3}$$

for all $X \in B_0(\mathbb{R})$, where $\mathcal{C}$ is the subdifferential $\partial\hat{\nu}(0)$ of $\hat{\nu}$ at 0 (see, e.g., Phelps, 1992). Two remarks are in order:

1. $\mathcal{C} = \partial\hat{\nu}(0)$ is weak* compact, i.e. τ_Σ compact, and convex.

2. For all $\mu \in \mathcal{C}$ and all $b \in \mathbb{R}$, $\mu(b\chi_S) \leq \hat{\nu}(b\chi_S) = b$, in particular $\mu(S) \leq 1$ and $-\mu(S) = \mu(-\chi_S) \leq -1$. That is $\mu(S) = 1$. Thus $\mathcal{C} \subseteq ba_1(\Sigma)$.

Assume $\mathcal{D}$ is another τ_Σ (weak*) compact and convex subset of $ba_1(\Sigma)$ such that

$$X \succsim Y \Leftrightarrow \max_{\mu \in \mathcal{D}} \mu(X) \geq \max_{\mu \in \mathcal{D}} \mu(Y).$$

For all $X \in B_0([0,1])$, since $\varepsilon X + (1-\varepsilon)\frac{1}{2} \sim \gamma_X$, then

$$\max_{\mu \in \mathcal{D}} \mu\left(\varepsilon X + (1-\varepsilon)\frac{1}{2}\right) = \gamma_X = \varepsilon\nu(X) + (1-\varepsilon)\frac{1}{2} = \max_{\mu \in \mathcal{C}} \mu\left(\varepsilon X + (1-\varepsilon)\frac{1}{2}\right),$$

whence $\max_{\mu \in \mathcal{D}} \mu(X) = \max_{\mu \in \mathcal{C}} \mu(X)$, it follows that $\max_{\mu \in \mathcal{D}} \mu(Y) = \max_{\mu \in \mathcal{C}} \mu(Y)$ for all $Y \in B_0(\mathbb{R})$ and $\mathcal{D} = \mathcal{C}$.

It remains to show that $\mathcal{C}$ consists of atomless measures. This will be done using Axiom 6 and (the monotone continuity descending from) Axiom 2 that have not been used up to this point.

Let $A_n \uparrow S$ and $\delta \in (0, 1/2)$. If $A_n \precsim 1-\delta$ for infinitely many n, then there exists an increasing subsequence A_{n_j} of A_n such that $A_{n_j} \uparrow S$ and $A_{n_j} \precsim 1-\delta$, hence, by Axiom 2, $1 \precsim 1-\delta$ and $1 \leq 1-\delta$, which is absurd. Therefore, eventually $A_n \succ 1-\delta$. If $\frac{1}{2}\chi_{A_n} \succsim \frac{1}{2} + \frac{\delta}{2}$ for infinitely many n, then there exists an increasing subsequence A_{n_j} of A_n such that $A_{n_j} \uparrow S$ and $\frac{1}{2}\chi_{A_{n_j}} \succsim \frac{1}{2} + \frac{\delta}{2}$, hence, by Axiom 2, $\frac{1}{2} \succsim \frac{1}{2} + \frac{\delta}{2}$ and $\frac{1}{2} \geq \frac{1}{2} + \frac{\delta}{2}$, which is absurd. Therefore, eventually $\frac{1}{2}\chi_{A_n} \prec \frac{1}{2} + \frac{\delta}{2}$. We can conclude that eventually

$$\max_{\mu \in \mathcal{C}} \mu(A_n) > 1-\delta \text{ and } \max_{\mu \in \mathcal{C}} \mu(A_n) < 1+\delta.$$

Since this is true for every $\delta \in (0, 1/2)$, then $\max_{\mu \in \mathcal{C}} \mu(A_n) \to 1$.

Let $B_n \downarrow \emptyset$, choose $\delta \in (0, 1/2)$. If $B_n \succsim \delta$ for infinitely many n, then there exists a decreasing subsequence B_{n_j} of B_n such that $B_{n_j} \downarrow \emptyset$ and $B_{n_j} \succsim \delta$, hence, by Axiom 2, $0 \succsim \delta$ and $0 \geq \delta$, which is absurd. Therefore, eventually $B_n \prec \delta$. If $\frac{1}{2} + \frac{1}{2}\chi_{B_n} \precsim \frac{1}{2} - \frac{\delta}{2}$ for infinitely many n, then there exists a decreasing subsequence B_{n_j} of B_n such that $B_{n_j} \downarrow \emptyset$ and $\frac{1}{2} + \frac{1}{2}\chi_{B_{n_j}} \precsim \frac{1}{2} - \frac{\delta}{2}$, hence, by Axiom 2, $\frac{1}{2} \precsim \frac{1}{2} - \frac{\delta}{2}$ and $\frac{1}{2} \leq \frac{1}{2} - \frac{\delta}{2}$, which is absurd. Therefore, eventually $\frac{1}{2} + \frac{1}{2}\chi_{B_n} \succ \frac{1}{2} - \frac{\delta}{2}$. We can conclude that eventually

$$\max_{\mu \in \mathcal{C}} \mu(B_n) < \delta \text{ and } \max_{\mu \in \mathcal{C}} \mu(B_n) > -\delta.$$

Since this is true for every $\delta \in (0, 1/2)$, then $\max_{\mu \in \mathcal{C}} \mu(B_n) \to 0$.

Lemma 3 guarantees that $\mathcal{C} \subseteq ca(\Sigma)$.

Notice that $E \in \Sigma$ is null for $\succsim$ if and only if $\max_{\mu \in \mathcal{C}} \mu(A \cup E) = \max_{\mu \in \mathcal{C}} \mu(A)$ for all $A \in \Sigma$, that is, if and only if E is null for the set function $\nu_{|\Sigma}$. As a consequence, $A \in \Sigma$ is an atom for $\succsim$ if and only if A is an atom for $\nu_{|\Sigma}$. Therefore, Axiom 6 guarantees that $\nu_{|\Sigma}$ has no atoms, and Lemma 4 implies that $\mathcal{C} \subseteq na(\Sigma)$.

The converse is a long, but relatively simple verification.

A.2 Proof of Theorem 2

Notice that Theorem 1 of Dubins and Spanier (1961) does not require the nonatomic measures $u_1, ..., u_n$ to be positive. It follows that the proof of their Corollary 1.1 holds unchanged provided $u_1, ..., u_n$ are nonatomic measures each with total mass 1; also Corollary 1.2 remains true in this case, but the proof needs some minor variation.[9]

[9] E.g., replace the sentence "Without loss of generality (by symmetry) we can suppose that $u_1(A)/\alpha_1 \geqq u_2(B)/\alpha_2$." with "By the Lyapunov Theorem, the range of the vector-valued measure (u_1, u_2) is convex and contains $(1,1)$ and $(u_1(A), u_2(A))$. Therefore, given a sequence β_k in $(0,1)$ which strictly increases to 1, for each

By Theorem 2, for all $i = 1, 2, ..., n$, there exists a product compact and convex set $\mathcal{C}_i \subseteq \mathcal{M}$ such that the function

$$\nu_i(X) = \max_{\mu \in \mathcal{C}_i} \mu(X)$$

for all X in $\mathcal{X}$ represents $\succsim_i$. Choose arbitrarily $\mu_i \in \mathcal{C}_i$. (The above extended version of) Corollary 1.1 of Dubins and Spanier (1961), guarantees the existence of a partition $\{A_1, A_2, ..., A_n\}$ of S in Σ such that $\mu_i(A_i) \geq \alpha_i$ for each $i = 1, 2, ..., n$, hence

$$\nu_i(A_i) \geq \mu_i(A_i) \geq \alpha_i = \nu_i(\alpha_i \chi_S) \text{ that is } A_i \succsim_i \alpha_i \chi_S$$

for each $i = 1, 2, ..., n$. While, if $\succsim_j \neq \succsim_k$, it must be $\mathcal{C}_j \neq \mathcal{C}_k$. Choose $\mu_j \in \mathcal{C}_j$ and $\mu_k \in \mathcal{C}_k$ such that $\mu_j \neq \mu_k$, and $\mu_i \in \mathcal{C}_i$ arbitrarily, if $i \neq j, k$. If $\alpha_1, \alpha_2, ..., \alpha_n > 0$, by (the above extended version of) Corollary 1.2 of Dubins and Spanier (1961), there exists a partition $\{A_1, A_2, ..., A_n\}$ of S in Σ such that $\mu_i(A_i) > \alpha_i$ for each $i = 1, 2, ..., n$, hence

$$\nu_i(A_i) \geq \mu_i(A_i) > \alpha_i = \nu_i(\alpha_i \chi_S) \text{ that is } A_i \succ_i \alpha_i \chi_S$$

for each $i = 1, 2, ..., n$.

A.3 Proof of Corollary 1

For all $i = 1, 2, ..., n$, set $\mathcal{C}_i = \{\mu \in \mathcal{M} : \mu(A) \leq \upsilon_i(A) \quad \forall A \in \Sigma\}$. Using the results of Marinacci and Montrucchio (2004) it can be shown that $\mathcal{C}_i$ is non-empty, product compact, convex, and

$$\upsilon_i(A) = \max_{\mu \in \mathcal{C}_i} \mu(A)$$

for all A in Σ. Define $\succsim_i$ on $\mathcal{X}$ by $X \succsim_i Y \Leftrightarrow \max_{\mu \in \mathcal{C}_i} \mu(X) \geq \max_{\mu \in \mathcal{C}_i} \mu(Y)$ for all $i = 1, 2, ..., n$, and apply Theorems 2 and 3.

A.4 On Polytopes

Let $n \geq 2$, and $\Delta = \Delta^n$ be the $(n-1)$-dimensional simplex, i.e. the convex hull of the orthonormal base $\{e^1, \ldots, e^n\}$ of $\mathbb{R}^n$. From now on, we will sometimes write N instead of $\{1, 2, \ldots, n\}$. If $T \subseteq N$ is non-empty, Δ^T is convex hull of $\{e^j : j \in T\}$.

Let δ^i denote the projection on the i-th component ($\delta^i(x) = x_i$), and set $H = \left[\sum_{i \in N} \delta^i = 1\right]$, then

$$\Delta = H \cap \left[\delta^1 \geq 0\right] ... \cap \left[\delta^n \geq 0\right],$$

and aff $\Delta = H$.

We recall that:

- A hyperplane in H is a set of the form $[f = 0]$ where f is a nonconstant affine functional $f : H \to \mathbb{R}$.
- If ψ and ϕ are nonzero linear functional on $\mathbb{R}^n$ and $a \in \mathbb{R}$, then $\phi_{|[\psi = a]}$ is nonconstant if and only if ϕ is not a scalar multiple of ψ.

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k there exists a measurable A_k such that

$$(u_1(A_k), u_2(A_k)) = (\beta_k + (1 - \beta_k) u_1(A), \beta_k + (1 - \beta_k) u_2(A)).$$

Clearly $(u_1(A_k), u_2(A_k)) \to (1, 1)$, and hence it is possible to choose k' large enough that, denoting by $B_{k'}$ the complement of $A_{k'}$,

$$u_1(A_{k'}) > u_2(A_{k'}) > \alpha_1 > 0, \quad \frac{u_2(B_{k'})}{u_1(A_{k'})} < \frac{\alpha_2}{\alpha_1}, \quad 0 < 1 + \frac{\alpha_1}{u_1(A_{k'})}(1 - u_1(A_{k'}) - u_2(B_{k'})) < 1.$$

Replace A with $A_{k'}$."

- If ψ is a nonzero linear functional on $\mathbb{R}^n$, $\phi_1, ..., \phi_m$ are nonzero linear functionals on $\mathbb{R}^n$ none of which is a scalar multiple of ψ, and $a, b_1, ..., b_m$ are real numbers, then

$$\text{int}_{[\psi=a]}\left([\psi = a] \cap [\phi_1 \geq b_1] \cap ... \cap [\phi_m \geq b_m]\right) = [\psi = a] \cap [\phi_1 > b_1] \cap ... \cap [\phi_m > b_m]. \quad (4)$$

In particular, since $[\psi = a] \cap [\phi_1 \geq b_1] \cap ... \cap [\phi_m \geq b_m]$ is closed in $[\psi = a]$,

$$\text{bdry}_{[\psi=a]}([\psi = a] \cap [\phi_1 \geq b_1] \cap ... \cap [\phi_m \geq b_m]) \subseteq \bigcup_{k=1}^{m} [\psi = a] \cap [\phi_k - b_k = 0]$$

and the latter set is a finite union of hyperplanes in $[\psi = a]$.

We now consider $p = (p_1, \ldots, p_N) \in \Delta$. For all $h \in N$, the polytope of Δ generated by p and $\{e^j : j \in N - \{h\}\}$ is

$$\begin{aligned}\Delta_h(p) = \text{conv}(p, \Delta^{N-\{h\}}) &= \left\{ t_h p + \sum_{j \neq h} t_j e^j : \sum_{j \in N} t_j = 1, t_j \geq 0 \quad \forall j \in N \right\} \\ &= \left\{ (t_h p_h, t_h p_{-h} + t_{-h}) : t \in \Delta^N \right\}\end{aligned}$$

where p_{-h} (resp. t_{-h}) is the vector obtained by eliminating the h-th component of p (resp. t).

The following lemmas determine whether a given point belongs to one of these polytopes. We need to consider two different cases, depending on the value of p_h.

Lemma 5 *Let $p_h \neq 0$, and $x = (x_1, \ldots, x_n) \in \Delta$. Then*

1. *$x \in \Delta_h(p)$ if and only if*
$$p_j x_h - p_h x_j \leq 0 \qquad \forall j \neq h. \quad (5)$$

2. *The following facts are equivalent:*

   (i) $x \in \text{int}_H \Delta_h(p)$;

   (ii) $x_h > 0$ and
$$p_j x_h - p_h x_j < 0 \qquad \forall j \neq h; \quad (6)$$

   (iii) $x = (t_h p_h, t_h p_{-h} + t_{-h})$ for some $t \in \text{int}_H \Delta$.[10]

Proof. Let
$$x = (t_h p_h, t_h p_{-h} + t_{-h}) \quad (7)$$
for some $t \in \mathbb{R}^N$.

For all $j \neq h$

$$p_j x_h - p_h x_j = p_j(t_h p_h) - p_h(t_h p_j + t_j) = p_j p_h t_h - p_j p_h t_h - p_h t_j = -p_h t_j \quad (8)$$

hence
$$p_j x_h - p_h x_j \leq 0 \Leftrightarrow t_j \geq 0, \quad (9)$$
and analogously
$$p_j x_h - p_h x_j < 0 \Leftrightarrow t_j > 0; \quad (10)$$
moreover, for $j = h$
$$x_h > 0 \Leftrightarrow t_h > 0. \quad (11)$$

[10]In particular, $\text{int}_H \Delta_h(p)$ is not empty since $\text{int}_H \Delta$ is not empty.

1. If $x \in \Delta_h(p)$, then $x = (t_h p_h, t_h p_{-h} + t_{-h})$ for some $t \in \Delta^N$, then, by (9), $p_j x_h - p_h x_j \leq 0$ for all $j \neq h$. Conversely, assume $x \in \Delta$ and $p_j x_h - p_h x_j \leq 0$ for all $j \neq h$. Define

$$t_h = \frac{x_h}{p_h} \text{ and } t_j = x_j - t_h p_j \qquad \forall j \neq h,$$

then

$$x_h = t_h p_h \text{ and } x_j = t_h p_j + t_j \qquad \forall j \neq h,$$

that is $x = (t_h p_h, t_h p_{-h} + t_{-h})$. Moreover, hypotheses (5) and (9) guarantee that $t_j \geq 0$ for all $j \neq h$, and clearly $t_h \geq 0$. A simple computation yields $\sum_i t_i = 1$. In fact

$$\begin{aligned}\sum_i t_i &= \frac{x_h}{p_h} + \sum_{j\neq h}\left(x_j - \frac{x_h p_j}{p_h}\right) = \frac{x_h}{p_h} + \sum_{j\neq h} x_j - \frac{x_h}{p_h}\sum_{j\neq h} p_j \\ &= \frac{x_h}{p_h} + (1 - x_h) - \frac{x_h}{p_h}(1 - p_h) = \frac{x_h}{p_h} + 1 - x_h - \frac{x_h}{p_h} + x_h = 1.\end{aligned}$$

Thus $x \in \Delta_h(p)$.

2. By 1.,

$$\Delta_h(p) = \Delta \cap \Delta_h(p) = H \cap \bigcap_i [x_i \geq 0] \cap \bigcap_{j\neq h} [p_j x_h - p_h x_j \leq 0]$$

by (4),

$$\text{int}_H \Delta_h(p) = H \cap \bigcap_i [x_i > 0] \cap \bigcap_{j\neq h} [p_j x_h - p_h x_j < 0]\,. \tag{12}$$

Then (i) $\Rightarrow$ (ii).

If (ii) holds, by 1., $x \in \Delta_h(p)$, hence $x = (t_h p_h, t_h p_{-h} + t_{-h})$ for some $t \in \Delta^N$; apply (10) and (11) to obtain $t_j > 0$ for all $j \in N$. That is (ii) $\Rightarrow$ (iii).

Finally, if $x = (t_h p_h, t_h p_{-h} + t_{-h})$ for some $t \in \text{int}_H \Delta$, it is obvious that $x \in H \cap \bigcap_i [x_i > 0]$, (10) delivers $x \in \bigcap_{j\neq h} [p_j x_h - p_h x_j < 0]$ and (12) yields $x \in \text{int}_H \Delta_h(p)$. That is (iii) $\Rightarrow$ (i). ■

Lemma 6 *If $p_h = 0$ then $\Delta_h(p) = \Delta^{N-\{h\}} = \{x \in \Delta : x_h = 0\}$.*

Proof. Just notice that in this case $p \in \Delta^{N-\{h\}}$. ■

In particular, if $p_h = 0$ then $\text{int}_H \Delta_h(p) = \emptyset$, in fact,

$$\Delta^{N-\{h\}} = H \cap \bigcap_i [x_i \geq 0] \cap [x_h \leq 0]$$

and by (4)

$$\text{int}_H \Delta^{N-\{h\}} = H \cap \bigcap_i [x_i > 0] \cap [x_h < 0] = \emptyset.$$

Proposition 2 *If $\{p^m\}_{m\in\mathbb{N}} \subseteq \Delta^N$ converges to p, then for each $h \in N$ and each $y \in \Delta - \text{bdry}_H \Delta_h(p)$*

$$\chi_{\Delta_h(p^m)}(y) \to \chi_{\Delta_h(p)}(y) \qquad \text{as } m \to +\infty. \tag{13}$$

Proof. Since $\Delta_h(p)$ is closed in H, then $\text{bdry}_H \Delta_h(p) = \Delta_h(p) - \text{int}_H (\Delta_h(p))$, therefore

$$\begin{aligned}\Delta - \text{bdry}_H \Delta_h(p) &= \Delta - (\Delta_h(p) - \text{int}_H (\Delta_h(p))) = \Delta \cap \left(\Delta_h(p) \cap [\text{int}_H (\Delta_h(p))]^c\right)^c \\ &= \Delta \cap (\Delta_h(p)^c \cup \text{int}_H (\Delta_h(p))) = (\Delta - \Delta_h(p)) \cup \text{int}_H (\Delta_h(p))\,.\end{aligned}$$

The proof is divided into cases depending on the position of y and the value of p_h. All parts use contradiction arguments. Throughout the proof h is fixed.

Case 1: $y \in \Delta - \Delta_h(p)$ and $p_h > 0$. By Lemma 5, this implies the existence of some $j^* \neq h$ for which

$$p_{j^*} y_h - p_h y_{j^*} > 0. \tag{14}$$

Now suppose that the convergence in the statement does not hold, so there exists a subsequence $\{p^{m_l}\}$ of $\{p^m\}$ for which

$$\chi_{\Delta_h(p^{m_l})}(y) = 1 \qquad \forall l \in \mathbb{N}$$

or, equivalently,

$$y \in \Delta_h(p^{m_l}) \qquad \forall l \in \mathbb{N}. \tag{15}$$

Lemma 5 then implies

$$p_j^{m_l} y_h - p_h^{m_l} y_j \leq 0 \qquad \forall j \neq h \text{ and } l \in \mathbb{N}. \tag{16}$$

Passing to the limit as $l \to +\infty$, for $j = j^*$, we obtain

$$p_{j^*} y_h - p_h y_{j^*} \leq 0$$

a contradiction.

Case 2: $y \in \Delta - \Delta_h(p)$ and $p_h = 0$. In this case, according to Lemma 6, $y_h > 0$. Now suppose that the convergence in the statement does not hold, so there exists a subsequence $\{p^{m_l}\}$ of $\{p^m\}$ for which

$$\chi_{\Delta_h(p^{m_l})}(y) = 1 \qquad \forall l \in \mathbb{N}$$

or, equivalently,

$$y \in \Delta_h(p^{m_l}) \qquad \forall l \in \mathbb{N}. \tag{17}$$

Lemma 5 then implies

$$p_j^{m_l} y_h - p_h^{m_l} y_j \leq 0 \qquad \forall j \neq h \text{ and } l \in \mathbb{N}. \tag{18}$$

Passing to the limit as $l \to +\infty$,

$$p_j y_h \leq 0 \qquad \forall j \neq h.$$

But since $p_{j^*} > 0$ for some $j^* \neq h$, then $y_h = 0$, which is absurd.

Case 3: $y \in \text{int}_H \Delta_h(p)$. Then $p_h > 0$ (otherwise $\text{int}_H \Delta_h(p) = \emptyset$). Lemma 5 implies

$$p_j y_h - p_h y_j < 0 \qquad \forall j \neq h. \tag{19}$$

Once again, suppose that the convergence in the statement does not hold, so there exists a subsequence $\{p^{m_l}\}$ of $\{p^m\}$ for which

$$\chi_{\Delta_h(p^{m_l})}(y) = 0 \qquad \forall l \in \mathbb{N}$$

that is

$$y \notin \Delta_h(p^{m_l}) \qquad \forall l \in \mathbb{N}.$$

Lemma 5 implies the existence of a sequence of indexes $\{j_{m_l}\} \in N - \{h\}$ such that

$$p_{j_{m_l}}^{m_l} y_h - p_h^{m_l} y_{j_{m_l}} > 0 \qquad \forall l \in \mathbb{N}.$$

We can further extract a subsequence $\{p^{m_{l_i}}\}$ of $\{p^{m_l}\}$ such that the previous inequality holds for a single index $j^* \in N - \{h\}$:

$$p_{j^*}^{m_{l_i}} y_h - p_h^{m_{l_i}} y_{j^*} > 0 \qquad \forall i \in \mathbb{N}.$$

Since $\{p^{m_{l_i}}\}$ converges to p, passing to the limit, we obtain

$$p_{j^*} y_h - p_h y_{j^*} \geq 0$$

again a contradiction. ∎

Proposition 3 *For all* $p \in \Delta^N$, $\{\Delta_h(p)\}_{h\in N}$ *is a partition of* Δ *into polytopes.*

Proof. Let $x \in \Delta$. If $x_h = 0$ for some $h \in N$, then, by Lemma 6, $x \in \Delta_h(p)$. Else $x_j > 0$ for all $j \in N$, consider the vector $\left(\frac{p_1}{x_1}, \frac{p_2}{x_2}, ..., \frac{p_n}{x_n}\right)$ and choose $h \in N$ such that

$$\frac{p_h}{x_h} \geq \frac{p_j}{x_j} \quad \forall j \neq h.$$

Clearly $p_h > 0$ and $p_j x_h - p_h x_j \leq 0$ for all $j \neq h$, by Lemma 5, $x \in \Delta_h(p)$. This shows that $\Delta = \bigcup_{h\in N} \Delta_h(p)$.

Let $k, i \in N$, $k \neq i$. If $p_k p_i = 0$, then either $\mathrm{int}_H \Delta_k(p) = \emptyset$ or $\mathrm{int}_H \Delta_i(p) = \emptyset$, in any case $\mathrm{int}_H \Delta_k(p) \cap \mathrm{int}_H \Delta_i(p) = \emptyset$. Else $p_k > 0$ and $p_i > 0$. If there exists $x \in \mathrm{int}_H \Delta_k(p) \cap \mathrm{int}_H \Delta_i(p)$, then by Lemma 5:

$$x_k > 0 \text{ and } p_i x_k - p_k x_i < 0$$

since $x \in \mathrm{int}_H \Delta_k(p)$, and

$$x_i > 0 \text{ and } p_k x_i - p_i x_k < 0$$

since $x \in \mathrm{int}_H \Delta_i(p)$, therefore $p_i x_k < p_k x_i$ and $p_k x_i < p_i x_k$, which is absurd. This shows that the interiors are disjoint. ■

A.5 On subintervals

Let $I = [0, 1]$. For all $p \in \Delta^N$ set $p_0 = 0$ and $I_h(p) = \left[\sum_{j=0}^{h-1} p_j, \sum_{j=0}^{h} p_j\right]$ for all $h \in N$.

Proposition 4 *If* $\{p^m\}_{m\in\mathbb{N}} \subseteq \Delta^N$ *converges to* p*, then for each* $h \in N$ *and each* $y \in I - \mathrm{bdry} I_h(p)$

$$\chi_{I_h(p^m)}(y) \to \chi_{I_h(p)}(y) \qquad \text{as } m \to +\infty.$$

Proof. Fix $h \in N$ arbitrarily. Let $a_m = \sum_{j=0}^{h-1} p_j^m$, $a = \sum_{j=0}^{h-1} p_j$, $b_m = \sum_{j=0}^{h} p_j^m$, $b = \sum_{j=0}^{h} p_j$, then $0 \leq a_m \leq b_m \leq 1$, $a_m \to a$, $b_m \to b$, $\chi_{I_h(p^m)} = \chi_{[a_m,b_m]}$, $\chi_{I_h(p)} = \chi_{[a,b]}$, and $\mathrm{bdry} I_h(p) = \{a, b\}$. It is sufficient to show that, for all $y \notin \{a, b\}$,

$$\chi_{[a_m,b_m]}(y) \to \chi_{[a,b]}(y) \qquad \text{as } m \to +\infty.$$

Clearly $a \leq b$. If $y < a$, eventually $y < a_m$, thus eventually $\chi_{[a_m,b_m]}(y) = 0 = \chi_{[a,b]}(y)$. If $a < y < b$, eventually $a_m < y$ and eventually $b_m > y$, thus eventually $a_m < y < b_m$ and $\chi_{[a_m,b_m]}(y) = 1 = \chi_{[a,b]}(y)$. Finally, if $y > b$, eventually $y > b_m$, thus eventually $\chi_{[a_m,b_m]}(y) = 0 = \chi_{[a,b]}(y)$. ■

Obviously, for all $p \in \Delta^N$, $\{I_h(p)\}_{h\in N}$ is a partition of I into polytopes.

A.6 Proof of Proposition 1

Notice that Axiom 6 only appears in the last paragraph of the proof of Theorem 2. Thus Axioms 1 - 5 guarantee that, for all $i \in N$, there exist a τ_Σ compact and convex set $\mathcal{C}_i \subseteq ca_1(\Sigma)$ such that

$$\nu_i(X) = \max_{\mu\in\mathcal{C}_i} \mu(X)$$

for all X in $\mathcal{X}$ represents $\succsim_i$. Denote by υ_i the restriction of ν_i to the Lebesgue σ-field Σ. For all $i \in N$, if E is contained in the intersection of S with any finite union of hyperplanes in aff$S$, then, by Axiom 7, $A \cup E \sim_i A$, hence

$$\upsilon_i(A \cup E) = \max_{\mu\in\mathcal{C}_i} \mu(A \cup E) = \max_{\mu\in\mathcal{C}_i} \mu(A) = \upsilon_i(A),$$

that is E is υ_i-null. For all $i \in N$, take $\lambda_i \in \mathcal{C}_i^{\#}$ such that $\mathcal{C}_i \lll \lambda_i$, then (by Lemma 4) $E \in \Sigma$ is υ_i-null iff it is λ_i-null. Set $\lambda = \sum_{i\in N} \lambda_i$. For all $i \in N$, $\mathcal{C}_i \ll \lambda \in ca^+(\Sigma)$ and if E is contained in the intersection of S with any finite union of hyperplanes in aff$S$, then $\lambda_i(E) = 0$ for all $i \in N$, hence $\lambda(E) = 0$.

Let $S = \Delta^N$. For all $i, j \in N$ consider the function from Δ to $\mathbb{R}$ defined by

$$u_i^j : p \mapsto \upsilon_i(\Delta_j(p)).$$

If $p^m \to p$, then Proposition 2 guarantees that, for all $y \in \Delta - \mathrm{bdry}_H \Delta_j(p)$,

$$\chi_{\Delta_j(p^m)}(y) \to \chi_{\Delta_j(p)}(y) \qquad \text{as } m \to +\infty,$$

but $\lambda(\mathrm{bdry}_H \Delta_j(p)) = 0$, that is, $\chi_{\Delta_j(p^m)} \to \chi_{\Delta_j(p)}$ λ-almost everywhere; Lemma 2 implies that

$$\lim_m \upsilon_i(\Delta_j(p^m)) = \lim_m \left(\sup_{\mu\in\mathcal{C}_i} \int \chi_{\Delta_j(p^m)} d\mu \right) = \sup_{\mu\in\mathcal{C}_i} \int \chi_{\Delta_j(p)} d\mu = \upsilon_i(\Delta_j(p)).$$

Therefore u_i^j is continuous.

Let $S = I$. For all $i, j \in N$ consider the function from Δ to $\mathbb{R}$ defined by

$$u_i^j : p \mapsto \upsilon_i(I_j(p)).$$

If $p^m \to p$, then Proposition 4 guarantees that, for all $y \in I - \mathrm{bdry} I_j(p)$,

$$\chi_{I_j(p^m)}(y) \to \chi_{I_j(p)}(y) \qquad \text{as } m \to +\infty,$$

but $\lambda(\mathrm{bdry} I_j(p)) = 0$, that is, $\chi_{I_j(p^m)} \to \chi_{I_j(p)}$ λ-almost everywhere; Lemma 2 implies that

$$\lim_m \upsilon_i(I_j(p^m)) = \lim_m \left(\sup_{\mu\in\mathcal{C}_i} \int \chi_{I_j(p^m)} d\mu \right) = \sup_{\mu\in\mathcal{C}_i} \int \chi_{I_j(p)} d\mu = \upsilon_i(I_j(p)).$$

Therefore u_i^j is continuous.

Set

$$C_i^j = \left\{ p \in \Delta^N : u_i^j(p) \geq u_i^k(p) \quad \text{for all } k \in N \right\}.$$

For each $i \in N$, and all $p \in \Delta^N$ consider the vector $\left(u_i^1(p), u_i^2(p), ..., u_i^n(p)\right)$ and choose $j_p \in N$ such that $u_i^{j_p}(p) \geq u_i^k(p)$ for all $k \in N$, then $p \in C_i^{j_p}$; therefore, the collection $\left\{C_i^j\right\}_{j\in N}$ is a cover of Δ^N. Moreover, for all $i, j \in N$

$$C_i^j = \bigcap_{k\in N} \left\{ p \in \Delta^N : u_i^j(p) - u_i^k(p) \geq 0 \right\},$$

continuity of the u_i^ks guarantees that C_i^j is closed.

Next we show that $C_i^j \cap \Delta^{N-\{j\}} = \emptyset$ for all $i, j \in N$.

Let $S = \Delta^N$. If $p \in \Delta^{N-\{j\}}$, then $\Delta_j(p) = \Delta^{N-\{j\}} = \Delta \cap [x_j = 0]$ is $\succsim_i$-null, hence

$$u_i^j(p) = \upsilon_i(\Delta_j(p)) = \upsilon_i(\Delta_j(p) \cup \emptyset) = \upsilon_i(\emptyset) = 0.$$

Moreover, since $\bigcup_{k\in N} \mathrm{bdry}_H \Delta_k(p)$ is $\succsim_i$-null, then

$$\Delta_k(p) = \mathrm{bdry}_H \Delta_k(p) \cup \mathrm{int}_H \Delta_k(p) \sim_i \mathrm{int}_H \Delta_k(p) \quad \forall k \in N \text{ and}$$
$$S = \bigcup_{k\in N} \mathrm{bdry}_H \Delta_k(p) \cup \bigsqcup_{k\in N} \mathrm{int}_H \Delta_k(p) \sim_i \bigsqcup_{k\in N} \mathrm{int}_H \Delta_k(p)$$

where $\bigsqcup$ denotes a disjoint union. Subadditivity of ν_i implies

$$1 = \upsilon_i(S) = \upsilon_i\left(\bigsqcup_{k\in N} \mathrm{int}_H \Delta_k(p) \right) \leq \sum_{k\in N} \upsilon_i(\mathrm{int}_H \Delta_k(p)) = \sum_{k\in N} \upsilon_i(\Delta_k(p)) = \sum_{k\in N} u_i^k(p)$$

therefore there exists $k \in N$ such that $u_i^k(p) > 0$, in particular $u_i^j(p) = 0 < u_i^k(p)$ and $p \notin C_i^j$.

Let $S = I$. If $p \in \Delta^{N-\{j\}}$, then $p_j = 0$ and $I_j(p)$ is a singleton, and hence it is $\succsim_i$-null. Therefore

$$u_i^j(p) = \upsilon_i(I_j(p)) = \upsilon_i(I_j(p) \cup \emptyset) = \upsilon_i(\emptyset) = 0.$$

Moreover, since $\bigcup_{k \in N} \text{bdry} I_k(p)$ is $\succsim_i$-null, then

$$I_k(p) = \text{bdry} I_k(p) \cup \text{int} I_k(p) \sim_i \text{int} I_k(p) \quad \forall k \in N \text{ and}$$
$$S = \bigcup_{k \in N} \text{bdry} I_k(p) \cup \bigsqcup_{k \in N} \text{int} I_k(p) \sim_i \bigsqcup_{k \in N} \text{int} I_k(p).$$

Subadditivity of ν_i implies

$$1 = \upsilon_i(S) = \upsilon_i\left(\bigsqcup_{k \in N} \text{int} I_k(p)\right) \leq \sum_{k \in N} \upsilon_i(\text{int} I_k(p)) = \sum_{k \in N} \upsilon_i(I_k(p)) = \sum_{k \in N} u_i^k(p)$$

therefore there exists $k \in N$ such that $u_i^k(p) > 0$, in particular $u_i^j(p) = 0 < u_i^k(p)$ and $p \notin C_i^j$.

Therefore, for each $i \in N$, $\Delta^T \subseteq \bigcup_{j \in T} C_i^j$ for all non-empty $T \subseteq N$. In fact, if $p \in \Delta^T$, then $p_j = 0$ for all $j \in N - T$, therefore $p \in \Delta^{N-\{j\}}$ for all $j \in N - T$, hence $p \notin C_i^j$ for all $j \in N - T$ and $p \in \bigcup_{j \in N} C_i^j = \Delta^N$, it must be the case that $p \in \bigcup_{j \in T} C_i^j$.

Summing up: for each $i \in N$, $\left\{C_i^j\right\}_{j \in N}$ is a closed cover of Δ^N, and $\Delta^T \subseteq \bigcup_{j \in T} C_i^j$ for all non-empty $T \subseteq N$. Gale (1984)'s Lemma implies that there exist a permutation σ of N and a point $p^* \in \Delta^N$ such that

$$p^* \in C_i^{\sigma(i)} \quad \forall i \in N. \tag{20}$$

Equation (20) says that for all $i \in N$, $u_i^{\sigma(i)}(p^*) \geq u_i^k(p^*)$ for all $k \in N$, that is

- $\upsilon_i\left(\Delta_{\sigma(i)}(p)\right) \geq \upsilon_i(\Delta_k(p))$ for all $i, k \in N$ if $S = \Delta^N$, in this case set $A_j = \Delta_{\sigma(j)}(p^*)$ for all $j \in N$;

- $\upsilon_i\left(I_{\sigma(i)}(p)\right) \geq \upsilon_i(I_k(p))$ for all $i, k \in N$ if $S = I$, in this case set $A_j = I_{\sigma(j)}(p^*)$ for all $j \in N$.

In any case, $\{A_1, A_2, \ldots, A_n\}$ is a partition of S into polytopes and, for all $i, k \in N$, $\upsilon_i(A_i) \geq \upsilon_i\left(A_{\sigma^{-1}(k)}\right)$, that is, $A_i \succsim_i A_{\sigma^{-1}(k)}$, or $A_i \succsim_i A_j$ for all $i, j \in N$, as wanted.

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