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Elisa Luciano
Patrizia Semeraro

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A Generalized Normal Mean-Variance Mixture for Return Processes in Finance[1]

Elisa Luciano[2] and Patrizia Semeraro[3]

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[2]Dipartimento di Statistica e Matematica Applicata D. De Castro, Università degli Studi di Torino, ICER and Collegio Carlo Alberto.

[3]Dipartimento di Statistica e Matematica Applicata D. De Castro, Università degli Studi di Torino

Abstract

Time-changed Brownian motions are extensively applied as mathematical models for asset returns in Finance. Time change is interpreted as a switch to trade-related business time, different from calendar time. Time-changed Brownian motions can be generated by infinite divisible normal mixtures. The standard multivariate normal mean variance mixtures assume a common mixing variable. This corresponds to a multidimensional return process with a unique change of time for all assets under exam. The economic counterpart is uniqueness of trade or business time, which is not in line with empirical evidence.

In this paper we propose a new multivariate definition of normal mean-variance mixtures with a flexible dependence structure, based on the economic intuition of both a common and an idiosyncratic component of business time. We analyze both the distribution and the related process.

We use the above construction to introduce a multivariate generalized hyperbolic process with generalized hyperbolic margins. We conclude with a stock market example to show the ease of calibration of the model.

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Introduction

We aim at providing a multidimensional model for financial asset pricing based on a generalization of the traditional multivariate normal mixtures and multivariate time-changed Brownian motions.

This class of processes has been introduced in the financial literature by Clark [11] to model the departure of returns from normality. The idea underlying his work is that, even though returns are normal in calendar time, the latter may not be appropriate to represent financial-market time. Business time depends on the arrival of information and can often be proxied by trade. A change is needed in order to go from business time to the calendar time needed in modelling. The generality of the models proposed by Clark is supported by the fact that any arbitrage free return process can be written as a time changed Brownian motion[1].

In the Lévy environment, univariate subordinators (see Sato [23] on this matter) are used to time change a Brownian motion and introduce a stochastic clock (see Geman, Madan and Yor [15]). Different Lévy processes, discussed in the financial literature, can be represented as time-changed Brownian motions: the variance gamma process, introduced in Madan and Seneta [20], the normal inverse Gaussian introduced by Barndorff-Nielsen [4], the CGMY in Carr et al. [10], the hyperbolic and generalized hyperbolic ones, defined by Barndorff-Nielsen [2] and applied to finance by Eberlein [12] and Eberlein and Prause [13].

The law at time one of a time-changed Brownian motion is a normal mean-variance distribution, that has been extensively studied from a statistical perspective. Among the others, Kelker [17] studied the infinite divisibility of such distributions, Barndorff-Nielsen et al. [7] focused on the n dimensional case.

Both normal mean-variance distributions and time-changed Lévy processes have been extended to the multivariate setting. The extensions proposed in the literature are based, respectively, on a common mixing distribution and a common time change. The financial meaning is that the corresponding assets have a common business time. This last assumption seems to be quite restrictive in the stock market setting (see for instance Harris [16] and Lo and Wang [18]). Since the change of time has trade as a proxy, a more realistic assumption is that each return has its own change of time (each marginal distribution its own mixing variable).

Here we propose to adopt multidimensional mixing distributions. We use a feature of trade which has been recently explored by the financial literature: the fact that trade over different stocks or assets presents a common component. This is the key ingredient to our modelling approach: since it stems from empirics, it seems to us a due base for

[1] Monroe [22] established that any semimartingale can be written as a time-changed Brownian motion; see also Ané and Geman [1] and references therein.

modelling. The above argument supports our choice of a one factor structure for the change of time. The construction is therefore based on a random-additive-effect model, as introduced in Barndorff-Nielsen et al [6]. The one factor change of time has been used in Semeraro [25] and Luciano and Semeraro [19] to generalize the multivariate variance gamma and other processes of interest in Finance. Their models can be derived as particular cases of the present one.

As an example of parametrical Lévy model arising from our general mean-variance mixture we study a generalized hyperbolic (GH) distribution. We propose a multivariate GH distribution different from the popular one, introduced by Barndorff-Nielsen [2]. We discuss the features of the (linear and non linear) dependence introduced and study a methodology for dependence calibration, once the marginal parameters are arbitrarily fixed. We conclude with a stock market example to show that the generalized GH process is easy to calibrate.

The paper is organized as follows. Section 1 recalls some notations. Section 2 defines the generalized normal mean-variance distribution, provides conditions for infinite divisibility and introduces the corresponding Lévy model. We prove that the latter is a subordinated Brownian motion and characterize the multidimensional subordinator. In Section 3 we focus on the generalized hyperbolic example. We give its characteristic function and provide a method to determine the Lévy triplet. We specify two subcases. In Section 4 we analyze the dependence structure of the model focusing on linear correlation. Linear correlation is indeed relevant for financial applications. For fixed margins, at least in the basic case, it identifies the joint distribution of the mixing variable and then of the whole mixture. Section 5 provides a method to calibrate the model on data and discusses an example. The proofs are in the appendix.

1 Notations

With capital upshape bold letters $\mathbf{X}$, we denote $\mathbb{R}^n$ - valued random variables $\mathbf{X} =: (\mathrm{X}_1, ..., \mathrm{X}_n)^T$, where T stands for the transpose and vectors are column vectors. We set $\sqrt{\mathbf{X}} = (\sqrt{\mathrm{X}_1}, ..., \sqrt{\mathrm{X}_n})^T$. $\psi_{\mathbf{X}}$ and $\Psi_{\mathbf{X}}$ represent respectively the characteristic function and the characteristic exponent of $\mathbf{X}$. $\mathcal{L}(\mathbf{X})$ stands for the law of $\mathbf{X}$ and $\mathbf{X} \overset{\mathcal{L}}{=} \mathbf{Y}$ means that $\mathbf{X}$ and $\mathbf{Y}$ have the same law. We denote with italic bold letters $\boldsymbol{X}$ the Lévy process $\{\boldsymbol{X}(t),\, t > 0\}$ which has the law of the vector $\mathbf{X}$ at time 1 $\mathcal{L}(\boldsymbol{X}(1)) = \mathcal{L}(\mathbf{X})$.

Let $\mathcal{M}_n$ be the set of $n \times n$ matrices and $\mathbb{I}_n$ be the $n \times n$ identity matrix; $\mathbb{X}$ stands for an element in $\mathcal{M}_n$.

Given a vector $\mathbf{X}$, $diag(\mathbf{X})$ stands for the diagonal matrix $\mathbb{X} = \begin{pmatrix} \mathrm{X}_1 & 0 & \dots & 0 \\ 0 & \mathrm{X}_2 & \dots & 0 \\ 0 & 0 & \dots & \mathrm{X}_n \end{pmatrix}$.

We recall here the definition of Lévy process and infinite divisibility, for a complete overview about this matter see Sato [23].

A càdlàg stochastic process $\boldsymbol{X} = \{\boldsymbol{X}(t), t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^n$ such that $\boldsymbol{X}(0) = \mathbf{0}$ is called a Lévy process if it has independent and stationary increments and it is stochastically continuous, i.e. $\forall \varepsilon > 0,\ \lim_{h\to 0} P(|\boldsymbol{X}(t+h) - \boldsymbol{X}(t)| \geq \varepsilon) = 0$.

A probability measure μ on $\mathbb{R}^n$ is infinitely divisible if, for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^n$ such that $\mu = \mu_n^n$, where μ_n^n represent the n-fold convolution of μ_n with itself.

Let $\boldsymbol{X}(t)$ be a Lévy process, it can be proved that for any t the random vector $\mathbf{X}(t)$ has an infinitely divisible distribution and conversely if F is an infinitely divisible distribution then there exists a Lévy process $\{\boldsymbol{X}(t), t \geq 0\}$ such that the distribution of $\boldsymbol{X}(1)$ is F, moreover if $\boldsymbol{X}(t)$ and $\boldsymbol{X}'(t)$ are Lévy processes on $\mathbb{R}^n$ such that $\boldsymbol{X}(1)$ and $\boldsymbol{X}'(1)$ have the same distributions then $\boldsymbol{X}(t)$ and $\boldsymbol{X}'(t)$ are identical in law (see Sato [23], Theorem 7.10).

The process $\boldsymbol{X} = \{(X_1(\boldsymbol{s}), ..., X_n(\boldsymbol{s}))^T, \boldsymbol{s} \in \mathbb{R}_+^n\}$ is an $\mathbb{R}_+^n$-parameter process (see Barndorff-Nielsen et al. [8]) if the following hold:

1. for any $m \geq 3$ and for any choice of $\boldsymbol{s}^1 \preceq ... \preceq \boldsymbol{s}^m$, the increments $\boldsymbol{X}(\boldsymbol{s}^j) - \boldsymbol{X}(\boldsymbol{s}^{j-1})$, $j = 1, ..., m$, are independent, where $\boldsymbol{s}^1 \preceq \boldsymbol{s}^2$ iff the all the component of $\boldsymbol{s}^1$ are smaller then the components of $\boldsymbol{s}^2$;
2. for any $\boldsymbol{s}^1 \preceq \boldsymbol{s}^2$ and $\boldsymbol{s}^3 \preceq \boldsymbol{s}^4$ satisfying $\boldsymbol{s}^2 - \boldsymbol{s}^1 = \boldsymbol{s}^4 - \boldsymbol{s}^3$, $\boldsymbol{X}(\boldsymbol{s}^2) - \boldsymbol{X}(\boldsymbol{s}^1) \stackrel{\mathcal{L}}{=} \boldsymbol{X}(\boldsymbol{s}^4) - \boldsymbol{X}(\boldsymbol{s}^3)$ (increments are stationary);
3. $\boldsymbol{X}(\mathbf{0}) = \mathbf{0}$ almost surely;
4. $\boldsymbol{X}(\boldsymbol{s})$ is almost surely right continuous with left limits in $\boldsymbol{s}$ in the partial ordering $\preceq$ of $\mathbb{R}_+^n$.

2 Generalized normal mean-variance mixture

In this section we recall the notion of normal mean-variance mixture ($Mnmv$), provide condition for infinite divisibility and introduce the corresponding time-changed Lévy process in a multidimensional environment. We therefore propose a generalization.

Definition 2.1. *A random vector* $\mathbf{Y}$ has a multivariate normal mean variance distribution (shortly $\mathbf{Y} \in Mnmv$) if

$$\mathbf{Y} \stackrel{\mathcal{L}}{=} \boldsymbol{\mu}_{\mathbf{0}} + \boldsymbol{\mu} G + \sqrt{G}\mathbb{Q}\mathbf{W}, \tag{2.1}$$

where $\boldsymbol{\mu_0}, \boldsymbol{\mu} \in \mathbb{R}^n$, $\mathbb{Q} \in \mathcal{M}_n$ and $\mathbb{Q}\mathbb{Q}^T$ is positive-definite, G is a positive random variable, $\mathbf{W} \sim N(\mathbf{0}, \mathbb{I}_n)$ and G is independent from (each element) in $\mathbf{W}$.

For simplicity from now on we assume $\boldsymbol{\mu_0} = 0$. The $Mnmv$ distributions are strictly related to type G distributions[2] on $\mathbb{R}^n$. Properties and examples of the former class of distributions are in Barndorff-Nielsen et al. [7].

If the random vector $\mathbf{Y}$ has an infinitely divisible (shortly i.d.) distribution, its law uniquely determines a Lévy motion.

These processes play a central role in representing returns of stock prices in financial applications. For this reason our interest is in i.d. $Mnmv$ distributions and the Lévy processes related to them. The infinite divisibility of this class is discussed for example in Kelker [17]. A sufficient condition for i.d. is that the mixing distribution is i.d. itself (see Barndorff-Nielsen et al. [7]). Under this condition Barndorff-Nielsen et al. [8] proved that the corresponding process is a time-changed Lévy motion, whose subordinator at time one has the law of the mixing distribution. In financial applications the subordinator represents economic time. Therefore the model assumes that each return has the same change of time. As explained in the introduction, the same clock for all stocks seems to be too restrictive, taking into consideration the empirical cross-sectional properties of information tested in Harris [16].

We therefore propose a generalization of the $Mnmv$ definition, using a multivariate mixing random vector instead of G. We then provide sufficient conditions for the distribution introduced to be infinitely divisible in order to introduce the corresponding Lévy process. The latter can also be represented as a subordinated Brownian motion. Our task is to provide a multidimensional model capable of describing the joint behavior of returns, which attaches to each single stock its own change of time.

Definition 2.2. *A random vector* $\mathbf{Y}$ has generalized normal mean-variance mixture distribution (shortly $\mathbf{Y} \in Gnmv$) if

$$\mathbf{Y} = \mathbb{A}\mathbb{G}\boldsymbol{\mu} + \mathbb{Q}\sqrt{\mathbb{G}}\mathbf{W}, \tag{2.2}$$

where $\mathbf{W} \sim N(\mathbf{0}, \mathbb{I}_n)$, $\mathbb{A}, \mathbb{Q} \in \mathcal{M}_n$, $\mathbb{Q}\mathbb{Q}^T$ is positive-definite, $\mathbb{G} = diag(\mathbf{G})$, $\mathbf{G}$ positive and independent from $\mathbf{W}$.

It is easy to verify that the model introduced covers a wide range of dependence and also allows to model independence. Moreover, Definition 2.1 can be derived as a subcase of Definition 2.2. The following theorem provides sufficient condition for i.d.

[2] $\mathbf{Y}$ is of type G if there is a standard Gaussian random $\mathbf{X}$ vector on $\mathbb{R}^n$ and a non negative i.d. random variable T, independent of $\mathbf{X}$, such that $\mathbf{Y} \stackrel{\mathcal{L}}{=} T^{1/2}\mathbf{X}$

Theorem 2.1. *If the mixing distribution* $\mathbf{G}$ *is infinitely divisible, the vector* $\mathbf{Y}$ *defined in 2.2 is i.d. with characteristic function*

$$\psi_{\mathbf{Y}}(\boldsymbol{z}) = \exp\left(\Psi_{\mathbf{G}}\left(i\, diag(\boldsymbol{\mu})\mathbb{A}^T\boldsymbol{z} - \frac{1}{2}\mathbb{Q}_{\boldsymbol{z}}\right)\right), \tag{2.3}$$

where $\mathbb{Q}_{\boldsymbol{z}} = ((\sum_{l=1}^{n} z_l q_{l1})^2, ..., (\sum_{l=1}^{n} z_l q_{ln})^2)^T = diag(\mathbb{Q}^T\boldsymbol{z})\mathbb{Q}^T\boldsymbol{z}$.

Under the condition of the previous theorem the vector $\mathbf{Y} \in Gnmv$ uniquely determines a Lévy process in law.

Definition 2.3. *The Lévy motion* $\mathbf{Y} = \{\mathbf{Y}(t), t \geq 0\}$ *is the (unique in law) process such that* $\mathcal{L}(\mathbf{Y}(1)) = \mathcal{L}(\mathbf{Y})$*, where* $\mathbf{Y} \in Gnmv$ *and* $\mathbf{Y}$ *is infinitely divisible.*

The following proposition shows that the Lévy motion $\boldsymbol{Y}$ can be constructed by multidimensional subordination. A complete treatment of the matter is in Barndorff-Nielsen et al. [8].

The following holds:

Proposition 2.1. *A random vector* $\mathbf{Y}$ is in *Gnmv* if and only if $\mathbf{Y} \stackrel{\mathcal{L}}{=} \boldsymbol{Y}(1)$, where $\boldsymbol{Y}$ is a Lévy process obtained by subordination of a $\mathbb{R}^n_+$-parameter Brownian motion $\boldsymbol{B}(\boldsymbol{s})$. Moreover the subordinator $\boldsymbol{G}$ is the Lévy process $\{\boldsymbol{G}(t) : t > 0\}$, such that $\mathcal{L}(\boldsymbol{G}(1)) = \mathcal{L}(\mathbf{G})$.

Since our task is to propose a multi dimensional normal mixture model for returns, we specify the structure of $\mathbf{G}$ in order to satisfy the following requirements:

- representing an idiosyncratic and a common component in the change of time, consistently with Lo and Wang [18];
- modelling different levels of dependence for fixed univariate marginal distributions;
- generating infinitely divisible mixing distributions with given margins (in order to be able to resort to popular processes to represent single returns).

We adopt the random-additive-effect distributions proposed in Barndorff-Nielsen et al. [7] to define the mixing vector $\mathbf{G}$.

Definition 2.4. *Let* $\mathbf{G}$ be

$$\mathbf{G} = (\mathrm{X}_1 + \gamma_1\mathrm{Z}, ..., \mathrm{X}_n + \gamma_n\mathrm{Z}), \tag{2.4}$$

where $\gamma_1, ..., \gamma_n$ are positive real parameters and $\mathrm{X}_i, i = 1, ..., n$ and Z are independent positive random variables.

If the margins of $\mathbf{G}$ have distributions closed under convolution and under scale transformations, it is possible to fix the marginal distributions G_i (and consequently the margins of $\mathbf{Y}$) and move the dependence structure of $\mathbf{G}$ from independence to maximal dependence.

If $\mathrm{X}_i, i = 1, ..., n$ and Z are i.d., then $\mathbf{G}$ is i.d. In this case, by Theorem 2.1, $\mathbf{Y} \in Gnmv$ is infinitely divisible. Since our task is to discuss the Lévy motion arising from such distributions, in the sequel $\mathrm{X}_i, i = 1, ...n$ and Z are assumed to be independent and i.d. random variables.

The resulting subordinator $\boldsymbol{G}$ can be also decomposed as the sum of an idiosyncratic and a common component:

$$\boldsymbol{G}(t) \overset{\mathcal{L}}{=} (X_1(t) + \gamma_1 Z(t), ..., X_n(t) + \gamma_n Z(t))^T,$$

for each $t > 0$ (the proof is straightforward, see Semeraro [25]).

The Lévy triplet of $\boldsymbol{Y}$ is derived from the ones of $\boldsymbol{G}$ and of the Brownian motion as stated in Theorem 4.7 in Barndorff-Nielsen [8]. It is easy to verify that the subcase with a common subordinator always has normal mean-variance marginal distributions. This property does not hold in general. Sufficient conditions are given in the following proposition.

Proposition 2.2. *Let $\mathbb{Q}^* = (q_{ij}^2)_{1 \le i,j \le n}$ with $rank(\mathbb{Q}^*) = n$. If either $\mu_i = 0$ (symmetric case), or $\mu_i = 1$ and $\mathbb{A} = \mathbb{Q}^*$, the marginal processes are time-changed Brownian motion. The change of time is a subordinator $\boldsymbol{G}_i^*$ whose distribution at time 1 is $\mathcal{L}(\boldsymbol{G}_i^*(1)) = \mathcal{L}(\sum_{j=1}^n q_{ij}^2 G_j(1))$.*

The previous proposition implies that the marginal laws of the subordinators at time one are $\mathcal{L}(\sum_{j=1}^n q_{ij}^2 G_j(1))$. Therefore they are generally not known. As a consequence, the subordinators of the marginal processes are unknown.

We provide an example based on the multivariate of α-Variance Gamma (α-VG) model, defined in Semeraro [25], whose subordinator has gamma margins.

Example 1. ***The VG case.*** *Let $\mathbf{Y}(t) = \boldsymbol{W}(\boldsymbol{G}(t))$ be a multivariate α-VG process, symmetric (the Brownian motions have no drift). Consider the process $\tilde{\mathbf{Y}}(t) = \mathbb{Q}\boldsymbol{W}(\boldsymbol{G}(t))$. It follows by construction that $\tilde{\mathbf{Y}}(1) \in Gnmv$ and Proposition 2.2 applies to $\tilde{\mathbf{Y}}$. Then $\mathcal{L}(\tilde{Y}_i(t)) = \mathcal{L}(\boldsymbol{W}(\sum_{j=1}^n q_{ij}^2 G_j(t)))$. If $\mathrm{G}_j := G_j(1) \sim \Gamma(a_j, b_j)$ then $q_{ij}^2 G_j(t) \sim \Gamma(a_j, \frac{b_j}{q_{ij}^2})$ and $\sum_{j=1}^n q_{ij}^2 G_j$ is gamma distributed if and only if $\frac{b_1}{q_{i1}^2} = \frac{b_2}{q_{i2}^2} = \ldots = \frac{b_n}{q_{in}^2}$. Since the rows of $\mathbb{Q}^*$ are pairwise different ($rank(\mathbb{Q}^*) = n$), the previous equations can only be fulfilled for at most one $i \in \{1, ..., n\}$. Therefore the process $\tilde{\mathbf{Y}}(t)$ has time-changed marginal processes and the economic idea of attaching to each return its own time change is preserved, but the time changes are no longer gamma distributed and the $Y_i(t)$ are no longer VG processes (apart from possibly one).*

The previous example shows that even if $\boldsymbol{G}$ has very simple marginal laws, the marginal law of $\boldsymbol{G}^*$, as defined in Proposition 2.2, may not have known distributions. Moreover by Proposition 2.1,

$$\mathcal{L}(\mathbb{Q}\sqrt{\mathbb{G}}\boldsymbol{W}(t)) = \mathcal{L}\left(\sum_{j=1}^{n} q_{1j} W_j(G_j(t)), ..., \sum_{j=1}^{n} q_{nj} W_j(G_j(t))\right)^T. \tag{2.5}$$

Each component of $\boldsymbol{Y}$ depends on more than one margin of $\boldsymbol{G}$, in that the conditional law of Y_j given $\boldsymbol{G} = \boldsymbol{s}$, i.e $\mathcal{L}(\sum_{j=1}^{n} q_{ij} W_j(s_j))$, depends on the whole multi-parameter $\boldsymbol{s}$. Since our aim is to model returns and to represent each single return as a time-changed Brownian motion, its dependence from different business times. We want to attach to each Brownian motion its own change of time. We therefore have to consider independent Brownian motions.

We therefore formally define the class independent generalized mean-variance distributions, $IGnmv$:

Definition 2.5. *A random vector* $\mathbf{Y}$ *has independent generalized mean-variance distribution* ($\mathbf{Y} \in IGnmv$), *if* $\mathbf{Y} \in Gnmv$, $\mathbb{Q} = \mathbb{A}$ *and they are diagonal.*

Let $\mathbf{Y} \in IGnmv$ and $\mathbb{Q} = diag(\sigma_j)$, we have

$$\mathbf{Y}^T = (\sqrt{\mathrm{G}_1}\sigma_1 W_1 + \mu_1\sigma_1\mathrm{G}_1, ..., \sqrt{\mathrm{G}_n}\sigma_n W_n + \mu_n\sigma_n\mathrm{G}_n). \tag{2.6}$$

Remark 1. *Random vectors* $\mathbf{Y} \in IGnmv$ *always have margins that are normal mean-variance mixtures, even if the restrictions of Proposition 2.2 are not fulfilled. This can be seen easily from equation (2.6). For example, if* $\tilde{\mathrm{G}}_i := \sigma_i^2\mathrm{G}_i$ *and* $\tilde{\mu}_i := \frac{\mu_i}{\sigma_i}$, *then* $\mathrm{Y}_i = \tilde{\mu}_i\tilde{\mathrm{G}}_i + \sqrt{\tilde{\mathrm{G}}_i}\mathrm{W}_i$, *that is,* $\mathbf{Y} = \tilde{\mathbb{G}}\tilde{\boldsymbol{\mu}} + \sqrt{\tilde{\mathbb{G}}}\mathbf{W}$ *where* $\boldsymbol{\mu} = \mathbb{Q}^{-1}\boldsymbol{\mu}$ *and* $\tilde{\mathbf{G}} = \mathbb{Q}^2\mathbf{G}$.

Observe that if $\mathbf{G}$ has independent components so does $\mathbf{Y} \in IGnmv$. Therefore the model allows to capture independence. Observe also that, since $\mathbb{Q}$ and $\mathbb{A}$ just imply a rescaling of the components and marginal distributions of $\mathbf{Y}$, one could assume $\mathbb{A} = \mathbb{Q} = \mathbb{I}_n$ without loss of generality.

We will consider also the process $\boldsymbol{Y}$ associated to $\mathbf{Y}$ by $\mathcal{L}(\boldsymbol{Y}(1)) = \mathcal{L}(\mathbf{Y})$. When $\mathbb{Q} = \mathbb{A} = \mathbb{I}_n$ and $\boldsymbol{\mu} = 0$ the law of $\boldsymbol{Y}$ is

$$\mathcal{L}(\sqrt{\mathbb{G}}\boldsymbol{W}(t)) = \mathcal{L}(\boldsymbol{W}(\boldsymbol{G}(t))). \tag{2.7}$$

The characterization of this process in terms of its Lévy triplet $(\boldsymbol{\gamma_Y}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$, can be obtained through Theorem 3.3 in Barndorff Nielsen et al. [8]. The Lévy triplet is

$$\begin{aligned} \boldsymbol{\gamma_Y} &= \textstyle\int_{\mathbb{R}_+^n} \nu_{\boldsymbol{T}}(d\boldsymbol{s}) \int_{|\boldsymbol{x}|\leq 1} \boldsymbol{x}\rho_{\boldsymbol{s}}(d\boldsymbol{x}), \\ \Sigma_{\boldsymbol{Y}} &= \mathbf{0}, \\ \nu_{\boldsymbol{Y}}(B) &= \textstyle\int_{\mathbb{R}_+^n} \rho_{\boldsymbol{s}}(B)\nu_{\boldsymbol{G}}(d\boldsymbol{s}), \end{aligned} \tag{2.8}$$

where $\rho_s = \mathcal{L}(\boldsymbol{W}(\boldsymbol{s}))$, $\boldsymbol{s} \in \mathbb{R}^n_+$, $\boldsymbol{x} = (x_1, ..., x_n)^T$, $B \in \mathbb{R}^n \setminus \{0\}$ and $\nu_{\boldsymbol{G}}$ is the Lévy measure of $\boldsymbol{G}$. Observe that $\boldsymbol{Y}$ is a pure jump process. $\boldsymbol{Y}$ has finite activity/variations if and only if the margins do.

Please notice that $\sum_{\boldsymbol{Y}} = 0$ holds true only if the drift component $\boldsymbol{c}$ in the L´evy-Khintchine representation of the characteristic function of the subordinator $\boldsymbol{G}$ vanishes (see Barndorff-Nielsen et. al. [8], Proposition 3.1). This is the case for the Gamma and GIG-distributions considered later in the paper.

In the following sections we apply the above described construction using a multivariate subordinator of random-additive type to get a GIG subordinator (mixing distribution in static case), which is not closed under convolution. We obtain a multidimensional GH distribution, and we use it in order to generate the corresponding time changed Lévy model.

3 The Multivariate GH model

We now focus on the generalized hyperbolic case. We have proved that $Gnmv$ are the distributions at time one of a subordinated Lévy process. Taken this into account, in this section we start by introducing a multivariate generalized hyperbolic distribution in order to investigate the associated process. The process we are going to introduce could be an alternative to the multidimensional GH process. The multivariate generalized hyperbolic distribution (MGH) is defined in the literature as a normal mean-variance distribution with mixing variable GIG distributed: see Barndorff-Nielsen [2], [6] and Barndorff-Nielsen at al. [7]. A first extensive survey about its properties was given in Blæsild and Jensen [9]. The GH process and its multidimensional extension are very popular in the financial literature to model stock returns, see Eberlein [13], [14]. The literature assumes one common business time, as discussed for the general $Mnmv$.

The goal of this section is to introduce a multivariate GH distribution such that:

- it has GH margins;
- it allows to calibrate easily both the margins and the dependence;
- it contains also non linear dependence;
- it answers our economic requirement to attach to each Brownian motion its own change of time.

The main difficulty in the construction is that the GIG distribution is not closed under convolution [3]. However, under a proper choice of the parameters, the convolution of a gamma and a GIG distribution is itself GIG distributed. We adopt the device of defining the change of time by means of a gamma distributed common component. For this reason we are not able to recover the multidimensional GH process analyzed by Eberlein [13] on stock market as a limit case. In fact, if the idiosyncratic component degenerates we find the variance gamma (VG) process. The peculiarity of our model is that both the distribution and the process are generalizations of the αVG model.

Definition 3.1. *Let $\lambda > 0$, $b \geq 0$, $\gamma_i > 0$, $0 < a \leq \lambda$. Let δ_i and $\frac{b}{\gamma_i}$ both nonnegative and not simultaneously zero. Let X_i be $GIG(-\lambda, \delta_i, \frac{b}{\gamma_i})$, V_i be $\Gamma(\lambda - a, \frac{b^2}{2\gamma_i^2})$ and $\mathrm{Z} \sim \Gamma(a, \frac{b^2}{2})$. Let $\mathbf{G}$ be*

$$\mathbf{G} = (\mathrm{X}_1 + \mathrm{V}_1 + \gamma_1^2 \mathrm{Z}, ..., \mathrm{X}_n + \mathrm{V}_n + \gamma_n^2 \mathrm{Z}). \tag{3.1}$$

Since the X_j, V_j, $j = 1, ..., n$ and Z have i.d. distributions so does $\mathbf{G}$. Moreover $\mathrm{X}_i + \mathrm{V}_i + \gamma_i^2 \mathrm{Z}$ is GIG with parameters $(\lambda, \delta_i, \frac{b}{\gamma_i})$, where $\gamma_i \geq 0$ since

$$GIG(-\lambda, \delta, \gamma) * \Gamma\left(\lambda, \frac{\gamma^2}{2}\right) = GIG(\lambda, \delta, \gamma),$$

as first stated, but not proven, in Barndorff-Nielsen et al. [5], in Barndorff-Nielsen [3] and proved in Eberlein and Hammerstein [14]. In the limiting case $\delta = 0$, one identifies $GIG\left(-\lambda, 0, \frac{b}{\gamma_i}\right) \cong \varepsilon_0$, where ε_0 is the Delta-Dirac function centered at zero, X_i vanishes and $\mathrm{G}_i \sim \Gamma\left(\lambda, \frac{b^2}{2\gamma_i^2}\right)$.

The characteristic function of $\mathbf{G}$ is

$$\psi_{\mathbf{G}}(\boldsymbol{z}) = \prod_{j=1}^{n} \psi_{\mathrm{X}_j}(z_j) \psi_{\mathrm{V}_j}(z_j) \psi_{\mathrm{Z}}(\sum_{j=1}^{n} \gamma_j^2 z_j), \tag{3.2}$$

The vector $\mathbf{G}$ is defined as the sum of three independent factors. Anyway, since both $\mathbf{X}$ and $\mathbf{V}$ have independent components, $\mathbf{G}$ has one common factor to satisfy our economic intuition of a common factor in the change of time. Let $\mathbf{T} = \mathbf{X} + \mathbf{V}$, $\mathbf{T}$ represent the idiosyncratic change of time [4].

We now define a multivariate distribution whose margins are GH distributed by means of the previous vector $\mathbf{G}$.

[3] This means that if both the independent and the common components of $\mathbf{G}$, respectively $\mathbf{X} = (\mathrm{X}_1, ..., \mathrm{X}_n)$ and $\mathbf{Z} = (\gamma_1 \mathrm{Z}, ..., \gamma_n \mathrm{Z})$, have GIG distributions, the margins of $\mathbf{G}$ are no longer necessarily GIG distributed.

[4] The construction given in Definition 3.1 can even be generalized by allowing different λ_i for each component G_i of $\mathbf{G}$ if the restrictions for a are modified as follows: $0 < a < \min_{1 \leq i \leq n} \lambda_i$. We prefer to stick to Definition 3.1 for parsimoniousness of the parameters. The usefulness of this choice will appear from the model calibration. We thank the referee for having pointed out the potential extension.

Definition 3.2. *We say that* $\mathbf{Y}$ *has a G-multidimensional generalized hyperbolic distribution (shortly* $\mathbf{Y} \in GMGH$*) if* $\mathbf{Y} \in IGnmv$ *(i.e.,* $\mathbb{A} = \mathbb{Q}$ *diagonal) and the mixing distribution has the law of* $\mathbf{G}$*.*

The following proposition is a consequence of our construction.

Proposition 3.1. *Let* $\mathbf{Y} \in GMGH$*. Let* $\lambda \geq 0$ *(we admit the degenerate case* $a = \lambda = 0$*),* $b > 0$*,* $\gamma_i > 0$*. Let* δ_i *and* $\frac{b}{\gamma_i}$ *both nonnegative and not simultaneously zero and*

$$\delta_i \geq 0, \ \ |\beta_i| < \alpha_i \ \ \text{if } \lambda > 0. \tag{3.3}$$

The distribution of $\mathbf{Y}$ *is infinitely divisible and it has GH margins with parameters* $\alpha_i, \beta_i, \delta_i|\sigma_i|, \lambda$*, where*

$$\begin{aligned} \beta_i &= \frac{\mu_j}{\sigma_i} \\ \sqrt{\alpha_j^2 - \beta_j^2} &= \frac{b}{\gamma_j|\sigma_i|}. \end{aligned} \tag{3.4}$$

Notice that $|\sigma_i| = 1$ if $\mathbb{A} = \mathbb{Q} = \mathbb{I}_n$. Observe that we do not allow $\lambda < 0$ because $\mathrm{V}_j +$ Z, $j = 1, ..., n$ are gamma distributed, and their first parameter is λ. The components Y_i are univariate normal mean-variance mixtures with GIG mixing variable.

The characteristic function of $\mathbf{Y}$ becomes

$$\begin{aligned} \psi_{\mathbf{Y}}(\boldsymbol{z}) = \prod_{j=1}^{n} \left(\frac{\alpha_j^2 - \beta_j^2}{\alpha_j^2 - (\beta_j + iz_j)^2} \right)^{\lambda/2} \frac{K_\lambda \delta_j \sqrt{\alpha_j^2 - (\beta_j + iz_j)^2}}{K_\lambda \left(\delta_j \sqrt{\alpha_j^2 - \beta_j^2} \right)} \cdot \\ \cdot \left(1 - \frac{-\frac{1}{2} z_j^2 + i\beta_j z_j}{(\alpha_j^2 - \beta_j^2)/2} \right)^{-(\lambda - a)} \left(1 - \frac{\sum_{j=1}^{n} (-\frac{1}{2} z_j^2 + i\beta_j z_j)\gamma_j^2}{(\alpha_j^2 - \beta_j^2)/2} \right)^{-a}. \end{aligned} \tag{3.5}$$

From the expression of $\psi_{\mathbf{Y}}$ we infer that $\mathbf{Y}$ is the convolution of a vector with independent GH margins, $\mathbf{Y}^X$, and a multivariate $\boldsymbol{\alpha}$-VG random vector, $\mathbf{Y}^Z$.

With this choice of the mixing distribution we can change the level of dependence moving a. Letting $a \to 0$, for fixed marginal distributions, we get independence. This happens because $\Gamma\left(a, \frac{b^2}{2}\right) \xrightarrow{\mathcal{L}} \varepsilon_0$ (i.e. Z degenerates), as can be seen from the convergence of the corresponding characteristic function. On the other hand we are not able to capture perfect correlation for the subordinator only through a: we should also let X_j, for $j = 1, ..., n$ degenerate. This limit case corresponds to a gamma mixing distribution and generates a VG distribution. Therefore as subclasses of this family we find both the α-VG distribution and the distribution with independent GH margins.

We now investigate the Lévy motion defined by the GMGH distribution.

Definition 3.3. *A Lévy process $\boldsymbol{Y}$ is said to be G- multidimensional generalized hyperbolic ($\mathbf{Y} \in GMGH$) if $\mathcal{L}(\mathbf{Y}(1)) = \mathbf{Y}$, where $\mathbf{Y} \in GMGH$.*

The characteristic function of $\boldsymbol{Y}(1)$ has been explicitly stated in (3.5).

From Proposition 3.1 the process $\boldsymbol{Y}$ has $GH(\alpha_i, \beta_i, \delta_i, \lambda)$ marginal processes. It is a time-changed Brownian motion and the change of time is a GIG process, in fact, as discussed in general, $\mathcal{L}(\boldsymbol{G}(1)) = \mathbf{G}$. We recall that the GIG distributions are not closed under convolution. As a consequence we do not know the distribution of a GIG process at any time different from one.

The dependence structure will be analyzed using linear correlation. It is possible, as we will show in the application, that the data have high correlation and we might need to add correlation in the Brownian motion. Since by considering the *Gnmv* mixture we would not have GH margins, we end this section by proposing a device to add correlation leaving GH margins.

In order to do that we split the process $\boldsymbol{Y}$ as the sum of two independent multivariate processes. From its characteristic function it can be argued that the addenda of $\boldsymbol{Y}$ are: a process with independent GH margins and a time-changed Brownian motion.

Proposition 3.2. *Let $\boldsymbol{Y} \in GMGH$. Then $\boldsymbol{Y} = \mathbf{Y}^{\boldsymbol{T}} + \mathbf{Y}^{\boldsymbol{Z}}$, where $\mathbf{Y}^{\boldsymbol{T}}$ has independent GH margins and $\mathbf{Y}^{Z}$, the VG component, is the α-VG process. It has both a common and an idiosyncratic time-change.*

The representation evidenced by the previous proposition can be derived from the characteristic function of $\mathbf{Y}$. Anyway it is a particular case of a more general result, stated and proved in the Appendix. The process $\boldsymbol{Y}$ can thus be expressed as $\boldsymbol{Y} \stackrel{\mathcal{L}}{=} \mathbf{Y}^{\boldsymbol{T}} + \mathbf{Y}^{\boldsymbol{Z}}$, where $\mathbf{Y}^{\boldsymbol{T}}$, and $\mathbf{Y}^{\boldsymbol{Z}}$ are independent time-changed Brownian motions with subordinators respectively $\boldsymbol{T}(t)$ and $\boldsymbol{\gamma}^2\boldsymbol{Z}(t)$. The processes $\boldsymbol{T}(t)$ and $\boldsymbol{\gamma}^2\boldsymbol{Z}(t)$ are defined by the vectors $\mathbf{T}$, $\boldsymbol{\gamma}^2\mathbf{Z}$ under Definition 3.1. In particular $\boldsymbol{\gamma}^2\mathbf{Z}$ has comonotone marginal distributions $\Gamma(a, \frac{b^2}{2\gamma_i^2})$. Thus $\mathbf{Y}^{\gamma^2 Z}$ is of VG type. This decomposition allows us to add correlation in the model leaving both the marginal processes fixed in law; furthermore the marginal returns depend only on their own time change.

Definition 3.4. *We name $\tilde{\mathbb{Q}} - GMGH$ the process $\tilde{\mathbf{Y}}$ defined by*

$$\tilde{\mathbf{Y}} = \mathbf{Y}^{\boldsymbol{T}} + \tilde{\mathbf{Y}}^{Z},$$

where $\tilde{\mathbf{Y}}^{Z} = \tilde{\mathbb{Q}}\mathbf{Y}^{Z}$ and $\mathbf{Y}^{Z}$ is a multivariate process of VG type with a common subordinator $Z(1) \sim \Gamma(a, \frac{b^2}{2})$.

Since $\tilde{\mathbf{Y}}^{Z} \in Mnmv$, i.e. it has a common subordinator, it follows that: $\mathcal{L}(\tilde{Y}_i^Z(t)) = \mathcal{L}(\sum_{j=1}^n \tilde{q}_{ij}\mu Z(t) + W_i^Z((\sum_{j=1}^n \tilde{q}_{ij})^2 Z(t)))$, $i = 1, ..., n$, where W_i^Z is a standard Brownian motion for each $i = 1, ..., n$ (we are investigating the marginal laws and not their dependence relationship).

Proposition 3.3. *Under the condition:*

$$\sum_{j=1}^{n} \tilde{q}_{ij}^2 = \gamma_i^2 \text{ and } \sum_{j=1}^{n} \tilde{q}_{ij}\mu_j = \gamma_i^2 \mu_i \qquad 1 \leq i \leq n \tag{3.6}$$

the process $\tilde{\boldsymbol{Y}}$ *has* $GH(\alpha_j, \beta_j, \delta_j, \lambda)$ *marginal processes.*

The process $\tilde{\mathbf{Y}}$ depends on the marginal parameters $(\beta_j = \mu_j, \alpha_j, \lambda, j = 1, ..., n)$ and on the parameter a, involved in the correlation between the subordinator margins and also on the matrix $\tilde{\mathbb{Q}}$. We underline that since $\sqrt{\alpha_j^2 - \beta_j^2} = \frac{b}{\gamma_j}$ is fixed once the marginal distributions are, moving b we change γ_j and the matrix $\tilde{\mathbb{Q}}$. This fact makes b relevant in correlation, as we will see in the sequel.

3.1 The general *Gnmv* model

For completeness we devote this section to discuss a multidimensional GH model arising from the general normal mixture. We start by considering the distribution of GH type arising from the general model. Formally let $\mathbf{Y} \in Gnmv$, with mixing distribution $\mathcal{L}(\boldsymbol{G})$ of Definition 3.1, the vector $\mathbf{Y}$ has infinitely divisible distribution. Moreover if $\mathbb{Q} = \mathbb{A}$, then $\mathbf{Y} = \mathbb{Q}\mathbf{Y}^*$, where $\mathbf{Y}^*$ is a *IGnmv* with $\mathbb{A} = \mathbb{Q} = \mathbb{I}_n$. Its characteristic function is

$$\psi_{\mathbf{Y}}(\boldsymbol{z}) = \prod_{j=1}^{n} \left(\frac{\alpha_j^2 - \beta_j^2}{\alpha_j^2 - (\beta_j + i\sum_{l=1}^{n} q_{lj} z_l)^2} \right)^{\lambda/2} \frac{K_\lambda \delta_j \sqrt{\alpha_j^2 - (\beta_j + i\sum_{l=1}^{n} q_{lj} z_l)^2}}{K_\lambda(\delta_j \sqrt{\alpha_j^2 - \beta_j^2})} \cdot$$
$$\cdot \left(1 - \frac{-\frac{1}{2}(\sum_l z_l q_{lj})^2 + i\sum_l \beta_l z_l q_{lj}}{(\alpha_j^2 - \beta_j^2)/2}\right)^{-(\lambda - a)} \left(1 - \frac{\sum_{j=1}^{n}(-\frac{1}{2}(\sum_l z_l q_{lj})^2 + i\sum_l \beta_l z_l q_{lj})\gamma_j^2}{(\alpha_j^2 - \beta_j^2)/2}\right)^{-a}. \tag{3.7}$$

The *Gnmv* family, under the condition $\mathbb{Q} = \mathbb{A}$, contains the affine generalized hyperbolic one proposed and studied by Schmidt [24], when $Z \to 0$. As we noticed at the beginning of this section, our model does not capture the MGH with a common GIG mixing distribution, since the common component of the subordinator is gamma distributed. If the independent part degenerates we indeed find a VG distribution with a common mixing law and correlated Brownian motions.

For completeness we also mention the process of GH type arising from the above general GH distribution. Let us consider now the Levy process $\{\boldsymbol{Y}(t), t \geq 0\}$ defined by $\mathcal{L}(\boldsymbol{Y}(1)) = \mathbf{Y}$. Proposition 2.1implies that $\boldsymbol{Y}$ is a subordinated Brownian motion with subordinator $\boldsymbol{G}$ defined by $\mathcal{L}(\boldsymbol{G}(1)) = \mathcal{L}(\mathbf{G})$, where $\mathcal{L}(\mathbf{G})$ is the one in 3.1. In general $\boldsymbol{Y}$ has neither *GH* margins, nor time-changed ones.

Even if the marginal processes do not have known distributions the subcase $\mathbb{A} = \mathbb{Q}$ can be restated as the case with GH margins through a linear transformation.

4 Dependence

Linear dependence is the major concern for calibration of return processes, since the corresponding coefficient is the measure adopted in theoretical asset pricing models and its estimates are easy to obtain from market data. In addition, Luciano and Semeraro [19] have shown that, at least for the α-VG case, non-linear dependence "fades away" over time. Given the mixture nature of the underlying distribution, the same could happen in the models studied here. Last but not least, in the simplest of our models, $GMGH$, correlation completely determines the joint distribution. Most of our theoretical dependence analysis is therefore focused on the linear case, on which also the calibration will be built. However, before attempting the analysis of linear dependence of the multidimensional generalized hyperbolic processes that we are going to calibrate, we want to make some considerations about non linear dependence.

The process $\boldsymbol{Y} \in GMGH$ has non linear dependence. To prove this, we observe that the process has dependent margins also in the symmetric case ($\rho = 0$): indeed the Lévy measure of $\boldsymbol{Y}$ is given by

$$\nu_{\boldsymbol{Y}}(B) = \int_{\mathbb{R}_+^n} \rho_{\boldsymbol{s}}(B)\nu_{\boldsymbol{G}}(d\boldsymbol{s}), \tag{4.1}$$

where $\nu_{\boldsymbol{G}}$ is the Lévy measure of the subordinator. Let ν_j, ν_Z be respectively the Lévy measures of the processes $\tilde{T}_j, j = 1, ..., n$ and $\tilde{Z}$, then the Lévy measure $\nu_{\boldsymbol{G}}$ of $\boldsymbol{G}$ satisfies

$$\nu_{\boldsymbol{G}}(E) = \sum_{j=1}^{n} \nu_j(E_j) + \nu_Z(E_\Delta), \tag{4.2}$$

where $E \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$, $E_j = E \cap A_j$ and $A_j = \{\boldsymbol{x} \in \mathbb{R}^n : x_k = 0, k \neq j, k = 1, ..., n\}$ (see Semeraro [25] for the proof).

From the expression of $\nu_{\boldsymbol{G}}$ it follows that the components of $\boldsymbol{Y}$ may jump together. Thus the processes $\sigma_j B_j(G_j(t))$ have non-linear dependence, unless the random variable Z is degenerate.

We now analyze linear dependence. In the asymmetric case, linear dependence allows us to fully characterize the parameters of the model, given the marginal ones. It is not exhaustive in describing the dependence structure of $\boldsymbol{Y}$. Anyway it always allows us to fully characterize the parameters of the subordinator $\boldsymbol{G}$, given the marginal ones.

4.1 Linear dependence

Let $\boldsymbol{Y} \in GMGH$, with $\mathbb{A} = \mathbb{Q} = \mathbb{I}_n$. We start from the correlation matrix $\rho_{\boldsymbol{G}} = (\rho_{\boldsymbol{G}}(l, j))$ of the subordinator.

Since

$$Cov(G_l, G_j) = \gamma_l^2\gamma_j^2 V(\mathrm{Z}) \text{ and } V(G_j) = V(X_j) + V(V_j) + \gamma_j^2 V(\mathrm{Z}), \tag{4.3}$$

we have

$$\rho_{\boldsymbol{G}}(l,j) = \frac{\gamma_l^2\gamma_j^2 V(\mathrm{Z})}{\sqrt{[V(G_l)][V(G_j)]}} = \frac{\gamma_l^2\gamma_j^2 4a}{b^4\sqrt{[V(G_l)][V(G_j)]}},$$

where the expression for $V(G_j)$ is given in (B.3) in the Appendix. Since $\mathcal{L}(G_j) = GIG(\lambda, \delta_j, \frac{b}{\gamma_j})$, given the marginal parameters the joint distribution of $\boldsymbol{G}$ is uniquely determined by the parameter a; in turn a is uniquely determined by ρ.

Let us assume now that the marginal parameters are fixed and such that the marginal distributions do not degenerate. Since the margins are independent iff $a = 0$ (iff $\rho = 0$), imposing $a = 0$ we can capture independence starting from no matter which marginal distribution. The same is not true for perfect correlation: a necessary condition for $\rho = 1$ is that X_j degenerates for each j. In this case the subordinator degenerates in a real gamma random variable and we get the VG model.

Since $\boldsymbol{Y}$ is a subordinated process, the variance of $\mathrm{Y}_j = \boldsymbol{Y}_j(1)$ is:

$$V[\mathrm{Y}_j] = E[V[\mathrm{Y}_j|\mathrm{G}_j]] + V[E[\mathrm{Y}_j|\mathrm{G}_j]] = E[\mathrm{G}_j] + \beta_j^2 V[\mathrm{G}_j]. \tag{4.4}$$

The lj -covariance of the process at time 1 is:

$$cov[\mathrm{Y}_l, \mathrm{Y}_j] = \beta_i\beta_j cov[\mathrm{G}_l, \mathrm{G}_j] = \beta_l\beta_j\gamma_l^2\gamma_j^2 V(\mathrm{Z}).$$

Therefore the linear correlation coefficients are

$$\rho_{\boldsymbol{Y}}(l,j) = \frac{\beta_l\beta_j\gamma_l^2\gamma_j^2 V(\mathrm{Z})}{\sqrt{V(\mathrm{Y}_l)V(\mathrm{Y}_j)}} = \frac{\beta_l\beta_j\gamma_l^2\gamma_j^2 4a}{b^4\sqrt{V(\mathrm{Y}_l)V(\mathrm{Y}_j)}}, \tag{4.5}$$

where the expression for the marginal variances (B.8) are in the Appendix.

Observe that the linear correlation coefficient is zero if β is zero, i.e. in the symmetric case, for each value of a. Therefore in the symmetric case the linear correlation coefficient does not determine uniquely the joint distribution of $\boldsymbol{Y}$ for each value of the marginal parameters. Anyway in the asymmetric case, which is more interesting for financial applications, it does. In the latter case in order to calibrate the parameter a we can use an estimate of the correlation coefficient. Since the subcase with a common subordinator leads to the VG process, to reach high correlation leaving the GH marginal distributions fixed we also investigate the $\tilde{\mathbb{Q}} - GMGH$ correlation coefficients.

Let $\tilde{\boldsymbol{Y}} \in \tilde{\mathbb{Q}} - GMGH$, its linear correlation coefficients are

$$\rho_{\tilde{\boldsymbol{Y}}}(i,j) = \frac{\sum_{h=1}^n \tilde{q}_{ih}\tilde{q}_{jh} V(Y_h^Z) + \sum_{\substack{k,l=1 \\ k\neq l}}^n \tilde{q}_{il}\tilde{q}_{jk} cov(Y_j^Z, Y_i^Z)}{\sqrt{V(Y_i)V(Y_j)}}, \tag{4.6}$$

where $Y_i^Z \sim VG$ with a gamma subordinator whose parameters are $(a, \frac{b^2}{2})$.

$$V[Y_i^Z] = \frac{2a}{b^2} + \mu_i^2 \frac{4a}{b^4}, \quad i = 1, ..., n; \tag{4.7}$$

and

$$cov[Y_i^Z, Y_j^Z] = \mu_i \left(\sum_{l=1}^{n} q_{il} \right) \mu_j \left(\sum_{l=1}^{n} q_{jl} \right) \frac{4a}{b^4} = \mu_i \mu_j \frac{4a}{b^4}. \tag{4.8}$$

5 A stock market application: the hyperbolic case

As usual, define a price process to be the exponential of the process $\boldsymbol{Y}$:

$$\boldsymbol{S}(t) = \boldsymbol{S}(0) exp(\boldsymbol{Y}(t)), \quad t \geq 0.$$

Let the process $\boldsymbol{Y}$ represent the stock returns under the historical measure[5].

In this section we will first discuss a calibration procedure that can be developed for the GMGH and $\tilde{\mathbb{Q}} - GMGH$ models. Using the first model, we then provide a simple numerical example in which the marginal parameters are calibrated on stock market data, and the remaining parameters are selected in order to calibrate dependence of the model.

The parameters involved in the GMGH model are:

- The marginal parameters of the returns: $\alpha_j, \beta_j, \delta_j, \lambda$;
- The parameters of the subordinator, involved in the dependence structure of the model: γ_j, a, b.

The relationship between the marginal parameters and the dependence ones is:

$$\frac{b}{\gamma_j} = \sqrt{\alpha_j^2 - \beta_j^2}. \tag{5.1}$$

The calibration procedure we apply is divided into two steps: first calibrate the marginal parameters, through the returns. Then the remaining ones, through correlation.

Once the marginal parameters are fixed we only have to find the common parameters a, b, since the γ_j are determined by (5.1). In order to calibrate a we look for the value which minimizes the distance between historical and theoretical correlation. The correlation coefficients depend on b only through the ratios $\frac{b}{\gamma_j}$: therefore for this kind of analysis we can fix $b = 1$. An analogous procedure could be developed for the $\tilde{\mathbb{Q}} - GMGH$ model[6].

[5] In this paper we only work with the historical measure; we do not discuss any choice of a risk neutral equivalent measure

[6] The parameters involved in the $\tilde{\mathbb{Q}} - GMGH$ model are:

5.1 Calibration example

In this section we investigate an application of the model $GMGH$ discussed above. We use the common parameter a to minimize the distance between the theoretical (model) and historical correlation matrix. The example shows that, once the marginal parameters are fixed, the model is very parsimonious and correlation can be changed moving a single parameter with a small error.

Step 1: data choice.

The data used for the calibration are weekly returns on nine stocks - American Express Co. (AXP), Boeing Co. (BA), Citigroup Inc. (C), Walt Disney Co. (DIS), Eastman Kodak Co. (EK), Intel Corp (INTC), JPMorgan Chase & Co. (JPM), Coca-Cola Co. (KO), Microsoft Corp. (MSFT) - which belong to the Dow Jones index.

The data set is the same discussed in McNeil et al. [21], ch. 3.2. Given that they already detected non-normality for returns below the monthly horizon, we focus on weekly returns. All the parameter values will therefore be weekly ones.

The time span covered by our time series is from January 1993 to December 2000, for a total of 416 observations. Dow Jones quotes are used.

Step 2: marginal parameters.

The marginal parameters can be calibrated using the same procedure as in the univariate case, stock by stock. Since the marginal fit is not our main concern in this calibration example, we fix $\lambda = 1$, which means restricting the marginal distributions to the hyperbolic distribution (GH with $\lambda = 1$.)[7] Estimation is obtained by maximum likelihood, as follows. We are interested in estimating the four parameters of each marginal H density based on the set of 416 return observations. We first write down the likelihood function of each stock sample, in terms of the corresponding GH density, then use numerical procedures in order to obtain the density function from the character-

- The marginal parameters of the returns: $\alpha_j, \beta_j, \delta_j, \lambda$;
- The parameters of the subordinator, involved in the dependence structure of the model: γ_j, a, b.
- The entries of the matrix $\tilde{\mathbb{Q}}$.

The relationships between the marginal parameters and the dependence one are (5.1) and (3.6), namely $\sum_{j=1}^{n} \tilde{q}_{ij}^2 = \gamma_i^2$ and $\sum_{j=1}^{n} \tilde{q}_{ij}\mu_j = \gamma_i^2 \mu_i$.

In this case we can fix a and b and use the matrix $\tilde{\mathbb{Q}}$ to get high correlation. The γ_j, $j = 1, ..., n$ are a consequence of (5.1). The usefulness of b in this generalization is clear from (3.6). Since $\sqrt{\alpha_j^2 - \beta_j^2} = \frac{b}{\gamma_j}$ is fixed once the marginal distributions are, moving b we change γ_j and the matrix $\tilde{\mathbb{Q}}$. This fact makes b relevant in correlation. Therefore we can look for the parameters a, b and the entries of $\tilde{\mathbb{Q}}$ that minimize the distance between the sample and theoretical correlation matrix under the constraints (5.1) and (3.6)

[7] Unreported calibrations (available from the Authors upon request) show that the increase in maximum likelihood, obtained by letting λ_i vary for each stock, is quite negligible for the data set at hand.

istic function by inverse Fast Fourier transform. These procedures are conducted in MATLAB environment and require initial guess values for the parameters. The possible influence on the result of the use of guess values has been smoothed by adopting the maximizing procedure iteratively. At each iteration step we use, as starting values, the maximizing ones in the previous iteration. Maximization is performed taking into account the parameter bounds and constraints.

The estimated parameters for our sample are given in Table 1. Together with the H parameters, the table presents: 1) a β' value which stays between $(-1, 1)$, as soon as $\alpha > |\beta|$, since it is defined as $\beta = \alpha\beta'$. It checks that (5.1) is well defined; 2) a μ value which was obtained as the expectation of the returns over the period of observation (we estimated μ + GH).

Table 1: Calibrated parameter values for each stock.

parameter	AXP	BA	C	DIS	EK	INTC	JPM	KO	MSFT
μ	0,0048	-0.0001	-0.0070	-0.0043	0.0033	0.0286	0.0090	0.0006	0.0035
α	49.3500	39.0285	35.7161	44.7035	37.2004	34.3269	38.6181	43.5423	38.7950
β'	0.0041	0.0513	0.1568	0.0803	-0.0375	-0.2230	-0.0640	0.0348	0.0165
β	0.2023	2.0022	5.6003	3.5897	-1.3950	-7.6549	-2.4716	1.5153	0.6401
δ	0.0584	0.0198	0.0366	0.0396	0.0134	0.0517	0.0407	0.0286	0.0467

In Table 2 we present, for each stock, the results of the Kolmogorov-Smirnov test, with H_0 representing acceptance of the hyperbolic distribution.

Table 2: Results of 5% KS test for each stock. cv is the 5% confidence value.

	AXP	BA	C	DIS	EK	INTC	JPM	KO	MSFT
H_0	0	0	0	0	0	0	0	0	0
p-value	0.9653	0.9890	0.9672	0.9895	0.9312	0.8601	0.9605	0.9150	0.9795
KS-stat	0.0243	0.0217	0.0241	0.0216	0.0264	0.0294	0.0246	0.0272	0.0230
cv 5 %	0.0662	0.0662	0.0662	0.0662	0.0662	0.0662	0.0662	0.0662	0.0662

The reader can certainly notice that the KS test is highly significant. The corresponding statistics is well below the confidence value at 5%.

Step 3: correlation

As explained above, we can choose $b = 1$: γ_j $j = 1, ..., n$ follow from (5.1). The remaining parameter to be calibrated is the parameter $a \in [0, 1]$. The maximal correlation allowed by the model corresponds to $a = max = 1$, as can be easily argued from the constraints of the parameters, $\lambda = 1$ together with $a \leq \lambda$. This is the correlation which minimizes the distance from the (estimate) of the observed correlation.

The theoretical correlation matrix for $a = \max = 1$ is in Table 3.

Table 3: Theoretical correlation matrix ($a = \max = 1$).

	AXP	BA	C	DIS	EK	INTC	JPM	KO	MSFT
AXP	1.0000								
BA	0.0003	1.0000							
C	0.0007	0.0117	1.0000						
DIS	0.0003	0.0056	0.0157	1.0000					
EK	-0.0002	-0.0032	-0.0090	-0.0043	1.0000				
INTC	-0.0009	-0.0155	-0.0434	-0.0208	0.0119	1.0000			
JPM	-0.0003	-0.0046	-0.0129	-0.0062	0.0035	0.0171	1.0000		
KO	0.0002	0.0026	0.0074	0.0035	-0.0020	-0.0098	-0.0029	1.0000	
MSFT	0.0001	0.0012	0.0032	0.0015	-0.0009	-0.0043	-0.0013	0.0007	1.0000

The mean square error of dependence calibration so obtained is 7.97%. This value is obtained taking the average of the squared differences between the entries of the model correlation matrix and those of the (estimated) historical one [8].

6 Conclusions

In this paper we provide a method to construct multidimensional normal mixtures and multidimensional time-changed Brownian motions based on the economic intuition of a common component in trade and consequently in business time. We couple the change of time technique, which has been by now extensively applied in Finance, with some novel results in the cross section of time-change, as represented by trade. The novel results put into evidence the low factor nature of trade over different assets. We use exactly such nature to construct a new, or generalized, time-changed process, under the form of normal mean-variance mixture.

Using a GIG distributed idiosyncratic component and a common gamma one, our construction gives rise to GH margins. We use such specification to show that the model maintains the marginal properties that characterize the GH motions, such as asymmetry and fat tails, which are commonly considered as desired features for asset returns. On top of these marginal features, we have linear and non-linear dependency at the multivariate - or portfolio - level. By means of a stock market example we show that indeed our model, once parametrized, is easy to calibrate in its basic version.

The calibration shows that, on a sample of nine stocks from the Dow Jones index, already studied by McNeil et al. [21], the model is able to capture the correlation through

[8] To lower further the error, one could proceed to calibration of the $\tilde{\mathbb{Q}} - GMGH$ model, according to footnote on pag 16.

a single parameter, even when the margins are restricted to be GH. Parsimoniousness preserves the possibility of describing dependence with a moderate error. The error can be further reduced, as explained in the theoretical model, trading off parsimoniousness for higher accuracy.

A Appendix

A.1 Proofs

Theorem 2.1. We begin computing the characteristic function of $\mathbf{Y}$. Let

$$\mathbf{Y}^j = (a_{1j}\mu_j g_j + q_{1j}\sqrt{g_j}\mathrm{W}_j, ..., a_{nj}\mu_j g_j + q_{nj}\sqrt{g_j}\mathrm{W}_j)^T, \ \ j = 1, ..., n;$$

where $\mathbf{g} \in \mathbb{R}^n_+$. The random variables Y^j are independent and $\mathcal{L}(\mathbf{Y}|\mathbf{G} = \mathbf{g}) = \mathcal{L}(\sum_{j=1}^n \mathbf{Y}^j)$. Consider

$$E[\exp\{i\left\langle \mathbf{Y}, \boldsymbol{z}\right\rangle\}|\mathbf{G} = \mathbf{g}] = E\left[\exp\left\{i\sum_{j=1}^{n}\left\langle \mathbf{Y}^j, \boldsymbol{z}\right\rangle\right\}\right] = \prod_{j=1}^{n}\psi_{\mathbf{Y}^j}(\boldsymbol{z}). \tag{A.1}$$

where

$$\begin{aligned}
\psi_{\mathbf{Y}^j}(\boldsymbol{z}) &= E\left[\exp\{i\left\langle \mathbf{Y}^j, \boldsymbol{z}\right\rangle\}\right] \\
&= E\left[\exp\left\{i\sum_{l=1}^{n} z_l a_{lj} g_j \mu_j + i\sum_{l=1}^{n} z_l q_{lj}\sqrt{g_j}W_j\right\}\right] \\
&= \exp\left\{i\sum_{l=1}^{n} z_l a_{lj} g_j \mu_j\right\}\psi_{\sqrt{g_j}W_j}(\sum_{l=1}^{n} z_l q_{lj}) \\
&= \exp\left\{i\sum_{l=1}^{n} z_l a_{lj} g_j \mu_j\right\}\exp\left\{-\frac{1}{2}g_j(\sum_{l=1}^{n} z_l q_{lj})^2\right\}.
\end{aligned} \tag{A.2}$$

The characteristic function of $\mathbf{Y}$ becomes

$$\begin{aligned}
\psi_{\mathbf{Y}}(\boldsymbol{z}) &= E\left[\prod_{j=1}^{n}\exp\left\{i\sum_{l} z_l a_{lj}\mathrm{G}_j\mu_j\right\}\exp\left\{-\frac{1}{2}\mathrm{G}_j(\sum_{l} z_l q_{lj})^2\right\}\right] \\
&= E\left[\exp\sum_{j=1}^{n}\mathrm{G}_j\left(\left\{i\sum_{l} z_l a_{lj}\mu_j - \frac{1}{2}(\sum_{l} z_l q_{lj})^2\right\}\right)\right] \\
&= E\left[\exp\left\langle \boldsymbol{G}, i\cdot diag(\boldsymbol{\mu})\mathbb{A}^T\boldsymbol{z} - \frac{1}{2}\mathbb{Q}_{\boldsymbol{z}}\right\rangle\right] = \exp\left(\Psi_{\mathbf{G}}\left(i\cdot diag(\boldsymbol{\mu})\mathbb{A}^T\boldsymbol{z} - \frac{1}{2}\mathbb{Q}_{z}\right)\right).
\end{aligned} \tag{A.3}$$

From the previous equation and the infinite divisibility of $\mathbf{G}$ it easily follows that $\mathbf{Y}$ is also infinitely divisible. ■

Proposition 2.1.

Define the Brownian motion

$$\boldsymbol{B}(\boldsymbol{s}) := (\sum_{j=1}^{n} \alpha_{1j} s_j + \sum_{j=1}^{n} \beta_{1j} W_j(s_j), ..., \sum_{j=1}^{n} \alpha_{nj} s_j + \sum_{j=1}^{n} \beta_{nj} W_j(s_j))^T, \tag{A.4}$$

where α_{ij}, $\beta_{ij} \in \mathbb{R}_+$, $i, j = 1, ...n$. $\boldsymbol{B}(\boldsymbol{s})$ is an $\mathbb{R}^n_+$-parameter process. Define $\boldsymbol{Z}_i(t) := (a_{1i} B_i(t), ..., a_{ni} B_i(t))^T$. The $\boldsymbol{Z}_i$ are independent Lévy processes on $\mathbb{R}^n$ and $\boldsymbol{B}(\boldsymbol{s}) = \sum_{i=1}^{n} \boldsymbol{Z}_i(s_i)$. The assert is now a direct consequence of Example 4.4 in Barndorff-Nielsen et al. [8].

The second part of the proof is similar to that of proposition 6.4 in Barndorff Nielsen et al. [8]. Let $\boldsymbol{B}(\boldsymbol{s})$ be a $\mathbb{R}^n_+$-parameter Brownian motion defined as in (A.4) with $\alpha_{ij} = a_{ij}\mu_j$ and $\beta_{ij} = q_{ij}$. Let $\boldsymbol{Y}(t)$ be the subordination of $\boldsymbol{B}(\boldsymbol{s})$ by a multivariate subordinator $\boldsymbol{G}(t)$ and let $\mathbf{G} := \boldsymbol{G}(1)$. Using the scaling property of Brownian motion, for every bounded measurable function f, we have

$$\begin{aligned} E[f(\mathbf{Y}(1))] &= E[E[f(\boldsymbol{B}(\boldsymbol{s}))|_{\boldsymbol{G}(1)=\boldsymbol{s}}]] \\ &= E[E[f(\sum_j a_{1j}\mu_j s_j + \sum_j q_{1j} W_j(s_j), ..., \sum_j a_{nj}\mu_j s_j + \sum_j q_{nj} W_j(s_j))|_{\boldsymbol{G}=\boldsymbol{s}}]] \\ &= E[E[f(\sum_j a_{1j}\mu_j s_j + \sum_j q_{1j}\sqrt{s_j} W_j(1), ..., \sum_j a_{nj}\mu_j s_j + \sum_j q_{nj}\sqrt{s_j} W_j(1))|_{\boldsymbol{G}=\boldsymbol{s}}]] \\ &= E[f(\sum_j a_{1j}\mu_j \mathrm{G}_j + \sum_j q_{1j}\sqrt{\mathrm{G}_j} W_j(1), ..., \sum_j a_{nj}\mu_j \mathrm{G}_j + \sum_j q_{nj}\sqrt{\mathrm{G}_j} W_j(1))]. \end{aligned} \tag{A.5}$$

Thus

$$\mathbf{Y}(1) \stackrel{\mathcal{L}}{=} (\sum_j a_{1j}\mu_j \mathrm{G}_j + \sum_j q_{1j}\sqrt{\mathrm{G}_j} W_j(1), ..., \sum_j a_{nj}\mu_j \mathrm{G}_j + \sum_j q_{nj}\sqrt{\mathrm{G}_j} W_j(1))^T, \tag{A.6}$$

and $\boldsymbol{Y}(1)$ is a generalized normal mean-variance mixture. On the other hand let $\mathbf{Y} \in$ *Gnmv* with mixing distribution $\mathbf{G}$. Define $\boldsymbol{G}(t)$ as the subordinator so that $\boldsymbol{G}(1) \stackrel{\mathcal{L}}{=} \mathbf{G}$ and define the process $\boldsymbol{Y}$ by $\boldsymbol{Y}(t) = \boldsymbol{B}(\boldsymbol{G}(t))$. An argument similar to the previous one shows that $\boldsymbol{Y}(1) \stackrel{\mathcal{L}}{=} \mathbf{Y}$. ■

Proposition 2.2. The proof is a consequence of the following Proposition. ■

Proposition A.1. *Let* $\mathbf{Y} \in$ *Gnmv. The following holds:*

1. *if* $\mu_i = 0, i = 1, ..., n$, *i.e. in the symmetric case, the marginal distributions of* $\mathbf{Y}$ *are normal mean-variance distributions with a mixing variable that is a linear combination of the components of* $\mathbf{G}$.

2. *if* $\mathbb{A} = \mathbb{Q}^* := (q_{ij}^2)_{ij}$, $rank(\mathbb{Q}^*) = n$ *and* $\mu_j = 1, j = 1, ..., n$ *then the marginal distributions of* $\mathbf{Y}$ *are normal mean-variance distributions with a mixing variable that is a linear combination of the components of* $\boldsymbol{G}$*.*

Proof. Since $rank(\mathbb{Q}^*) = \max$, define $\mathbf{G}^* = \mathbb{Q}^* \boldsymbol{G}(1)$.

1. Since $\boldsymbol{Y}_i(1) =: \mathrm{Y}_i = \sum_{j=1}^n q_{ij}\sqrt{\mathrm{G}_j}\mathrm{W}_j$, where $\mathrm{G}_j =: \boldsymbol{G}_j(1)$ and W_j are i.i.d $N(0,1)$, from the scaling property of the normal distribution, it follows that $\mathcal{L}\left(\sum_{j=1}^n q_{ij}\sqrt{g_j}\mathrm{W}_j\right) = \mathcal{L}\left(\sqrt{\sum_{j=1}^n q_{ij}^2 g_j}\mathrm{W}\right)$, where W is a $N(0,1)$.

$$\begin{aligned}\mathcal{L}(\mathrm{Y}_i|\mathbf{G}=\boldsymbol{g}) &= \mathcal{L}\left(\sum_{j=1}^n q_{ij}\sqrt{\mathrm{G}_j}\mathrm{W}_j|\mathbf{G}=\boldsymbol{g}\right) = \mathcal{L}\left(\sum_{j=1}^n q_{ij}\sqrt{g_j}\mathrm{W}_j\right)\\ &= \mathcal{L}\left(\sqrt{\sum_{j=1}^n q_{ij}^2 g_j \mathrm{W}_j}\right) = \mathcal{L}\left(\sqrt{\sum_{j=1}^n q_{ij}^2 \mathrm{G}_j \mathrm{W}|\mathbf{G}=\boldsymbol{g}}\right)\end{aligned}. \tag{A.7}$$

$\mathbf{G} = \boldsymbol{g}$ iff $\mathbf{G}^* = \mathbb{Q}^*\boldsymbol{g}$, then

$$\mathcal{L}\left(\sqrt{\sum_{j=1}^n q_{ij}^2 \mathrm{G}_j \mathrm{W}_j|\mathbf{G}=\boldsymbol{g}}\right) = \mathcal{L}\left(\sqrt{\sum_{j=1}^n q_{ij}^2 \mathrm{G}_j \mathrm{W}|\mathbf{G}^*=\mathbb{Q}^*\boldsymbol{g}}\right) =$$

$$= \mathcal{L}\left(\sqrt{\sum_{j=1}^n q_{ij}^2 \mathrm{G}_j \mathrm{W}|\mathrm{G}_j^* = g_j^*}\right) = \mathcal{L}\left(\sqrt{g_j^*}\mathrm{W}|\mathrm{G}_j^* = g_j^*\right),$$

where $g_j^* = \sum_j q_{ij}^2 g_j$.

Therefore $\mathcal{L}(Y_j) = \mathcal{L}(\sqrt{\mathrm{G}_j^*}\mathrm{W})$ and the statement is proved.

2. If $\mathbb{A} = \mathbb{Q}^*$ and $\mu_i = 1$, $1 \le i \le n$, then $\sum_{j=1}^n a_{ij}\mathrm{G}_j\mu_j = \sum_{j=1}^n q_{ij}^*\mathrm{G}_j = \mathrm{G}_i^*$ and $\mathcal{L}(Y_i) = \mathcal{L}(\mathrm{G}_i^* + \sqrt{\mathrm{G}_i^*}\mathrm{W})$

□

Proposition 3.2. The statement is a direct application of the following Lemma that applies to $\boldsymbol{Y}$. ■

Lemma A.1. *Let* $\boldsymbol{Y} \in IGnmv$ *and assume* $\boldsymbol{G}$ *as in Definition 2.4. Then* $\mathbf{Y} \stackrel{\mathcal{L}}{=} \mathbf{Y}^X + \mathbf{Y}^Z$*, where* $\mathbf{Y}^X$ *has independent unidimensional normal mean-variance margins and* $\mathbf{Y}^Z$ *is a multivariate normal mean-variance mixture.* $\mathbf{Y}^X$ *and* $\mathbf{Y}^Z$ *are independent.*

Proof. Since (A.12) below holds, we give the proof for the vectors corresponding to $\mathbf{Y}(1)$ and $\boldsymbol{G}(1)$. The characteristic function of $\mathbf{G} =: \boldsymbol{G}(1)$ is

$$\psi_{\mathbf{G}}(\boldsymbol{z}) = \prod_{j=1}^{n} \psi_j(z_j)\psi_{\mathrm{Z}}\left(\sum_{j=1}^{n} \gamma_j z_j\right), \tag{A.8}$$

where ψ_j and ψ_{Z} are respectively the characteristic functions of X_j and Z, then that of $\mathbf{Y} =: \boldsymbol{Y}(1)$ becomes:

$$\begin{aligned}\psi_{\mathbf{Y}}(\boldsymbol{z}) &= \exp\left(\Psi_{\mathbf{G}}\left(i \cdot diag(\boldsymbol{\mu})\mathbb{A}^T \boldsymbol{z} - \frac{1}{2}\mathbb{Q}_{\boldsymbol{z}}\right)\right) \\ &= \prod_{j=1}^{n} \exp\left(\Psi_j\left(i\sum_{l=1}^{n} a_{lj} z_l \mu_j - \frac{1}{2}(\sum_{l=1}^{n} z_l q_{lj})^2\right)\right) \\ &\cdot \exp\left(\Psi_{\mathrm{Z}}\left(\sum_{j=1}^{n}\gamma_j\left\{i\sum_{l=1}^{n} a_{lj} z_l \mu_j - \frac{1}{2}(\sum_{l=1}^{n} z_l q_{lj})^2\right)\right\}\right).\end{aligned} \tag{A.9}$$

From the expression of $\psi_{\mathbf{Y}}$ we infer that $\mathbf{Y}$ is the convolution of two generalized mean-variance distributions, which we denote as $\mathbf{Y}^X$ and $\mathbf{Y}^Z$. Moreover if $\mathbf{Y} \in IGnmv$, its characteristic function reduces to

$$\begin{aligned}\psi_{\mathbf{Y}}(\boldsymbol{z}) &= \exp(\Psi_{\mathbf{G}}(\log\psi_{\mathrm{W}_1}(z_1), ..., \log\psi_{\mathrm{W}_n}(z_1))) \\ &= \prod_{j=1}^{n} \exp(\Psi_{\mathrm{X}_j}(\log\psi_{\mathrm{W}_j}(z_j)))\exp(\Psi_{\mathrm{Z}}(\log\sum_{j=1}^{n}\gamma_j\psi_{\mathrm{W}_j}(z_j))),\end{aligned} \tag{A.10}$$

where, as it is well known, $\prod_{j=1}^{n}\exp(\Psi_{\mathrm{X}_j}(\log\psi_{\mathbf{W}_j}(z_j)))$ is the characteristic function of a random vector with independent normal mean-variance mixture components and $\exp(\Psi_{\mathrm{Z}}(\log\sum_{j=1}^{n}\gamma_j\psi_{\mathrm{W}_j}(z_j)))$ is the characteristic function of a *Mnmv* distribution. □

For completeness we also prove that the previous results apply to the general case discussed in Section 2.

Proposition A.2. *Let* $\mathbf{Y} \in Gnmv$. *Let* $\mathbf{X}^j$, $j = 1, ..., n$ *be independent non negative infinitely divisible random vectors and* $\mathbf{G} = \sum_{j=1}^{n}\mathbf{X}^j$, *then*

$$\mathbf{Y}(t) \stackrel{\mathcal{L}}{=} \sum_{j=1}^{n}(\mathbf{X}^j\boldsymbol{\mu} + \mathbb{Q}\sqrt{\mathbf{X}^j}\boldsymbol{W})(t), \tag{A.11}$$

moreover the processes $\mathbf{X}^j\boldsymbol{\mu} + \mathbb{Q}\sqrt{\mathbf{X}^j}\boldsymbol{W}$, $j = 1, ..., n$ *are independent.*

Proof. Let $\mathbf{Y}^j := \mathbf{X}^j\boldsymbol{\mu} + \mathbb{Q}\sqrt{\mathbf{X}^j}\mathbf{W}$, and let $\mathbf{Y}^j(t)$ be the Lévy process such that $\mathcal{L}(\mathbf{Y}^j(1)) = \mathcal{L}(\mathbf{Y}^j)$ for $j = 1, ..., n$. Since $\mathbf{Y}(t)$ is a Lévy process, its characteristic function is

$$\psi_{\mathbf{Y}(t)}(\boldsymbol{z}) = (\psi_{\mathbf{Y}}(\boldsymbol{z}))^t. \tag{A.12}$$

Since $\Psi_{\mathbf{G}}(\boldsymbol{z}) = \sum_j \Psi_{\mathbf{X}_j}(\boldsymbol{z})$ holds we have

$$\begin{aligned}\psi_{\mathbf{Y}}(\boldsymbol{z}) &= \exp\left(\Psi_{\mathbf{G}}\left(i \cdot diag(\boldsymbol{\mu})\mathbb{A}^T\boldsymbol{z} - \frac{1}{2}\mathbb{Q}_{\boldsymbol{z}}\right)\right) \\ &= \exp\left(\sum_{j=1}^{n}\Psi_{\mathbf{X}_j}\left(i \cdot diag(\boldsymbol{\mu})\mathbb{A}^T\boldsymbol{z} - \frac{1}{2}\mathbb{Q}_z\right)\right) = \prod_{j=1}^{n}\psi_{\mathbf{Y}^j}(\boldsymbol{z}),\end{aligned} \tag{A.13}$$

where for each $j = 1, ..., n$, $\psi_{\mathbf{Y}^j}(\boldsymbol{z}) = \exp(\Psi_{\mathbf{X}_j}(i\boldsymbol{\mu}^T\boldsymbol{z}\mathbb{A} - \frac{1}{2}\mathbb{Q}_z))$ is the characteristic function of a *Gnmv* distribution. It follows that

$$\begin{aligned}\psi_{\mathbf{Y}(t)}(\boldsymbol{z}) &= (\exp(\Psi_{\mathbf{G}}(i \cdot diag(\boldsymbol{\mu})\mathbb{A}^T\boldsymbol{z} - \frac{1}{2}\mathbb{Q}_{\boldsymbol{z}})))^t \\ &= (\exp(\sum_{j=1}^{n}\Psi_{\mathbf{X}_j}(i \cdot diag(\boldsymbol{\mu})\mathbb{A}^T\boldsymbol{z} - \frac{1}{2}\mathbb{Q}_{\boldsymbol{z}})))^t = (\prod_{j=1}^{n}\psi_{\mathbf{Y}^j}(\boldsymbol{z}))^t = \prod_{j=1}^{n}(\psi_{\mathbf{Y}^j}(\boldsymbol{z}))^t,\end{aligned} \tag{A.14}$$

where $(\psi_{\mathbf{Y}^j}(\boldsymbol{z}))^t$ is the characteristic function of $\mathbf{Y}^j(t)$. Thus the thesis. □

Proposition 3.3. It is sufficient to show that $\tilde{\mathbf{Y}} := \tilde{\mathbf{Y}}(1)$ admits the representation $\tilde{\mathbf{Y}} = \mathbb{G}\boldsymbol{\mu} + \sqrt{\mathbb{G}}\mathbf{W}$ where $\mathbf{G}$ is given by Definition 3.1 and equation (3.1). The claim then immediately follows from Proposition 3.1. By Definitions 3.1 and 3.4 we have:

$$\tilde{\mathrm{Y}}_i \stackrel{\mathcal{L}}{=} (\mathrm{X}_i + \mathrm{V}_i)\mu_i + \sqrt{\mathrm{X}_i + \mathrm{V}_i}\tilde{\mathrm{W}}_i + \sum_{j=1}^{n} q_{ij}\mathrm{Z}\mu_j + \sum_{j=1}^{n} q_{ij}\sqrt{\mathrm{Z}}\mathrm{W}_j$$

where $\tilde{\mathbf{W}}_i \sim N(0, 1)$ is independent from $(\mathrm{W}_j)_{1\leq j\leq n}$. The scaling property of the normal distribution implies

$$\tilde{\mathrm{Y}}_i \stackrel{\mathcal{L}}{=} (\mathrm{X}_i + \mathrm{V}_i)\mu_i + \mathrm{Z}\sum_{j=1}^{n} q_{ij}\mu_j + \mathrm{W}\sqrt{\mathrm{X}_i + \mathrm{V}_i + \mathrm{Z}\sum_{j=1}^{n} q_{ij}^2}$$

where $\mathrm{W} \sim N(0, 1)$. A comparison of the last equation with (3.1) shows that the desired representation $\tilde{\mathbf{Y}} = \mathbb{G}\boldsymbol{\mu} + \sqrt{\mathbb{G}}\mathbf{W}$ holds if and only if $\sum_{j=1}^{n} q_{ij}\mu_j = \gamma_i^2\mu_i$ and $\sum_{j=1}^{n} q_{ij}^2 = \gamma_i^2$ for all $1 \leq i \leq n$. ■

Proof of equation (3.5).

By Theorem 2.1 we have for $\mathbb{A} = \mathbb{Q} = \mathbb{I}_n$

$$\psi_{\mathbf{Y}}(\boldsymbol{z}) = \psi_{\mathbf{G}}\left(i \cdot diag(\boldsymbol{\mu})\boldsymbol{z} - \frac{1}{2}diag(\boldsymbol{z})\boldsymbol{z}\right)$$

where $\psi_{\mathbf{G}}(\boldsymbol{w})$ can be derived with help of equations equations (3.2) and (B.1):

$$\psi_{\mathbf{G}}(\boldsymbol{w}) = \prod_{j=1}^{n} \psi_{\mathrm{X}_j}(w_j)\psi_{\mathrm{V}_j}(w_j) \cdot \psi_{\mathrm{Z}}\left(\sum_{j=1}^{n} \gamma_j^2 w_j\right) =$$

$$= \prod_{j=1}^{n} \frac{(1 - 2\frac{w_j}{b_j^2})^{-\frac{\lambda}{2}}}{K_\lambda(\delta_j b_j)} K_\lambda\left(\delta_j b_j \sqrt{1 - 2w_j b_j^2}\right)\left(1 - \frac{2w_j}{b_j^2}\right)^{-(\lambda - a)}\left(1 - \frac{2\sum_{j=1}^{n}\gamma_j^2 w_j}{b_j^2}\right)^{-a}.$$

Inserting $\boldsymbol{w} = diag(\boldsymbol{\mu})\boldsymbol{z} - \frac{1}{2}diag(\boldsymbol{z})\boldsymbol{z}$ and setting $\mu_j =: \beta_j$, $b_j = \sqrt{\alpha_j^2 - \beta_j^2}$ with respect to equation (3.4) of proposition 3.1 yields

$$\psi_Y(\boldsymbol{z}) = \prod_{j=1}^{n}\left(\frac{\alpha_j^2 - \beta_j^2}{\alpha_j^2 - (\beta_j + iz_j)^2}\right)^{\frac{\lambda}{2}} \frac{K_\lambda\left(\delta_j\sqrt{\alpha_j^2 - (\beta_j + iz_j)^2}\right)}{K_\lambda\left(\delta_j\sqrt{\alpha_j^2 - \beta_j^2}\right)}\left(\frac{\alpha_j^2 - \beta_j^2}{\alpha_j^2 - (\beta_j + iz_j)^2}\right)^{\lambda - \alpha} \cdot$$

$$\cdot\left(1 - \frac{\sum_{j=1}^{n}(2i\beta_j z_j - z_j^2)\gamma_j^2}{\alpha_j^2 - \beta_j^2}\right)^{-a}.$$

B Appendix

B.1 Generalized Inverse Gaussian distribution

Let $\lambda \in \mathbb{R}$, $a, b \in \mathbb{R}_+$ and neither zero. A generalized inverse Gaussian distribution is a three parameter distribution defined on the positive half line (shortly $GIG(\lambda, a, b)$). It is an infinitely divisible distribution and it generates a subordinator. Its characteristic function is

$$\psi_{GIG}(u) = \frac{1}{K_\lambda(ab)}\left(1 - \frac{2iu}{b^2}\right)^{-\frac{\lambda}{2}} K_\lambda(ab\sqrt{1 - 21ub^{-2}}), \qquad \text{(B.1)}$$

where $K_\lambda(x)$ denotes the modified Bessel function of the third kind with index λ.

The GIG mean and variance are:

$$\frac{aK_{\lambda+1}(ab)}{bK_\lambda(ab)} \qquad \text{(B.2)}$$

$$a^2 b^{-2} K_\lambda^{-2}(ab)(K_{\lambda+2}(ab)K_\lambda(ab) + K_{\lambda+1}^2(ab)). \qquad \text{(B.3)}$$

B.2 Generalized Hyperbolic distribution

Let $\lambda, \beta \in \mathbb{R}$, $\alpha, \delta \in \mathbb{R}_+$, with

$$\begin{aligned} &\delta \geq 0, \ \ |\beta| < \alpha \ \text{ if } \lambda > 0 \\ &\delta > 0, \ \ |\beta| < \alpha \ \text{ if } \lambda = 0 \\ &\delta > 0, \ \ |\beta| \leq \alpha \ \text{ if } \lambda < 0. \end{aligned} \tag{B.4}$$

The Generalized hyperbolic distribution - shortly $GH(\alpha, \beta, \delta, \lambda)$ - has been introduced in literature by Barndorff-Nielsen [2]. He also showed that it is a normal mean-variance mixture with mixing distribution GIG. If G $\sim GIG(\lambda, a, b)$ (positive distribution), W is standard normal and they are independent, then $\sqrt{\mathrm{G}}\mathrm{W} + \mu\mathrm{G}$ has a GH distribution, with parameters $\gamma, \beta, \delta, \lambda$ where:

$$\begin{aligned} a &= \delta \\ \mu &= \beta \\ b &= \sqrt{\alpha^2 - \beta^2}. \end{aligned} \tag{B.5}$$

The GH characteristic function is:

$$\psi_{GH}(u) = \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2} \right)^{\lambda/2} \frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta + iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}. \tag{B.6}$$

The GH distribution mean is

$$\frac{\beta\delta}{\alpha^2 - \beta^2} \frac{K_{\lambda+1}(\delta\sqrt{\alpha^2 - \beta^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}. \tag{B.7}$$

Its variance is

$$\delta^2 \left(\frac{K_{\lambda+1}(\delta\sqrt{\alpha^2 - \beta^2})}{\delta\sqrt{\alpha^2 - \beta^2} K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})} + \frac{\beta^2}{\alpha^2 - \beta^2} \left(\frac{K_{\lambda+2}(\delta\sqrt{\alpha^2 - \beta^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})} - \frac{K^2_{\lambda+1}(\delta\sqrt{\alpha^2 - \beta^2})}{K^2_\lambda(\delta\sqrt{\alpha^2 - \beta^2})} \right) \right) \tag{B.8}$$

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References

[1] Ané, T., Geman, H. (2000) Order flow, transaction clock, and normality of asset returns. *The Journal of Finance*, 55, 2259-2284.

[2] Barndorff-Nielsen, O.E. (1977) Exponential decreasinG distributions for the logarithm of particle size. *Proceedings of the Royal Society of London*, A 353 401-419.

[3] Barndorff-Nielsen, O.E. (1978) Hyperbolic distributions and distributions on hyperbolae. *Scandinavian Journal of Statistics*, 5, 151-157.

[4] Barndorff-Nielsen, O.E. (1995) Normal inverse Gaussian distributions and the modeling of stock returns. *Research report* no. 300, Department of Theoretical Statistics, Aarhus University.

[5] Barndorff-Nielsen, O.E., Jensen, J.L., and Halgreen, C. (1987) First hitting time models for the generalized inverse Gaussian distribution. *Stochastic Processes and their Applications*, 7, 49–54.

[6] Barndorff-Nielsen, O.E., Blaesild, P., Seshadri, V. (1992) Multivariate distributions with generalized inverse Gaussian marginals, and associated Poisson mixtures. *The Canadian Journal of Statistics*, 20, 2, 109-120.

[7] Barndorff-Nielsen, O.E., Kent, J. Sorensen, M. (1982) Normal Variance-mean mixtures and z distributions *International Statistical Review* 50 No. 2, 145-159.

[8] Barndorff-Nielsen, O.E., Pedersen, J. Sato, K.I. (2001) Multivariate Subordination, Self-Decomposability and Stability. *Adv. Appl. Prob.* 33, 160-187.

[9] Blæsild, P., Jensen, J.L. (1981) Multivariate distributions of hyperbolic type. In C. Taillie, G.B. Patil, and B.A. Baldessari (Eds.). *Statistical Distributions in Scientific Work*, Vol. 4, Reidel (Dordrecht), 45–66.

[10] Carr, P., Geman, H., Madan, D. B., Yor, M. (2002) The fine structure of asset returns: an empirical investigation. *Journal of Business* 75, 305-332.

[11] Clark, P.K. (1973) A subordinated stochastic process model with finite variance for speculative prices *Econometrica* 41, 135-155.

[12] Eberlein, E. Keller, U. (1995) Hyperbolic distributions in Finance. *Bernoulli*, 1(3), 281-299.

[13] Eberlein, E., Prause, K. (2002) The generalized hyperbolic model: financial derivatives and risk measures. *Mathematical finance—Bachelier Congress* (Paris), 245–267, Springer Finance, Springer, Berlin.

[14] Eberlein, E.; v. Hammerstein, E. A. (2004) Generalized hyperbolic and inverse Gaussian distributions: limiting cases and approximation of processes. Seminar on Stochastic Analysis, Random Fields and Applications IV, 221–264, Progr. Probab., 58, Birkhäuser, Basel.

[15] Geman, H., Madan, D.B., Yor, M. (2001) Time changes for Lévy processes. *Mathematical Finance* 11, 1, 79-96.

[16] Harris, L. (1986) Cross-security tests of he mixture distribution *The Journal of Financial and Quantitative Analysis* 21, 39-46.

[17] Kelker, D. (1971) Infinite divisibility and variance mixture of the normal distribution. *The Annals of Mathematical statistics*, 42, 2, 802-808.

[18] Lo, A. W., Wang, J. (2000) Trading volume: definitions, data analysis, and implications of portfolio theory *Review of Financial Studies*, 13, 257-300.

[19] Luciano, E., Semeraro, P. (2007) *Extending Time-Changed Lévy Asset Models Through Multivariate Subordinators*, working paper 42/2007, Collegio Carlo Alberto.

[20] Madan, D. B., Seneta, E. (1990) The v.g. model for share market returns. *Journal of Business* 63, 511-524.

[21] McNeil, A.J., Frey, R., Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools.* Princeton University Press.

[22] Monroe, I. (1978) Process that can be embedded in Brownian motion *Annals of Probability* 6, 42-56.

[23] Sato, K.I. (2003) *Lévy processes and Infinitely divisible distributions* Cambridge Studies in Advanced Mathematics Cambridge University Press.

[24] Schmidt, R. Hrycej, T., Stützle, E. (2006) Multivariate distribution models with generalized hyperbolic margins. *Comput. Statist. Data Anal.*, 50, 2065-2096.

[25] Semeraro, P. (2006) A Multivariate Variance Gamma Model for Financial Application. *Journal of Theoretical and Applied Finance*, 11: 1-18.