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A general theory of subjective mixtures*

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Abstract

We provide a framework for constructing subjective mixtures which requires neither the Certainty Independence nor the Monotonicity axiom, replacing them with much weaker "local" properties. As we show by means of examples, this framework provides a purely subjective foundation to most of the recent preference models which employ an Anscombe-Aumann setting. It also allows disentagling the notions of ambiguity aversion and preference for randomization. The scope of our framework is further demonstrated by discussing how subjective mixtures can be employed in modelling choice between menus of consequences, and also by providing a fully subjective axiomatization of Recursive Variational Preferences.

1 Introduction

L.J. Savage's axiomatization of the Subjective Expected Utility (SEU) model (Savage, 1954) has justly been hailed as a "crowning achievement" of decision theory. Its conciseness and beauty, however, came at a cost, and its extensions formulated to capture non-expected utility behavior have been very few (the closest being Gilboa's axiomatization of Choquet Expected Utility in Gilboa (1987)). On the other hand, the SEU axiomatization of Anscombe and Aumann (1963) based on the presence of an external randomizing device, as reformulated by Fishburn (1970), has been extended in many different directions. The reason for this popularity is that the vector space structure assumed in the Anscombe-Aumann (AA) approach makes it much easier to formulate generalizations of axioms such as indepedence, which correspond directly to properties of the preference representation. However, a conceptual problem with the AA approach is that the vector space structure is built upon the assumption of a randomizing device with the feature that all actors agree on its "mixing" results. For this reason, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2001, 2003, henceforth GMMS) proposed a framework to construct "subjective

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mixtures" without assuming the presence of such randomizing device. Within their framework, it is possible to obtain a (subjective) vector space structure that can be used to formulate general AA-style axioms. For instance, one can obtain a model which accounts for both the Ellsberg and the Allais paradoxes (see Dean and Ortoleva, 2017), something that is impossible in the standard AA setting.

However, the proposal of GMMS suffers from some limitations that over time have become increasingly apparent. First, the GMMS construction was developed in the context of preferences satisfying the "Certainty Independence" axiom of Gilboa and Schmeidler (1989). It is by now well understood that Certainty Independence imposes strong restrictions on the decision maker's attitudes as they change over wealth levels, ruling out important models such as Variational Preferences of Maccheroni, Marinacci, and Rustichini (2006a) and the "Smooth Ambiguity" model of Klibanoff, Marinacci, and Mukerji (2005) (see, e.g., the discussion in Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011), henceforth CGMMS). Second, the GMMS construction employed the traditional "Monotonicity" axiom of Gilboa and Schmeidler (1989), a condition which also entails a separability property of preferences, as we shall discuss below.[1] This rules out several recent decision models, such as the Mean-Dispersion preference model of Grant and Polak (2013), the dual utility model of Bommier (2017), the generalized convex representation of Cheridito, Delbaen, Drapeau, and Kupper (2015), and the correlation misperception model of Ellis and Piccione (2017).

In this paper we provide a framework for constructing subjective mixtures which requires neither Certainty Independence nor Monotonicity, replacing these axioms with much weaker "local" properties. This framework provides a purely subjective foundation to all the mentioned models, on top of those discussed in GMMS. As we illustrate by means of several examples (Section 4), the basic preference assumptions needed to apply our construction really add very limited structure on top of that assumed by the model that one wants to implement.

While the need to relax Certainty Independence —which stipulates that independence should hold when mixing with constant acts— is clear, there has been less discussion about the restrictiveness of Monotonicity —which stipulates that[2] if $f(s) \succcurlyeq g(s)$ for every $s \in S$, then $f \succcurlyeq g$. Consider a (non-null) event A and outcomes x, y and z; denote by xAz the bet that pays x if A ob-

[1] For SEU preferences, it is also equivalent to AA's original state independence axiom.

[2] An act is a function from states of the world to outcomes $f : S \to X$. We denote by $f(s)$ the payoff $f(s) \in X$ of act f in state $s \in S$.

tains and z otherwise. Then the Monotonicity axiom implies that $xAz \succcurlyeq yAz$ if $x \succcurlyeq y$, so that the evaluation of the bets xAz and yAz is determined by the preference between the payoffs x and y, *independently of* z and of A. The fact that such separability is restrictive is openly argued by Schneider and Schonger (2015), who provide experimental evidence in support of their view. Our axiomatic framework imposes monotonicity and independence conditions only for *one* specific event, and is therefore much less restrictive than the one employed in GMMS.[3]

An advantage of a subjective construction of mixtures is the possibility of operating a distinction between sensitivity to ambiguity and attitudes with respect to (objective) randomizations. Ever since the seminal work of Schmeidler (1989), aversion of uncertainty has been linked to a preference for randomization: "substituting objective mixing for subjective mixing makes the decision maker better off" (Schmeidler, 1989, p. 582). There are however doubts as to the necessity of such link, both from a theoretical perspective (see, e.g. Eichberger and Kelsey, 1996) and from an experimental perspective (Dominiak and Schnedler, 2011). Our framework allows us to severe the ties between ambiguity aversion and preference for randomization, thus providing the necessary flexibility to discuss these questions.

In brief, our analysis proceeds as follows. We first (Section 3) state axioms which allow us to derive a cardinally unique utility u and a representation of preferences which is separable with respect to bets on a "nontrivial" event E (more details below). Then, following GMMS, we introduce the main tool for defining subjective mixtures: the notion of *preference midpoint* of two outcomes x and y. Given the event E and a bet vEw —which pays outcome v if event E obtains and outcome w otherwise— let c_{vEw} denote its certainty equivalent.[4] We call the outcome z a *preference midpoint* of x and y, and denote $z = \frac{1}{2}x \oplus \frac{1}{2}y$, if z satisfies the following indifference:

$$xEy \text{ is indifferent to } \; c_{xEz}Ec_{zEy}$$

That is, since $xEy \sim c_{xEx}Ec_{yEy}$, we can replace z for x and y in the bets on the right-hand side of the indifference without affecting the DM's *evaluation* of the bet xEy. It is readily shown that in the representation mentioned above, the preference midpoint z satisfies $u(z) = \frac{1}{2}u(x) + \frac{1}{2}u(y)$. Therefore, z is also a *utility midpoint*, as expected. In view of this, the mixture operator $\oplus$ over out-

[3]In the extended version of 2001, GMMS obtain as an intermediate result (Lemma 1) a representation which holds only for one event, but under stronger conditions than those used in this paper.

[4]The existence of such certainty equivalent is guaranteed by our axioms.

comes thus defined has the same properties of the "objective" mixture of Anscombe and Aumann, and, once extended to acts state by state, it can be analogously applied to leverage functional analytic techniques as in the traditional AA setting. However, its nature is completely subjective.

As mentioned above, the construction of preference/utility midpoints, and hence subjective mixtures, revolves on the possibility of obtaining a locally separable representation of the preferences, with a cardinal utility. For, if u were just an ordinal index, the conclusion that z is a utility midpoint would be nonsensical. The challenge is how to do so while imposing minimal restrictions on preferences. We assume some basic (and standard) preference axioms, plus a bisymmetry and a weak separability axiom on the bets "on" the event E; i.e., bets of the form vEw with outcome v *weakly preferred* to outcome w. Given these axioms, there is a utility function u on outcomes and a number $\rho \in (0,1)$ such that the preference over the bets on E takes the form

$$V(vEw) = u(v)\rho + u(w)(1-\rho).$$

This representation only holds for bets "on" E, but not for bets on other events, or for non-binary acts. This is why we call it a "locally biseparable" representation. We provide (Section 4) several examples of preferences having one such representation for some event, and therefore satisfying our axioms, while violating full Monotonicity and/or Certainty Independence.

In principle, the utility index thus obtained –and hence also the notion of preference midpoint– could be dependent on the specific event E used in the axioms. We next show that requiring preference midpoints to be independent of the event used to identify them, a natural restriction that we call *Invariance*, guarantees that the utility index will also be identified independently of E. We also discuss (in Section 5) further restrictions on the global properties of the preferences, showing that —while such restrictions have no bearing on the construction of subjective mixtures— they provide more structure to the locally biseparable representation we obtain. This allows us to relate our representation theorem to those contained in CGMMS and Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011). Section 6 introduces an objective mixture structure and discusses the relationship between attitudes to ambiguity and objective randomization.

We close the paper (Section 7) with some applications of subjective mixtures to different issues in decision theory. We begin by showing how to introduce an "ex-ante" notion of randomization in our setting. This allows us to model DMs who may not be indifferent toward the timing of resolution of uncertainty. Then, we discuss how to use our subjective mixtures to mix menus of consequences, in the spirit of e.g., Dekel, Lipman, and Rustichini (2001) and Gul and Pesendorfer (2001). Lastly, we provide a complete fully subjective axiomatization of the recursive multiple prior of Epstein and Schneider (2003) and the recursive variational preferences of Maccheroni, Marinacci, and Rustichini (2006b). Next, we The Appendix contains all the proofs of the results in the paper.

2 Preliminaries

We consider an arbitrary state space S, an algebra Σ of subsets of S which represents events, and a set of consequences X. $\mathscr{F}$ denotes the set of all *simple* acts; i.e., finite-valued Σ-measurable functions $f : S \to X$. We make the following assumption on the set X:

Structural assumption: X is a connected and separable topological space with topology τ.

As it is well-known, τ induces the product topology on the set X^S of all functions from S into X. In this topology, a net $\{f_\alpha\}_{\alpha\in D} \subseteq X^S$ converges to $f \in X^S$ if and only if $f_\alpha(s) \xrightarrow{\tau} f(s)$ for all $s \in S$ (remember that S is arbitrary). For this reason it is also called the topology of *pointwise convergence*.

As customary, we notationally identify a consequence $x \in X$ with the constant act f such that $f(s) = x$ for all $s \in S$. Given a functional $V : \mathscr{F} \to \mathbb{R}$, we say that: V *represents* a binary relation $\succcurlyeq$ on $\mathscr{F}$ if $f \succcurlyeq g$ iff $V(f) \geq V(g)$, that V is *constant-bounded* if for any $f \in \mathscr{F}$, $V(x) \geq V(f) \geq V(y)$ for some $x, y \in X$, and finally that V is *subcontinuous* if $\lim_\alpha V(f_\alpha) = V(f)$ whenever $\{f_\alpha\}_{\alpha\in D} \subseteq \mathscr{F}$ is a net that pointwise converges to $f \in \mathscr{F}$, and such that all f_α's and f are measurable with respect to the same finite partition. (Notice that this implies that $V|_X$ is τ-continuous.) Given representations V and V', we write $V \approx V'$, if there are $a, b \in \mathbb{R}$ such that $a > 0$ and $V' = aV + b$. In this case, we say that V and V' are *cardinally equivalent*.[5]

We denote by $B_0(\Sigma,\Gamma)$ the set of simple Σ-measurable functions on S taking value in $\Gamma \subseteq \mathbb{R}$. A functional $I : B_0(\Sigma,\Gamma) \to \mathbb{R}$ is said to be: *monotonic* if $I(a) \geq I(b)$ when $a \geq b$, *continuous* if it is sup-norm continuous, *normalized* if $I(\gamma 1_S) = \gamma$ for all $\gamma \in \Gamma$. I is said to be *convex* (resp. *concave*) if, for all $a, b \in B_0(\Sigma,\Gamma)$ and all $\lambda \in [0,1]$, $I(\lambda a + (1-\lambda)b) \leq \lambda I(a) + (1-\lambda) I(b)$ (resp. $I(\lambda a + (1-\lambda)b) \geq$

[5] We will also use the same terminology for utility functions.

$\lambda I(a) + (1-\lambda)I(b))$, *affine* if it is both convex and concave. A set function $\rho : \Sigma \to [0,1]$ is called a *capacity* if it is normalized ($\rho(\emptyset) = 0$ and $\rho(S) = 1$), and monotone ($\rho(A) \geq \rho(B)$ whenever $A \subseteq B$).

3 A foundation for subjective mixtures

3.1 Locally biseparable preferences

As previewed in the Introduction, the definition of preference midpoint hinges on the possibility of representing the preference in a specific way. The following basic assumptions on preferences are clearly necessary:

Axiom (Preference Order - P)**.** *$\succsim$ is a complete and transitive relation on $\mathscr{F}$ which also satisfies: 1) there are $f, g \in \mathscr{F}$ such that $f \succ g$, 2) for each $f \in \mathscr{F}$, there exist $x, y \in X$ such that $x \succsim f \succsim y$.*

This axiom bundles some standard requirements on preferences, including nontriviality and boundedness.[6] The next axiom is a continuity property, taken from Ghirardato and Marinacci (2001). Notice that all acts involved are measurable with respect to the same finite partition of S, making the condition rather weak.

Axiom (Continuity - C)**.** *Let $\{f_\alpha\}_{\alpha \in A}$ be a net in $\mathscr{F}$ that pointwise converges to $f \in \mathscr{F}$ and such that all f_α and f are measurable w.r.t. the same finite partition of (S, Σ). If $f_\alpha \succsim g$ (resp. $g \succsim f_\alpha$) for all $\alpha \in A$, then $f \succsim g$ (resp. $g \succsim f$).*

These two axioms (and the Structural Assumption) imply the existence of certainty equivalents (see Lemma 5 in the Appendix). Precisely, given any $f \in \mathscr{F}$, there is a $c_f \in X$ such that $c_f \sim f$. Indeed, standard arguments imply the existence of a representation V of $\succsim$. Formally:

Lemma 1. *The binary relation $\succsim$ satisfies axioms P and C if and only if there exists a non-constant, constant bounded sub-continuous $V : \mathscr{F} \to \mathbb{R}$ such that $V(f) \geq V(g)$ if and only if $f \succsim g$. Moreover, V is unique up to monotonic nondecreasing trasformations.*

This is not quite enough, as cardinality is necessary in order to meaningfully define a 1/2:1/2 mixture of two outcomes. As in our setting the consequence space X has no algebraic structure,[7]

[6] Boundedness is usually omitted, as (for simple acts) it is implied by Monotonicity.

[7] For an alternative approach where subjective mixtures are derived by adding some structure to the space of outcomes see (Ghirardato and Pennesi, 2019).

deriving a cardinally unique utility requires a representation with a degree of separation between utility and beliefs. The next definition provides one such representation:

Definition 1. *A representation V of a binary relation $\succcurlyeq$ on $\mathscr{F}$ is said to be E-*biseparable *if V is nontrivial, constant-bounded and subcontinuous, and for some $E \in \Sigma$ there exists a $\rho_E \in [0,1]$ such that for any $x, y \in X$ satisfying $x \succcurlyeq y$,*

$$V(xEy) = \rho_E V(x) + (1-\rho_E)V(y). \tag{1}$$

The binary relation $\succcurlyeq$ is said to be locally biseparable *if for some $E \in \Sigma$, it has a E-biseparable representation which is unique up to positive affine transformations.*

The features of a locally biseparable preference which are important for our objective are the following: 1) The representation $V \equiv V_E$ identified via Eq. (1) is cardinal; thus, the restriction of V_E to the constants, denoted u_E, is also cardinal and the 1/2:1/2 mixtures it produces are well-defined. 2) Eq. (1) also allows us to identify the DM's *willingness to bet on E*, denoted ρ_E, independently of the bet's payoffs x and y and of (the positive affine transformations of) V_E. Eq. (1) also guarantees that the representation V_E is "state-independent" within the sets of the bets "on" the event E (i.e., acts xEy such that $x \succcurlyeq y$), as $V_E(x) > V_E(x')$ if and only if $V_E(xEy) > V_E(x'Ey)$, and symmetrically for y.[8]

In order to show how to characterize locally biseparable preferences behaviorally, we start with an observation that formalizes an insight of Ellsberg (1954) on the necessity of nontrivial uncertainty to obtain the cardinality of utility (see also Remark 8 in Ghirardato and Marinacci (2001)). Even if the preference satisfies Axioms P and C, when all events are either null or universal the representation cannot be cardinal. An event $A \in \Sigma$ is said to be *null* (resp. *universal*) if $y \sim xAy$ (resp. $x \sim xAy$) for every $x \succ y$.

Fact 1. *Suppose that $\succcurlyeq$ satisfies Axioms P and C. If all events in Σ are either null or universal, then any functional $V : \mathscr{F} \to \mathbb{R}$ representing $\succcurlyeq$ is such that for all $x \succ y$ and all $A \in \Sigma$, $V(xAy) = \rho(A)V(x) + (1-\rho(A))V(y)$, with $\rho(A) = 0$ if A is null and $\rho(A) = 1$ if A is universal.*

It follows that a locally biseparable preference $\succcurlyeq$ must be such that some $E \in \Sigma$ is neither null nor universal for $\succcurlyeq$, which implies $\rho_E \in (0,1)$. For, let $E \in \Sigma$ be such that $\succcurlyeq$ has a cardinal E-

[8] As we show immediately after Fact 1, cardinality of V_E implies $\rho_E \in (0,1)$.

biseparable representation denoted V_E. If E were null or universal, then either $V_E(xEy) = V_E(y)$ or $V_E(xEy) = V_E(x)$. Hence, the corresponding $\rho_E \in \{0,1\}$. But then, any monotonic nondecreasing transformation of V_E would also satisfy Eq. (1), contradicting the cardinality of V_E.

The existence of an event E which is neither null nor universal is, however, not sufficient to establish local biseparability. In order for local biseparability to obtain for the event E, the willingness to bet on E must be assessed independently of the utility of the bet's payoffs. This is the purpose of our third axiom.

To state the axiom, we start with a condition that, following the theory of measurement, we call "ordered bisymmetry" (see Pfanzagl, 1959). In the formulation, we use $x \succcurlyeq \{z, z'\}$ (resp. $\{z, z'\} \succcurlyeq y$) as a short-hand for $x \succcurlyeq z$ and $x \succcurlyeq z'$ (respectively $z \succcurlyeq y$ and $z' \succcurlyeq y$).

Definition 2. *An event $E \in \Sigma$ is* ordered-bisymmetric *if, for all $x, y, z, z' \in X$ such that $x \succcurlyeq \{z, z'\} \succcurlyeq y \in X$, we have:*

$$c_{xEz} E c_{z'Ey} \sim c_{xEz'} E c_{zEy} \tag{2}$$

That is, the event E is ordered-bisymmetric if receiving the two "internal" outcomes z and z' in symmetric "combinations" of E and E^c does not affect the DM's preference.

Our third axiom requires the existence of an event E which is: a) ordered bisymmetric, and b) such that all bets on E satisfy the "state independence" property mentioned above.

Axiom (Ordered Niceness - ON)**.** *There exists $E \in \Sigma$ which satisfies:*

*(a) (*Ordered E-bisymmetry*) E is ordered-bisymmetric;*

*(b) (*Ordered E-monotonicity*) For all $x, y, z \in X$: $x \succcurlyeq y$ if and only if $xEz \succcurlyeq yEz$ for any $z \preccurlyeq \{x, y\}$; $x \succcurlyeq y$ if and only if $zEx \succcurlyeq zEy$ for any $z \succcurlyeq \{x, y\}$.*

It follows readily from property (b) above (and Axiom P) that the event E satisfies $x \succ xEy \succ y$ for all $x \succ y$. That is, E is an "essential" event in the sense of Ghirardato and Marinacci (2001); in particular, it is neither null nor universal.

It is also easy to check that (alongside Axioms P and C) both properties in Axiom ON are necessary for the existence of a representation $V \equiv V_E$ of $\succcurlyeq$ which is E-biseparable. Roughly, Ordered E-bisymmetry is linked to the fact that the weight ρ_E is independent of the payoffs $x \succcurlyeq y$,[9] and

[9] In Appendix B, we provide an example of a preference with payoff-dependent willingness to bet on an event E which satisfies Axioms P, C, and Ordered E-monotonicity, but violates Ordered E-bisymmetry.

Ordered E-monotonicity is linked to the fact that $V_E(xEy)$ depends on x (resp. y) only through $V_E(x)$ (resp. $V_E(y)$).

Leveraging on results of Nakamura (1990), we next show that (alongside Axioms P and C) the two conditions in Axiom ON are also sufficient for E-biseparability. They also imply that given any positive affine transformation V' of V_E, Eq. (1) holds for V' —so that local biseparability follows— and that it holds with *the same weight* ρ_E. That is, the willingness to bet ρ_E is independent of the cardinal transformations of V_E. We thus get our basic representation result:

Proposition 1. *$\succcurlyeq$ is a binary relation satisfying Axioms P, C, and ON if and only if there exists $E \in \Sigma$ for which there exists a E-biseparable representation of $\succcurlyeq$ which is unique up to a positive affine transformation; i.e., $\succcurlyeq$ is locally biseparable.*

Suppose $\succcurlyeq$ is a locally biseparable preference, with a cardinal E biseparable representation V_E. As our notation makes clear, V_E depends on the event E. If F is another event for which the conditions of Axiom ON are satisfied, then we cannot rule out that the corresponding representation V_F, although ordinally equivalent to V_E, will be cardinally *non-equivalent* to V_E on the constant acts in X. That is, with our three axioms we cannot rule the existence of multiple "event-dependent" cardinal representations and utilities.

3.2 Preference midpoints and subjective mixtures

Recall the notion of preference midpoint discussed in the Introduction (assuming that the preference satisfies Axioms P and C):

Definition 3. *Given $x \succcurlyeq y$ and an event E for which $\succcurlyeq$ satisfies Axiom ON, we call $z \in X$ a E-*preference midpoint *of x and y if z satisfies $x \succcurlyeq z \succcurlyeq y$ and*

$$xEy \sim c_{xEz}Ec_{zEy}.$$

Let u_E denote the restriction of V_E to the constant acts; i.e., $u_E(x) \equiv V_E(x)$ for all $x \in X$. It follows immediately from the representation in Proposition 1 that such z satisfies $u_E(z) = \frac{1}{2}u_E(x) + \frac{1}{2}u_E(y)$; that is, z is the midpoint of x and y according to the utility function u_E.

Suppose now that $F \neq E$ is an event such that $\succcurlyeq$ is also F-biseparable. Specifically, $F \in \mathscr{E}$, where

$$\mathscr{E} \equiv \{F \in \Sigma : \succcurlyeq \text{ satisfies Axiom ON for } F\}.$$

In general, midpoints could depend on the event in $\mathscr{E}$ used to identify them. Our next axiom rules out this possibility.

Axiom (Invariance - NV)**.** *For all $E, F \in \mathscr{E}$ and $x, y \in X$ and $z \in X$ satisfying $x \succcurlyeq z \succcurlyeq y$,*

$$xEy \sim c_{xEz}Ec_{zEy} \iff xFy \sim c_{xFz}Fc_{zFy}.$$

If, as mentioned earlier, the representing utility u_F (where analogously $u_F \equiv V_F|_X$) were to be cardinally non-equivalent to u_E, some utility midpoints according to u_F would not coincide with those induced by u_E, even up to indifference. Therefore, (see also Lemma 3 and Corollary 6 in GMMS (2001)) Axiom NV also guarantees the cardinal equivalence of u_E and u_F (and therefore V_E and V_F).[10]

Lemma 2. *Suppose $\succcurlyeq$ satisfies Axioms P, C, ON, and NV. Then:*

(a) For any $x \succ y$, there exists a unique (up to indifference) $z \in X$ that is a E-preference midpoint of x and y. Moreover, for any $F \in \mathscr{E}$, z is also a F-preference midpoint of x and y.

(b) If $E, F \in \mathscr{E}$, and u_E (resp. V_E) and u_F (resp. V_F) are the respective cardinal utilities (resp. representations) obtained via Proposition 1, then $u_E \approx u_F \equiv u$ (resp. $V_E \approx V_F \equiv V$).

In light of Lemma 2, we henceforth refer to the z identified in item (a) as *the preference midpoint* of x and y, and denote it by

$$z = \frac{1}{2}x \oplus \frac{1}{2}y. \tag{3}$$

We can now present our second representation result:

Proposition 2. *$\succcurlyeq$ is a binary relation satisfying Axioms P, C, ON and NV if and only if there exist a nontrivial, constant-bounded and subcontinuous functional $V : \mathscr{F} \to \mathbb{R}$ which represents $\succcurlyeq$ and a*

[10] This result is also of use for the purpose of distiguishing differences in traditional risk attitude from comparative definitions of ambiguity aversion in fully subjective environments; see the discussion in Ghirardato and Marinacci (2002).

function $\rho : \mathscr{E} \to (0,1)$, such that for all $x \succcurlyeq y$,

$$V(xEy) = \rho(E)V(x) + (1 - \rho(E))V(y). \tag{4}$$

Moreover, V is unique up to positive affine transformations.

Two comments on the representation in Proposition 2 are in order. First, because of the lack of Monotonicity of $\succcurlyeq$, it may not be the case that the set-function ρ is itself monotonic: given $E, F \in \mathscr{E}$, it is not necessarily true that $E \subseteq F$ implies $\rho(E) \le \rho(F)$ (see Proposition 4 below). Second, even if both E and E^c belong to $\mathscr{E}$, it may not be the case that $\rho(E) + \rho(E^c) = 1$ (see Example 1 in Section 4). That is, the preference over bets on E is not necessarily represented by expected utility.[11]

Given the notion of preference midpoint of Eq. (3), we can consider iterated midpoints such as $\frac{1}{2}x \oplus \frac{1}{2}\left[\frac{1}{2}x \oplus \frac{1}{2}y\right]$, which corresponds to a $\frac{3}{4} : \frac{1}{4}$ mixture of x and y. More generally, since a dyadic rational $\lambda \in (0,1)$ can be written as $\lambda = \sum_i^N \frac{a_i}{2^i}$ (where N is finite, $a_i \in \{0,1\}$, and $a_N = 1$), we can define $\lambda x \oplus (1-\lambda)y$ as an iterated preference average

$$\frac{1}{2}z_1 \oplus \frac{1}{2}\left(\ldots\left(\frac{1}{2}z_{N-1} \oplus \frac{1}{2}\left(\frac{1}{2}z_N \oplus \frac{1}{2}y\right)\right)\ldots\right)$$

where for every i, $z_i = x$ if $a_i = 1$ and $z_i = y$ otherwise. The subjective mixture of acts f and g is now defined in the usual, state-by-state, fashion:

Definition 4. *Given $f, g \in \mathscr{F}$ and a dyadic rational λ, the* subjective (dyadic) mixture $\lambda f \oplus (1-\lambda)g$ *is an act $h \in \mathscr{F}$ such that $h(s) = \lambda f(s) \oplus (1-\lambda)g(s)$ for every $s \in S$.*

It follows from continuity of the u function and this definition (see Lemma 13 in GMMS (2001)) that for every $f, g \in \mathscr{F}$ and $s \in S$ and for all $\lambda \in [0,1]$,

$$u[(\lambda f \oplus (1-\lambda)g)(s)] = \lambda u(f(s)) + (1-\lambda)u(g(s)). \tag{5}$$

Hence, subjective mixtures behave as the AA "objective" mixtures in the traditional AA model: they allow us to obtain the convex combination of any two utility profiles. Differently from "objective" mixtures, the vector space structure that they induce is robust to violations of expected utility on

[11] The comment applies a fortiori to Proposition 1.

the objective probabilities (for instance, those that would be displayed by a preference that obeys the rank-dependent EU model).

4 Some examples

In this section, we present some examples which help assessing the scope of our foundation of subjective mixtures and shed some light on the empirical content of our Axioms ON and NV. The examples also provide a comparison with the earlier work of GMMS, as they involve preferences which satisfy the axioms of Proposition 2 while violating full Monotonicity and/or Certainty Independence.[12]

Example 1. Given a finite S, let $\succcurlyeq$ be represented by a mean-standard deviation preference with respect some prior $p \in \Delta(S)$ and a Bernoulli utility u, with parameter $\tau \geq 0$:

$$V(f) = \mathbb{E}_p[u \circ f] - \tau \sqrt{\sum_s p(s) \left(u(f(s)) - \mathbb{E}_p[u \circ f] \right)^2}.$$

Assume that for some event E in Σ, $p(E) = \frac{1}{2}$, then for all $x \succcurlyeq y$,[13]

$$V(xEy) = \frac{1}{2}(1-\tau)\, u(x) + \frac{1}{2}(1+\tau)\, u(y).$$

If $0 \leq \tau < 1$, the preference $\succcurlyeq$ is locally biseparable with $\rho(E) = \frac{1}{2}(1-\tau)$. Since $\succcurlyeq$ belongs to the class of dispersion aversion model of Grant and Polak (2013), it violates Certainty Independence. More importantly, it can violate Monotonicity. Indeed, $\succcurlyeq$ satisfies Monotonicity only if $p(s) > \frac{\tau^2}{1+\tau^2}$ for all $s \in S$ (see Grant and Kajii, 2007). Taking $|S| = 10$, $p(s) = \frac{1}{10}$, and $\tau = \frac{1}{2}$, the preference satisfies the axioms of Proposition 2 but is not monotone, as $p(s) < \frac{\tau^2}{1+\tau^2}$ for all $s \in S$. Consider now the bets

[12] See also Casadesus-Masanell, Klibanoff, and Ozdenoren (2000) and Alon and Schmeidler (2014).

[13] This is shown as follows:

$$\begin{aligned} V(xEy) &= \frac{1}{2}u(x) + \frac{1}{2}u(y) - \tau \sqrt{\frac{1}{2}\left(u(x) - \frac{1}{2}u(x) - \frac{1}{2}u(y)\right)^2 + \frac{1}{2}\left(u(y) - \frac{1}{2}u(x) - \frac{1}{2}u(y)\right)^2} \\ &= \frac{1}{2}u(x) + \frac{1}{2}u(y) - \tau \sqrt{\frac{1}{2}\left(\frac{1}{2}u(x) - \frac{1}{2}u(y)\right)^2 + \frac{1}{2}\left(\frac{1}{2}u(y) - \frac{1}{2}u(x)\right)^2} \\ &= \frac{1}{2}u(x) + \frac{1}{2}u(y) - \tau \sqrt{\left(\frac{1}{2}u(x) - \frac{1}{2}u(y)\right)^2} = \frac{1}{2}(1-\tau)\, u(x) + \frac{1}{2}(1+\tau)\, u(y). \end{aligned}$$

on E^c, xE^cy with $x \succcurlyeq y$. Since p is a probability, $p(E^c) = \frac{1}{2}$ and

$$V(xE^cy) = \frac{1}{2}(1-\tau)u(x) + \frac{1}{2}(1+\tau)u(y)$$

for all $x, y \in X$. It follows that $\rho(E) + \rho(E^c) = \frac{1}{2}(1-\tau) + \frac{1}{2}(1-\tau) \neq 1$, unless $\tau = 0$.

Example 2. Consider a preference $\succcurlyeq$ represented by a non-monotonic Choquet integral as in Gilboa (1989) and De Waegenaere and Wakker (2001, Eq. 4). Assume $|S| = n$ and denote by x_i the payoff of an act f in state s_i. The preferences $\succcurlyeq$ is represented by

$$V(f) = p(s_1)u(x_1) + \sum_{i=2}^{n} p(s_i)u(x_i) - \tau_i(u(x_{i-1}) - u(x_i))^+$$

for some $p \in \Delta(S)$, with $(a)^+ = \max(a, 0)$ and $\tau_i \in \mathbb{R}$. The function V could be interpreted as representing the value of a stream of consumption when the DM has a weak form of reference dependence. If the payoff of period i is smaller than the payoff of period $i-1$, there is a utility penalty if $\tau_i > 0$ or a utility premium if $\tau_i < 0$. Hence, the value of a sequence of payoffs is sensitive to intertemporal variations in consumption.

As shown in De Waegenaere and Wakker (2001), the representation V can be written as a non-monotone Choquet integral, i.e. a Choquet integral with respect to a capacity that does not necessarily satisfy the monotonicity property.[14] Consider a bet on E, xEy for some $x \succcurlyeq y$ where $E = \{s_k, \ldots, s_n\}$ for some $1 < k < n$. Then $V(xEy) = p(E^c \setminus s_k)u(y) + p(s_k)u(x) - \tau_k(u(y) - u(x))^+ + \sum_{j=k+1} p(s_j)u(x)$. Since $x \succcurlyeq y$, we obtain

$$V(xEy) = p(E)u(x) + (1 - p(E))u(y).$$

The preference represented by V is E-biseparable and it satisfies the Certainty Independence axiom (being represented by a Choquet integral). However, it does not satisfy AA Monotonicity but only a weaker form (see De Waegenaere and Wakker, 2001, p. 52). Notice that V is locally biseparable for all events containing the states from k to n, i.e. $E_k = \bigcup_{i=k}^{n} s_i$.

Example 3. Consider a preference $\succcurlyeq$ represented by the Smooth Ambiguity model of Klibanoff

[14] A signed capacity is a set function $v : \Sigma \to \mathbb{R}$ such that $v(\emptyset) = 0$ and $v(S) = 1$. The non-monotone Choquet integral is a Choquet integral with respect to a signed capacity v.

et al. (2005) (see also Seo, 2009):

$$V(f) = \phi^{-1}\left(\int_{\Delta(S)} \phi\left(\int_S (u \circ f)\, dp\right) d\pi(p)\right).$$

As it is well-known, smooth ambiguity preferences in general violate Certainty Independence. Assume that event $E \in \Sigma$ satisfies $p(E) = q(E) = k \in (0,1)$ for all $p, q \in \operatorname{supp} \pi$. Then, for all $x \succcurlyeq y$,

$$V(xEy) = p(E)u(x) + (1 - p(E))u(y)$$

for some (all) $p \in \operatorname{supp} \pi$. Locally biseparability is satisfied for bets on E. The existence of an event E over which all the first-order beliefs agree is not a very strong condition. It may for instance be satisfied in the classical Ellsberg's 3-color urn experiment when E denotes the event of a draw of a red ball from the urn.

On the other hand, while the example shows that the Smooth Ambiguity model is not incompatible with the construction of our subjective mixture, whether one can use our framework to supplt it with an axiomatic foundation is still an open question. The known axiomatizations rely either on multiple sources of uncertainty (Nau, 2006; Ergin and Gul, 2009), or on multiple "randomization devices" (Klibanoff et al., 2005; Seo, 2009).

Example 4. Consider a preference $\succcurlyeq$ satisfying the Vector Expected Utility model of Siniscalchi (2009). There is a Bernoulli utility u, a baseline prior p, an adjustment factor[15] (a random variable) ζ with $\mathbb{E}_p(\zeta) = 0$, and a function $A : \mathbb{R} \to \mathbb{R}$, with $A(0) = 0$ and $A(r) = A(-r)$ for all $r \in \mathbb{R}$ such that

$$V(f) = \mathbb{E}_p[u \circ f] + A\big(\mathbb{E}_p[\zeta \cdot u \circ f]\big).$$

Assume that for an event $E \in \Sigma$, $p(E) = \frac{1}{2}$, $\zeta(E) = \frac{1}{2} = -\zeta(E^c)$ and $A(r) = -|r|$. Then, for all[16] $x \succcurlyeq y$,

$$V(xEy) = \frac{1}{4}u(x) + \frac{3}{4}u(y).$$

Hence, for bets on E, $\succcurlyeq$ is locally biseparable and satisfies all the axiom of Proposition 2. However, it does not satisfy Certainty Independence.

[15] For simplicity we assume a single adjustment factor, but the model allows for any finite number of adjustment factors (see Siniscalchi, 2009).

[16] $V(xEy) = \frac{1}{2}u(x) + \frac{1}{2}u(y) - \left|\frac{1}{2}\frac{1}{2}u(x) - \frac{1}{2}\frac{1}{2}u(y)\right|$. Since $x \succcurlyeq y$, $\frac{1}{4}u(x) - \frac{1}{4}u(y) \geq 0$, then $V(xEy) = \frac{1}{4}u(x) + \frac{3}{4}u(y)$.

Example 5. We conclude with a negative result. While the previous examples show that the axioms of Proposition 2 allow a wide range of preference models, there is a class of preferences that are not locally biseparable: the second-order subjective expected utility (SOSEU) of Grant, Polak, and Strzalecki (2009). The value of an act f is given by:

$$V(f) = \phi^{-1}\left(\int_S \phi\left(u \circ f\right) dp\right)$$

for some monotone function $\phi : u(X) \to \mathbb{R}$ and a cardinally unique $u : X \to \mathbb{R}$. The function ϕ determines ambiguity attitude, while u the attitude toward "risk." The SEU model corresponds to a linear ϕ. It is well-known that, in Savage's setting, the two models are observationally indistinguishable (see, for example, Strzalecki, 2011). This is because it is difficult to disentangle risk attitude from the curvature of the function ϕ. Subjective mixtures provide a way to circumvent this impossibility, as they allow to cardinally identify risk attitude. However, the axioms of Proposition 1 imply that the SOSEU preference must be SEU. (Specifically, the Ordered E-bisymmetry requirement is violated by a SOSEU with a non-affine ϕ.)

Fact 2. *A SOSEU preference satisfies Ordered E-bisymmetry for some $E \in \Sigma$ if and only if it is SEU (i.e. $\phi(k) = ak + b$, for some $a > 0, b \in \mathbb{R}$).*

A simple consequence of Fact 2 is that Multiplier preferences, a subclass of variational preferences commonly used in financial applications, cannot be axiomatized using our technique. Indeed, multiplier preferences lie at the intersection between the variational and the SOSEU preferences (see Strzalecki, 2011). The intuition behind the result comes from the relation between probabilistic sophistication and non-expected utility. Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2012), for example, shows that in the presence of an unambiguous event[17] a probabilistically sophisticated and uncertainty averse preference collapses to SEU. Though not directly comparable, our result is in the same spirit. However, it builds on the non-affinity of ϕ, rather than on its concavity (which corresponds to uncertainty aversion in their sense). There is even an "uncertainty loving" SOSEU preference that does not satisfy the axioms of Lemma 1.

[17] An event A is unambiguous if, for all $x \succ y$, there is $z \in X$ with $x \succ z \succ y$ and $\lambda xAy \oplus (1-\lambda)h \sim \lambda z \oplus (1-\lambda)h$, for all $h \in \mathscr{F}$ and all $\lambda \in (0,1]$.

5 Some global properties

In this section, we show the effects of imposing some global behavioral restrictions to preferences satisfying our basic axioms (P, C, ON and NV). We start by considering a mild condition, called "decomposability," and then move on to consider the effects of adding full Monotonicity and a form of Uncertainty Aversion.

5.1 Decomposable preferences

Consider the following property, which stipulates that two acts be indifferent whenever they are indifferent state-by-state.

Axiom (Decomposability - D)**.** *For all $f, g \in \mathscr{F}$, $f(s) \sim g(s)$ for all $s \in S$ implies $f \sim g$.*

This is a weak form of separability across states, implied by standard Monotonicity. It is called indifference substitution in Grant and Polak (2013). Decomposable but non-monotone preferences are the Mean-standard deviation preferences of example 1 and the Signed Choquet preferences of example 2.

Adding Decomposability to the previous axioms allows us to transform the representation V of $\succcurlyeq$ defined on $\mathscr{F}$ into a representation I defined on $B_0(\Sigma, u(X))$. An act f is then identified with its utility profile $u \circ f$ and a subjective mixture of acts $\frac{1}{2} f \oplus \frac{1}{2} g$ with the mixture of utility profiles $\frac{1}{2} u \circ f + \frac{1}{2} u \circ g$. Thius allows us to leverage all the functional-analytic techniques that make the AA framework more advantageous than Savage's.

Proposition 3. *$\succcurlyeq$ satisfies Axioms P, C, ON, NV, and D if and only if the representation V obtained in Proposition 2 can be written as $V = I \circ u$, where $u : X \to \mathbb{R}$ is a τ-continuous, nonconstant function, and $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ is a normalized, continuous functional, such that for all binary acts xAy with $x \succcurlyeq y$ and $A \in \Sigma$,*

$$I(u \circ (xAy)) = \rho_{x,y}(A)u(x) + (1 - \rho_{x,y}(A))u(y)$$

for some $\rho_{\cdot,\cdot} : \Sigma \to \mathbb{R}$ which satisfies $\rho_{x,y}(F) = \rho(F) \in (0,1)$ for all $x, y \in X$ and $F \in \mathscr{E}$.[18]

This representation generalizes the one given by CGMMS, as it shows that a Bernoullian and Archimedean representation of preferences can be obtained by (Decomposability and) a form of

[18]As to uniqueness: If $J \circ v$ also represents, with $v : X \to \mathbb{R}$ and $J : B_0(\Sigma, v(X)) \to \mathbb{R}$, there exist $\mu \in \mathbb{R}, \lambda > 0$, such that, $v(x) = \lambda u(x) + \mu$, and $J(b) = \lambda I(\lambda^{-1}[b - \mu]) + \mu$ for all $b \in B_0(\Sigma, v(X))$.

separability on a single event E. It can be seen that the examples provided in Section 4 are all special cases of the representation in Proposition 3.

5.2 Monotonic and "Uncertainty Averse" preferences

If we strengthen Decomposability to full Monotonicity, we derive a fully subjective version of the Monotone, Bernoullian and Archimedean (MBA) preferences of CGMMS.

Axiom (Monotonicity - M)**.** *For all $f, g \in \mathscr{F}$, $f(s) \succsim g(s)$ for all $s \in S$ implies $f \succsim g$.*

This axiom immediately implies monotonicity of the functional I. It also implies that $\rho_{x,y}$ is a capacity: given $x \succ y$, and $A \subseteq B$, $xBy \succsim xAy$, hence $I(u \circ (xBy)) = \rho_{x,y}(B)(u(x) - u(y)) + u(y) \geq \rho_{x,y}(A)(u(x) - u(y)) + u(y) = I(u(xAy))$, yielding $\rho_{x,y}(B) \geq \rho_{x,y}(A)$. We thus obtain:

Proposition 4. *$\succsim$ satisfies Axioms P, C, ON, NV, and M if and only if the functional I obtained in Proposition 3 is also monotone and $\rho_{x,y} : \Sigma \to [0,1]$ is a capacity for all $x \succ y$.*

Proposition 4 also shows that an MBA representation can be obtained without imposing the Risk Independence axiom with respect to an objective mixture structure.[19] This allows us to model non-expected utility behavior over payoffs, and thus to jointly account for the Ellsberg and Allais paradoxes, as in Dean and Ortoleva (2017).[20] On the other hand, imposing a version of the Independence Axiom with respect to subjective mixtures allows us to show that the functional I obtained in Proposition 4 is linear. By standard arguments, we can then prove that I is an expectation. We thus obtain an alternative axiomatization of subjective expected utility (see Section 7.1 for further details).

One particularly interesting class of monotone preferences is the class of "Uncertainty Averse" preferences (UAP) axiomatized by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) in the AA setting. It generalizes most of the pre-existing models, such as MEU, Variational and Confidence preferences. The only restriction of the DM ambiguity attitude is given by the *Uncertainty Aversion* (UA) axiom of Schmeidler (1989), which stipulates that the (objective) random-

[19] That is, if $x, y, z \in X$ and $\lambda \in [0,1]$, then (with $+$ denoting the objective mixture operation)

$$x \succsim y \Longleftrightarrow \lambda x + (1-\lambda)z \succsim \lambda y + (1-\lambda)z.$$

[20] Our construction allows for a wider spectrum of attitudes with respect to ambiguity than Dean and Ortoleva (2017), as we do not impose any additional independence assumption (such as their Degenerate Independence axiom).

ization between two indifferent acts may be strictly preferred to each act (see the discussion in Section 6). In our context, one can analogously allow the DM to prefer the subjective mixture of two indifferent acts to either act:

Axiom (Ambiguity Hedging - AH)**.** *For all* $f, g \in \mathscr{F}$*,*

$$f \sim g \implies \frac{1}{2} f \oplus \frac{1}{2} g \succcurlyeq g$$

As our terminology suggests, in our setting this axiom has a natural interpretation: when f and g are indifferent, an act whose utility profile is the arithmetic average of the utility profiles of f and g will be weakly preferred to either act as it provides a hedge to the ambiguity connected to each act.

In the presence of the axioms of Proposition 4 (respectively, Proposition 2) Axiom AH is equivalent to quasi-concavity of I (respectively, of V). Leveraging on the results of Cerreia-Vioglio et al. (2011), it is then straightforward to obtain a fully subjective axiomatization of UAP (in which we formulate the representation directly in terms of u and a functional G, for ease of comparison with their result):

Proposition 5. $\succcurlyeq$ *satisfies Axioms P, C, M, ON, NV, and AH if and only if there exist a nonconstant* $u : X \to \mathbb{R}$ *and a linearly continuous*[21] $G : u(X) \times \Delta(S) \to (-\infty, \infty]$ *such that, for all* $f, g \in \mathscr{F}$*,*

$$f \succcurlyeq g \iff \inf_{p \in \Delta(S)} G\left(\int u \circ f dp, p\right) \geq \inf_{p \in \Delta(S)} G\left(\int u \circ g dp, p\right). \tag{6}$$

The function u *is cardinally unique and, for given* u*, there exists a (unique) minimal* $G^\star : u(X) \times \Delta(S) \to (-\infty, \infty]$ *satisfying Eq.* (6)*, given by*

$$G^\star(t, p) = \sup_{f \in \mathscr{F}} \left\{ u(c_f) : \int u \circ f dp \leq t \right\}, \quad \forall (t, p) \in u(X) \times \Delta.$$

Lastly, we may want to find out when, as in the biseparable preferences of Ghirardato and Marinacci (2001), the capacity $\rho_{x,y}$ is independent of the payoffs. Since we dropped Certainty Independence, we do not have constant linearity of the functional I. From a behavioral viewpoint, constant linearity means that the DM's valuation of an act is independent of the scale and position

[21] A function $G : T \times \Delta(S) \to (-\infty, \infty]$ with $T \subseteq \mathbb{R}$ is said to be *linearly continuous* if the map $a \mapsto \inf_{p \in \Delta(S)} G\left(\int a dp, p\right)$ from $B_0(\Sigma, T)$ is extended-valued and continuous.

of its outcomes. Without this property, the DM's willingness to bet on an event depends on the magnitude of the prizes, which is often considered a behavioral bias (Bénabou, 2015), but arises naturally in our world. It turns out that outcome independence can be restored by assuming a version of the Certainty Independence axiom that applies only to binary acts. We refer the interested reader to the discussion in Section 4.1 of CGMMS, which extends with little work to our more general setting.

6 Discussion: Subjective and objective mixtures

Our subjective mixture operator can be defined even when an "objective" mixture is available to make X a mixture space. For instance, when consequences are monetary payoffs, or when the elements of X are objective lotteries over a set of prizes Z. In this framework, it is possible to formulate the traditional uncertainty aversion axiom of Schmeidler (1989):

Axiom (Uncertainty Aversion - UA). *For all $f, g \in \mathscr{F}$,*

$$f \sim g \implies \frac{1}{2}f + \frac{1}{2}g \succcurlyeq g.$$

As mentioned earlier, the natural interpretation of this axiom is a preference for objective randomization, in the spirit of Raiffa (1968). In constrast, our Axiom AH is interpreted as a preference for ambiguity hedging. In our context, the two properties are independent, as there may be no relation between (the ranking of) objective and subjective mixtures.

Indeed, the comparison of midpoints obtained via $\oplus$ and $+$ conveys information about the properties of the utility u with respect to the objective mixture structure. Consider two outcomes $x, y \in X$, with $x \succcurlyeq y$, and $\frac{1}{2}x + \frac{1}{2}y$ and $\frac{1}{2}x \oplus \frac{1}{2}y$. By construction, the utility u_E of Proposition 1 is affine with respect to $\oplus$; therefore, $\frac{1}{2}x + \frac{1}{2}y \succcurlyeq$ (resp. $\preccurlyeq$)$\frac{1}{2}x \oplus \frac{1}{2}y$ for any such x and y if and only if u_E is concave (resp. convex) with respect to $+$. In the case $X \subseteq \mathbb{R}$, with $+$ representing standard addition, concavity (resp. convexity) represents risk aversion (resp. risk seeking). If $X = \Delta(Z)$, with $+$ representing probability mixture, strict concavity or convexity imply *non-linear utility* over lotteries; namely, violations of expected utility. As mentioned earlier, such nonlinearity can be exploited to model preferences which display *both* the Allais and Ellsberg modal choices, as in Dean and Ortoleva (2017).

Differences in the "construction" of midpoints translate into differences in the ranking of mixtures of acts. For instance, a preference satisfying the assumptions of Proposition 4 (Axioms P, C, ON, NV, and M) with a strictly monotonic functional I could display either

$$\frac{1}{2}f + \frac{1}{2}g \sim f \sim g \text{ and } \frac{1}{2}f \oplus \frac{1}{2}g \succ f \tag{7}$$

or

$$\frac{1}{2}f + \frac{1}{2}g \succ f \sim g \text{ and } \frac{1}{2}f \oplus \frac{1}{2}g \sim f \tag{8}$$

depending on whether the utility u is strictly convex or strictly concave.[22] In Eq. (7), the DM displays ambiguity hedging while she displays neutrality with respect to the objective randomization (i.e., uncertainty neutrality in the sense of Schmeidler). In Eq. (8), the DM is ambiguity (hedging) neutral and displays a preference for objective randomization (i.e., uncertainty aversion).

For a stark example of a preference satisfying Eq. (8), let $X = \mathbb{R}_{++}$, $u(x) = \ln x$, and $V(f) = \mathbb{E}_p[\ln(f)]$. It is immediate to see that $\succcurlyeq$ represented by V satisfies Eq. (8). It is natural to interpret V as an expected utility preference with a logarithmic Bernoulli index, hence not sensitive to ambiguity. However, as it satisfies Axiom UA, it would traditionally be interpreted as reflecting aversion to ambiguity.

The above discussion shows that in our setting it is possible to model distinct reactions to the presence of objective randomization and ambiguity, differently from what happens with models tied to the traditional AA setting. This flexibility is relevant in light of some existing experimental evidence on the relationship between ambiguity aversion and attitudes to randomization, which is at odds with the traditional view that a preference for objective randomization is a behavioral marker for the presence of aversion to ambiguity. Dominiak and Schnedler (2011) find that the connection between ambiguity sensitivity and preference for randomization is rather weak: a majority of their subjects are intuitively ambiguity averse, but they display indifference to randomization, and some are *both* ambiguity averse and randomization averse.[23] As we discuss below in Section 7.3, analogous considerations can be made in dynamic environments pertaining the distinction between ambiguity sensitivity and preference for consumption smoothing.

[22] For instance, if u is strictly convex, then $I(u \circ f) = I\left(u \circ \left(\frac{1}{2}f + \frac{1}{2}g\right)\right) < I\left(\frac{1}{2}u \circ f + \frac{1}{2}u \circ g\right) = I\left(u \circ \left(\frac{1}{2}f \oplus \frac{1}{2}g\right)\right)$. The argument for strictly concave u is symmetrical.

[23] Also Agranov and Ortoleva (2017) observe that preference for randomization may arise for reasons that are orthogonal to ambiguity sensitivity.

The distinction between preference for objective randomization and ambiguity hedging disappears if we impose the following stronger form of Axiom NV:

Axiom (Objective Invariance - OI)**.** *For all $F \in \mathscr{E}$ and $x \succcurlyeq y$, $z \in X$ satisfies $z \sim \frac{1}{2}x + \frac{1}{2}y$ if and only if $x \succcurlyeq z \succcurlyeq y$ and*

$$xFy \sim c_{xFz}Fc_{zFy}$$

As mentioned, Axiom OI implies Axiom NV, as it implies that the consequence $\frac{1}{2}x + \frac{1}{2}y$ can be used in both preferences in the Invariance axiom. It follows from Axiom OI that $\frac{1}{2}x + \frac{1}{2}y \sim \frac{1}{2}x \oplus \frac{1}{2}y$; i.e., objective mixtures are indifferent to subjective mixtures. When this axiom holds, Axiom AH becomes equivalent to Axiom UA.[24] Most of the standard axiomatizations based on the traditional AA setting implictly assume the validity of Axiom OI, thus forcing the equivalence of AH and UA.

7 Extensions and applications

7.1 Modeling "ex-ante" randomization

As mentioned earlier, we define subjective mixtures of acts state-by-state. In the traditional environment envisioned by AA, such state-by-state mixtures are interpreted as follows: it is as if the DM has decided to commit to a certain randomization over the acts, which is performed *after* the revelation of the state of nature s. For this reason, it is customary to call them "ex-post" mixtures. AA also considered the possibility of "ex-ante" randomizations over acts, which correspond to the more traditional idea of performing the randomization over the acts *before* the revelation of s. As discussed by Saito (2015) and Ke and Zhang (2018) (see also Seo (2009)), such distinction between "ex-ante" and "ex-post" mixtures allows for a more nuanced analysis of the attitudes toward ambiguity. For example, the extent to which "ex-ante" randomization can be used to eliminate the effects of ambiguity (Raiffa's critique).

An obvious question is whether a similar distinction is possible in our subjectuive setting; i.e., how to provide a subjective notion of act mixture corresponding to AA's "ex-ante" randomizations.[25] In the traditional AA setting, when the DM performs an "ex-ante" 1/2:1/2 randomization

[24]Since the representing u is by construction linear with respect to subjective mixtures, OI also implies that the preference $\succcurlyeq$ satisfies the Risk Independence axiom (with respect to objective mixtures).

[25]While we adopt the usage of "ex-ante" and "ex-post" for the sake of brevity, it is important to stress that in our setting such terminology is purely suggestive. Our subjective mixtures do not *actually* involve uncertainty and its

between acts f and g, she ends up performing either f or g, and subjecting herself to all the variability (over S) entailed by each act. It is as if she envisions receiving either c_f or c_g with probability 1/2, as c_f (respectively c_g) describes the DM's *global* evaluation of the act f (respectively g). Thus, the mixture $\frac{1}{2}c_f + \frac{1}{2}c_g$ could be indifferently substituted to the "ex-ante" mixture.

This intuition remains valid in our subjective setting. Given $f, g \in \mathscr{F}$, we call the "ex-ante" 1/2:1/2 subjective mixture of f and g a prize $z \in X$ such that $z = \frac{1}{2}c_f \oplus \frac{1}{2}c_g$, with c_f (resp. c_g) a certainty equivalent of f (resp. g). Given the representation functional V obtained in Proposition 2 (and its restriction u), the utility of z is

$$u(z) = \frac{1}{2}u(c_f) + \frac{1}{2}u(c_g) = \frac{1}{2}V(f) + \frac{1}{2}V(g),$$

a 1/2:1/2 mixture of the "ex-ante" utilities of f and g.

The prize $\frac{1}{2}c_f \oplus \frac{1}{2}c_g$ is not necessarily indifferent to the "ex-post" subjective mixture $\frac{1}{2}f \oplus \frac{1}{2}g$. The comparison of $\frac{1}{2}c_f \oplus \frac{1}{2}c_g$ and $\frac{1}{2}f \oplus \frac{1}{2}g$ reflects a DM's difference in attitudes with respect to "ex-ante" or "ex-post" randomizations, transporting the analysis in Saito (2015) and Ke and Zhang (2018) to our subjective setting. The following axiom spells out three possible attitudes with respect to the "timing" of mixtures:

Axiom. *For all $f, g \in \mathscr{F}$ and all $\lambda \in [0,1]$, if $c_f, c_g \in X$ satisfy $f \sim c_f$ and $g \sim c_g$:*

(a) (Preference for Ex-ante Mixtures - PEM) $\lambda c_f \oplus (1-\lambda)c_g \succcurlyeq \lambda f \oplus (1-\lambda)g$;

(b) (Aversion to Ex-ante Mixtures - AEM) $\lambda f \oplus (1-\lambda)g \succcurlyeq \lambda c_f \oplus (1-\lambda)c_g$;

(c) (Mixture Timing Indifference - MTI) $\lambda c_f \oplus (1-\lambda)c_g \sim \lambda f \oplus (1-\lambda)g$.

Besides the suggestive interpretation behind its name, Axiom MTI is a quite transparent separability condition. It stipulates that for the DM it is equivalent to evaluate the act $\lambda f \oplus (1-\lambda)g$ by considering it as a state-by-state mixture of f and g, or by evaluating f and g in isolation and then finding the subjective mixture of their certainty equivalents. That is, the acts of evaluating f and g (finding their certainty equivalent) are performed separately, and they are also separated/independent from the act of evaluating their mixture. In turn, Axioms PEM and AEM classify DMs whose preferences are a bit less separable, but are still consistent in terms of "timing" considerations. Notice that AEM implies Ambiguity Hedging.

resolution: $\frac{1}{2}f \oplus \frac{1}{2}g$ is an act which in state s pays the certain prize $\frac{1}{2}f(s) \oplus \frac{1}{2}g(s)$; there is no uncertainty further to the state s.

The following simple result shows that these properties translate immediately into properties of the representation functional V.

Proposition 6. *Given a preference* $\succcurlyeq$ *satisying the assumptions of Prop. 2, it satisfies Axiom MTI (resp. PEM, AEM) if and only if* V *is also affine (resp. convex, concave) with respect to* $\oplus$. *Under the assumptions of Prop. 3, so that* $V = I \circ u$, $\succcurlyeq$ *satisfies Axiom MTI (resp. PEM, AEM) holds if and only if* I *is an affine (resp. convex, concave) functional on* $B_0(\Sigma, u(X))$.

Notice that the results of Proposition 6 do not require AA monotonicity. Consider now the adaptation to subjective mixtures of the standard independence axiom:

Axiom (Subjective mixture Independence - SI)**.** *For all* $f, g, h \in \mathscr{F}$ *and all* $\lambda \in [0,1]$,

$$f \succcurlyeq g \Longrightarrow \lambda f \oplus (1-\lambda) h \succcurlyeq \lambda g \oplus (1-\lambda) h. \tag{9}$$

It is straightforward to apply Proposition 6 to show that, in the presence of the basic axioms of our representation, Axiom SI is equivalent to Axiom MTI.

Corollary 1. *Given a preference* $\succcurlyeq$ *satisfying the assumptions of Prop. 2, it satisfies Axiom MTI if and only if it also satisfies Axiom SI.*

Recalling the discussion after Proposition 4, Corollary 1 can be used to prove that the axioms of Prop. 4 and Axiom SI characterize subjective expected utility preferences.

This result shows that imposing Axiom MTI precludes non-neutral attitudes toward ambiguity. More generally, imposing restrictions to the preference for the timing of randomization (PEM, AEM and MTI) may be unduly restrictive. We stress that these restrictions are not embedded in our analysis.[26] Indeed, it is easy to see that all the results in Ke and Zhang (2018), including their representation theorems, can be generalized to our subjective setting.

7.2 A general framework for mixtures of menu-acts

We now briefly show that our approach to subjective mixtures can also be used in settings where acts map states of the world to *menus* of consequences. Such settings can be used to model,

[26] Other restrictions may be. For instance, Ke and Zhang (2018) introduce an axiom called "Indifference to Mixture Timing of Constant Acts," which is automatically satisfied given our definition of "ex-ante" subjective mixture.

among others, preference for flexibility Dekel et al. (2001, henceforth DLR), dynamic (in)consistency due to temptation and other behavioral problems (Gul and Pesendorfer, 2001), or costly learning about the decision framework (e.g. Ergin and Sarver, 2010). Our approach allows us to provide a more general theory, as it does not restrict subjective states to be vNM expected utilities. We replace the Independence axiom, that is common in models of menu-choice (e.g. DLR, Gul and Pesendorfer, 2001; Dekel, Lipman, and Rustichini, 2009), with a more transparent assumption concerning subjective mixtures

Working in the setting described in Section 2, we follow Nehring (1999) and Ghirardato (2001) and let acts be maps $f : S \to \mathscr{K}$, where $\mathscr{K}$ is a suitably chosen family of non-empty subsets of X (called *menus*). We impose first of all the Preference Order axiom, thus asking the DM to be able to rank all acts in $\mathscr{F}$. Call *crisp* any act f such that $f(s)$ is a singleton for all $s \in S$; i.e., a traditional Savage act. Denote by $\mathscr{F}_c$ the set of all crisp acts. We next impose a key assumption on preferences:

Axiom (Crisp Equivalence - CE)**.** *For all $A \in \mathscr{K}$, there exists $x \in X$ such that $x \sim A$.*

Using intuitive notation, we denote by c_A the crisp equivalent of menu A. It should be observed that we are not making specific assumptions on the relation between c_A and the elements of A: c_A could belong to A, as it is the case for instance in the Desire for Commitment axiom of Dekel et al. (2009) or in the Contingencywise Dominance axiom of Ghirardato (2001), but that is not necessary. If we add Decomposability to Preference Order and Crisp Equivalence, we obtain that for any $f \in \mathscr{F}$, there is a crisp act $f_c \in \mathscr{F}_c$ such that $f_c \sim f$. Because of this result, if we now impose Continuity, Ordered Niceness and Invariance (restricted to the set $\mathscr{F}_c$), we can follow the steps in the previous sections to endow $\mathscr{F}$ with a subjective mixture structure defined by $\frac{1}{2} f_c \oplus \frac{1}{2} g_c$, for every $f, g \in \mathscr{F}$. This structure allows us to recast in our framework existing axiomatizations, providing a fully subjective representations of preferences over menu-acts.

One issue that arises in this context is how to define subjective mixtures of menus. On one hand, given two menus $A, B \in \mathscr{K}$ and our notion of subjective mixture $\oplus$, we can follow DLR in defining the mixture $\bar{\oplus}$ of A and B as follows: $\lambda A \bar{\oplus} (1-\lambda) B \equiv \{\lambda x \oplus (1-\lambda) y : x \in A, y \in B\}$; i.e., a menu consisting of the $z \in X$ that are subjective mixtures of elements of A and B. On the other hand, treating A and B as constant acts, we can also employ a mixture operation in the spirit of Section 7.1, $\lambda c_A \oplus (1-\lambda) c_B$. In general, these two notions of (subjective) menu mixture will be different. However, there is a natural restriction on preferences which makes them equivalent: Axiom MTI for constant menu-acts.

Axiom (Menu Mixture Timing Indifference - MMTI)**.** *For all* $A, B \in \mathcal{K}$ *and all* $\lambda \in [0,1]$,

$$\lambda c_A \oplus (1-\lambda) c_B \sim \lambda A \bar{\oplus} (1-\lambda) B \tag{10}$$

The discussion of axiom MTI in the previous section applies *mutatis mutandis*: the evaluations of A and B (i.e., determining their crisp equivalent) are performed separately, and they are also performed separately from the act of evaluating their mixture.

To provide a more structured interpretation, we recall that preferences over menus are typically used to model situations where the DM may not perfectly know —say, because of "unforeseen contingencies"— her future tastes (see, e.g., DLR). That is, the DM faces two types of uncertainty: the objective states (described by $s \in S$) as well as "subjective states" that correspond to her final preference orderings. There are therefore two notions of "timing" that one can envision: timing with respect to the resolution of the objective uncertainty, and timing with respect to the resolution of the "subjective-state" uncertainty. Axiom MTI in Section 7.1 stipulates that the DM be indifferent between (subjective) mixtures that are resolved before the observation of s and mixtures that are resolved after the observation of s.[27] Symmetrically, Axiom MMTI requires that the DM be indifferent between mixtures that resolve before (left-hand side) and after (right-hand side) the "realization" of the DM's "subjective state." The indifference in MMTI can also be interpreted as "neutrality to contingent planning," in the spirit of Ergin and Sarver (2010). Indeed, the right-hand side of the indifference does not entail the anticipation of (state-contingent) second-period choice, since it is the mixture of two singletons. In contrast, the evaluation of the mixture of two menus in the left-hand side forces the individual to consider second-period choice contingent on each subjective state.

Reasoning along the same lines of Corollary 1, it is simple to show that imposing Axiom MMTI is equivalent to requiring that the DM satisfies DLR's Independence Axiom with respect to (either notion of) menu mixtures. The advantage of MMTI lies in its more direct interpretation. DLR observe that in their setting with a given objective mixture structure, the Independence Axiom

[27] Once again, our usage of the terms "uncertainty" and "resolution" in reference to subjective mixtures is purely suggestive.

forces the DM to be indifferent between objective ex-ante and ex-post randomization, thus restricting their "subjective states" to expected utility preferences. In contrast, in our setting one could obtain a *fully subjective* DLR-style representation. In such representation, the "subjective states" would be preferences whose representing utilities are not necessarily affine (i.e. non-EU) with respect to an objective mixture structure, but are affine with respect to the subjective mixture structure $\oplus$. Thus, the adoption of subjective mixtures allows for a wider scope of representations, even in the presence of strong separability conditions like Axiom MMTI.

Our approach also allows weakening the DLR independence axiom, hence non-indifference towards the timing of randomization. Analogously to Section 7.1, relaxations of MMTI which capture preference for early or late randomization are directly mapped to global properties (i.e. concavity and convexity) of the representation. For example, following Ergin and Sarver (2010), consider the function:

$$u(A) = \max_{G \in \mathscr{G}} \int_{\Omega} \max_{x \in A} \mathbb{E}_{\mu}(U|G)(\omega)(x)\mu(d\omega) - c(G) \tag{11}$$

where $A \in \mathscr{K}$ and $\omega \in \Omega$ is a "subjective state" *à la* DLR, which entails the preference representation $U(\omega)(\cdot)$ on X. Each $G \in \mathscr{G}$ represents a contemplation strategy (i.e., a σ-algebra of the subjective state space Ω) and, given the subjective probability $\mu \in \Delta(\Omega)$, $\mathbb{E}_{\mu}(U|G)(\omega)(x)$ is the conditional expectation of $U(\omega)(x)$ with respect to μ and the σ-algebra G. In a setting with objective randomization $+$, Ergin and Sarver (2010) show that a preference over menus represented by Eq. (11) with *convex* u is characterized by a form of translation invariance and the following property, dubbed "Aversion to Contingent Planning": for all $A, B \in \mathscr{K}$, $A \sim B$ implies $A \succcurlyeq \lambda A + (1-\lambda)B$.

In our subjective mixture setting, the convexity of u can be guaranteed by a single axiom relaxing MMTI which can be analogously interpreted as aversion to contingent planning: a preference for early mixture $\lambda c_A \oplus (1-\lambda)c_B \succcurlyeq \lambda A \bar{\oplus} (1-\lambda)B$. Moreover, as for the extension of DLR discussed above, in our setting subjective states are not necessarily EU preferences.

7.3 Dynamic choice

The subjective mixture operator can be used to provide a fully subjective foundation to some popular dynamic models of ambiguity aversion that have been formulated in the AA framework.

In particular, here we focus on the dynamic variational preferences model of Maccheroni et al. (2006b, henceforth DVP), of which the recursive MEU model of Epstein and Schneider (2003) is a special case. Our analysis could be similarly extended to other dynamic recursive models; e.g., Siniscalchi (2011).[28]

Borrowing the notation and terminology from DVP, time is finite and discrete with dates $\mathscr{T} = \{0,1,\ldots,T\}$, there is a set of states of the world Ω, and information is represented by a filtration $\{\mathscr{G}_t\}_{t=0}^{T}$. We assume that $\mathscr{G}_0$ is trivial, while $\mathscr{G}_t$ is generated by a finite partition G_t, with $G_t(\omega)$ denoting the cell of the partition containing ω. That is, if ω is the true state, at time t the decision maker knows that event $G_t(\omega)$ is true. Given an outcome space X satisfying our Structural Assumption, an act is an X-valued process $h = (h_0, h_1, h_2, \ldots, h_T)$, where each $h_t : \Omega \to X$ is $\mathscr{G}_t$-measurable (hence simple). Clearly, each act h can also be seen as a function $h : \Omega \to X^{T+1}$. $\mathscr{H}$ denotes the set of all acts. An important subset of $\mathscr{H}$ is the set of the the *certain* acts, i.e. the elements of $\mathscr{H}$ that pay the same *stream* of payoffs in all states ($h(\omega) = h(\omega')$ for all $\omega, \omega' \in \Omega$). With the usual abuse of notation, each such act h is identified with the vector $\mathbf{x} = (x_0, x_1, \ldots, x_T)$, and the set of certain acts is identified with the set X^{T+1}. Finally, for any $x \in X$, we let $\bar{x}$ denote the certain *and constant* act $\mathbf{x} = (x, x, x, \ldots, x)$.

The binary relation $\succcurlyeq_{t,\omega}$ represents the preferences of the individual at time t and state ω. The following axiom specifies that such preferences do not convey more information than $\mathscr{G}_t$ and it also assumes a form of *consequentialism*; i.e., preferences at t (with information $\mathscr{G}_t$) only depend on an act's performance from t onwards.

Axiom (Conditional preference - CP)**.** *For all* $(t,\omega) \in \mathscr{T} \times \Omega$*,*

1. $\succcurlyeq_{t,\omega}$ *coincides with* $\succcurlyeq_{t,\omega'}$ *if* $G_t(\omega) = G_t(\omega')$.

2. *If* $h(\tau,\omega') = h'(\tau,\omega')$ *for all* $\tau \geq t$ *and* $\omega' \in G_t(\omega)$*, then* $h \sim_{t,\omega} h'$.

In light of this axiom $\succcurlyeq_{0,\omega} = \succcurlyeq_{0,\omega'}$ for every $\omega, \omega' \in \Omega$, so we henceforth just denote it $\succcurlyeq_0$.

We now turn to the axioms which deliver a locally biseparable representation in this setting. Notice that here the constants are the elements of X^{T+1}; i.e., the certain acts. Hence, given a (suitably defined) event E, a *bet* $\mathbf{x}E\mathbf{y}$ is an act paying $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_T)$ if E occurs, and $\mathbf{y} = (y_0, y_1, y_2, \ldots, y_T)$ otherwise. Also, given an act $h \in \mathscr{H}$, its $\succcurlyeq_0$-certainty equivalent $\mathbf{c}_h$ is some

[28]Another approach to the construction of dynamic ambiguity-sensitive preferences is Kochov (2015). We discuss it in some detail at the end of the section.

$\mathbf{x} \in X^{T+1}$ such that $\mathbf{x} \sim_0 h$. As a consequence, the equation(s) in ordered bisymmetry read(s) as follows: $\mathbf{c}_{\mathbf{x}E\mathbf{z}}E\mathbf{c}_{\mathbf{z}'E\mathbf{y}} \sim_0 \mathbf{c}_{\mathbf{x}E\mathbf{z}'}E\mathbf{c}_{\mathbf{z}E\mathbf{y}}$ with $\mathbf{x} \succsim_0 \{\mathbf{z}, \mathbf{z}'\} \succsim_0 \mathbf{y}$.

Axiom (Locally Biseparable Monotonic - LBM)**.** *For all $(t,\omega) \in \mathscr{T} \times \Omega$:*

1. *$\succsim_{t,\omega}$ is complete, transitive and non-trivial.*
2. *Let $\{h_\alpha\}_{\alpha \in A}$ be a net in $\mathscr{F}$ that pointwise converges to $h \in \mathscr{F}$ and such that all h_α and h are measurable w.r.t. the same finite partition of Ω. If $h_\alpha \succsim_0 g$ (resp. $g \succsim_0 h_\alpha$) for all $\alpha \in A$, then $h \succsim_0 g$ (resp. $g \succsim_0 h$).*
3. *$\succsim_0$ satisfies Ordered Niceness for some $E \in \mathscr{A}(G_T)$, where $\mathscr{A}(G_T)$ is the algebra generated by $\mathscr{G}_T$.*
4. *For all $h, h' \in \mathscr{H}$, if $(h_0(\omega'), h_1(\omega'), \ldots, h_T(\omega')) \succsim_{t,\omega} (h'_0(\omega'), h'_1(\omega'), \ldots, h'_T(\omega'))$ for all $\omega' \in \Omega$, then $h \succsim_{t,\omega} h'$.*

With respect to the axioms employed in Proposition 1, we have followed DVP in adding the Monotonicity axiom (item 4), which allows us to forego the boundedness requirement in item 1. Notice that only the preference $\succsim_0$ satisfies Ordered Niceness in item 3.

The axioms stated so far allow us to obtain the following straightforward adaptation of Proposition 1 to the intertemporal setting:

Lemma 3. *Given Axiom CP, the preference $\succsim_0$ satisfies items 1-4 in LBM if and only if there exist a cardinally unique $U_0 : X^{T+1} \to \mathbb{R}$, $\rho \in (0,1)$ such that $V : \mathscr{H} \to \mathbb{R}$, defined by $V(h) \equiv U_0(\mathbf{c}_h)$ for any $h \in \mathscr{H}$, represents $\succsim_0$, and it satisfies for every $\mathbf{x} \succsim \mathbf{y}$,*

$$V(\mathbf{x}E\mathbf{y}) = \rho U_0(\mathbf{x}) + (1-\rho) U_0(\mathbf{y}).$$

We now follow DVP in imposing a rationality property on the preference on certain acts. Our axiom is stronger as it explicitly requires additive separability (item 2).

Axiom (Risk Preference - RP)**.**

1. *For any $\mathbf{y} \in X^{T+1}$ and all $x, x', x'', x''' \in X$, if*

$$(\mathbf{y}_{-(\tau,\tau+1)}, x, x') \succsim_{t,\omega} (\mathbf{y}_{-(\tau,\tau+1)}, x'', x''')$$

holds for some $(t,\omega) \in \mathscr{T} \times \Omega$ *and some* $\tau \geq t$*, then it holds for all* $(t,\omega) \in \mathscr{T} \times \Omega$ *and all* $\tau \geq t$*.*

2. *For all* $(t,\omega) \in \mathscr{T} \times \Omega$*, all* $\mathbf{y}, \mathbf{y}' \in X^{T+1}$ *and all* $x, x', x'', x''' \in X$*,*

$$(\mathbf{y}_{-(t,t+1)}, x, x') \succcurlyeq_{t,\omega} (\mathbf{y}_{-(t,t+1)}, x'', x''') \quad \textit{if and only if} \quad (\mathbf{y}'_{-(t,t+1)}, x, x') \succcurlyeq_{t,\omega} (\mathbf{y}'_{-(t,t+1)}, x'', x''').$$

When added to the assumptions of Lemma 3, Axiom RP is shown by standard arguments to imply that U_0 has an additively separable (and continuous) representation with geometric discounting; that is, there exist a cardinal and continuous $u_0 : X \to \mathbb{R}$ and $\delta \in (0,1]$ such that $U_0(\mathbf{x}) = \sum_t \delta^t u_0(x_t)$. It similarly implies that for every $(t,\omega) \in \mathscr{T} \times \Omega$ there exist cardinal $u_{t,\omega} : X \to \mathbb{R}$ and $\delta_{t,\omega} \in (0,1]$ such that $U_{t,\omega}(\mathbf{x}) = \sum_{t'} \delta_{t,\omega}^{t'} u_{t,\omega}(x_{t'})$ represents $\succcurlyeq_{t,\omega}$ on X^{T+1}. Finally, it also implies all the u_0 and $u_{t,\omega}$ are cardinally equivalent (which also implies that $\delta = \delta_{t,\omega}$); that is, all the "instantaneous" utility functions on X are the same.

Assuming Axioms CP, LBM and RP, we can now define a subjective mixture operator $\oplus_0$ on X based on the preference $\succcurlyeq_0$. We start by making a straightforward observation.

Lemma 4. *If* $\succcurlyeq_0$ *satisfies Axioms CP, LBM and RP, for any* $x, y \in X$ *such that* $\bar{x} \succcurlyeq_0 \bar{y}$ *there exists* $w \in X$ *such that* $\bar{x}E\bar{y} \sim_0 \bar{w}$ *(which we denote* $\bar{c}_{\bar{x}E\bar{y}}$*) and there exists* $z \in X$ *such that* $\bar{x}E\bar{y} \sim_0 \bar{c}_{\bar{x}E\bar{z}}E\bar{c}_{\bar{z}E\bar{y}}$*.*

If we now denote the z in the lemma $z \equiv \frac{1}{2}x \oplus_0 \frac{1}{2}y$, then it is follows (from the proof of the lemma) that

$$u_0\left(\frac{1}{2}x \oplus_0 \frac{1}{2}y\right) = \frac{1}{2}u_0(x) + \frac{1}{2}u_0(y).$$

The operator $\oplus_0$ can depend on the nice event $E \in \mathscr{E}_0$ used to define it. As we did in the a-temporal case, we rule this out by imposing an invariance axiom, in which

$$\mathscr{E}_0 \equiv \left\{E \in \mathscr{A}(\mathscr{G}_T) : \ \succcurlyeq_0 \text{satisfies Ordered E-bisymmetry for } E\right\}.$$

Axiom (Constant Invariance - CI)**.** *For all* $E, F \in \mathscr{E}_0$ *and all* $x, y \in X$ *such that* $\bar{x} \succcurlyeq_0 \bar{y}$*,*

$$\bar{x}E\bar{y} \sim_0 c_{\bar{x}E\bar{z}}Ec_{\bar{z}E\bar{y}} \iff \bar{x}F\bar{y} \sim_0 \bar{c}_{\bar{x}F\bar{z}}F\bar{c}_{\bar{z}F\bar{y}}.$$

Taking into account the consequentialist property imposed by CP, it is now straightforward to show that the subjective mixture operation can be extended to the conditional preferences $\succcurlyeq_{t,\omega}$.

Since all the $\succsim_{t,\omega}$ are represented by u_0, z is the utility midpoint of x and y at time zero if and only if it is the utility midpoint of x and y at any time t and state ω. If the nice event E used to define $\oplus_0$ belongs to $\mathscr{A}(G_t(\omega))$, the algebra generated by $G_t(\omega)$, then $\bar{x}E\bar{y} \sim_0 \bar{c}_{\bar{x}E\bar{z}}E\bar{c}_{\bar{z}E\bar{y}} \iff \bar{x}E\bar{y} \sim_{t,\omega} \bar{c}_{\bar{x}E\bar{z}}E\bar{c}_{\bar{z}E\bar{y}}$, so that z can be also identified from the preference $\succsim_{t,\omega}$. But the utility midpoint is well-defined even if $E \notin \mathscr{A}(G_t(\omega))$. That is, it is not necessary to identify a nice event with respect to every preference $\succsim_{t,\omega}$.

Given $f, g \in \mathscr{H}$, we now define for any $(t,\omega) \in \mathscr{T} \times \Omega$,

$$[(1/2)f \oplus (1/2)g]_t(\omega) \equiv (1/2)f_t(\omega) \oplus_0 (1/2)g_t(\omega).$$

That is, the 50:50 mixture of f and g is an act whose state and time contingent payoffs are utility midpoints of f and g. The extension to $\lambda f \oplus (1-\lambda)g$, with λ any dyadic rational in $[0,1]$, is then obtained as in the static setting.

The axioms stated so far in the section can be applied to any dynamic separable decision model with a finite horizon. The next axiom, however, specifically endows the preference representation with a variational structure:

Axiom (Variational Preferences - VP)**.** *For all* $(t,\omega) \in \mathscr{T} \times \Omega$*:*

1. *For all* $h, h' \in \mathscr{H}$*, if* $h \sim_{t,\omega} h'$*, then* $(1/2)h \oplus (1/2)h' \succsim_{t,\omega} h$.

2. *For all* $h, h' \in \mathscr{H}$*,* $\mathbf{y}, \mathbf{y}' \in X^{T+1}$*, and for all dyadic rationals* $\lambda \in (0,1)$*, if* $\lambda h \oplus (1-\lambda)\mathbf{y} \succsim_{t,\omega} \lambda h' \oplus (1-\lambda)\mathbf{y}$*, then* $\lambda h \oplus (1-\lambda)\mathbf{y}' \succsim_{t,\omega} \lambda h' \oplus (1-\lambda)\mathbf{y}'$.

3. *There exist* $x, x' \in X$ *such that* $\bar{x} \succ_{t,\omega} \bar{x}'$ *and for any dyadic rational* $\lambda \in (0,1)$ *there exists* $x'' \in X$ *satisfying either* $\bar{x}' \succ_{t,\omega} \lambda \bar{x}'' \oplus (1-\lambda)\bar{x}$ *or* $\lambda \bar{x}'' \oplus (1-\lambda)\bar{x}' \succ_{t,\omega} \bar{x}$.

4. *For no state* $\omega \in \Omega$*, it is the case that* $h(\omega') = h'(\omega')$ *and* $\omega \neq \omega'$ *for all* $\omega' \in \Omega$ *implies* $h \sim_0 h'$.

The above properties are well-known from the static axiomatization of Variational Preferences in Maccheroni et al. (2006a), to which we refer for interpretation.[29] However, there is a difference in the interpretation of the Ambiguity Hedging property (item 1), analogously to the discussion in Section 6. In a setting with an objective mixture operation $(1/2)\mathbf{x} + (1/2)\mathbf{y}$, a decision maker might display a preference for "consumption smoothing" (with respect to the objective mixture

[29] The only (non-consequential) difference being that all the mixtures employed are with dyadic rational weights.

operation) by expressing $(1/2)\mathbf{x} + (1/2)\mathbf{y} \succ_0 \mathbf{y}$ even if $\mathbf{x} \sim_0 \mathbf{y}$. This preference would be explicitly ruled out if the utility function u_0 were to satisfy (as in traditional AA axiomatizations) affinity with respect to objective mixtures. In contrast, having formulated axiom VP in terms of the *subjective* mixture operation, we do not rule out the possibility of preference for consumption smoothing.

Remark 1. The axioms stated so far (CP, LBM, RP, CI, and VP) are necessary and sufficient to obtain an intertemporal version of the Variational Preferences model (analogous to Proposition 1 in DVP) in a fully subjective setting. Precisely, under these axioms each $\succsim_{t,\omega}$ can be represented by[30]

$$V_{t,\omega}(h) = \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau=t}^{T} \delta^{\tau-t} u(h_\tau)\, d p_{G_t(\omega)} + \gamma_t(\omega, p_{G_t(\omega)}) \right)$$

where $\gamma_t : \Omega \times \Delta^{++}(\Omega) \to [0,\infty]$ is a dynamic ambiguity index; see DVP for the precise definition.

Analogously, it can be seen that the axioms LBM, CI and VP, once reformulated for an a-temporal setting, are necessary and sufficient to obtain the (static) Variational Preferences representation of Maccheroni et al. (2006a), thus providing an axiomatization of Variational Preferences in a fully subjective setting.

The last axiom is the standard dynamic consistency requirement:

Axiom (Dynamic Consistency - DC)**.** *For all $(t,\omega) \in \mathcal{T} \times \Omega$ with $t < T$, and all $h, h' \in \mathcal{H}$, if $h_\tau = h'_\tau$ for all $\tau \leq t$ and $h \succsim_{t+1,\omega} h'$ for all $\omega' \in \Omega$, then $h \succsim_{t,\omega} h'$.*

We can now obtain a fully subjective version of the DVP representation:

Theorem 1. *The preferences $\{\succsim_{t,\omega}\}_{(t,\omega)\in\mathcal{T}\times\Omega}$ satisfy CP, LBM, CI, RP, VP, and DC if and only if there exist a scalar $\delta \in (0,1]$, and unbounded $u : X \to \mathbb{R}$ and $\gamma_t : \Omega \times \Delta^{++}(\Omega) \to [0,\infty]$ such that for every $(t,\omega) \in \mathcal{T} \times \Omega$, $V_{t,\omega} : \mathcal{H} \to \mathbb{R}$ represents $\succsim_{t,\omega}$, where*

$$V_{t,\omega}(h) = u(h(t,\omega)) + \min_{p \in \Delta^{++}(\Omega, \mathcal{G}_{t+1})} \left(\delta \int V_{t+1,\omega'}(h)\, p(d\omega') + \gamma_t(\omega, p) \right)$$

$u : X \to \mathbb{R}$ is unbounded and affine with respect to $\oplus$, and γ_t is a dynamic ambiguity index.

The discussion so far illustrates how our "bet-wise" mixture construction can be fruitfully used in dynamic models. An alternative, approach to modelling mixtures in a dynamic and fully subjective setting is presented by Kochov (2015), who also provides an axiomatization of intertemporal

[30] $\Delta^{++}(\Omega)$ denotes the set of all probabilities on Ω with full support.

Variational Preferences, such as that mentioned in Remark 1. He obtains mixtures by exploiting the multi-dimensionality of the consumption space and the additive separability of preferences. His results are different from ours in two respects: 1) his results require an infinite horizon, 2) his key assumption, called "Intertemporal Hedging," while reminiscent of Ambiguity Hedging, is different from it as it applies to uncertainty over time rather than to single-period uncertainty. So his model is conceptually orthogonal to the one presented above.

Appendices

A Proofs

Lemma 5. *Suppose that $\succcurlyeq$ satisfies P and C. Then for any $f \in \mathscr{F}$, there exists $c_f \in X$ such that $c_f \sim f$.*

Proof. By P, for each $f \in \mathscr{F}$, there are $x, y \in X$ with $x \succcurlyeq f \succcurlyeq y$, hence $B = \{z \in X : z \succcurlyeq f\}$ and $W = \{z \in X : f \succcurlyeq z\}$ are both non-empty. By C both sets are closed, hence connectedness of X implies non-emptyness of $B \cap W$. So, for all $f \in \mathscr{F}$ there exists $c_f \in X$ with $c_f \sim f$. □

Proof. of Fact 1. Given P and C, Lemma 5 implies that for each $f \in \mathscr{F}$ there exists $x \in X$ with $x \sim f$ and denote it c_f. By standard arguments, there exists a continuous $u : X \to \mathbb{R}$ representing $\succcurlyeq$ on X, unique up to a monotonic nondecreasing transformation. Define $V : \mathscr{F} \to \mathbb{R}$ as $V(f) \equiv u(c_f)$. Then, V represents $\succcurlyeq$. Consider $A \in \Sigma$ and $x \succ y$, then, either $xAy \sim x$ or $xAy \sim y$. In the former case, $V(xAy) = V(x)$, so, letting $\rho(A) = 1$, we have $V(xAy) = \rho(A)V(x) + (1 - \rho(A))V(y)$. A similar argument holds for the latter case with $\rho(A) = 0$. Clearly, V is unique up to a monotonic nondecreasing transformation and $\rho(A) \in \{0, 1\}$ for all $A \in \Sigma$. □

Proof. of Proposition 1. To prove sufficiency, we show that our axioms implies axioms A1, A2, A3, A5 and A6 of Nakamura (1990) for bets on E. P implies A1. A2 and A5 follow from C and the Structural Assumption. A3 is equal to our Ordered E-Monotonicity. Axiom A6 is equivalent to our Ordered E-Bisymmetry. Notice that the proof of Nakamura (1990, Lemma 3) does not require axiom A4, therefore, applying Lemma 3 in Nakamura (1990), there exist a unique $\rho_E \in (0, 1)$ and a cardinally unique function $u_E : X \to \mathbb{R}$ such that the functional $V_E : \mathscr{F} \to \mathbb{R}$ defined by $V_E(f) = u(c_f)$ (certainty equivalents exist by Lemma 5) represents $\succcurlyeq$ and, for all $x \succcurlyeq y \in X$, $V_E(xEy) = \rho_E u_E(x) + (1 - \rho_E)u_E(y) = \rho_E V_E(x) + (1 - \rho_E)V_E(y)$. Nontriviality and constant-boundedness of V_E follow immediately from Axiom P, while its subcontinuity is proved as in the proof of Lemma 31 in Ghirardato and Marinacci (2001). The proof of necessity is straightforward from the properties of V_E. □

Proof. of Lemma 2. The first statement follows from the structural assumption and Axioms P, C, and ON along the same lines as Proposition 1 in GMMS (2003). As to the second, take $x \succcurlyeq y$, if E is the unique event satisfying ON, there is nothing to prove. Take a $F \in \Sigma$, and $F \neq E$, $F \in \mathscr{E}$ implies that there exists a cardinal utility $u_F : X \to \mathbb{R}$, by NV and Equation (1),

$$u_E(z) = \frac{1}{2}u_E(x) + \frac{1}{2}u_E(y) \iff u_F(z) = \frac{1}{2}u_F(x) + \frac{1}{2}u_F(y)$$

Since u_E and u_F represent the same preferences over X, there exists a monotone and continuous function $\phi : u_F(X) \to u_E(X)$ such that $u_E = \phi(u_F)$. Then, $u_E(z) = \phi(u_F(z)) = \frac{1}{2}\phi(u_F(x)) + \frac{1}{2}\phi(u_F(y))$ but also $\phi(u_F(z)) = \phi\left(\frac{1}{2}u_F(x) + \frac{1}{2}u_F(y)\right)$, Then $\frac{1}{2}\phi(r) + \frac{1}{2}\phi(s) = \phi\left(\frac{1}{2}r + \frac{1}{2}s\right)$ and the continuous solution to this standard Jensen's functional equation is $\phi(r) = \alpha r + \beta$ (see Aczél, 1966), for some $\alpha \neq 0$, $\beta \in \mathbb{R}$. In our case, $\alpha > 0$ since u_F and u_E represent the same preferences. Then $u_E = \alpha u_F + \beta$. □

Proof. of Proposition 2. Necessity is straightforward. For sufficiency, if there is a unique $E \in \mathscr{E}$, it is sufficient to define $\rho_E = \rho(E)$ and $u = u_E$. Suppose the existence of $F \in \mathscr{E}$ with $F \neq E$. Take $x \succcurlyeq y$ and consider $V_E(xEy) = \rho_E u_E(x) + (1-\rho_E)u_E(y)$ and $V_F(xEy)$. By Lemma 2, axioms P, C, ON and MV implies that u_E and u_F (resp V_E and V_F) are cardinally equivalent, hence we can normalize them to be equal $u = u_E = u_F$ (resp $V_E = V_F = V$). Therefore, for any $x \succcurlyeq y$, $V(xEy) = V_F(xEy) = u(x)\rho_E + (1-\rho_E)u(y)$. Defining $\rho : \mathscr{E} \to (0,1)$ by $\rho(E) \equiv \rho_E$ concludes the proof. □

Proof. of Proposition 3. Necessity is straightforward. For sufficiency, by Proposition 2, P, C, ON and NV imply the existence of a cardinally unique u representing $\succcurlyeq$ on X. Since it is non-constant we can choose u s.t. $[-1,1] \subseteq u(X) = \mathbb{R}$. Denote by $B_0(\Sigma, u(X))$ the subset of $B_0(\Sigma)$ consisting of simple measurable functions with range in $u(X)$. It is simple to show that $B_0(\Sigma, u(X)) = \{u \circ f : f \in \mathscr{F}\}$. Let define $I(u \circ f) = u(c_f)$ for all $f \in \mathscr{F}$. Clearly, $u \circ f = u \circ g$, if and only if, $f(s) \sim g(s)$ for all $s \in S$, by Decomposability, $f \sim g$ and $I(u \circ f) = u(c_f) = u(c_g) = I(u \circ g)$. Hence $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ is well defined. It is also normalized, pick $k \in u(X)$. By assumption, there exists $x \in X$ such that $u(x) = k$, letting $a = k1_S$, $I(a) = u(c_f) = u(x) = k$. To prove continuity, consider the set $L \equiv \{\phi \in B_0(\Sigma, u(X)) : I(\phi) \leq k\}$, and assume it is open. Then there exists a sequence $\phi_n \to \phi$, such that $I(\phi_n) \leq k$ but $I(\phi) > k$. It follows that there exists a net $f_\alpha \in \mathscr{F}$, with $\phi_\alpha = u \circ f_\alpha \to u \circ f = \phi$ with $I(u \circ f_\alpha) = u(c_{f_\alpha}) \leq u(z)$ where $I(u \circ z) = u(z) = k$ and $I(u \circ f) = u(c_f) > u(z)$. Since u represents preferences on X, $u(c_{f_\alpha}) \to u(c_f)$ and $u(c_{f_\alpha}) \leq u(z) < u(c_f)$ for all α, contradicts τ-continuity of u. Hence, I is lower-semicontinuous. Upper-semicontinuity follows from a symmetric argument. So I is continuous. Hence, the first part is proved. For the second we need to introduce a preference relation derived from $\succcurlyeq$. Define:

$$f \succcurlyeq^* g \Longleftrightarrow \lambda f \oplus (1-\lambda) h \succcurlyeq \lambda g \oplus (1-\lambda) h$$

for all dyadic rationals $\gamma \in (0,1]$ and all $h \in \mathscr{F}$. Since I represents $\succcurlyeq$, we have:

$$f \succcurlyeq^* g \;\Leftrightarrow\; I(u \circ (\lambda f \oplus (1-\lambda) h)) \geq I(u \circ (\lambda g \oplus (1-\lambda) h))$$

by the properties of u it is equal to

$$I(\lambda u \circ f + (1-\lambda) u \circ h)) \geq I(\lambda u \circ g + (1-\lambda) u \circ h))$$

hence it induces a continuous (by continuity of I), conic (by the previous property), preorder on $B_0(\Sigma, u(X))$. By a simple modification of a standard result (see Ghirardato and Marinacci (2002)) and denoting $ba_1(\Sigma)$ the space of signed charges on Σ with $\mu(S) = 1$, there exists a non-empty, unique, convex and weak*-closed set $C \subseteq ba_1(\Sigma)$ such that:

$$f \succcurlyeq^* g \Longleftrightarrow \int u \circ f \, d\mu \geq \int u \circ g \, d\mu, \quad \forall \mu \in C$$

In the special case of a binary act, $f = xAy$, this representation has a local biseparable form given by

$$I(u \circ (xAy)) = u(x)\rho_{x,y}(A) + u(y)(1 - \rho_{x,y}(A)) \tag{12}$$

Where

$$\rho_{x,y}(A) \equiv \alpha(u \circ (xAy))(\underline{C}(1_A) - \overline{C}(1_A)) + \overline{C}(1_A)$$

and

$$\alpha(a) \equiv \frac{\overline{C}(a) - I(a)}{\overline{C}(a) - \underline{C}(a)}$$

Now, consider an act xFy with $x \succcurlyeq y$ and $F \in \mathscr{E}$, then by Proposition 2, $V(xFy) = I(u \circ (xFy)) = u(x)\rho(F) + u(y)(1 - \rho(F))$, for some $\rho(F) \in (0,1)$. Then,

$$\rho(F) = \frac{I(u \circ (xAy)) - u(y)}{u(x) - u(y)} = \rho_{x,y}(F)$$

for all $x \succcurlyeq y \in X$. The uniqueness part is standard and follows from Proposition 2. □

Proof. of Fact 2. We begin with an additional fact:

Fact 3. *If a function $\phi : [a,b] \to \mathbb{R}$ is not affine, for all $p \in (0,1)$, there are at least two real numbers $r \neq s \in [a,b]$ such that $\phi(pr + (1-p)s) \neq p\phi(r) + (1-p)\phi(s)$.*

Proof. Suppose not, then for all $r, s \in [a,b]$, $\phi(pr + (1-p)s) = p\phi(r) + (1-p)\phi(s)$, the fact that $p \in (0,1)$ allows to apply Aczél (1966, Th. 1, p. 66) and conclude that $\phi(r) = \alpha r + \beta$ for some $\alpha \neq 0$, $\beta \in \mathbb{R}$, a contradiction to the assumption that ϕ is not affine.

Now, consider a SOSEU preference with a non-affine ϕ. By Fact 3, for all $p \in (0,1)$ there are $r, s \in u(X)$ with $\phi(pr + (1-p)s) \neq p\phi(r) + (1-p)\phi(s)$. W.l.o.g. let $r > s$ and take $x, z, z', y \in X$ with $u(x) = u(z) = r$ and $u(y) = u(z') = s$. By Ordered E-bisymmetry:

$$
\begin{aligned}
V(c_{xEz}Ec_{z'Ey}) =& \phi^{-1}\left(p\phi(pu(x) + (1-p)u(z)) + (1-p)\phi(pu(z') + (1-p)u(y))\right) \\
=& \phi^{-1}\left(p\phi(pu(x) + (1-p)u(z')) + (1-p)\phi(pu(z) + (1-p)u(y))\right) \\
=& V(c_{xEz'}Ec_{zEy})
\end{aligned}
$$

The definitions of $u(x), u(y), u(z), u(z')$ and the second equality above imply:

$$
p\phi(r) + (1-p)\phi(s) = p\phi(pr + (1-p)s) + (1-p)\phi(pr + (1-p)s) = \phi(pr + (1-p)s)
$$

A contradiction. □

Proof. of Proposition 5. Given axioms P,C, ON, MV and M, we can apply Proposition 4, there exists a continuous and cardinally unique function u and a monotone, continuous and normalized functional I that represents preferences. Axiom AH is equivalent to $I(u \circ f) = I(u \circ g)$ implies $I(u \circ (\frac{1}{2} f \oplus \frac{1}{2} g)) \geq I(u \circ g)$, by the properties of $\oplus$, we have $I(\frac{1}{2} u \circ f + \frac{1}{2} u \circ g) \geq I(u \circ g)$, by induction

$$
I\left(\frac{k}{2^n} u \circ f + \frac{2^n - k}{2^n} u \circ g\right) \geq I(u \circ g)
$$

$n = 0, 1, 2, \ldots$ and $0 \leq k \leq 2^n$. The set $\left\{\frac{k}{2^n}, n = 0, 1, 2, \ldots, 0 \leq k \leq 2^n\right\}$ is dense in $[0,1]$, so for all $\lambda \in (0,1)$ it contains a sequence converging to γ, namely $\lim_{j \to \infty} \frac{k_j}{2^{n_j}} = \gamma$. By continuity of I, for all

$\lambda \in (0,1)$,

$$I(\lambda u \circ f + (1-\lambda) u \circ g) = \lim_{j\to\infty} I\left(\frac{k_j}{2^{n_j}} u \circ f + \left(1 - \frac{k_j}{2^{n_j}}\right) u \circ g\right) \geq I(u \circ g)$$

Then I is quasiconcave, so the result follows from Cerreia-Vioglio et al. (2011). The opposite direction is standard. □

Proof. of Proposition 6. Assume that $\succcurlyeq$ satisfies the axioms of Proposition 2 and let $u(x) = V(x)$ for all $x \in X$, so that Eq. (5) holds. By Eq. (5) and Axiom PEM, $\lambda V(f) + (1-\lambda)V(g) = \lambda u(c_f) + (1-\lambda)u(c_g) = u(\lambda c_f \oplus (1-\lambda)c_g) = V(\lambda c_f \oplus (1-\lambda)c_g) \geq V(\lambda f \oplus (1-\lambda)g)$ for any $\lambda \in [0,1]$. Hence V is convex w.r.t. $\oplus$. The opposite implication follows by reversing the argument. The proofs that AEM is equivalent to convexity and MTI to affinity of V are analogous. For the second part of Proposition 6, by PEM and Eq. (5) $\lambda I(u \circ f) + (1-\lambda)I(u \circ g) = \lambda V(f) + (1-\lambda)V(g) = \lambda u(c_f) + (1-\lambda)u(c_g) = u(\lambda c_f \oplus (1-\lambda)c_g) = V(\lambda c_f \oplus (1-\lambda)c_g) \geq V(\lambda f \oplus (1-\lambda)g) = I(u \circ (\lambda f \oplus (1-\lambda)g)) = I(\lambda u \circ g + (1-\lambda)u \circ g)$ for any $\lambda \in [0,1]$. Hence V is convex w.r.t. $\oplus$. The other proofs are again analogous. □

Proof. of Corollary 1. If MTI holds, $u(c_{\lambda f \oplus (1-\lambda)g}) = u(\lambda c_f \oplus (1-\lambda)c_g) = \lambda u(c_f) + (1-\lambda)u(c_g)$. So, by the definition of V, $V(\lambda f \oplus (1-\lambda)g) = \lambda V(f) + (1-\lambda)V(g)$. Suppose now that $f \succcurlyeq g$. Then, $u(c_f) \geq u(c_g)$ and therefore $\lambda u(c_f) + (1-\lambda)u(c_h) \geq \lambda u(c_g) + (1-\lambda)u(c_h)$ for any $\lambda \in [0,1]$ and any $h \in \mathscr{F}$. By definition of subjective mixture, $u(\lambda c_f \oplus (1-\lambda)c_h) \geq u(\lambda c_g \oplus (1-\lambda)c_h)$; by MTI, $u(c_{\lambda f \oplus (1-\lambda)h}) = u(\lambda c_f \oplus (1-\lambda)c_h) \geq u(\lambda c_g \oplus (1-\lambda)c_h) = u(c_{\lambda g \oplus (1-\lambda)h})$, and therefore $\lambda f \oplus (1-\lambda)h \succcurlyeq \lambda g \oplus (1-\lambda)h$. For the opposite direction, Independence w.r.t. $\oplus$, Preference Order, and the definitions of c_f, c_g imply $\lambda f \oplus (1-\lambda)g \sim \lambda c_f \oplus (1-\lambda)g \sim \lambda c_f \oplus (1-\lambda)c_g$. □

Proof. Of Lemma 4. By Lemma 3 and Axiom RP, $U_0(\bar{x}E\bar{y}) = (\rho u_0(x) + (1-\rho)u_0(y))\sum_{t=0}^{T}\delta^t$. As u_0 is continuous and X connected, $u_0(X)$ is connected. Using the the representation in Lemma 3 and taking $w \in X$ such that $u_0(w) = \rho u_0(x) + (1-\rho)u_0(y)$ gives $\bar{x}E\bar{y} \sim_0 \bar{w}$. Analogously, taking $z \in X$ such that $u_0(z) = (1/2)u_0(x) + (1/2)u_0(y)$ and using repeatedly the representation in Lemma 3 (alonf the same as the proof of Prop. 1 in GMMS (2003)) yields $\bar{x}E\bar{y} \sim_0 \bar{c}_{\bar{x}E\bar{z}}E\bar{c}_{\bar{z}E\bar{y}}$. □

Proof. of Theorem 1. Notice that Axioms CP, LBM, CI and RP, Lemmas 3 and 4, and an application of Proposition 4 allow us to define a monotone, normalized and continuous functional $I_{t,\omega}$ representing $\succcurlyeq_{t,\omega}$. It is equal to $U_0(\mathbf{x}) = \sum_{\tau=t}^{T}\delta^{\tau-t}u_0(x_\tau)$ when representing the restriction of $\succcurlyeq_{t,\omega}$ to X^{T+1}. Then, as discussed in Remark 1, adding Axiom VP yields $I_{t,\omega}$ the variational representation of Maccheroni et al. (2006a). Lastly, Axiom DC makes the representation recursive along the same lines as Maccheroni et al. (2006b). □

B Payoff-dependent willingness to bet and bisymmetry

Let $S = \{s_1, s_2\}$, $u(X) = [0, 10]$, and define

$$V(xEy) = 1_{u(x)\leq 5}[0.3u(x) + 0.7u(y)] + 1_{u(x)>5}[\rho_x u(x) + (1-\rho_x)u(y)]$$

with $\rho_x = 0.2 + 0.02(10 - x)$. We first show that the preference represented by V satisfies Ordered E-monotonicity. There are three cases to check:

- $(5 \geq x, y, z)$: Then $V(xEz) = 0.3x + 0.7z$, hence $x \succcurlyeq y$ if and only if $xEz \succcurlyeq yEz$ for all $x, y \succcurlyeq z$. If $z \succcurlyeq x, y$, $V(zEx) = 0.3z + 0.7x$ and $x \succcurlyeq y$ if and only if $zEx \succcurlyeq zEy$.
- $(x > 5 \geq y)$: $V(xEz) = \rho_x x + (1-\rho_x)z$ and $V(yEz) = 0.3y + 0.7z$ for all $x, y \succcurlyeq z$. For $x = 10$, $0.2 * 10 + 0.8 * z \geq 0.3y + 0.7z$, for all $z \leq y \leq 5$. For $x = 5$, $0.3 * 5 + 0.7z \geq 0.3y + 0.7z$ for all $z \leq y \leq 5$.
- $(x, y > 5)$: $V(xEz) = \rho_x x + (1-\rho_x)z$ and $V(yEz) = \rho_y y + (1-\rho_y)z$ for all $x, y \succcurlyeq z$. Let $x \succcurlyeq y$ and we need to prove $V(xEz) - V(yEz) = \rho_x x - \rho_y y + z(\rho_y - \rho_x) \geq 0$, By definition $\rho_y \geq \rho_x$, so it is sufficient to prove $\rho_x x - \rho_y y \geq 0$, but this can be verified applying the definition of ρ_x and ρ_y.

The preference $\succcurlyeq$ violates bisymmetry. For instance, take $x > 5$ and $z, z' < 5$ such that $c_{xEz}, c_{xEz'} > 5$. Then for $5 \geq z' \geq y$,

$$\begin{aligned} V(c_{xEz}Ec_{z'Ey}) &= \rho_{c_{xEz}}(\rho_x x + (1-\rho_x)z) + (1-\rho_{c_{xEz}})(0.3z' + 0.7y) \\ &\neq \rho_{c_{xEz'}}(\rho_x x + (1-\rho_x)z') + (1-\rho_{c_{xEz'}})(0.3z + 0.7y) \\ &= V(c_{xEz'}Ec_{zEy}), \end{aligned}$$

a violation of bisymmetry.

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