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Mean-variance dynamic optimality for DC pension schemes

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Abstract

It is well known that the mean-variance portfolio selection is a time-inconsistent optimization problem. In the current literature, this time inconsistency is often tackled with either a game theoretical approach (Basak and Chabakauri, 2010, and Björk and Murgoci, 2010) or a so-called precommitment approach (Zhou and Li, 2000). The framework of a defined contribution (DC) pension scheme, which we deal with in this work, makes no exception, with a number of papers computing either the Nash equilibrium or the precommitment strategy in the presence of a variety of financial markets. Here, we solve a mean-variance portfolio selection problem for a DC pension fund through the dynamically optimal approach introduced by Pedersen and Peskir (2017), and we compare the dynamically optimal strategy with the precommitment one. We show that both strategies are the solution to target-based problems. The precommitment strategy has a constant target, while the dynamically optimal strategy has a time-varying target whose expectation coincides with the constant target of the previous case. We also show that the expected wealth is the same under the two approaches. Numerical simulations show that, with respect to the precommitment strategy, the dynamically optimal strategy provides: (i) a larger variance of wealth,

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(ii) a less volatile asset allocation, and (iii) a larger effectiveness in reacting against most unfavorable and persistent market conditions.

Keywords. Time inconsistency, dynamic programming, martingale approach, pre-commitment approach, mean-variance portfolio selection.

JEL classification: C61, D81, G11.

1 Introduction and motivation

The risk management of defined contribution (DC) pension schemes is gaining increasing importance in industrialized countries. The population ageing is threatening the solvability of Pay As You Go public pension systems, and the largely adopted solution is to enhance the so-called second pillar, with an overall preference towards defined contribution schemes rather than defined benefit schemes. The search of the most appropriate portfolio strategy in the accumulation phase is the subject of extensive research in the actuarial and financial literature.

Two common optimization criteria are the maximization of the expected utility of the fund's wealth at retirement, and the mean-variance approach. In this paper we focus on the latter. It is well known that the mean-variance portfolio selection is a time-inconsistent problem due to the presence of the variance of final wealth in the performance criterion (Zhou and Li, 2000 and Basak and Chabakauri, 2010).

The problem of time-inconsistency is commonly approached in three ways: (i) the pre-commitment approach (Strotz, 1956), (ii) the game-theoretical approach (e.g. Basak and Chabakauri, 2010), and (iii) the dynamically optimal approach introduced by Pedersen and Peskir (2017), which is a continuous-time version of the so-called naive approach described by Pollak (1968).[1]

While the first two approaches have been widely investigated in the portfolio selection problem for DC pension schemes, the third one has not been adopted in this context. Since the risk management problem in a DC pension scheme is a crucial topic in the agenda of

[1]In the remaining of the paper, the third approach (and the corresponding investment strategy) will be either called *dynamically optimal*, or *dynamically optimal naive*, or simply *naive*.

welfare systems of many countries, the relevance of investigating these alternative strategies is clear.

This paper aims to fill this gap in the literature by adopting the dynamically optimal approach to solve a mean-variance portfolio selection problem in a DC pension scheme. This approach turns out to be equivalent to a target-based approach with stochastic target that moves over time in response to renovated circumstances. A DC pension scheme is characterized by a non self-financing portfolio, in which contributions from workers/subscribers are received and managed over time. The pension fund managers have no control over the contributions that are stochastic state variables (because they are a percentage of stochastic salaries) whose risk must be managed with a suitable portfolio allocation, together with the usual market risk and interest rate risk. In particular, here we assume that: (i) there exist a set of stochastic state variables that drive all the relevant financial variables on the market, and (ii) the financial market is complete or, in other words, it is possible to hedge against any risk through a suitable portfolio.

The remainder of the paper is structured as follows. In Section 2 we define the financial market. In Section 3 we set the mean-variance problem. In Section 4 we present and compare the precommitment and the dynamically optimal approaches. In Section 5 we derive the optimal portfolios for the two approaches, while in Section 6 we provide further theoretical results about their comparison. Section 7 is devoted to numerical simulations and Section 8 concludes.

2 The model

At time $t_0 \geq 0$ a worker joins a DC pension scheme whose initial fund is $x_0 \geq 0$. We further assume that the retirement time T is legally compulsory and, accordingly, is not a choice variable. From t_0 till $T > t_0$ the worker pays periodic contributions into the fund, as a percentage of his salary. The pension fund manager decides the asset allocation at any time $t \in [t_0, T]$, but has no control over the contribution.

We present a general model, in which the uncertainty on an arbitrage free and complete financial market is driven by a set of s stochastic state variables. On the market one riskless asset and n risky assets are listed. As a particular case, we also present a framework where all the state variables are constant and there is only one risky asset following a geometric Brownian motion (so-called Black and Scholes model, Björk, 1998).

2.1 General model

The financial market is arbitrage free, complete, frictionless, and continuously open at any time $t \in [t_0, T]$. The risk is described by a set of n independent Brownian motions $W(t)$, defined on the complete filtered probability space $\{\mathbf{\Omega}, \mathcal{F}(t), \mathbb{P}\}$, where $\mathcal{F}(t)$ is the filtration generated by the Brownian motions and $\mathbb{P}$ is the real-world probability measure. The financial market is described by the following variables:

- s state variables $z(t)$ (with $z(t_0) = z_0 \in \mathbb{R}^s$ known) whose values solve the matrix stochastic differential equation (SDE)

$$\underset{s\times 1}{dz(t)} = \underset{s\times 1}{\mu_z(t,z)}dt + \underset{s\times n}{\Omega(t,z)}\underset{n\times 1}{dW(t)}, \tag{1}$$

- one riskless asset whose price $G(t)$ solves the (ordinary) differential equation

$$dG(t) = G(t)\, r(t,z)\, dt,$$

  where $r(t,z)$ is the spot instantaneously riskless interest rate;

- n risky assets whose prices $S(t)$ (with $S(t_0) = s_0 \in \mathbb{R}^n$ known) solve the matrix stochastic differential equation

$$\underset{n\times 1}{dS(t)} = \underset{n\times n}{I_S}\left[\underset{n\times 1}{\mu(t,z)}dt + \underset{n\times n}{\Sigma(t,z)}\underset{n\times 1}{dW(t)}\right], \tag{2}$$

  where I_S is the $n \times n$ square diagonal matrix gathering the prices $S_1, S_2, ..., S_n$.

The drift and diffusion terms in (1) and (2) are assumed to satisfy the usual conditions for the existence and uniqueness of a strong solution to the SDEs.

The absence of arbitrage and completeness imply the existence of a unique risk-neutral equivalent martingale measure $\mathbb{Q}$. This also implies the existence and uniqueness of a market prices of risk $\xi(t,z)\in\mathbb{R}^n$ which solves the linear system $\Sigma(t,z)\,\xi(t,z)=\mu(t,z)-r(t,z)\,\mathbf{1}$, where $\mathbf{1}$ is a vector of 1's (i.e. $\exists\Sigma(t,z)^{-1}$). Assuming that $\xi(t,z)$ satisfies the Novikov's condition, the Girsanov theorem applies and the Wiener processes $dW(t)$ can be rewritten under $\mathbb{Q}$ as follows:

$$dW^{\mathbb{Q}}(t)=\xi(t,z)\,dt+dW(t)\,. \tag{3}$$

The Radon-Nikodym derivative is (the prime denotes transposition):

$$m(t_0,t)=e^{-\frac{1}{2}\int_{t_0}^{t}\xi(u,z)'\xi(u,z)du-\int_{t_0}^{t}\xi(u,z)'dW(u)}\iff\begin{cases}dm(t_0,t)=-m(t_0,t)\,\xi(t,z)'\,dW(t)\,,\\ m(t_0,t_0)=1.\end{cases}$$

Thus, given any $t-$measurable random variable $\Xi(t)$, the following relationship holds true

$$\mathbb{E}_{t_0}^{\mathbb{Q}}[\Xi(t)]=\mathbb{E}_{t_0}[\Xi(t)\cdot m(t_0,t)]\,, \tag{4}$$

where $\mathbb{E}_{t_0}^{\mathbb{Q}}[\bullet]$ and $\mathbb{E}_{t_0}[\bullet]$ are the expected values conditional to $\mathcal{F}(t_0)$ and computed under the risk neutral or the real world probabilities, respectively.

Remark 1. Throughout the paper, the notation $\mathbb{E}_{t_0}[\bullet]$ denotes $\mathbb{E}[\bullet|\mathcal{F}_{t_0}]$ and is a compact version of the more complete notation $\mathbb{E}_{t_0,z_0,x_0}[\bullet]$.

Let $B(t,T)$ be the price in t of a zero-coupon bond expiring in T, and $\sigma_B(t,T)$ the (vector) diffusion term of $\frac{dB(t,T)}{B(t,T)}$. It is well known that the so-called "forward probability measure" ($\mathbb{F}_T$) can be defined as follows

$$dW^{\mathbb{Q}}(t)=\sigma_B(t,T)\,dt+dW^{\mathbb{F}_T}(t)\,, \tag{5}$$

and, given any $T-$measurable random variable $\Xi(T)$, we can write

$$\mathbb{E}_t^{\mathbb{Q}}\left[\Xi(T)\,e^{-\int_t^T r(u,z)du}\right]=\mathbb{E}_t^{\mathbb{F}_T}[\Xi(T)]\,\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u,z)du}\right]=\mathbb{E}_t^{\mathbb{F}_T}[\Xi(T)]\,B(t,T)\,, \tag{6}$$

where the new *numéraire* of the economy is $B(t,T)$ (Björk, 1998). $\mathbb{F}_T$ is useful for simplifying the role of contributions in the evolution of the pension fund's wealth.

Remark 2. The forward probability measure is needed to split the expected value of a product into the product of two expected values, as in (6). In this way, also the derivative of the expected value can be written in a much simpler way.

A stochastic contribution $c(t,z) > 0$ is continuously paid by the member into the fund's wealth $X(t)$. If $w(t) \in \mathbb{R}^n$ contains the monetary amount invested at time t in each risky asset (i.e. a *portfolio*) and satisfies the usual "admissible" properties (Karatzas and Shreve, 1998), the wealth dynamics are given by the following SDE:

$$dX(t) = \big(X(t)\, r(t,z) + c(t,z) + w(t)' \left(\mu(t,z) - r(t,z)\,\mathbf{1}\right)\big)\, dt + w(t)' \,\Sigma(t,z)\, dW(t)\,. \quad (7)$$

2.2 Black and Scholes model, constant salary

Our general model collapses into the Black and Scholes framework if we assume $\mu_z = \mathbf{0}$ and $\Omega = \mathbf{0}$ (where $\mathbf{0}$ is a matrix/vector of zeros), i.e. all the state variables are constant. Accordingly, both the interest rate and the contributions are constant and positive: $r \geq 0$, $c \geq 0$. Furthermore, in the financial market, we have $n = 1$ and both μ and $\Sigma = \sigma$ are constant. Thus, we can write

$$dG(t) = G(t)\, r dt,$$

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW(t)\,,$$

and the wealth dynamics is accordingly

$$dX(t) = (X(t)\, r + c + w(t)(\mu - r))\, dt + w(t)\, \sigma dW(t)\,. \quad (8)$$

3 The mean-variance problem

At time t_0 with initial state variables z_0 and initial wealth x_0 the pension fund manager wants to maximize the expected final wealth at retirement, adjusted by the wealth variance that can be interpreted like a risk measure. Accordingly, he wants to solve the following mean-variance problem:[2]

$$\left[\mathcal{P}^{MV}_{t_0,z_0,x_0}\right] \qquad \sup_w J^{MV}\left(t_0, z_0, x_0, w\right) = \sup_w \left\{\mathbb{E}_{t_0}\left[X\left(T\right)\right] - \alpha \mathbb{V}_{t_0}\left[X\left(T\right)\right]\right\}, \tag{9}$$

where the optimization is done over some set of admissible controls, and $\alpha > 0$ is a measure of the agent's risk aversion.

It is well known (e.g. Zhou and Li, 2000) that it is not possible to solve the mean-variance problem $\left[\mathcal{P}^{MV}_{t_0,z_0,x_0}\right]$ with dynamic programming, because of the presence of a non-linear function of expected final wealth in the performance criterion (Björk, Khapko, and Murgoci, 2017). Thus, according to the existing literature, the problem is said to be time-inconsistent.

There are three possible ways to tackle this time-inconsistency: (i) a precommitment approach; (ii) a game theoretical approach; (iii) a dynamically optimal or naive approach. The fist gives raise to a time-inconsistent policy, while the last two approaches lead to time-consistent policies.

In the current literature on defined contribution pension schemes, only the first and the second approaches have been thoroughly investigated, while the third one has neither been adopted nor analyzed.

In this paper we fill this gap of the literature. In particular, we investigate the adoption of the dynamically optimal naive approach in a defined contribution pension scheme and we make a comparison with the precommitment approach.

4 The precommitment and the dynamically optimal naive approaches

4.1 Precommitment approach

Given the initial point (t_0, z_0, x_0), the so-called precommitment strategy that solves the mean-variance problem $\left[\mathcal{P}^{MV}_{t_0,z_0,x_0}\right]$ in (9) is the control plan $\hat{w}$ that maximizes just $J^{MV}\left(t_0, z_0, x_0, w\right)$.

[2] As we have highlighted in Remark 1 for the expectation, also for the variance operator $\mathbb{V}\left[\bullet\right]$ we have that $\mathbb{V}_{t_0}\left[\bullet\right]$ is a short notation for $\mathbb{V}_{t_0,z_0,x_0}\left[\bullet\right]$.

More formally, we can write what follows.

Definition 1. Given the mean-variance problem (9), if there exists a strategy $\hat{w}_{t_0,z_0,z_0}(t,z,x)$, with $(t,z,x) \in [t_0,T] \times \mathbb{R}^s \times \mathbb{R}$, that maximizes $J^{MV}(t_0,z_0,x_0,w)$, i.e., a control map

$$\hat{w}_{t_0,z_0,x_0} : [t_0,T] \times \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}^n, \tag{10}$$

such that

$$J^{MV}(t_0,z_0,x_0,\hat{w}_{t_0,z_0,x_0}) = \sup_w J^{MV}(t_0,z_0,x_0,w),$$

then the strategy $\hat{w}_{t_0,z_0,x_0}(t,z,x)$ for $(t,z,x) \in [t_0,T] \times \mathbb{R}^s \times \mathbb{R}$ is called *precommitment strategy.*

The precommitment strategy for $\left[\mathcal{P}^{MV}_{t_0,z_0,x_0}\right]$ in both models of Sections 2.1 and 2.2 exists and is known in closed form (Vigna, 2014, and Menoncin and Vigna, 2017).

4.2 Dynamically optimal or naive approach

The dynamically optimal approach introduced by Pedersen and Peskir (2017) is the continuous-time version of the naive approach described by Pollak (1968). This approach can be easily defined from the precommitment approach. We illustrate the construction of the dynamically optimal strategy for the mean-variance problem $\left[\mathcal{P}^{MV}_{t_0,z_0,x_0}\right]$ in three steps.

Step 1. Assume that for the initial point (t_0,z_0,x_0) there exists the precommitment strategy

$$\hat{w}_{t_0,z_0,x_0} : [t_0,T] \times \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}^n \tag{11}$$

that maximizes the criterion $J^{MV}(t_0,z_0,x_0,w)$.

Step 2. Define the new control map

$$\tilde{w} : [t_0,T] \times \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}^n \tag{12}$$

as follows

$$\tilde{w}(t,z,x) := \hat{w}_{t,z,x}(t,z,x) \qquad \forall (t,z,x) \in [t_0,T] \times \mathbb{R}^s \times \mathbb{R}, \tag{13}$$

where the right hand side of (13) is obtained by replacing (t_0, z_0, x_0) with (t, z, x) in the function (11).

Step 3. The strategy $\tilde{w}(t, z, x)$ for $(t, z, x) \in [t_0, T] \times \mathbb{R}^s \times \mathbb{R}$ given by (13) is called the *dynamically optimal or naive strategy.*

To put it in simple terms, the dynamically optimal strategy is obtained by replacing (t_0, z_0, x_0) with (t, z, x) in the precommitment strategy. Therefore, the calculation of the dynamically optimal naive strategy for the models of Sections 2.1 and 2.2 is straightforward.

4.3 Link between the two approaches

There is a strict **link** between the dynamically optimal naive and the precommitment approaches:[3]

- At time t_0 with wealth x_0 the dynamically optimal naive investor and the precommitted investor play the same strategy $\hat{w}_{t_0,z_0,x_0}(t_0, z_0, x_0)$, and they face the same problem $\left[\mathcal{P}^{MV}_{t_0,z_0,x_0}\right]$.
- When time passes by, at time $t \in]t_0, T]$ with wealth x the naive investor faces problem $\left[\mathcal{P}^{MV}_{t,z,x}\right]$ and solves it with the precommitment approach at time t, *as if* his initial point were (t, z, x).
- In fact, the naive investor plays $\hat{w}_{t,z,x}(t, z, x)$, that would be the initial control played by an investor who, starting at time t with wealth x, wants to solve Problem $\left[P^{MV}_{t,z,x}\right]$ over the time horizon $[t, T]$ with the precommitment approach.
- Then, the intuition is that the dynamically optimal investor can be seen as the *continuous reincarnation* of the precommitted investor.
- Such an investor is called either *dynamically optimal* by Pedersen and Peskir (2017) or *naive* by Pollak (1968).

[3]To improve the readability and the interpretation, in what follows we will sometimes refer only to wealth (that is the only controlled state variable), and will ignore the remaining state variables. In the special case of the Black and Scholes market, where there are no state variables other than wealth, this turns out to be correct.

4.4 A target-based approach

Both strategies that we have presented in the previous sections, can be interpreted as two particular cases of target-based problems. Indeed, it is possible to prove (Zhou and Li, 2000, and Vigna, 2014) that a target $\gamma_{t_0,z_0,x_0} \in \mathbb{R}$ exists such that at any time $t \in [t_0, T]$ with wealth x the precommitted investor plays the strategy that minimizes the following criterion

$$\mathbb{E}_t \left[\left(X(T) - \gamma_{t_0,z_0,x_0} \right)^2 \right].$$

In other words, the precommitted investor plays the strategy that makes the final wealth as close as possible to the target γ_{t_0,z_0,x_0}. In this case the target is **constant** over time. Instead, it can be proved that the naive investor at time t with wealth x plays the strategy that minimizes the following criterion

$$\mathbb{E}_t \left[\left(X(T) - \gamma_{t,z,x} \right)^2 \right].$$

In other words, the dynamically optimal naive investor plays the strategy that makes the final wealth as close as possible to the target $\gamma_{t,z,x}$. Nevertheless, in this case the target is **time-varying** and it also depends on the wealth x achieved at time t (i.e. $\gamma_{t,z,x} \neq \gamma_{t_0,z_0,x_0}$).

In the next section we show the value of the target for the two approaches.

5 The precommitment and the naive portfolios

Since the naive portfolio is obtained from the precommitment one, we first present the precommitment strategy.

5.1 The precommitment portfolio

After Zhou and Li (2000), it is known that the mean-variance problem (9) can be recast as a target problem in the following form

$$\inf_w J(t_0, z_0, x_0, w) = \inf_w \mathbb{E}_{t_0} \left[\frac{1}{2} \left(X(T) - \gamma_{t_0,z_0,x_0} \right)^2 \right], \tag{14}$$

in which

$$\gamma_{t_0,z_0,x_0} = \frac{x_0 + \int_{t_0}^{T} \mathbb{E}_{t_0}^{\mathbb{F}_s}\left[c\left(s,z\right)\right] B\left(t_0,s\right) ds}{B\left(t_0,T\right)} + \frac{1}{2\alpha}\frac{\mathbb{E}_{t_0}\left[e^{2\Phi(t_0,T)}\right]}{B\left(t_0,T\right)^2}, \tag{15}$$

$$\Phi\left(t_0,T\right) = -\int_{t_0}^{T} r\left(u,z\right) du - \frac{1}{2}\int_{t_0}^{T} \xi\left(u,z\right)' \xi\left(u,z\right) du - \int_{t_0}^{T} \xi\left(u,z\right)' dW\left(u\right). \tag{16}$$

The form of the target is worth a comment. The first term of (15) coincides with the forward price of a floating versus fix swap. Let us assume that the fund wants to exchange its future stochastic cash flows till time T with a fixed amount x_0^T of money to be paid in T. This forward price x_0^T is a kind of certain equivalent for the whole cash flows of the fund, and, with a light abuse of terminology, we will call it *certain equivalent in* T, or T–certain equivalent. The value of x_0^T must satisfy the following pricing equation:

$$0 = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[x_0^T e^{-\int_{t_0}^{T} r(u,z)du} - \left(x_0 + \int_{t_0}^{T} c\left(s,z\right) e^{-\int_{t_0}^{s} r(u,z)du} ds\right)\right]. \tag{17}$$

Through this contract, the fund pays its initial wealth x_0 and all the subsequent payments $c\left(t,z\right)$ to its counterpart, and, in exchange, it receives, at maturity T, a constant amount of money x_0^T that solves (17). If we simplify this equation, the final result is

$$x_0^T = \frac{x_0 + \int_{t_0}^{T} \mathbb{E}_{t_0}^{\mathbb{F}_s}\left[c\left(s,z\right)\right] B\left(t_0,s\right) ds}{B\left(t_0,T\right)}. \tag{18}$$

The target γ_{t_0,z_0,x_0} is greater than the T–certain equivalent, in fact:

$$\gamma_{t_0,z_0,x_0} = x_0^T + \frac{1}{2\alpha}\frac{\mathbb{E}_{t_0}\left[e^{2\Phi(t_0,T)}\right]}{B\left(t_0,T\right)^2},$$

and the amount that is added to x_0^T for obtaining the target is a function of the risk aversion α. If the risk aversion is very high, then the fund will try to stay as close as possible to the T–certain equivalent, while if α is sufficiently low, the target will depart substantially from the T–certain equivalent.

In the Black and Scholes case, with c and r constant, the T–certain equivalent becomes

$$x_0^T = \frac{x_0 + c\int_{t_0}^{T} e^{-r(s-t_0)}ds}{e^{-r(T-t_0)}} = x_0 e^{r(T-t_0)} + c\frac{e^{r(T-t_0)} - 1}{r},$$

which coincides with the compounded value at the riskless rate r of initial wealth and contributions (i.e., the amount of money that could be obtained at time T by investing initial wealth x_0 and future contributions in the riskless asset). In the same framework, the target is

$$\gamma_{t_0,x_0} = x_0 e^{r(T-t_0)} + c\frac{e^{r(T-t_0)} - 1}{r} + \frac{1}{2\alpha}e^{\xi^2(T-t_0)}, \tag{19}$$

where $\xi = \frac{\mu - r}{\sigma}$ is the Sharpe ratio of the risky asset.

The solution to problem (14) is provided in the following proposition.

Proposition 1. *The optimal strategy of Problem* (14)*, that coincides with the precommitment solution to Problem* (9)*, is*

$$\begin{aligned} \hat{w}_{t_0,z_0,x_0}(t,z,x) = {} & B(t,T)\left(\gamma_{t_0,z_0,x_0} - x_t^T\right)(\Sigma')^{-1}\xi \\ & + (\Sigma')^{-1}\Omega' \frac{\partial\left(B(t,T)\left(\gamma_{t_0,z_0,x_0} - x_t^T\right)\right)}{\partial z(t)} \\ & - B(t,T)\left(\gamma_{t_0,z_0,x_0} - x_t^T\right)(\Sigma')^{-1}\Omega' \frac{\partial \ln \mathbb{E}_t\left[e^{2\Phi(t,T)}\right]}{\partial z(t)} \end{aligned} \tag{20}$$

where x_t^T is the T–certain equivalent at time t:

$$x_t^T = \frac{x + \int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c(s,z)\right] B(t,s)\, ds}{B(t,T)} \tag{21}$$

and γ_{t_0,z_0,x_0} and $\Phi(t_0,T)$ are given in (15) *and* (16)*, respectively.*

Proof. See Appendix (A). □

The optimal portfolio is formed by three components.

1. A speculative component proportional to the ratio between the market price of risk ξ and the diffusion matrix Σ. This component also contains the distance between the

initial target γ_{t_0,z_0,x_0} and the T–certain equivalent (21) at time t.

2. A hedging component that hedges the fund against the stochastic changes in the T–certain equivalent x_t^T. Actually, x_t^T is a stochastic variable since it is a conditional expected value. This portfolio component is needed because the state variables are stochastic (i.e. $\Omega \neq \mathbf{0}$) and is proportional to the correlation between the asset prices and the state variables. In fact, the matrix $(\Sigma')^{-1}\,\Omega'$ can be written as

$$(\Sigma')^{-1}\,\Omega' = (\Sigma')^{-1}\,\Sigma^{-1}\Sigma\Omega' = (\Sigma\Sigma')^{-1}\,\Sigma\Omega',$$

where $(\Sigma\Sigma')^{-1}$ is the inverse of the variance covariance matrix, while $\Sigma\Omega'$ is the matrix that contains the covariances between the asset prices and the state variables. Thus, the term $(\Sigma')^{-1}\,\Omega'$ can be interpreted as a kind of beta ratio between the market and the state variables.

3. The last hedging component is again proportional to the distance between the initial target γ_{t_0,z_0,x_0} and the T–certain equivalent x_t^T. Nevertheless, this time, the portfolio component hedges against the stochastic changes in the discount factor $\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]$. In particular, the portfolio contains the semi-elasticity of this discount factor with respect to the state variables.

Black and Scholes model

The precommitment strategy for the Black and Scholes model of Section 2.2 is[4]

$$\hat{w}_{t_0,x_0}(t,x) = \frac{\xi}{\sigma}\left[\gamma_{t_0,x_0}\; e^{-r(T-t)} - x - c\frac{1-e^{-r(T-t)}}{r}\right], \tag{22}$$

in which γ_{t_0,x_0} is given by (19).

Note that in the Black and Scholes case, the hedging portfolio components do not play any role, since all the state variables (r and c) are constant (i.e. $\Omega = \mathbf{0}$).

[4] In the Black and Scholes model $s = 0$ and there is no z variable.

5.2 Dynamically optimal naive approach

The dynamically optimal naive strategy for the general model of Section 2.1 is obtained by substituting γ_{t_0,z_0,x_0} with $\gamma_{t,z,x}$. In particular, given the value of γ_{t_0,z_0,x_0} in (15), we can write

$$\gamma_{t,z,x} = \frac{1}{2\alpha}\frac{\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]}{B\left(t,T\right)^2} + \frac{x + \int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c\left(s,z\right)\right]B\left(t,s\right)ds}{B\left(t,T\right)},$$

or

$$\gamma_{t,z,x} = x_t^T + \frac{1}{2\alpha}\frac{\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]}{B\left(t,T\right)^2},$$

and, accordingly, the dynamically optimal naive strategy is

$$\begin{aligned}\tilde{w}\left(t,z,x\right) =& \frac{1}{2\alpha}\frac{\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]}{B\left(t,T\right)}\left(\Sigma'\right)^{-1}\xi \\ &+ \left(\Sigma'\right)^{-1}\Omega'\left(\gamma_{t,x}\frac{\partial B\left(t,T\right)}{\partial z\left(t\right)} - \frac{\partial\int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c\left(s,z\right)\right]B\left(t,s\right)ds}{\partial z\left(t\right)}\right) \\ &- \frac{1}{2\alpha}\frac{\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]}{B\left(t,T\right)}\left(\Sigma'\right)^{-1}\Omega'\frac{\partial\ln\mathbb{E}_t\left[e^{2\Phi(t,T)}\right]}{\partial z\left(t\right)}.\end{aligned} \tag{23}$$

It is important to stress what follows.

1. The speculative portfolio component does not depend any longer on the contributions. In fact, the state variable $c\left(t,z\right)$ does not appear in the portfolio component containing $\left(\Sigma'\right)^{-1}\xi$.

2. The second portfolio component that hedges against the stochastic changes in the T–certain equivalent x_t^T, instead, still depends on contributions.

3. The last hedging portfolio component does not contain contributions.

Black and Scholes model

The dynamically optimal naive strategy for the Black and Scholes model of Section 2.2 is obtained by substituting (t,x) to (t_0,x_0) in (22), and is:

$$\tilde{w}\left(t,x\right)=\frac{\xi}{\sigma}\frac{1}{2\alpha}e^{\left(\xi^{2}-r\right)\left(T-t\right)},\quad\forall\left(t,x\right)\in\left[t_{0},T\right]\times\mathbb{R},\tag{24}$$

in which we see that the *amount* invested in the risky asset at time t does depend on t but does not depend on the fund level x at time t.

In this case the portfolio does not depend on contributions since it contains only the speculative component. Actually, also in the general case, the naive strategy implies that the speculative portfolio component does not depend on contributions.

Under the hypothesis that the variables ξ and σ are positive, we can see that the amount of money optimally invested in the risky asset $\tilde{w}\left(t,x\right)$, is either increasing or decreasing over time depending on the sign of the difference $\xi^{2}-r$. The sign of this difference is not obvious (recall that $\xi=\frac{\mu-r}{\sigma}$). When the interest rate is "sufficiently" high (low) the difference $\xi^{2}-r$ is negative (positive), and the amount of money invested in the risky asset increases (decreases) over time.

6 Black and Scholes case: two theoretical results

In this section we present two theoretical results holding in the Black and Scholes case that shed further light on the interactions between the precommitment and the dynamically optimal approaches.

The first result is that, although the precommitment and the naive strategies are substantially different, the expected value of the corresponding wealth is the same at any time.

Proposition 2. *In the Black and Scholes financial market, let $\hat{X}(t)$ and $\tilde{X}\left(t\right)$ denote the wealth at time t under adoption of the precommitment and dynamically optimal naive strategy, respectively. Then,*

$$\mathbb{E}_{t_{0}}\left[\hat{X}\left(t\right)\right]=\mathbb{E}_{t_{0}}\left[\tilde{X}\left(t\right)\right]\qquad\forall t\in\left[t_{0},T\right].\tag{25}$$

Proof. See Appendix (B). □

The second result refers to the connection between the constant target of the precommitment approach and the moving target of the dynamically optimal naive approach. In the precommitment approach the fund's wealth at time T is optimally set as close as possible to a constant target decided at time t_0 and given the initial wealth x_0. Instead, in the naive strategy the fund's wealth at time T is optimally set as close as possible to a time-varying target that depends on both the current time t and the current wealth x. Interestingly, we find that the expectation at time t_0 of the stochastic time-varying target relative to the dynamically optimal strategy coincides with the constant target.

Proposition 3. *In the Black and Scholes financial market, if $\tilde{X}(t)$ denotes the wealth at time t under adoption of the dynamically optimal naive strategy with $\tilde{X}(t_0) = x_0$, then*

$$\mathbb{E}_{t_0}\left[\gamma_{t,\tilde{X}(t)}\right] = \gamma_{t_0,x_0} \qquad \forall t \in [t_0, T]\,. \tag{26}$$

In other words, the target process $\gamma_{t,\tilde{X}(t)}$ is a martingale.

Proof. See Appendix (C). □

Then, the similarities between the two strategies are twofold. Standing at time t_0, over time the two strategies produce the same expected wealth. Moreover, standing at time t_0, the final target pursued at every time t remains on average the same.

It is then important to simulate and compare the actual behavior over time of the two strategies and the corresponding wealths, in order to identify those differences that cannot be captured *on average*.

7 Simulations

In the Black and Scholes model we have run 1000 Monte Carlo simulations with weekly discretisation for both the precommitment and the naive strategy and made a comparison between the two strategies with respect to the behavior over time of several quantities, such as the optimal portfolio and the optimal wealth. We have also investigated the distribution of the time-varying targets relative to the naive approach and compared them with the constant target of the precommitment approach.

The parameters for the simulations are

- $t_0 = 0$, and $T = 20$: we assume that the financial horizon is 20 years;

- $x_0 = 1$: the initial wealth can of course be scaled for taking into account any other wealth level;

- $c = 0.1$: we assume that the contribution is a percentage of the initial wealth (in this example 10%);

- $r = 3\%$, $\mu = 8\%$, and $\xi = \frac{1}{3}$, which imply a volatility $\sigma = 0.15$.

With these data, the T–certain equivalent x_0^T is

$$x_0^T = x_0 e^{r(T-t_0)} + c\frac{e^{r(T-t_0)} - 1}{r} = 4.562515,$$

while

$$\frac{1}{2\alpha} e^{\xi^2(T-t_0)} = \frac{4.613907}{\alpha}.$$

Thus, the initial target is

$$\gamma_{t_0,x_0} = x_0^T + \frac{1}{2\alpha} e^{\xi^2(T-t_0)} = 4.562515 + \frac{4.613907}{\alpha}.$$

If we want the target to be 1.2 times the T–certain equivalent x_0^T, we have

$$4.562515 + \frac{4.613907}{\alpha} = 1.2 \times 4.562515,$$

from which $\alpha = 5.0563$, and so

$$\gamma_{t_0,x_0} = \gamma_{0,1} = 5.475.$$

In what follows, we report the results of the simulations for the wealth, the optimal portfolio, and the time-varying target.

Figure 1 reports the statistics for the precommitment (PC) and the naive (N) wealth. The graph on top reports the mean and mean plus/minus twice the standard deviation for both approaches. The bottom left graph reports the minimum, the maximum, and the 5^{th}, 25^{th}, 50^{th}, 75^{th} and 95^{th} percentiles of PC-wealth, while the graph on the right reports the same measures for the N-wealth.

Figure 2 reports the statistics for the precommitment and the naive optimal investments in the risky asset (i.e. the optimal strategies). The graph on top reports the median for both strategies. The bottom left graph reports the minimum, the maximum, and the 5^{th}, 25^{th}, 50^{th}, 75^{th} and 95^{th} percentiles of PC-strategy, while the graph on the right reports the same measures for the N-strategy.

Finally, Figure 3 reports the statistics of the time-varying targets for the naive approach, $\gamma_{t,x}$, as well as the constant target for the precommitment approach, γ_{t_0,x_0}.

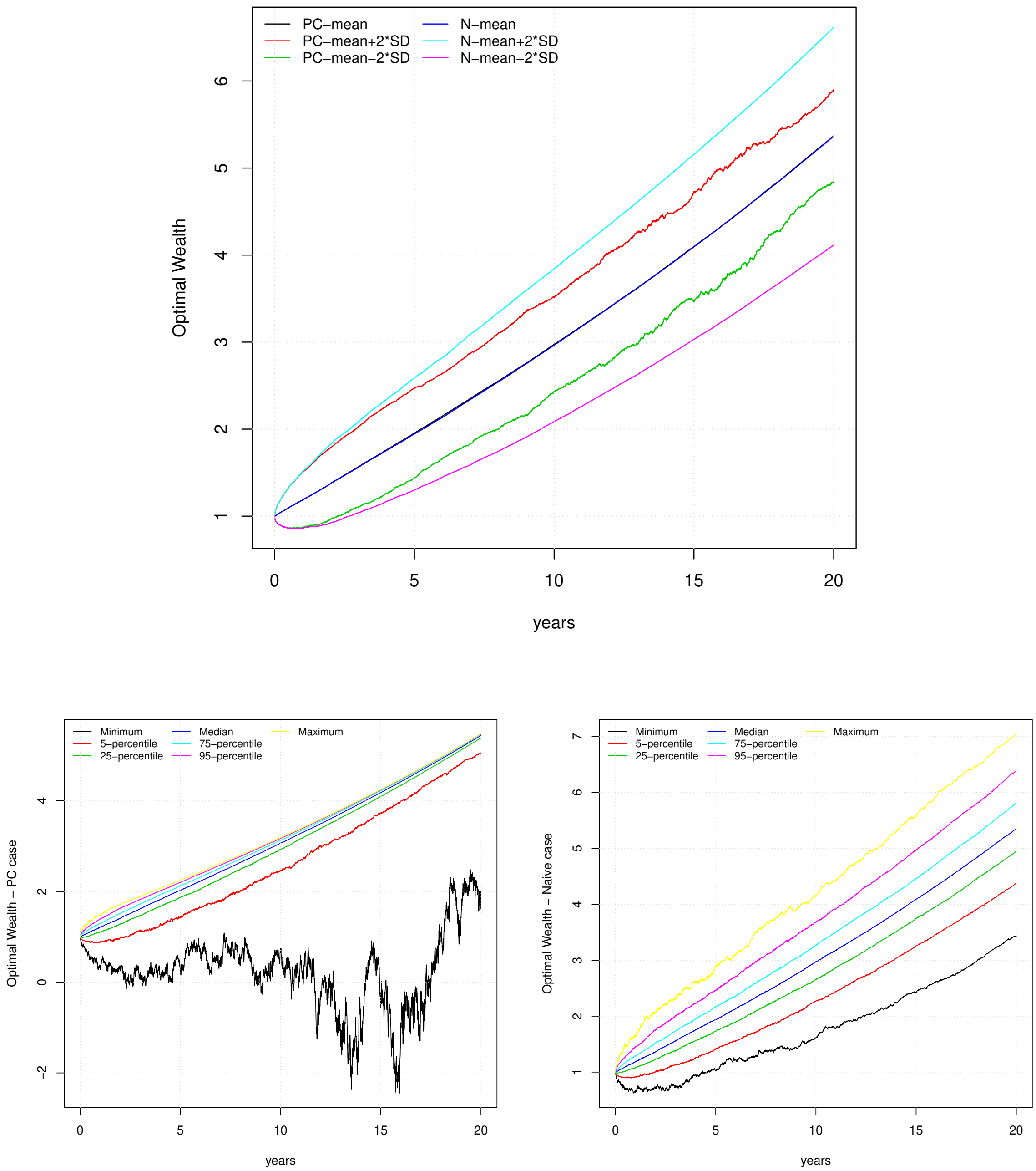

Figure 1: Precommitment (PC) and naive (N) wealths. Top graph: mean and mean $\pm$ 2 stand deviation. Bottom-left graph: statistics of PC-wealth. Bottom-right graph: statistics of N-wealth.

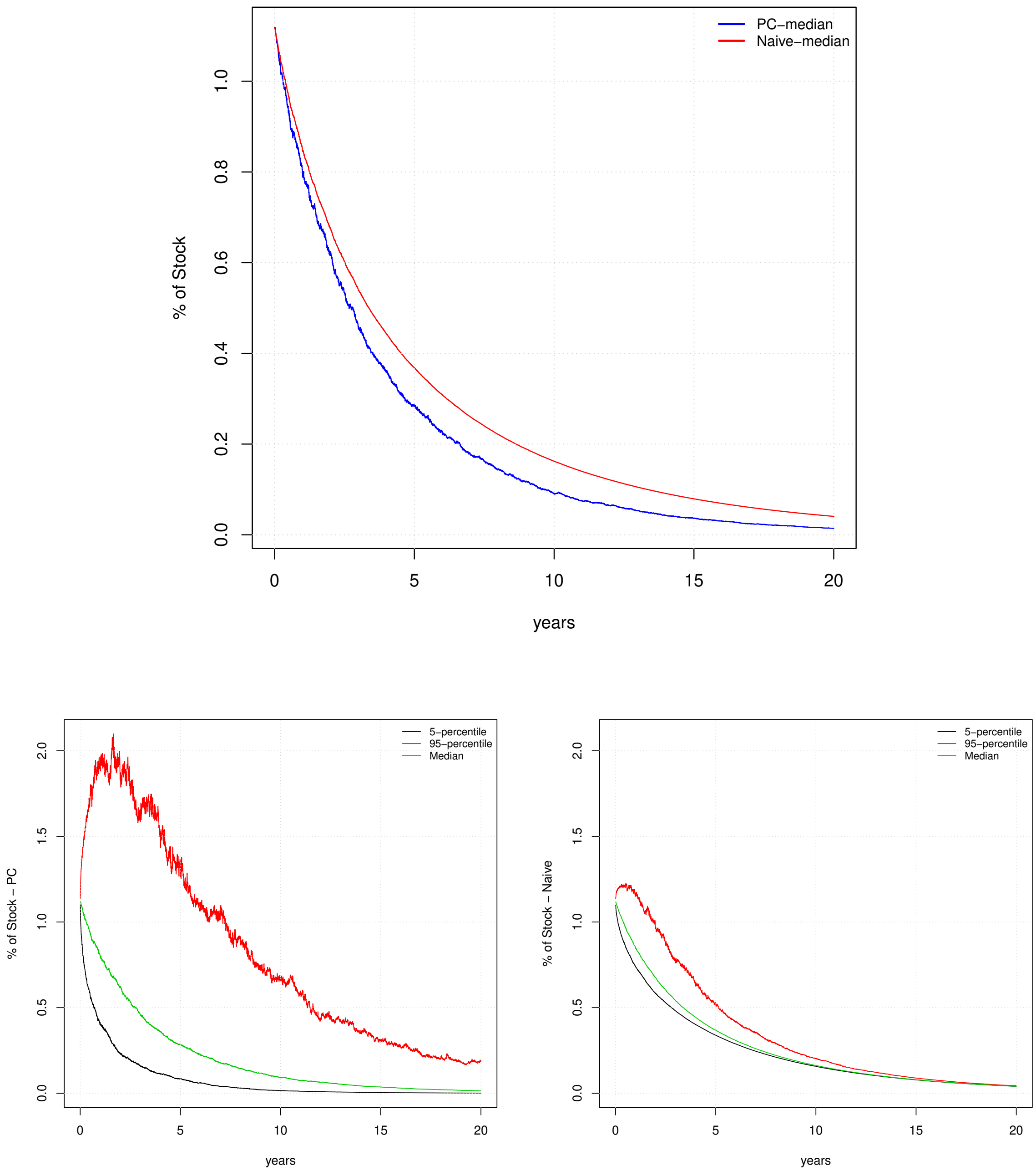

Figure 2: Precommitment (PC) and naive (N) strategies. Top graph: median of both PC and N. Bottom-left grah: statistics of PC-strategy. Bottom-right graph: statistics of N-strategy.

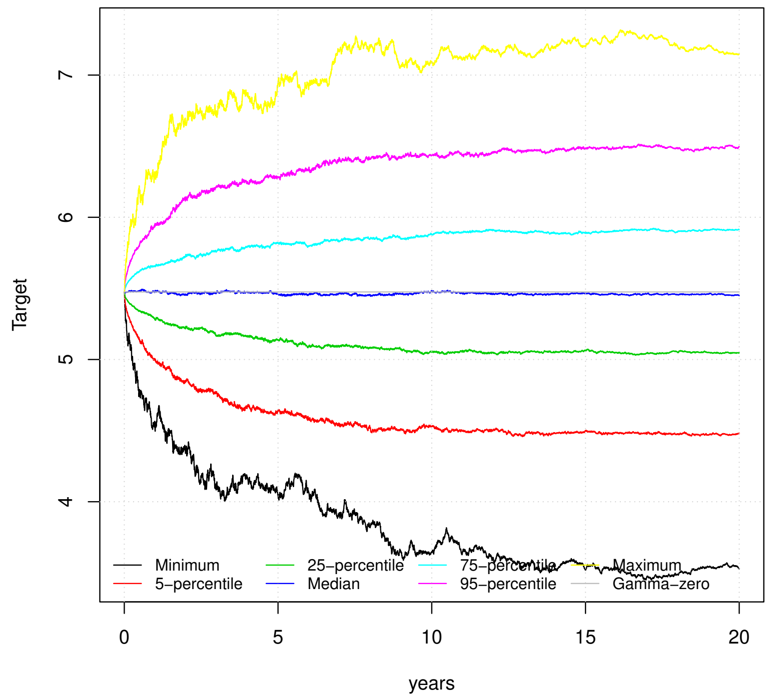

Figure 3: Statistics of time-varying targets for the naive approach, $\gamma_{t,x}$, and constant target for precommitment approach, γ_{t_0,x_0}

From Figures 1–3 we observe what follows.

- On average the wealth growth over time is exactly *the same* under the two approaches, that is consistent with Proposition 2.
- The standard deviation of wealth is *lower* with precommitment than with the naive approach.
- In the worst cases, the precommitment wealth behaves much *worse* than the naive wealth.
- On average, the optimal portfolio of the precommitment strategy contains less risky asset than the naive strategy.

- The amount of money invested in the risky asset according to the precommitment strategy is highly *more volatile* than in the naive case.

- As expected from Propositions 3, the time-varying targets are on average equal to the constant precommitment target. Furthermore, the time-varying targets are symmetrically distributed around the mean.

The fact that the variance of wealth is lower with the precommitment strategy is consistent with the theory: the precommitment strategy minimizes the variance of the final wealth given the same expected final wealth; therefore, any other strategy/portfolio that produces the same expected final wealth should produce a larger variance of final wealth.

Quite interestingly, however, we observe that in the worst cases, the precommitment strategy produces a much lower wealth than the naive strategy. This outcome is both interesting and unexpected, given that the precommitment strategy provides the lowest variance of final wealth. This only seeming contradiction is due to the fact that the naive strategy adjusts the final target at each time according to renovated circumstances. Instead, the precommitment strategy keeps the final target fixed over time. Indeed, when the market performance is bad, the fund is low; if the target remains fixed and too high compared to current wealth, the investment in the risky asset becomes important and this pushes the fund further down if the unfavorable market returns persist. If, instead, the target is regularly adjusted to current wealth and decreases when the fund falls down, the investment in stocks does not need to be so remarkable, and the potential loss from persisting bad market performance is reduced. Thus, the dynamically optimal strategy, which accounts for a time-varying target, seems to allow a more effective reaction against unfavorable and persistent market conditions, while the precommitment strategy does not.

The heavier investment in stocks of the precommitment strategy in bad scenarios is also confirmed by the higher volatility of the precommitment investment strategy with respect to the naive strategy, that instead turns out to be more stable.

8 Concluding remarks

In this work we have solved the mean-variance portfolio allocation problem for a defined contribution pension scheme in an arbitrage free and complete market driven by any number of stochastic state variables and listing any number of risky assets. We have solved the asset allocation problem with two common methods, the precommitment and the dynamically optimal naive approaches. To the best of our knowledge, this is the first paper that thoroughly investigates the dynamically optimal naive approach in the framework of a DC pension fund.

The precommitment method is equivalent to fix a given target at the initial time and keep it unchanged over time. The dynamically optimal method is based on this precommitment strategy, where we modify the target at each instant in time, as if we were solving a newly starting optimization problem.

From the theoretical point of view, we prove that in the Black and Scholes market the expected wealth under the two strategies is the same. Moreover, we prove that the expected value of the time-varying target of the dynamically optimal strategy coincides with the constant target of the precommitment strategy.

Merits and weaknesses of the two strategies are further investigated via numerical simulations, which show that: (i) the precommitment portfolio contains less risky asset than the naive strategy; (ii) the amount of money invested in the precommitment risky portfolio is highly more volatile than in the naive case; (iii) as expected, the variance of wealth is lower with the precommitment strategy than with the naive one; and (iv) although the precommitment wealth's variance is lower than the naive one, the naive strategy allows a more effective reaction to bad and persistent market scenarios because of the continuous adjustment of the final target.

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A Proof of Proposition (1)

Problem (14) can be recast as a static problem where the choice variable is the final wealth:

$$\begin{aligned} &\inf_{X(T)} \mathbb{E}_{t_0}\left[\tfrac{1}{2}\left(X\left(T\right)-\gamma\right)^2\right] \\ &\text{s.t.}\quad \mathbb{E}^{\mathbb{Q}}_{t_0}\left[-\int_{t_0}^{T} c\left(s,z\right)e^{-\int_{t_0}^{s} r(u,z)du}ds + X\left(T\right)e^{-\int_{t_0}^{T} r(u,z)du}\right] \leq x_0. \end{aligned} \tag{27}$$

The Lagrangian function of this problem is

$$\mathcal{L} = \mathbb{E}_{t_0}\left[\frac{1}{2}\left(X\left(T\right) - \gamma_{t_0,x_0}\right)^2 + \lambda X\left(T\right) e^{-\int_{t_0}^{T} r(u,z)du} m\left(t_0, T\right)\right]$$
$$- \lambda\left(x_0 + \int_{t_0}^{T} \mathbb{E}_{t_0}^{\mathbb{F}_s}\left[c\left(s, z\right)\right] B\left(t_0, s\right) ds\right),$$

where λ is the Lagrangian multiplier. The derivative of $\mathcal{L}$ with respect to $X\left(T\right)$ must be set to zero for each state of the world, i.e.

$$X^*\left(T\right) = \gamma_{t_0,x_0} - \lambda e^{-\int_{t_0}^{T} r(u,z)du} m\left(t_0, T\right). \tag{28}$$

Now, λ is computed from the constraint in (27) where $X^*\left(T\right)$ is substituted, and the inequality is replaced by the equality (since we want the solution to be compatible with the minimum amount of initial wealth):

$$\lambda = \frac{\gamma_{t_0,x_0} B\left(t_0, T\right) - \int_{t_0}^{T} \mathbb{E}_{t_0}^{\mathbb{F}_s}\left[c\left(s, z\right)\right] B\left(t_0, s\right) ds - x_0}{\mathbb{E}_{t_0}\left[e^{-2\int_{t_0}^{T} r(u)du} m^2\left(t_0, T\right)\right]}.$$

By defining the stochastic process $\Phi\left(t, T\right)$ as in (16), we can write

$$\lambda = \frac{\gamma_{t_0,x_0} B\left(t_0, T\right) - \int_{t_0}^{T} \mathbb{E}_{t_0}^{\mathbb{F}_s}\left[c\left(s, z\right)\right] B\left(t_0, s\right) ds - x_0}{\mathbb{E}_{t_0}\left[e^{-2\Phi(t_0,T)}\right]},$$

or

$$\lambda = \frac{B\left(t_0, T\right)}{\mathbb{E}_{t_0}\left[e^{-2\Phi(t_0,T)}\right]}\left(\gamma_{t_0,x_0} - \frac{x_0 + \int_{t_0}^{T} \mathbb{E}_{t_0}^{\mathbb{F}_s}\left[c\left(s, z\right)\right] B\left(t_0, s\right) ds}{B\left(t_0, T\right)}\right).$$

In the optimal solution, the constraint (27) must hold at any instant in time:

$$X^*\left(t\right) = -\int_{t}^{T} \mathbb{E}_{t}^{\mathbb{F}_s}\left[c\left(s, z\right)\right] B\left(t, s\right) ds + \mathbb{E}_t\left[X^*\left(T\right) e^{-\int_{t}^{T} r(u,z)du} m\left(t, T\right)\right].$$

If the optimal final wealth (28) is plugged into this equation we have:

$$X^*(t) = -\int_t^T \mathbb{E}_t^{\mathbb{F}_s}\left[c(s,z)\right] B(t,s)\, ds + \gamma_{t_0,x_0} B(t,T) - \lambda m(t_0,t)\, e^{-\int_{t_0}^t r(u,z)du} \mathbb{E}_t\left[e^{2\Phi(t,T)}\right]. \tag{29}$$

Now, the passages are as follows:

1. $dX^*(t)$ is found through Itō's lemma on (29) (differentiating w.r.t. $m(t_0,t)$ and $z(t)$);
2. $\lambda m(t_0,t)\, e^{-\int_{t_0}^t r(u,z)du} \mathbb{E}_t\left[e^{2\Phi(t,T)}\right]$ is substituted into the diffusion term of $dX^*(t)$ from (29);
3. this diffusion term is set equal to the diffusion term of investor's equation in (7) in order to find the portfolio which replicates the optimal wealth. Such a portfolio is given by (20).

B Proof of Proposition 2

If we plug the precommitment strategy (22) in the wealth (8)we get the following dynamics for the precommitment wealth $\hat{X}(t)$

$$d\hat{X}(t) = \left\{\hat{X}(t)\, r + c + \xi^2\left[x_0 e^{r(t-t_0)} + \frac{c}{r}\left(e^{r(t-t_0)} - 1\right) - \hat{X}(t) + \frac{1}{2\alpha} e^{\xi^2(T-t_0) - r(T-t)}\right]\right\} dt + (...)\, dW(t). \tag{30}$$

By taking the expectation at time t_0 given the wealth x_0, we get the following linear ordinary differential equation (ODE) for $\mathbb{E}_{t_0}\left[\hat{X}(t)\right]$:

$$\frac{d\mathbb{E}_{t_0}\left[\hat{X}(t)\right]}{dt} = \left(r - \xi^2\right)\mathbb{E}_{t_0}\left[\hat{X}(t)\right] + b(t), \tag{31}$$

where we have swapped the expected value and the derivative operators, and in which

$$b(t) = c + \xi^2 x_0 e^{r(t-t_0)} + \xi^2 \frac{c}{r}\left(e^{r(t-t_0)} - 1\right) + \frac{\xi^2}{2\alpha} e^{\xi^2(T-t_0) - r(T-t)},$$

with initial condition

$$\hat{X}\left(t_0\right) = x_0. \tag{32}$$

The solution of the linear ODE (31) with the initial condition (32) is the expectation of the precommitment wealth:

$$\mathbb{E}_{t_0}\left[\hat{X}\left(t\right)\right] = x_0 e^{r(t-t_0)} + \frac{c}{r}\left(e^{r(t-t_0)} - 1\right) + \frac{1}{2\alpha}e^{-r(T-t)}\left(e^{\xi^2(T-t_0)} - e^{\xi^2(T-t)}\right). \tag{33}$$

By following the same procedure for the dynamically optimal wealth (plugging (24) into (8), and taking expectation), we get the following ODE for the expected dynamically optimal wealth

$$\frac{d\mathbb{E}_{t_0}\left[\tilde{X}\left(t\right)\right]}{dt} = r\mathbb{E}_{t_0}\left[\tilde{X}\left(t\right)\right] + c + \xi^2\frac{1}{2\alpha}e^{\left(\xi^2-r\right)(T-t)}, \tag{34}$$

with the same initial condition (32). By solving (34) with initial condition (32), we get the following expectation of the dynamically optimal naive wealth:

$$\mathbb{E}_{t_0}\left[\tilde{X}\left(t\right)\right] = x_0 e^{r(t-t_0)} + \frac{c}{r}\left(e^{r(t-t_0)} - 1\right) + \frac{1}{2\alpha}e^{-r(T-t)}\left(e^{\xi^2(T-t_0)} - e^{\xi^2(T-t)}\right), \tag{35}$$

and comparing (33) with (35), we get the claim (25).

C Proof of Proposition 3

The analogue of γ_{t_0,x_0} (19) at time t with wealth x is

$$\gamma_{t,x} = \frac{1}{2\alpha}e^{\xi^2(T-t)} + xe^{r(T-t)} + \frac{c}{r}\left(e^{r(T-t)} - 1\right).$$

Therefore

$$\mathbb{E}_{t_0}\left[\gamma_{t,\tilde{X}(t)}\right] = \frac{1}{2\alpha}e^{\xi^2(T-t)} + \frac{c}{r}\left(e^{r(T-t)} - 1\right) + \mathbb{E}_{t_0}\left[\tilde{X}\left(t\right)\right]e^{r(T-t)}. \tag{36}$$

By plugging (35) into (36), and recalling (19), we get

$$\mathbb{E}_{t_0}\left[\gamma_{t,\tilde{X}(t)}\right] = x_0 e^{r(T-t_0)} + \frac{c}{r}\left(e^{r(T-t_0)} - 1\right) + \frac{1}{2\alpha}e^{\xi^2(T-t_0)} = \gamma_{t_0,x_0}, \tag{37}$$

that is claim (26).