Keeping the Listener Engaged: a Dynamic Model of Bayesian Persuasion*

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March 17, 2020

Abstract

We consider a dynamic model of Bayesian persuasion. Over time, a sender performs a series of experiments to persuade a receiver to take a desired action. Due to constraints on the information flow, the sender must take real time to persuade, and the receiver may stop listening and take a final action at any time. In addition, persuasion is costly for both players. To incentivize the receiver to listen, the sender must leave rents that compensate his listening costs, but neither player can commit to her/his future actions. Persuasion may totally collapse in Markov perfect equilibrium (MPE) of this game. However, for persuasion costs sufficiently small, a version of a folk theorem holds: outcomes that approximate Kamenica and Gentzkow (2011)'s sender-optimal persuasion as well as full revelation (which is most preferred by the receiver) and everything in between are obtained in MPE, as the cost vanishes.

Keywords: Bayesian persuasion, general Poisson experiments, Markov perfect equilibria, folk theorem.

JEL Classification Numbers: C72, C73, D83

*We thank Martin Cripps, Faruk Gul, Emir Kamenica, Stephan Lauermann, George Mailath, Meg Meyer, Sven Rady, Sönje Reiche, Nikita Roketskiy, Hamid Sabourian, Larry Samuelson, and audiences in various seminars and conferences for helpful comments and discussions.

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1 Introduction

Persuasion is a quintessential form of communication in which one individual (the sender) pitches an idea, a product, a political candidate, or a point of view, to another individual (the receiver). Whether the receiver ultimately accepts that pitch—or is "persuaded"—depends on the underlying truth (the state of world) but also importantly on the information the sender manages to communicate. In remarkable elegance and generality, Kamenica and Gentzkow (2011) show how the sender should communicate information in such a setting, when she can perform *any* (Blackwell) experiment *instantaneously*, *without any cost incurred* by her or by the receiver. This frictionlessness gives full commitment power to the sender, as she can publicly choose any experiment and reveal its outcome, all before the receiver can act.

In practice, however, persuasion is rarely frictionless. Imagine a salesperson pitching a product to a potential buyer. The buyer has some interest in buying the product but requires some evidence that it matches his needs. To convince the buyer, the salesperson might demonstrate certain features of the product or marshal customer testimonies or sales records, any of which takes real time and effort. Likewise, to process information, the buyer must pay attention, which is costly.

In this paper, we study the implications of these realistic frictions. Importantly, with the friction that real information takes time to generate, the sender no longer automatically enjoys full commitment power. Plainly, she cannot promise to the receiver what experiments she will perform in the future, so her commitment power is reduced to her current "flow" experiment. Given the lack of commitment by the sender, the receiver may stop listening and take an action at any time if he does not believe that the sender's future experiments are worth waiting for; the buyer in the example above may walk away at any time when he becomes pessimistic about the product or about the prospect of the salesperson eventually persuading him. We will examine *how well* and *in what manner* the sender can persuade the receiver in this limited commitment environment. As will become clear, the key challenge facing the sender is to instill the belief that she will be worth listening to, namely, to *keep the receiver engaged.*

We develop a dynamic version of the canonical persuasion model: the *state* is binary, L or R, and the receiver can take a binary action, ℓ or r. The receiver prefers to match the state, by taking action ℓ in state L and r in state R, while the sender prefers the receiver to choose r regardless of the state. Time is continuous. At each point in time, the sender may perform some "flow" experiment (unless the game has ended). In response, the receiver may take an action and end the game, or he could wait, in which case the game continues. Both the sender's choice of experiment and its outcome are publicly observable. Therefore, the two players always keep the same belief about the state.

For information generated by the sender, we consider a rich class of Poisson experiments. Specifically, we assume that at each instant in time the sender can generate a collection of Poisson signals that arrive at rates depending on their accuracy. The possible signals are flexible in their *directionalities*: a signal can be either good-news (inducing a higher posterior belief than the current belief), or bad-news (inducing a lower posterior than the current belief), and the news can be of arbitrary *accuracy*: the sender can choose any target posterior, although more accurate news (with targets closer to either 0 or 1) takes longer to arrive. Our model generalizes the existing Poisson models in the literature which considered either a good-news or bad-news Poisson experiment of a given accuracy (see, e.g., Keller, Rady, and Cripps, 2005; Keller and Rady, 2015; Che and Mierendorff, 2019).

Any real experiment, regardless of its accuracy, requires a fixed cost $c > 0$ per unit time for the sender to perform and for the receiver to process (or to comprehend). Our model of information allows for the flexibility and richness of Kamenica and Gentzkow (2011), but adds the friction that information takes time to generate. This serves to isolate the effect of the friction we introduce.

We study **Markov perfect equilibria (MPE)** of this game—namely, subgame perfect equilibrium strategy profiles prescribing the flow experiment chosen by the sender and the action (ℓ, r, "wait") taken by the receiver as a function of the belief p that the state is R. We are particularly interested in the equilibrium outcomes when the frictions are sufficiently small (i.e., in the limit as the flow cost c converges to zero). In addition, we investigate the persuasion dynamics or the "type of pitch" the sender uses to persuade the receiver in equilibrium of our game.

Is persuasion possible? If so, to what extent? How well the sender can persuade depends on, among other things, whether the receiver finds her worth listening to, or more precisely on his belief about the sender providing enough information to justify his listening costs. That belief depends on the sender's future experimentation strategy, which in turn rests on what the receiver will do if the sender betrays her trust and reneges on her information provision. The multitude of ways in which the players can coordinate on these choices yield a version of a folk theorem. There is an MPE in which no persuasion occurs. However, we also obtain a set of MPEs that range from ones that approximate Kamenica and Gentzkow (2011)'s sender-optimal persuasion to ones that approximate full revelation, and covers everything in between, when the cost c becomes arbitrarily small.

In the "persuasion failure" equilibrium, the receiver is pessimistic about the sender generating sufficient information, so he simply takes an action immediately without waiting for information. Up against that pessimism, the sender becomes desperate and maximizes her chance of once-and-for-all persuasion, which turns out to be the sort of strategy that the receiver fears the sender would employ, justifying her pessimism.

In a "persuasion" equilibrium, the receiver expects the sender to deliver sufficient information that would compensate his listening costs. This optimism in turn motivates the sender to deliver on her "promise" of informative experimentation; if she reneges on her experimentation, the ever optimistic receiver would simply wait for resumption of experimentation an instant later, instead of taking the action that the sender would hope she takes. In short, the receiver's optimism begets the sender's generosity in information provision, which in turn justifies that optimism. As will be shown, an equilibrium of this "virtuous cycle" of beliefs can support outcomes that approximate KG's optimal persuasion, full-revelation and anything in between, as the flow cost c tends to 0.

Persuasion dynamics. Our model informs us what kind of pitch the sender should make at each point in time, how long it takes for the sender to persuade, if ever, and how long the receiver listens to the sender before taking his action. The dynamics of the persuasion strategy adopted in equilibrium unpacks rich behavioral implications that are absent in the static persuasion model.

In our MPEs, the sender optimally makes use of the following three strategies: (i) *confidence-building*, (ii) *confidence-spending*, and (iii) *confidence-preserving*. The *confidence-building* strategy involves a bad-news Poisson experiment that induces the receiver's belief (that the state is R) to either drift upward or jump to zero. This strategy triggers upward movement of the belief when the state is R, but quite likely even when it is L; in fact, it minimizes the probability of bad-news, by insisting that the news be conclusive. In this sense, the sender can be seen as "R-biasing" or "overselling" the desired action. The sender finds it optimal to use this strategy when the receiver's belief is already close to the target belief that would lead the receiver to choose r.

The *confidence-spending* strategy involves a good-news Poisson experiment that generates an upward jump to some target belief, either one inducing the receiver to choose r, or at least one inducing him to listen to the sender. Such a jump arises rarely, however, and absent that jump, the receiver's belief drifts downward. In that sense, this strategy is a risky one that "spends" the receiver's confidence over time. This strategy is used in general when the receiver is already quite pessimistic about R, so that either the confidence-building strategy would take too long or the receiver would simply not listen. In particular, it is used as a "last ditch" effort, when the sender is close to giving up on persuasion or when the receiver is about to choose ℓ.

The *confidence-preserving* strategy combines the above two strategies—namely, a good-news Poisson experiment inducing the belief to jump to a persuasion target, and a bad-news Poisson experiment inducing the belief to jump to zero. This strategy is effective if the receiver is sufficiently skeptical relative to the persuasion target (i.e., the belief that will trigger him to choose r) so that the confidence-building strategy will take too long. Confidence spending could accomplish persuasion fast and thus can be used for a range of beliefs, but the sender would be running down the receiver's confidence in the process.

Hence, at some point the sender finds it optimal to switch to the confidence-preserving strategy, which prevents the receiver's belief from deteriorating further. Technically, the belief where the sender switches to this strategy constitutes an absorbing point of the belief dynamics; from then on, the belief does not move, unless either a sudden persuasion breakthrough or persuasion breakdown occurs.

The equilibrium strategy of the sender combines these three strategies in different ways under different economic conditions, thereby exhibiting rich and novel persuasion dynamics. Our equilibrium characterization in Section 5 describes precisely how the sender does this.

Related literature. This paper relates to several strands of literature. First, it contributes to the Bayesian persuasion literature that began with Kamenica and Gentzkow (2011) and Aumann and Maschler (1995), by studying the problem in a dynamic environment. Several recent papers also consider dynamic models (e.g., Brocas and Carrillo, 2007; Kremer, Mansour, and Perry, 2014; Au, 2015; Ely, 2017; Renault, Solan, and Vieille, 2017; Bizzotto, Rudiger, and Vigier, 2018; Che and Hörner, 2018; Henry and Ottaviani, 2019; Ely and Szydlowski, 2020; Orlov, Skrzypacz, and Zryumov, forthcoming). In most of these papers, there are no restrictions on the set of feasible experiments, and full commitment is assumed outright. When there are restrictions on the set of feasible experiments (e.g., Brocas and Carrillo, 2007; Henry and Ottaviani, 2019), the receiver cannot choose to stop listening. Hence, our central issues—namely, a lack of commitment by the sender to persuade and by the receiver to listen—do not arise in those papers.[1]

Second, the receiver's problem in our paper involves a stopping problem, which has been studied extensively in the single agent context, beginning with Wald (1947) and Arrow, Blackwell, and Girshick (1949). In particular, Nikandrova and Pancs (2018), Che and Mierendorff (2019) and Mayskaya (2016) study an agent's stopping problem when she acquires information through Poisson experiments.[2] Che and Mierendorff (2019) introduced the general class of Poisson experiments adopted in this paper. However, the generality is irrelevant in their model, because the decision-maker optimally chooses only between two conclusive experiments (i.e., never chooses a non-conclusive experiment).

Finally, the current paper is closely related to repeated/dynamic communication models. Margaria and Smolin (2018), Best and Quigley (2017), and Mathevet, Pearce, and

[1] Henry and Ottaviani (2019) consider a version of non-commitment problem but one in which the receiver has a stronger commitment power than in our model: the receiver in their model (e.g., a drug approver) can effectively force the sender (e.g., a drug company) to experiment by not approving the sender's application (e.g., for a new drug). The sender's desire for the receiver to "wait" arises in Orlov, Skrzypacz, and Zryumov (forthcoming), but "waiting" is a payoff-relevant action in their context of exercising a real option; that is, it is desired in its own merit and not as a means for persuasion as in the current paper.

[2] The Wald stopping problem has also been studied with drift-diffusion learning (e.g., Moscarini and Smith, 2001; Ke and Villas-Boas, 2016; Fudenberg, Strack, and Strzalecki, 2017), and in a model that allows for general endogenous experimentation (see Zhong, 2019).

Stachetti (2019) study repeated cheap-talk communication, and some of them establish versions of folk theorems. Their models consider repeated actions by receiver(s), serially independent states and feedbacks on the veracity of the sender's communication, based on which non-Markovian punishment can be levied to support a cooperative outcome. By contrast, the current model considers a fixed state, once-and-for-all action by the receiver (and hence no feedback), and Markov perfect equilibria.

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 3 illustrates the main ideas of our equilibria. Section 4 states our folk theorem. Section 5 explores the dynamics of our MPE strategies and their implications. Section 6 concludes.

2 Model

We consider a game in which a *Sender* ("she") wishes to persuade a *Receiver* ("he"). There is an unknown state ω which can be either L ("left") or R ("right"). The receiver ultimately takes a binary action ℓ or r, which yields the following payoffs for the sender and the receiver:

Payoffs for the sender and the receiver

state/actions	ℓ	r
L	$(0, u_\ell^L)$	(v, u_r^L)
R	$(0, u_\ell^R)$	(v, u_r^R)

The receiver gets u_a^ω if he takes action $a \in \{\ell, r\}$ when the state is $\omega \in \{L, R\}$. The sender's payoff depends only on the receiver's action: she gets $v > 0$ if the receiver takes r and zero otherwise. We assume $u_\ell^L > \max\{u_r^L, 0\}$ and $u_r^R > \max\{u_\ell^R, 0\}$, so that the receiver prefers to match the action with the state, and also $v > 0$, so that the sender prefers action r to action ℓ. Notice that this payoff structure corresponds to the leading examples considered by Kamenica and Gentzkow (2011) (KG, hereafter) and Bergemann and Morris (2019), where a prosecutor seeks to persuade a judge to convict a defendant, or a regulator seeks to dissuade a depositor from running on a bank. Both players begin with a common prior p_0 that the state is R, and use Bayes rule to update their beliefs.

KG Benchmark. By now, it is well understood how the sender may optimally persuade the receiver if she can commit to an experiment without any restrictions. For each $a \in \{\ell, r\}$, let $U_a(p)$ denote the receiver's expected payoff when he takes action a with belief p. In addition, let $\hat{p}$ denote the belief at which the receiver is indifferent between actions ℓ and r, that is, $U_\ell(\hat{p}) = U_r(\hat{p})$.[3]

[3] Specifically, for each $p \in [0,1]$, $U_\ell(p) := pu_\ell^R + (1-p)u_\ell^L$ and $U_r(p) := pu_r^R + (1-p)u_r^L$. Therefore, $\hat{p} = \left(u_\ell^L - u_r^L\right) / \left(u_r^R - u_\ell^R + u_\ell^L - u_r^L\right)$, which is well-defined in $(0,1)$ under our assumptions on the receiver's payoffs.

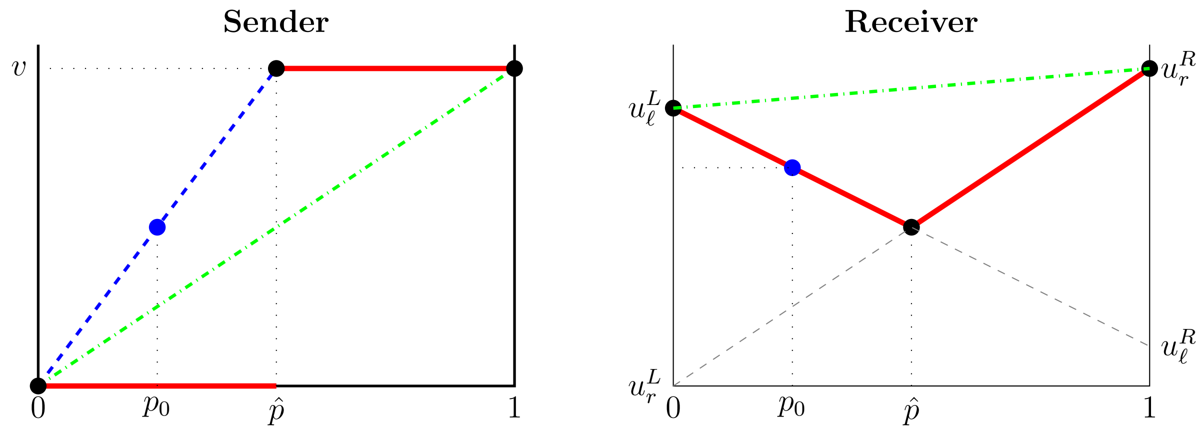


Figure 1: Payoffs from static persuasion. Solid red curves: payoffs without persuasion (information). Dashed blue curve: the sender's expected payoff in the KG solution. Blue dots: payoffs in the KG solution at prior p_0. Dash-dotted green curves: payoffs under a fully revealing experiment.

If the sender provides no information, then the receiver takes action r if and only if $p_0 \geq \hat{p}$. Therefore, persuasion is necessary only when $p_0 < \hat{p}$. In this case, the KG solution prescribes an experiment that induces only two posterior beliefs, $q_- = 0$ and $q_+ = \hat{p}$. The former leads to action ℓ, while the latter results in action r. This experiment is optimal for the sender, because $\hat{p}$ is the minimum belief necessary to trigger action r, and setting $q_- = 0$ maximizes the probability of generating $\hat{p}$, and thus action r. The resulting payoff for the sender is $p_0 v/\hat{p}$, as given by the dashed blue line in the left panel of Figure 1. The flip side is that the receiver enjoys no rents from persuasion; his payoff is $\mathcal{U}(p) := \max\{U_\ell(p), U_r(p)\}$, the same as if no information were provided, as depicted in the right panel of Figure 1.

Dynamic model. We consider a dynamic version of the above Bayesian persuasion problem. Time flows continuously starting from 0. Unless the game has ended, at each point in time $t \geq 0$, either the sender performs an informative experiment from a feasible set at the flow cost of $c > 0$, or she simply "passes," in which case she incurs no cost. Both the feasible experiments and the nature of flow costs will be made precise below. If the sender experiments, then the receiver also pays the same flow cost c and observes the experiment and its outcome. If the sender passes, then the receiver incurs no flow cost and obtains no information.[4] Then, the receiver decides whether to take an irreversible action $a \in \{\ell, r\}$ or to "wait." The former ends the game, while the latter lets the game continue to the next instant.

There are two notable modeling assumptions. First, the receiver can take a game-

[4]In this sense, flow cost c is interpreted as a "listening cost" rather than a waiting cost. This distinction does not matter in the continuous time game; our analysis below will not change even if the receiver incurs cost c, regardless of whether the sender passes or not. However, it is relevant in the discrete-time version of our model.

ending action at any point in time, that is, he is not restricted to listen to the sender. This is the fundamental difference from KG, in which the receiver is allowed to take an action only after the sender finishes her information provision. Second, the players' flow (information) costs are assumed to be the same. This, however, is just a normalization, which allows us to directly compare the players' payoffs, and all subsequent results can be reinterpreted as relative to each player's individual information cost.[5]

Feasible experiments. We endow the sender with a class of Poisson experiments. Specifically, at each point in time, the sender may expend one unit of a resource (attention) across different experiments that generate Poisson signals. The experiments are indexed by $i \in \mathbb{N}$.[6] Each Poisson experiment $i \in \mathbb{N}$ generates breakthrough news that moves the belief to a target posterior $q_i \in [0, 1]$ of the sender's choosing. The sender also chooses the share $\alpha_i \in [0, 1]$ of her resources allocated to experiment i, subject to the (budget) constraint that $\sum_{i=1}^{\infty} \alpha_i \leq 1$. We call a collection of experiments $(\alpha_i, q_i)_{i\in\mathbb{N}}$ an *information structure*.

Given an information structure $(\alpha_i, q_i)_{i\in\mathbb{N}}$, a Poisson jump to posterior to $q_i \neq p$ occurs at the arrival rate of[7]

$$\alpha_i \lambda \frac{p(1-q_i)}{|q_i - p|} \text{ if } \omega = L, \qquad \text{and} \qquad \alpha_i \lambda \frac{q_i(1-p)}{|q_i - p|} \text{ if } \omega = R.$$

The *unconditional* arrival rate is then given by

$$(1-p) \cdot \alpha_i \lambda \frac{p(1-q_i)}{|q_i - p|} + p \cdot \alpha_i \lambda \frac{q_i(1-p)}{|q_i - p|} = \alpha_i \lambda \frac{p(1-p)}{|q_i - p|}. \tag{1}$$

These arrival rates are micro-founded via a class of binary experiments in a discrete time model, as we show in Section 6.1. Further, they provide a natural generalization of the Poisson models considered in the existing literature.[8] To see this, suppose that the

[5]Suppose that the sender's cost is given by c_s, while that of the receiver is c_r. Such a model is equivalent to our normalized one in which $c'_r = c'_s = c_r$ and $v' = v(c_r/c_s)$. When solving the model for a fixed set of parameters $(u^{\omega}_a, v, c, \lambda)$, this normalization does not affect the results. If we let c tend to 0, we are implicitly assuming that the sender's and receiver's (unnormalized) costs, c_s and c_r, converge to zero at the same rate.

[6]One can extend this to an uncountable set of experiments. However, in our model, the sender never mixes over an infinite number of experiments, and thus such extra generality is unnecessary.

[7]For $q_i = p$ the experiment is uninformative and we set the arrival rate to zero in both states. This has the same effect on information as setting $\alpha_i = 0$.

[8]The class of feasible information structures is formulated in terms of the current belief p and jump-target beliefs q_i. We emphasize, however, that there is an underlying class of information structures that is independent of beliefs. Hence, which experiments are feasible does not depend on the current belief of the players. Section 6.1 makes this clear in a discrete time foundation, and Appendix A.2 states the class of feasible information structures in continuous time without reference to beliefs. This feature distinguishes our approach from the rational inattention model (Sims, 2003; Matejka and McKay, 2015), in which costs or constraints are based on a measure of information that quantifies the uncertainty in the posterior beliefs induced by an information structure (see also Frankel and Kamenica, forthcoming).

sender allocates the entire unit resource to one Poisson experiment with q. The jump to posterior q then occurs at the rate of $\lambda p(1-p)/|q-p|$. Conclusive R-evidence ($q=1$) is obtained at the rate of λp, as is assumed in "good" news models (see, e.g., Keller, Rady, and Cripps, 2005). Likewise, conclusive L-evidence ($q=0$) is obtained at the rate of $\lambda(1-p)$, as is assumed in "bad" news models (see, e.g., Keller and Rady, 2015). Our model allows for such conclusive news, but it also allows for arbitrary non-conclusive news with $q \in (0,1)$, as well as any arbitrary mixture among them. Further, our arrival rate assumption captures the intuitive idea that more accurate information takes longer to generate. For example, assuming $q > p$, the arrival rate increases as the news becomes less precise (q decreases), and it approaches infinity as the news becomes totally uninformative (i.e., in the limit as q tends to p). Lastly, limited arrival rates, together with the budget constraint $\sum_i \alpha_i \leq 1$, capture the important feature of our model that any meaningful persuasion takes time and requires delay.

For our purpose, it suffices to consider either informative information structures where the constraint is binding $\sum_i \alpha_i = 1$, or "passing" which corresponds to $\alpha_i = 0$ for all $i \in \mathbb{N}$. If the sender uses an informative information structure, both players incur a flow cost of c. If the sender passes, neither player incurs any flow cost.[9]

If no Poisson jump arrives when the sender uses the information structure $(\alpha_i, q_i)_{i\in\mathbb{N}}$, the belief drifts according to the following law of motion:[10]

$$\dot{p} = -\left(\sum_{i:q_i>p} \alpha_i - \sum_{i:q_i<p} \alpha_i\right) \lambda p(1-p). \tag{2}$$

Note that the drift rate depends only on the difference between the fractions of resources allocated to "right" versus "left" Poisson signals. In particular, the rate does not depend on the precision q_i of the news in the individual experiments. The reason is that the precision of news and its arrival rate offset each other, leaving the drift rate unaffected. This feature makes the analysis tractable while at the same time generalizing conclusive Poisson models in an intuitive way.

Among many feasible experiments, the following three, visualized in Figure 2, will

[9] A more general formulation would allow for "partial passing" by assuming that each player incurs flow cost c per unit experiment that is *informative*. Specifically, given $(\alpha_i, q_i)_{i\in\mathbb{N}}$, each player incurs the flow cost of $(\sum_{i:q_i\neq p} \alpha_i)c$. In other words, positive costs are incurred only when a nontrivial experiment is performed and are proportional to the total share of such experiments. We do not explicitly model partial passing since it is never optimal for the sender.

[10] Since the belief is a martingale, we have

$$\sum_{i=1}^{\infty} q_i \alpha_i \lambda \frac{p_t(1-p_t)}{|q_i - p_t|} dt + \left(1 - \sum_{i=1}^{\infty} \alpha_i \lambda \frac{p_t(1-p_t)}{|q_i-p_t|} dt\right)(p_t + \dot{p}_t dt) = p_t$$

$$\iff \quad \left(1 - \sum_{i=1}^{\infty} \alpha_i \lambda \frac{p_t(1-p_t)}{|q_i-p_t|} dt\right) \dot{p}_t = \left(\sum_{i=1}^{\infty} \frac{p_t - q_i}{|q_i - p_t|} \alpha_i\right) \lambda p_t (1-p_t).$$

Letting $dt \to 0$, we obtain the updating formula.

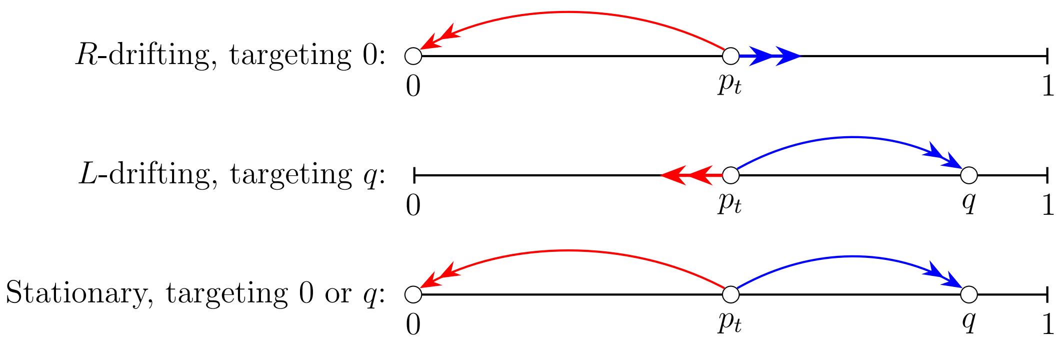


Figure 2: Three prominent feasible experiments.

prove particularly relevant for our purpose and will be frequently referred to. They formalize the three modes of persuasion discussed in the introduction:

- ***R*-drifting experiment** (confidence building): $\alpha_1 = 1$ with $q_1 = 0$. The sender devotes all resources to a Poisson experiment with the (posterior) jump target $q_1 = 0$. In the absence of a jump, the posterior drifts to the right, at rate $\dot{p} = \lambda p(1-p)$.
- ***L*-drifting experiment** (confidence spending): $\alpha_1 = 1$ with $q_1 = q$ for some $q > p$. The sender devotes all resources to a Poisson experiment with jumps targeting some posterior $q > p$. The precise jump target q will be specified in our equilibrium construction. In the absence of a jump, the posterior drifts to the left, at rate $\dot{p} = -\lambda p(1-p)$.
- **Stationary experiment** (confidence preserving): $\alpha_1 = \alpha_2 = 1/2$ with $q_1 = 0$ and $q_2 = q$ for some $q > p$. The sender assigns equal resources to an experiment targeting $q_1 = 0$ and one targeting $q_2 = q$. Absent jumps, the posterior remains unchanged.

Solution concept. We study (pure-strategy) Markov Perfect equilibria (MPE, hereafter) of this dynamic game in which both players' strategies depend only on the current belief p. Formally, a profile of Markov strategies specifies for each belief $p \in [0, 1]$, an information structure $(\alpha_i, q_i)_{i \in \mathbb{N}}$ chosen by the sender, and an action $a \in \{\ell, r, \text{wait}\}$ chosen by the receiver. An MPE is a strategy profile that, starting from any belief $p \in [0, 1]$, forms a subgame perfect equilibrium.[11] Naturally, this solution concept limits the use of (punishment) strategies depending solely on the payoff-irrelevant part of the histories, and serves to discipline the strategies off the equilibrium path.

We impose a restriction that captures the spirit of "perfection" in our continuous time framework. Suppose that at some p, the receiver would choose action ℓ immediately, unless a Poisson signal causes a discrete jump in beliefs. In continuous time, the latter event occurs with probability 0, and therefore, the sender's strategy at p is inconsequential

[11] There are well known technical issues in defining a game in continuous time (see Simon and Stinchcombe, 1989; Bergin and MacLeod, 1993). In Appendix A.1, we formally define admissible strategy profiles that guarantee a well defined outcome of the game and define Markov perfect equilibria.

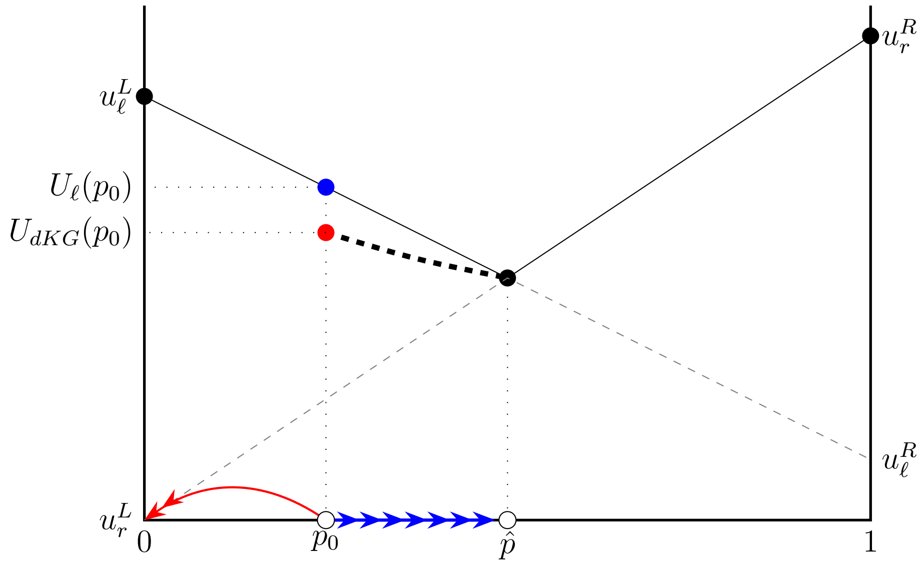


Figure 3: Replicating the KG outcome through R-drifting experiments.

for the players' payoffs. We require the sender to choose a strategy that maximizes her *flow payoff* in such a situation. This can be seen as selecting an MPE that is robust to a discrete-time approximation: in discrete time, a Poisson jump would occur with a positive probability, and thus the sender's strategy would have non-trivial payoff consequences. See Appendix A.1 for a formal definition.

3 Illustration: *Persuading the Receiver to Listen*

We begin by illustrating the key issue facing the sender: *persuading the receiver to listen.* To this end, consider any prior $p_0 < \hat{p}$, so that persuasion is not trivial, and suppose that the sender repeatedly chooses R-drifting experiments with jumps targeting $q = 0$ until the posterior either jumps to 0 or drifts to $\hat{p}$, as depicted on the horizontal axis in Figure 3. This strategy exactly replicates the KG solution (in the sense that it yields the same probabilities of reaching the two posteriors, 0 and $\hat{p}$, as the KG solution), provided that the receiver listens to the sender for a sufficiently long time.

But will the receiver wait until a target belief of $\hat{p}$ *or* 0 *is reached?* The answer is no. The KG experiment leaves no rents for the receiver even without listening costs, and thus listening will make the receiver strictly worse off compared with choosing ℓ immediately: in Figure 3, the receiver's expected gross payoff from the static KG experiment is $U_\ell(p_0)$. Due to listening costs, the receiver's expected payoff under the dynamic KG strategy, denoted here by $U_{dKG}(p_0)$, is strictly smaller than $U_\ell(p_0)$. In other words, the dynamic strategy implementing the KG solution cannot persuade the receiver to wait and listen, so it does not permit any persuasion.[12] Indeed, this problem leads to the existence of a

[12]The KG outcome can also be replicated by other dynamic strategies. For instance, the sender could

no-persuasion MPE, regardless of the listening cost $c > 0$.

Theorem 1 (Persuasion Failure)**.** *For any $c > 0$, there exists an MPE in which no persuasion occurs, that is, for any p_0, the receiver immediately takes either action ℓ or r.*

Proof. Consider the following strategy profile: *the receiver chooses ℓ for $p < \hat{p}$ and r for $p \geq \hat{p}$;* and *the sender chooses the L-drifting experiment with jump target $\hat{p}$ for all $p \in (\hat{\pi}_{\ell L}, \hat{p})$ and passes for all $p \notin (\hat{\pi}_{\ell L}, \hat{p})$*, where the cutoff $\hat{\pi}_{\ell L}$ is the belief at which the sender is indifferent between the L-drifting experiment and stopping (so that the receiver chooses ℓ).[13]

In order to show that this strategy profile is indeed an equilibrium, first consider the receiver's incentives given the sender's strategy. If $p \notin (\hat{\pi}_{\ell L}, \hat{p})$, then the sender never provides information, so the receiver has no incentive to wait, and will take an action immediately. If $p \in (\hat{\pi}_{\ell L}, \hat{p})$, then the sender never moves the belief into the region where the receiver *strictly* prefers to take action r (i.e., strictly above $\hat{p}$). This implies that the receiver's expected payoff is equal to $U_\ell(p_0)$ minus any listening cost she may incur. Therefore, again, it is optimal for the receiver to take an action immediately.

Now consider the sender's incentives given the receiver's strategy. If $p \geq \hat{p}$, then it is trivially optimal for the sender to pass. Now suppose that $p < \hat{p}$. Our refinement, discussed at the end of Section 2, requires that the sender choose an information structure that maximizes her flow payoff, which is given by[14]

$$\max_{(\alpha_i, q_i)_{i\in\mathbb{N}}} \sum_{q_i \neq p} \alpha_i \lambda \frac{p(1-p)}{|q_i - p|} \mathbf{1}_{\{q_i \geq \hat{p}\}} v - c \text{ subject to } \sum_{\alpha_i} \alpha_i = 1.$$

If the sender chooses any nontrivial experiment, its jump target must be $q_i = \hat{p}$. Hence the optimal information structure is either $(\alpha_1 = 1, q_1 = \hat{p})$ or $\alpha_i = 0$ for all i. The former is optimal if and only if $\frac{\lambda p(1-p)}{\hat{p}-p} v \geq c$, or equivalently $p \geq \hat{\pi}_{\ell L}$. ☐

The no-persuasion equilibrium constructed in the proof showcases a total collapse of trust between the two players. The receiver does not trust the sender to convey valuable information (e.g., an experiment targeting $q > \hat{p}$), so she refuses to listen to her. This attitude makes the sender desperate for a quick breakthrough; she tries to achieve

repeatedly choose a stationary strategy with jumps targeting $q_1 = \hat{p}$ and $q_2 = 0$ until either jump occurs. However, this (and in fact, any other) strategy would not incentivize the receiver to listen, for the same reason as in the case of repeating R-drifting experiments.

[13]Specifically, $\hat{\pi}_{\ell L}$ satisfies

$$c = \frac{\lambda \hat{\pi}_{\ell L}(1 - \hat{\pi}_{\ell L})}{\hat{p} - \hat{\pi}_{\ell L}} v \iff \hat{\pi}_{\ell L} = \frac{1}{2} + \frac{c}{2\lambda v} - \sqrt{\left(\frac{1}{2} + \frac{c}{2\lambda v}\right)^2 - \frac{c\hat{p}}{\lambda v}}.$$

[14]The equation follows from the fact that under the given strategy profile, the sender's value function is $V(p) = v$ if $p \geq \hat{p}$ and $V(p) = 0$ otherwise; and when the target posterior is q_i, a Poisson jump occurs at rate $\lambda p(1-p)/|q_i - p|$.

persuasion targeting just $\hat{p}$, which is indeed not enough for the receiver to be willing to wait.

Can trust be restored? In other words, can the sender ever persuade the receiver to listen to her? She certainly can, if she can commit to a dynamic strategy, that is, if she can credibly promise to provide more information in the future. Consider the following modification of the dynamic KG experiment discussed above: the sender repeatedly chooses R-drifting experiments with jumps targeting zero, until either the jump occurs or the belief reaches $p^* > \hat{p}$. If the receiver waits until p either jumps to 0 (in which case she takes action ℓ) or reaches p^* (in which case she takes action r), then her expected payoff is equal to[15]

$$U_R(p_0) = \frac{p^* - p_0}{p^*} u_\ell^L + \frac{p_0}{p^*} U_r(p^*) - \left(p_0 \log\left(\frac{p^*}{1-p^*} \frac{1-p_0}{p_0} \right) + 1 - \frac{p_0}{p^*} \right) \frac{c}{\lambda}.$$

Importantly, if p^* is sufficiently larger than $\hat{p}$ (and c is sufficiently small), then $U_R(p)$ (the dotted curve in Figure 4) stays above $\max\{U_\ell(p), U_r(p)\}$ (the black kinked curve) while p drifts toward p^*, so the receiver prefers to wait. Intuitively, unlike in the KG solution, this "more generous" persuasion scheme promises the receiver enough rents that make it worth listening to.

If c is sufficiently small, the required belief target p^* need not exceed $\hat{p}$ by much. In fact, p^* can be chosen to converge to $\hat{p}$ as $c \to 0$. In this fashion, a dynamic persuasion strategy can be constructed to virtually implement the KG solution when c is sufficiently small.

At first glance, this strategy seems unlikely to work without the sender's commitment power. *How can she credibly continue her experiment even after the posterior has risen past $\hat{p}$? Why not simply stop at the posterior $\hat{p}$—the belief that should have convinced the receiver to choose r?* Surprisingly, however, the strategy works even without commitment. The reason lies with the fact that the equilibrium beliefs generated by the Markov strategies themselves can provide a sufficient incentive for the sender to go above $\hat{p}$. We already argued that, with a suitably chosen $p^* > \hat{p}$, the receiver is incentivized to wait past $\hat{p}$, due to the "optimistic" equilibrium belief that the sender will continue to experiment

[15] To understand this explicit solution, first notice that under the prescribed strategy profile, the receiver takes action ℓ when p jumps to 0, which occurs with probability $(p^* - p_0)/p^*$, and action r when p reaches to p^*, which occurs with probability p_0/p^*. The last term captures the total expected listening cost. The length of time τ it takes for p to reach p^* absent jumps is derived as follows:

$$p^* = \frac{p_0}{p_0 + (1-p_0)e^{-\lambda\tau}} \Leftrightarrow \tau = \frac{1}{\lambda} \log\left(\frac{p^*}{1-p^*} \frac{1-p_0}{p_0} \right).$$

Hence, the total listening cost is equal to

$$(1-p_0) \int_0^\tau ct d\left(1 - e^{-\lambda t}\right) + \left(p_0 + (1-p_0)e^{-\lambda\tau}\right) c\tau = \left(p_0 \log\left(\frac{p^*}{1-p^*} \frac{1-p_0}{p_0} \right) + 1 - \frac{p_0}{p^*} \right) \frac{c}{\lambda}.$$

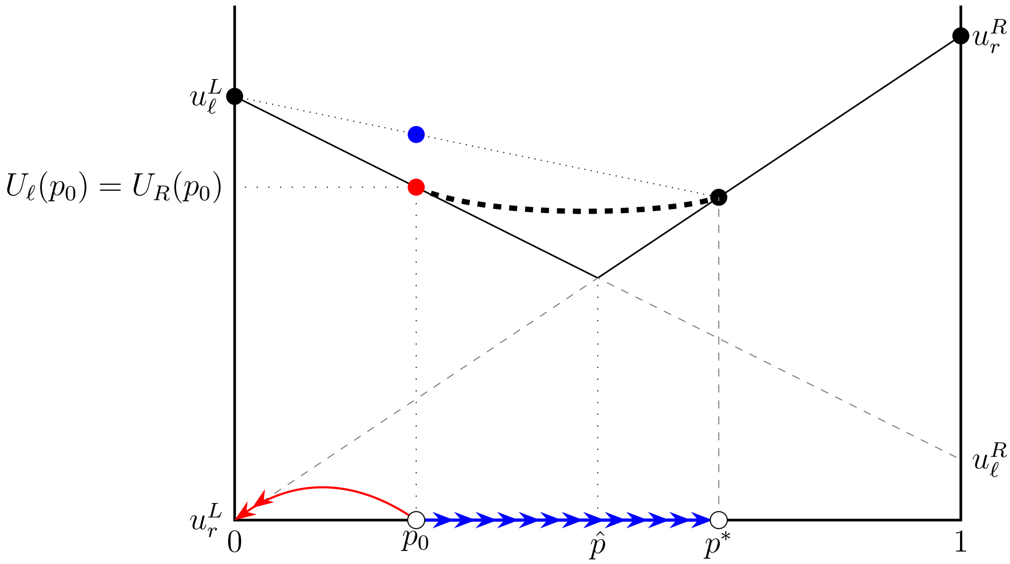


Figure 4: Persuasive R-drifting experiments

until a much higher belief p^* is reached. Crucially, this optimism in turn incentivizes the sender to carry out her strategy:[16] were she to deviate and, say, "pass" at $q = \hat{p}$, the receiver would simply wait (instead of choosing r), believing that the sender will shortly resume her R-drifting experiments after the "unexpected" pass.[17] Given this response, the sender cannot gain from deviating: she cannot convince the receiver to "prematurely" choose r. To sum up, the sender's strategy instills optimism in the receiver to wait and listen to the sender, and this optimism, or the *power of beliefs*, in turn incentivizes the sender to carry out the strategy.

4 Folk Theorem

The equilibrium logic outlined in the previous section applies not just to strategy profiles that approximate the KG solution, but also to other strategy profiles with a target belief $p^* \in (\hat{p}, 1)$. Building upon this observation, we establish a version of a folk theorem: *any payoff for the sender between the KG solution and her payoff from full revelation can be*

[16]We will show in Section 5.2 that under certain conditions, using R-drifting experiments is not just better than passing but also the optimal experiment (best response), given that the receiver waits. Here, we illustrate the possibility of persuasion for this case. The logic extends to other cases where the sender optimally uses different experiments to persuade the receiver.

[17]To be formal, suppose that the current belief is slightly below p^* (say, $p = p^* - \lambda dt$), and the sender deviates and passes. Given the belief that she will continue her R-drifting strategy in the next instant, the receiver prefers waiting to taking action r immediately if and only if

$$c \leq \lambda(1-p^*)(u_\ell^L - U_r(p^*)) + \lambda p^*(1-p^*)U_r'(p^*) \Leftrightarrow p^* \leq \overline{p} := \frac{(u_\ell^L - u_r^L)\,\lambda - c}{(u_\ell^L - u_r^L)\,\lambda}.$$

In other words, even if p is close to p^* (and strictly above $\hat{p}$), the receiver is willing to wait as long as p^* is not too high. Importantly, $\overline{p}$ approaches 1 as c tends to 0.

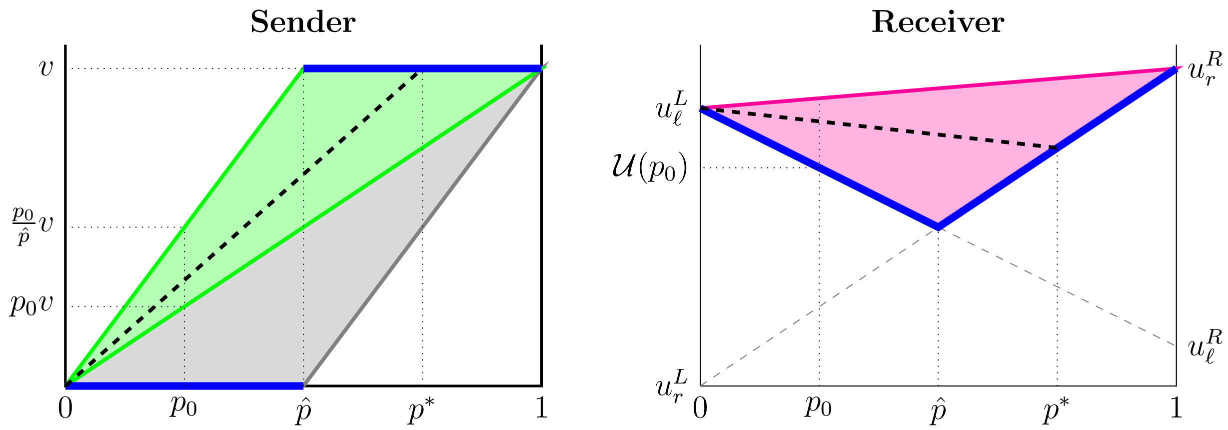


Figure 5: Implementable payoff set for each player at each p_0.

virtually supported as an MPE payoff.

Theorem 2 (Folk theorem)**.** *Fix any prior $p_0 \in (0,1)$. For any sender payoff $V \in (p_0 v, \min\{p_0/\hat{p}, 1\}v)$, if c is sufficiently small, there exists an MPE in which the sender obtains V; likewise, for any receiver payoff $U \in \big(\mathcal{U}(p_0), p_0 u_r^R + (1-p_0)u_\ell^L\big)$, if c is sufficiently small, there exists an MPE in which the receiver achieves U.*

Figure 5 depicts how the set of implementable payoffs for each player varies according to p_0 in the limit as c tends to 0. Theorem 2 shows that any payoffs in the green and red shaded areas can be implemented in an MPE, provided that c is sufficiently small. In the left panel, the upper bound for the sender's payoff is given by the KG-optimal payoff $\min\{p_0/\hat{p}, 1\}v$, and the lower bound of the green shaded area is given by the sender's payoff from full revelation $p_0 v$. For the receiver, by contrast, full revelation defines the upper bound $p_0 u_r^R + (1-p_0)u_\ell^L$, whereas the KG-payoff, which leaves no rent for the receiver, is given by $\mathcal{U}(p_0)$. In both panels, the thick blue lines correspond to the players' payoffs in the no-persuasion equilibria of Theorem 1.

Note that Theorem 2 is silent about payoffs in the gray shaded region. In the static KG environment, these payoffs can be achieved by the (sender-pessimal) experiment that splits the prior p into two posteriors, 1 and $q \in (0, \hat{p})$. The following theorem shows that the sender's payoffs in this region cannot be supported as an MPE payoff, for a sufficiently small $c > 0$.

Theorem 3. *If $p_0 < \hat{p}$, then the sender's MPE payoff is either equal to 0 or at least $p_0 v - 2c/\lambda$. If $p_0 \geq \hat{p}$, then the sender's MPE payoff cannot be smaller than $p_0 v - 2c/\lambda$.*

Proof. Fix $p_0 < \hat{p}$, and consider any MPE. If the receiver's strategy is to wait at p_0, then the sender can always adopt the stationary strategy with jump-targets 0 and 1, which will guarantee her the payoff of $p_0 v - 2c/\lambda$.[18] If the receiver's strategy is to stop at p_0,

[18]In order to understand this payoff, notice that the strategy fully reveals the state, and thus the sender gets v only in state R. In addition, in each state, a Poisson jump occurs at rate $\lambda/2$, and thus the expected waiting time equals $2/\lambda$, which is multiplied by c to obtain the expected cost.

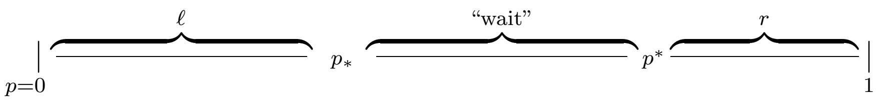
then the receiver takes action ℓ immediately, in which case the sender's payoff is equal to 0. Therefore, the sender's expected payoff is either equal to 0 or above $p_0 v - 2c/\lambda$.

Now suppose $p_0 \geq \hat{p}$, and consider any MPE. As above, if p_0 belongs to the waiting region, then the sender's payoff must exceed at least $p_0 v - 2c/\lambda$. If p belongs to the stopping region, then the sender's payoff is equal to v.[19] In either case, the sender's payoff is at least as much as $p_0 v - 2c/\lambda$. ☐

We prove the folk theorem by constructing MPEs with a particularly simple structure:

Definition 1. A Markov perfect equilibrium is a *simple MPE* (henceforth, SMPE) if there exist $p_* \in (0, \hat{p})$ and $p^* \in (\hat{p}, 1)$ such that the receiver chooses action ℓ if $p < p_*$, waits if $p \in (p_*, p^*)$, and chooses action r if $p \geq p^*$.[20]

In other words, in an SMPE, the receiver waits for more information if $p \in W$ and takes an action, ℓ or r, otherwise, where $W = (p_*, p^*)$ or $W = [p_*, p^*)$ denotes the *waiting region:*



While this is the most natural equilibrium structure, we do not exclude the possibility of MPEs that violate this structure. Our folk theorem, as well as Theorem 3, is independent of whether such non-simple equilibria exist or not.

To prove the folk theorem, we begin by fixing $p^* \in (\hat{p}, 1)$. Then, for each c sufficiently small, we identify a unique value of p_* that yields an SMPE. We then show that as c tends to 0, p_* approaches 0 as well. This implies that given p^*, the limit SMPE spans the sender's payoffs on the line that connects $(0, 0)$ and (p_*, v) (the dashed line in the left panel of Figure 5) and the receiver's payoffs on the line that connects $(0, u_\ell^L)$ and $(p^*, U_r(p^*))$ (the dashed line in the right panel). By varying p^* from $\hat{p}$ to 1, we can cover the entire shaded areas in Figure 5. Note that with this construction, we also obtain a characterization of feasible *payoff vectors* (V, U) for the sender and receiver that can arise in an SMPE in the limit as c tends to 0. We state this in the following corollary.

Corollary 1. *For any prior* $p_0 \in [0, 1]$*, in the limit as* c *tends to* 0*, the set of SMPE payoff vectors* (V, U) *is given by*

$$\left\{ (V, U) \middle| \exists p^* \in [\max\{p_0, \hat{p}\}, 1] : \, V = \frac{p_0}{p^*} v, \, U = \frac{p_0}{p^*} U_r(p^*) + \frac{p^* - p_0}{p^*} u_\ell^L \right\},$$

[19] At $\hat{p}$, the receiver is indifferent between ℓ and r. *In any MPE*, however, she must take action r if $\hat{p}$ is not in the waiting region. This is necessary for the existence of a best response of the sender. For example, in the no-persuasion equilibrium, if the receiver takes action ℓ at $\hat{p}$, then the sender has no optimal jump-target for $p < \hat{p}$.

[20] We do not restrict the receiver's decision at the lower bound p_*, so that the waiting region can be either $W = (p_*, p^*)$ or $W = [p_*, p^*)$. Requiring $W = (p_*, p^*)$ can lead to non-existence of an SMPE (see Proposition 2 below, as well as the discussion in Footnote 19). Requiring $W = [p_*, p^*)$ can lead to non-admissibility of the sender's best response in Proposition 3 (see the discussion of admissibility in Appendix A.1).

with the addition of the no-persuasion payoff vector $(0, U(p_0))$ *for* $p_0 < \hat{p}$.

5 Persuasion Dynamics

While the folk theorem in Section 4 is of clear interest, it is equally interesting to tease out the behavioral implications from our dynamic persuasion model. In this section, we provide a full description of SMPE strategy profiles and illustrate the resulting equilibrium persuasion dynamics. We first explain why the sender optimally uses the three modes of persuasion discussed in the Introduction and Section 2. Then, using them as building blocks, we construct full SMPE strategy profiles.

5.1 Modes of Persuasion

Suppose that the sender runs a flow experiment that targets q_i when the current belief is p. Then, the belief jumps to q_i at rate $\lambda p(1-p)/|q_i - p|$. Absent jumps, it moves continuously according to (2). Therefore, given the sender's value function $V(\cdot)$, her flow benefit is given by[21]

$$v(p; q_i) := \lambda \frac{p(1-p)}{|q_i - p|}(V(q_i) - V(p)) - \text{sgn}(q_i - p)\lambda p(1-p)V'(p).$$

At each point in time, the sender can choose any countable mixture over different experiments. Therefore, at each p, her flow benefit from *optimal* persuasion is equal to

$$v(p) := \max_{(\alpha_i, q_i)_i} \sum_{q_i \neq p} \alpha_i v(p; q_i) \text{ subject to } \sum_{i \in \mathbb{N}} \alpha_i = 1. \tag{3}$$

The function $v(p)$ represents the gross flow value from experimentation. If $p \notin W$ then the sender simply compares this to the flow cost c. Specifically, if $p > p^*$, then the receiver takes action r immediately, and thus $V(p) = v$ for all $p > p^*$. It follows that $v(p) = 0 < c$, so it is optimal for the sender to pass, which is intuitive. If $p < p_*$ then the sender has only one instant to persuade the receiver, and therefore experiments only when $v(p) \geq c$: if $v(p) < c$ then persuasion is so unlikely that she prefers to pass. If $p \in W$, then in equilibrium, the sender must continue to experiment, which suggests that $v(p) \geq c$. When the sender's optimal solution involves experimentation, her value function $V(\cdot)$ is adjusted so that her flow benefit $v(p)$ coincides with the corresponding flow cost c. More formally, $v(p) = c$ is the HJB equation that the sender's value function must satisfy.

[21] $\text{sgn}(x)$ denotes the sign function that assigns 1 if $x > 0$, 0 if $x = 0$, and -1 if $x < 0$. Note that the sender's value function is not everywhere differentiable in all equilibria. Here, we ignore this to give a simplified argument illustrating the properties of the optimal strategy for the sender. The formal proofs can be found in Appendix B.

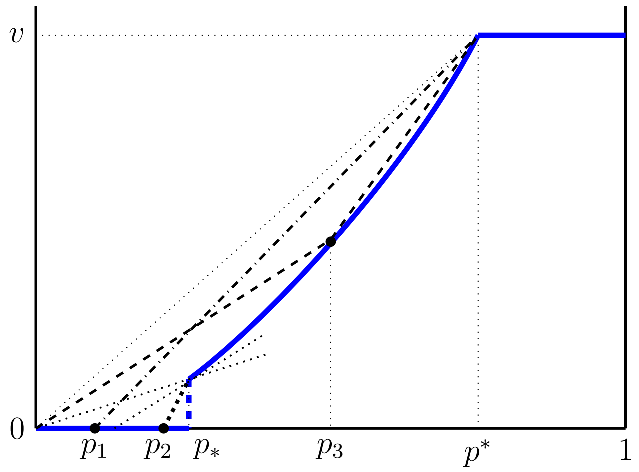


Figure 6: Optimal Poisson jump targets for different values of p. The solid curve represents the sender's value function in an SMPE with p_* and p^*.

The following proposition shows that the potentially daunting task of characterizing the sender's equilibrium strategy reduces to searching among a small set of experiments, instead of all feasible experiments, at each belief.

Proposition 1. *Consider an SMPE where the receiver's strategy is given by $p_* < \hat{p} < p^*$.*

(a) For all $p \in (0,1)$, there exists a best response that uses at most two experiments, (α_1, q_1) and (α_2, q_2).

(b) Suppose that $V(\cdot)$ is non-negative, increasing, and strictly convex over $(p_*, p^*]$, and $V(p_*)/p_* \le V'(p_*)$.

(i) If $p \in (p_*, p^*)$, then $\alpha_1 + \alpha_2 = 1$, $q_1 = p^*$, and $q_2 = 0$.

(ii) If $p < p_*$, then either $\alpha_1 = \alpha_2 = 0$ (i.e., the sender passes), or $\alpha_1 = 1$ and $q_1 = p_*$ or $q_1 = p^*$.

For part (a) of Proposition 1, notice that the right-hand side in equation (3) is linear in each α_i and the constraint $\sum_{i \in \mathbb{N}} \alpha_i \le 1$ is also linear. Therefore, by the standard linear programming logic, there exists a solution that makes use of at most two experiments, one below p and the other above p.[22] This result implies that $v(p)$ can be written as

$$v(p) = \max_{(\alpha_1,q_1),(\alpha_2,q_2)} \lambda p(1-p)\left[\alpha_1 \frac{V(q_1)-V(p)}{q_1 - p} + \alpha_2 \frac{V(q_2)-V(p)}{p-q_2} - (\alpha_1 - \alpha_2)V'(p)\right],$$

subject to $\alpha_1 + \alpha_2 = 1$ and $q_2 < p < q_1$.

Part (b) of Proposition 1 states that if $V(\cdot)$ satisfies certain properties, which will be shown to hold in equilibrium later, then there are only three candidates for optimal Poisson jump targets, 0, p_*, and p^*, regardless of $p \in (0, p^*)$. To see this, observe that

[22] One may wonder why we allow for two experiments. In fact, linearity implies that there exists a maximizer that puts all weight on a single experiment. Mixing, however, is necessary to obtain an *admissible* Markov strategy. For example, if p is an absorbing belief, then admissibility requires that the stationary strategy be used at that belief, requiring two experiments. See Section A.1 for details.

q_1 maximizes $(V(q) - V(p))/(q - p)$ (the slope of V between p and q_1) subject to $q > p$, while q_2 minimizes $(V(q) - V(p))/(p - q)$ (the slope between q_2 and p) subject to $q < p$. As shown in Figure 6, under the given assumptions on $V(\cdot)$, if $p \in (p_*, p^*)$ then $q_1 = p^*$ and $q_2 = 0$ are optimal (see p_3 and the dotted lines). Similarly, if $p < p_*$ then $q_2 = 0$ is optimal and q_1 is either p_* (see p_2 and the dotted line) or p^* (see p_1 and the dash-dotted line).

Proposition 1 implies that the sender makes use of the following three modes of persuasion while $p < p^*$.

***R*-drifting experiment:** This corresponds to choosing $\alpha_2 = 1$ and $q_2 = 0$, that is, the sender uses an experiment that generates Poisson jumps to 0 at (unconditional) rate $\lambda(1-p)$, upon which the receiver immediately takes action ℓ. In the absence of Poisson jumps, p continuously drifts rightward at rate $\dot{p} = \lambda p(1-p)$. Targeting $q_2 = 0$ is explained by the same logic as selecting zero as the belief that induces action ℓ in the static model: the jump to zero is less likely than jump to any other $q < p$, as the arrival rate $\lambda p(1-p)/(p-q)$ is increasing in q. Intuitively, this experiment can be interpreted as the strategy of building the receiver's confidence *slowly but steadily*. This strategy has low risks of losing the receiver's attention and, therefore, is particularly useful when the current belief is already close to p^*, in which case the sender can achieve persuasion relatively quickly and at low cost.

***L*-drifting experiment:** This corresponds to choosing $\alpha_1 = 1$ and $q_1 = p^*$ or possibly $q_1 = p_*$ if $p < p_*$. In other words, the sender generates rightward Poisson jumps that lead to either p_* or p^*. In the absence of Poisson jumps, the belief continuously drifts leftward at rate $\dot{p} = -\lambda p(1-p)$. This strategy is the polar opposite of the R-drifting experiment. It can yield fast success, but the success is unlikely to happen. In addition, when there is no success, the receiver's confidence diminishes. As is intuitive, this strategy is useful when the current belief is significantly away from the target belief and, therefore, the sender has a strong incentive to take risks.

Stationary experiment: This arises when the sender targets two beliefs, $q_1 = p^*$ and $q_2 = 0$, with equal weights ($\alpha_1 = \alpha_2$).[23] In this case, unless the belief jumps to 0 or p^*, it stays constant (thus, "stationary"). This can be interpreted as a mixture between R-drifting and L-drifting experiments and, therefore, combines their strengths and weaknesses. Naturally, this strategy is useful when p is not so close to p^* or p_*.

[23] We will show that any other mixture (in which $\alpha_1 \neq \alpha_2$) never arises in equilibrium.

5.2 Equilibrium Characterization

We now explain how the sender optimally combines the three modes of persuasion introduced in Section 5.1, and provide a full description of the unique SMPE strategy profile for each set of parameter values.

The structure of the sender's equilibrium strategy depends on two conditions. The first condition concerns how demanding the persuasion target p^* is:

$$p^* \leq \eta \approx 0.943. \tag{C1}$$

As explained later, this condition determines whether the sender always prefers the R-drifting strategy to the stationary strategy or not; η is the largest value of p^* such that the sender prefers the former strategy to the latter for all $p < p^*$. Notice that this condition holds for p^* close to $\hat{p}$, as with a strategy that approximates the KG solution (as long as $\hat{p} \leq \eta$).

The structure of the sender's equilibrium strategy also depends on the following condition:

$$v > U_r(p^*) - U_\ell(p^*). \tag{C2}$$

The left-hand side quantifies the sender's gains when she successfully persuades the receiver and induces action r, while the right-hand side represents the corresponding gains by the receiver.[24] If (C2) holds, then the sender has a stronger incentive to experiment than the receiver, and thus p_* (the belief below which some player wishes to stop) is determined by the receiver's incentives. Conversely, if (C2) fails, then the sender is less eager to experiment, and thus p_* is determined by the sender's incentives.

We first provide an equilibrium characterization for the case where (C2) is satisfied.

Proposition 2. *Fix $p^* \in (\hat{p}, 1)$ and suppose that $v > U_r(p^*) - U_\ell(p^*)$. For each c sufficiently small, there exists a unique SMPE such that the waiting region has upper bound p^*. The waiting region is $W = [p_*, p^*)$ for some $p_* < \hat{p}$, and the sender's equilibrium strategy is as follows:*[25]

(a.i) In the waiting region when $p^* \in (\hat{p}, \eta)$: the sender plays the R-drifting strategy with left-jumps to 0 for all $p \in [p_*, p^*)$.

(a.ii) In the waiting region when $p^* \in (\eta, 1)$:[26] *there exist cutoffs $p_* < \xi < \overline{\pi}_{LR} < p^*$ such that for $p \in [p_*, \xi) \cup (\overline{\pi}_{LR}, p^*)$, the sender plays the R-drifting strategy with*

[24] As explained in Section 2, the payoffs of the two players are directly comparable, because their information cost c is normalized to be the same. When they have different information costs, it suffices to interpret the payoffs in (C2) as relative to each player's individual information cost.

[25] It is important for existence of the sender's best response that the waiting region is $W = [p_*, p^*)$. See Lemma 15 in Appendix B.

[26] Notice that in the knife-edge case when $p^* = \eta$, there are two SMPEs, one as in (a.i) and another as in (a.ii). In the latter, however, $\overline{\pi}_{LR} = \xi$ and the L-drifting strategy is not used in the waiting region. The two equilibria are payoff-equivalent but exhibit very different dynamic behavior when $p_0 \in [p_*, \xi]$.

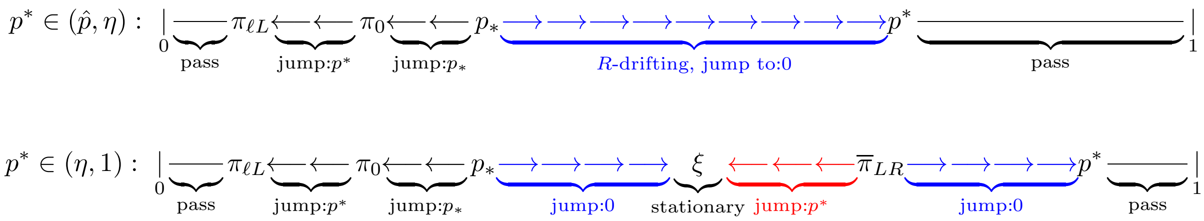


Figure 7: The sender's SMPE strategies in Proposition 2, that is, when $v > U_r(p^*) - U_\ell(p^*)$.

left-jumps to 0*; for* $p = \xi$*, she uses the stationary strategy with jumps to* 0 *or* p^**; and for* $p \in (\xi, \overline{\pi}_{LR}]$*, she adopts the L-drifting strategy with right-jumps to* p^**.*

(b) Outside the waiting region: there exist cutoffs $0 < \pi_{\ell L} < \pi_0 < p_*$ *such that for* $p < \pi_{\ell L}$*, the sender passes; for* $p \in (\pi_{\ell L}, \pi_0)$*, she uses the L-drifting strategy with jumps to* $q = p^*$*; and for* $p \in [\pi_0, p_*)$*, she uses the L-drifting strategy with jumps to* $q = p_*$*.*

The lower bound of the waiting region p_* *converges to zero as* $c \to 0$*.*

Figure 7 summarizes the sender's SMPE strategy in Proposition 2, depending on whether $p^* < \eta$ or not. If $p^* \in (\hat{p}, \eta)$, then the sender uses only R-drifting experiments in the waiting region $[p_*, p^*)$. If $p^* > \eta$, then she also employs L-drifting experiments and the stationary experiment for a range of beliefs.[27] In order to understand this difference, recall from Section 5.1 that R-drifting experiments are particularly useful when p is close to p^*. If $p^* < \eta$, then the sender would not want to deviate from R-drifting at any $p < p^*$. If $p^* > \eta$, however, R-drifting is no longer uniformly optimal for the sender: if p is sufficiently small, then it would take too long for the sender to gradually move p to p^* through R-drifting. In that case, other experiments that are more risky but can generate faster success than R-drifting experiments can be preferred. The structure of our model then implies that there exists a point $\xi \in (p_*, p^*)$ around which L-drifting is optimal if $p > \xi$, while R-drifting is optimal if $p < \xi$. In other words, ξ is an absorbing point towards which the belief converges from both sides in the absence of jumps. Consequently, at $p = \xi$, the sender plays the stationary strategy and the belief p does not drift any longer.

For an economic intuition, consider a salesperson courting a potentially interested buyer. If the buyer needs only a bit more reassurance to buy the product, then the salesperson should adopt a conservative low-risk pitch that slowly "works up" the buyer. The salesperson may still "slip off" and lose the buyer (i.e., p jumps down to 0). But most likely, the salesperson "weathers" that risk and gets the buyer over the last hurdle (i.e., $q = p^*$ is reached). This is exactly what our equilibrium persuasion dynamics describes when p_0 is close to p^*. If $p^* < \eta$, this holds for all $p_0 \in [p_*, p^*)$ since the buyer does not

[27] Although the sender plays the stationary strategy only at one belief ξ, in the absence of Poisson jumps, p reaches ξ in finite time whenever it starts from $p_0 \in [p_*, \overline{\pi}_{LR}]$. Therefore, the sender's strategy at ξ has a significant impact on the players' expected payoffs.

require an extreme belief to be convinced. By contrast, if $p^* > \eta$, the buyer requires a lot of convincing and there are beliefs where the buyer is rather uninterested (as in a "cold" call). Then the salesperson must offer a big pitch that can quickly overcome the buyer's skepticism. It is likely to fail, but may generate quick, unexpected success, which is better than to spend a significant amount of time in order to slowly convince a skeptical buyer.

Condition (C2) implies that p_* is determined by the receiver's incentive: p_* is the point at which the receiver is indifferent between taking action ℓ immediately and waiting (i.e., $U_\ell(p_*) = U(p_*)$). Intuitively, (C2) suggests that the receiver gains less from experimentation, and is thus less willing to continue, than the sender. Therefore, at the lower bound p_*, the receiver wants to stop, even though the sender wants to continue persuading the receiver (i.e., $V(p_*) > 0$).

When $p < p_*$, the sender plays only L-drifting experiments, unless she prefers to pass (below $\pi_{\ell L}$). This is intuitive, because the receiver would take action ℓ immediately unless the sender generates an instantaneous jump. It is intriguing, though, that the sender's target posterior can be either p_* or p^*, depending on how close p is to p_*: in the sales context used above, if the buyer is fairly skeptical, then the salesperson needs to use a big pitch. But, depending on how skeptical the buyer is, she may try to get enough attention only for the buyer to stay engaged (targeting $q = p_*$) or use an even bigger pitch to convince the prospect to buy outright (targeting $q = p^*$). If p is just below p_* (see p_2 in Figure 6), then the sender can jump into the waiting region at a high rate: note that the arrival rate approaches ∞ as p tends to p_*. In this case, it is optimal to target p_*, thereby maximizing the arrival rate of Poisson jumps: the salesperson is sufficiently optimistic about her chance of grabbing the buyer's attention, so she only aims to make the buyer stay. If p is rather far away from p_* (see p_1 in Figure 6), then the sender does not enjoy a high arrival rate. In this case, it is optimal to maximize the sender's payoff conditional on Poisson jumps, which she gets by targeting p^*: the salesperson tries to sell her product right away and if it does not succeed, then she just lets it go.

Next, we provide an equilibrium characterization for the case when (C2) is violated.

Proposition 3. *Fix $p^* \in (\hat{p}, 1)$ and assume that $v \leq U_r(p^*) - U_\ell(p^*)$. For each c sufficiently small, there exists a unique SMPE such that the waiting region has upper bound p^*. The waiting region is $W = (p_*, p^*)$ for some $p_* < \hat{p}$, and the sender's equilibrium strategy is as follows:*[28]

(a.i) In the waiting region when $p^* \in (\hat{p}, \eta)$: there exists a cutoff $\pi_{LR} \in W$ such that for $p \in (\pi_{LR}, p^*)$, the sender uses the R-drifting strategy with left-jumps to 0; and for $p \in (p_*, \pi_{LR})$, she uses the L-drifting strategy with right-jumps to p^*.

(a.ii) In the waiting region when $p^* \in (\eta, 1)$: there exist cutoffs $p_* < \underline{\pi}_{LR} < \xi < \overline{\pi}_{LR} < p^*$, such that for $p \in [\underline{\pi}_{LR}, \xi) \cup [\overline{\pi}_{LR}, p^*)$, the sender plays the R-drifting strategy with

[28]It is important that the waiting region is an open interval. Otherwise the strategy profile violates admissibility at p_*. See Appendix A.1.

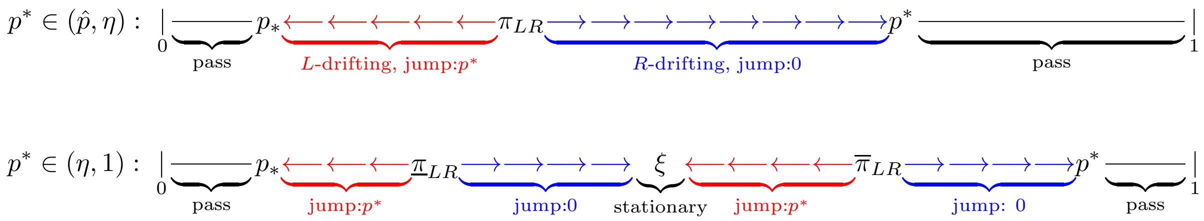


Figure 8: The sender's SMPE strategy in Proposition 3, that is, when $v \leq U_r(p^*) - U_\ell(p^*)$.

left-jumps to 0*; for* $p = \xi$*, she adopts the stationary strategy with jumps to* 0 *or* p^**; and for* $p \in (p_*, \underline{\pi}_{LR}) \cup (\xi, \overline{\pi}_{LR})$*, she uses the* L*-drifting strategy with right-jumps to* p^**.*

(b) Outside the waiting region the sender passes.

The lower bound of the waiting region p_* *converges to zero as* c *tends to* 0*.*

Figure 8 describes the persuasion dynamics in Proposition 3. There are two main differences from Proposition 2. First, if $p < p_*$ then the sender simply passes: recall that in Proposition 2, there exists $\pi_{\ell L} \in (0, p_*)$ such that the sender plays L-drifting experiments whenever $p \in (\pi_{\ell L}, p_*)$. Second, when p is just above p_*, the sender adopts L-drifting experiments, and thus the game may stop at p_*: recall that in Proposition 2, the sender always plays R-drifting experiments just above p_*, and the game never ends by the belief reaching p_*. Both these difference are precisely due to the failure of (C2): if $v \leq U_r(p^*) - U_\ell(p^*)$ then the sender is less willing to continue than the receiver, and thus p_* is determined by the sender's participation constraint (i.e., $V(p_*) = 0$). This implies that the sender has no incentive to experiment once p falls below p_*. In addition, when p is just above p_*, the sender goes for a big pitch by targeting p^* and, therefore, play L-drifting experiments.[29]

Invoking the salesperson's problem again, (C2) fails (i.e., $v \leq U_r(p^*) - U_\ell(p^*)$ holds) when either the salesperson is not so motivated, perhaps because of her compensation structure, or the stake to the buyer is sufficiently large. In this case, the salesperson prefers to make big sales pitches that can sell the product quickly (i.e., targeting p^*) until she runs down the buyer's confidence to p_*.

[29] To provide further intuition for the difference, we note that in Proposition 2, p_* is determined by the receiver's participation constraint. In other words, the receiver does not let the sender experiment for all beliefs where the sender would like to. When using the confidence spending L-drifting strategy, this implies that the sender is stopped earlier than she would like, which diminishes its value. By contrast, the R-drifting strategy's value is unaffected by a high p_*, since it builds confidence and moves the belief away from p_*. Hence, the sender does not use the L-drifting strategy stopping at p_* in Proposition 2.

6 Concluding Discussions

We conclude by discussing two important modeling assumptions and suggesting some directions for future research.

6.1 Discrete Time Foundation

Our timing structure as well as feasible information structures can be micro-founded by the following discrete time model. Time consists of discrete periods $k = 0, 1, ...$, with period length Δ. We consider only short period lengths $\Delta \in (0, 1/\lambda)$. At the beginning of period $k = 0, 1, ...,$ (unless the game has ended before), the sender performs experiments whose outcomes are realized after the elapse of time Δ. At the end of that period, after observing the experiments and their outcomes, the receiver takes an action $a \in \{\ell, r, w\}$. Either of the first two actions ends the game, whereas w moves the game to the next period $k + 1$.

In any period k, the sender allocates a share α_i of resources to experiment i, which has the following binary form:

Binary-signal experiment i

state/signal	L-signal	R-signal
L	x_i	$1 - x_i$
R	$1 - y_i$	y_i

An information structure in discrete time is given by a countable set of experiments $\{(\alpha_i, x_i, y_i)\}_{i\in\mathbb{N}}$ with weights α_i, as in the continuous time formulation. We impose the constraints $(\alpha_i, x_i, y_i) \in [0, 1]^3$ and $1 \leq x_i + y_i \leq 1 + \alpha_i\lambda\Delta$ for all i, and $\sum_{i\in\mathbb{N}} \alpha_i \leq 1$.[30] The shares could be interpreted as the intensities of messages sent simultaneously.

Observe that as $\Delta \to 0$, $x_i + y_i \to 1$ for all i (i.e., the experiments become uninformative), which again captures the idea that information takes time to generate and communicate. It is routine to show that a mixture of Poisson experiments $(\alpha_i, q_i)_{i\in\mathbb{N}}$ satisfying the arrival rates (1) and the drift rates (2) can be obtained in the limit as $\Delta \to 0$ from feasible discrete time information structures.

6.2 Discounting

The players in our model incur flow costs but they do not discount their future payoffs. This assumption simplifies the analysis significantly. In particular, it allows us to derive the value functions associated with the three main modes of persuasion in closed form. The assumption, however, has no qualitative impact on our main results. Specifically, consider

[30] This discrete-time foundation for our class of Poisson experiments appeared in Che and Mierendorff (2019). Arguably, the richness of the experiments did not play an important role in that paper, since conclusive experiments with $q_i = 0$ or $q_i = 1$ always prove optimal for a single decision maker.

an extension of our model in which there are both flow costs and discounting. SMPEs in such a model would converge to those in our model as the discount rate converges to zero. Therefore, all of our results are robust to the introduction of a small amount of discounting.

If one considers a model with only discounting (i.e., without flow cost c), however, the persuasion dynamics presented in Section 5 needs some modification. Among other things, the sender will never voluntarily stop experimentation: recall that in our flow-cost model, the sender never experiments if the belief p is sufficiently close to 0 (below $\pi_{\ell L}$). When there is only discounting, the opportunity cost of experimentation is 0, no matter how small p is (i.e., no matter how unlikely persuasion is to succeed). This implies that the lower bound p_* of the waiting region is always determined by the receiver's participation constraint, as in our Proposition 2. In addition, at the lower bound p_*, the sender will play either R-drifting experiments or the stationary strategy, because she always prefers playing the stationary strategy (which ensures the belief to stay within the waiting region) to L-drifting experiments at p_* (which may move the belief out of the waiting region).

Nevertheless, the main economic lessons from our flow-cost model are likely to apply to the discounting version. Specifically, all three theorems in Section 4 would continue to hold.[31] Furthermore, the advantages of the three main modes of persuasion remain unchanged. Therefore, the persuasion dynamics is also likely to be similar to those described in Propositions 2 and 3, provided that the belief is not so close to p_*. In particular, if p is rather close to p^*, then the sender will play only R-drifting experiments. But, if p is sufficiently far away from p^*, then the sender combines the three modes of persuasion as in the case of $p^* > \eta$ in Propositions 2 and 3.

6.3 Directions for Future Research

We studied a dynamic persuasion model in which real information takes time to generate and neither the sender nor the receiver has commitment power over future actions. There are several variations of our model that could be worth investigating. For example, one may consider a model in which the sender faces the same flow information constraint as in our model but has full commitment power over her dynamic strategy: given our discussion in Section 3, it is straightforward that the sender can approximately implement the KG outcome. However, it is non-trivial to characterize the sender's optimal dynamic strategy.

[31]The proofs of Theorems 1 and 3 can be readily modified. For Theorem 2, one can show that the main economic logic behind it (namely, "the power of beliefs" explained at the end of Section 3) holds also with discounting. To be formal, let ρ denote the common discount rate. Then, it suffices to modify the argument in footnote 17 as follows:

$$\rho U_r(p^*) \leq \lambda(1-p^*)(u_\ell^L - U_r(p^*)) + \lambda p^*(1-p^*)U_r'(p^*) \Leftrightarrow p^* \leq \overline{p} := \frac{\left(u_\ell^L - u_r^L\right)\lambda - u_r^L \rho}{\left(u_\ell^L - u_r^L\right)\lambda + (u_r^R - u_r^L)\rho}.$$

Notice that $\overline{p}$ approaches 1 as ρ tends to 0.

More broadly, the rich persuasion dynamics found in our equilibrium owes a great deal to the general class of Poisson experiments we allow for. At first glance, allowing for the information to be chosen from such a rich class of experiments *at each point in time* might appear extremely complex to analyze, and a clear analysis might seem unlikely. Yet, the model produced a remarkably precise characterization of the sender's optimal choice of information—namely, not just *when* to stop acquiring information but more importantly *what type* of information to search for. This modeling innovation may fruitfully apply to other dynamic settings.

A Continuous Time Formulation

A.1 Markov Strategies in Continuous time

An information structure was defined as a collection $(\alpha_i, q_i)_{i\in\mathbb{N}}$ of experiments, where each (α_i, q_i) specifies a Poisson experiment with jump-targets q_i and associated weight α_i. The set of feasible information structures is thus given by

$$\mathcal{I} = \left\{ (\alpha_i, q_i)_{i\in\mathbb{N}} \middle| \alpha_i \geq 0; \sum_{i=1}^{\infty} \alpha_i \leq 1; \, q_i \in [0,1] \right\}.$$

We define a game in Markov strategies. The sender's strategy is a measurable function $\sigma^S : [0,1] \to \mathcal{I}$ that maps the belief p to an information structure $\sigma^S(p)$.[32] The receiver's strategy is a measurable function $\sigma^R : [0,1] \to A$, that maps the belief p to an action $\sigma^R(p) \in A := \{\ell, r, w\}$. We impose the following *admissibility* restrictions in order to ensure that a strategy profile $\sigma = (\sigma^S, \sigma^R)$ yields a well defined outcome.

Admissible Strategies for the Sender. Our first restriction ensures that σ^S gives rise to a well-defined evolution of the (common) belief about the state.[33] For a Markov strategy $\sigma^S(p)$, with experiments $(\alpha_i(p;\sigma^S), q_i(p;\sigma^S))_{i\in\mathbb{N}}$, Bayesian updating leads to the following integral equation for the belief p_t:[34]

$$p_t = \frac{p_0 e^{-\lambda \int_0^t (\alpha_s^+ - \alpha_s^-) ds}}{p_0 e^{-\lambda \int_0^t (\alpha_s^+ - \alpha_s^-) ds} + (1-p_0)}, \tag{4}$$

where $\alpha_t^+ = \sum_{i:q_i(p_t;\sigma^S)>p_t} \alpha_i(p_t;\sigma^S)$ is the total weight on upward jumps at time t, and $\alpha_t^- = \sum_{i:q_i(p_t;\sigma^S)<p_t} \alpha_i(p_t;\sigma^S)$ is the total weight on downward jumps at time t. To define admissibility formally, we also introduce the following discrete time approximation. For period length $\Delta > 0$, let

$$\tilde{p}_{(k+1)\Delta} = \frac{\tilde{p}_k e^{-\lambda\Delta(\alpha_{k\Delta}^+ - \alpha_{k\Delta}^-)}}{\tilde{p}_k e^{-\lambda\Delta(\alpha_{k\Delta}^+ - \alpha_{k\Delta}^-)} + (1-\tilde{p}_k)}.$$

This can be used to define $\tilde{p}_{k\Delta}$ recursively for each $\tilde{p}_0 = p_0$, and yields a step-function $p_t^\Delta := \tilde{p}_{\lfloor t/\Delta \rfloor \Delta}$.

Definition 2. A measurable function $\sigma^S : [0,1] \to \mathcal{I}$ is an *admissible strategy for the*

[32] We can take $\mathcal{I}$ to be a subset of $\mathbb{R}^{2\mathbb{N}}$, the set of sequences $(\alpha_1, q_1), (\alpha_2, q_2), \ldots$ in $\mathbb{R}^2$, with the product σ-algebra $\mathcal{B}(\mathbb{R}^{2\mathbb{N}}) = \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(\mathbb{R}^2) \otimes \ldots$, where $\mathcal{B}(\mathbb{R}^2)$ is the Borel σ-algebra on $\mathbb{R}^2$.

[33] In this part, we follow Klein and Rady (2011), with the difference that in their model, the evolution of beliefs is jointly controlled by two players. Given that in our model, only the sender controls the information structures, we can dispense with their assumption that Markov strategies are constant on the elements of a finite interval partition of the state space.

[34] The corresponding differential equation is given by $\dot{p}_t = -\left(\alpha_t^+ - \alpha_t^-\right) \lambda p_t (1-p_t)$.

sender if for all $p_0 \in [0,1]$,

(a) there exists a solution to (4), and

(b) if there are multiple solutions to (4), then the pointwise limit $\lim_{\Delta \to 0} p_t^\Delta$ exists and solves (4).

This definition imposes two restrictions on Markov strategies. First, there must be a solution to (4). Indeed, there are Markov strategies for which no solution exists. Consider, for example, a strategy of the following form:

$$\sigma^S(p) = \begin{cases} (\alpha = 1, q = 1), & \text{if } p \geq p', \\ (\alpha = 1, q = 0), & \text{if } p < p'. \end{cases}$$

This strategy does not lead to a well-defined evolution of the belief if p_0 is given by the "absorbing belief" $p_0 = p'$. To satisfy admissibility, we can set $\sigma^S(p) = ((\alpha_1 = 1/2, q_1 = 1), (\alpha_2 = 1/2, q_1 = 0))$, while keeping the strategy otherwise unchanged.

The second restriction guarantees that if there are multiple solutions, we can select one of them by taking the pointwise limit of the discrete time approximation. Consider, for example, the following strategy:

$$\sigma^S(p) = \begin{cases} (\alpha = 1, q = 0), & \text{if } p \geq p', \\ (\alpha = 1, q = 1), & \text{if } p < p'. \end{cases} \tag{5}$$

If $p_0 = p'$, then there is an "obvious" solution $p_t^1 = \frac{p'e^{\lambda t}}{p'e^{\lambda t}+(1-p')} > p'$ for $t > 0$. However, there exists another solution $p_t^2 = \frac{p'e^{-\lambda t}}{p'e^{-\lambda t}+(1-p')}$ consistent with $p_0 = p'$. But, in discrete time, $\tilde{p}_\Delta > p'$ for any $\Delta > 0$, and thus $\lim_{\Delta \to 0} p_t^\Delta = p_t^1$. This means that the strategy in (5) is admissible, while the latter strategy with p_t^2 is not. In general, when there are multiple solutions, admissibility enables us to select the "obvious" one that would be obtained from the discrete time approximation. With this selection, admissibility of the sender's strategy guarantees a well defined belief for all $t > 0$ and all prior beliefs p_0.

Admissible Strategy Profiles. In addition to a well defined evolution of beliefs, we need to ensure that a strategy profile $\sigma = (\sigma^S, \sigma^R)$ leads to a well defined stopping time for any initial belief p_0. Consider for example the function

$$\sigma^R(p) = \begin{cases} w & \text{if } p \leq p', \\ r & \text{if } p > p'. \end{cases}$$

If the sender uses the (admissible) Markov strategy given by $\sigma^S(p) = (\alpha = 1, q = 0)$ for all p, and the prior belief is $p_0 < p'$, then the function $\sigma^R(p)$ does not lead to a well-defined stopping time. To be concrete, suppose that the true state is $\omega = R$. In this case, no

Poisson jumps occur, and the belief drifts upwards. Let t' denote the time at which the belief reaches p'. The receiver's strategy implies that for any $t \leq t'$, the receiver plays w and for any $t > t'$, the receiver has stopped before t. Hence, the stopping time is not well defined. Clearly, the following modified strategy fixes the problem:

$$\hat{\sigma}^{R\prime}(p) = \begin{cases} w & \text{if } p > p', \\ r & \text{if } p \leq p'. \end{cases}$$

This example demonstrates that we need a joint restriction on the sender's and the receiver's strategies to ensure a well defined outcome.

To formally define admissibility, we need the following notation: for a given strategy of the receiver σ^R, let $W = \{p \in [0,1] \big| \sigma^R(p) = w\}$ and $S = [0,1] \setminus W$ be the receiver's *waiting region* and *stopping region*, respectively, and denote the closures of these sets by $\overline{W}$ and $\overline{S}$.

Definition 3. A strategy profile $\sigma = (\sigma^S, \sigma^R)$ is *admissible* if (i) σ^S is an admissible strategy for the sender, and (ii) for each $p \in \overline{W} \cap \overline{S}$, either $p \in S$, or if $p \notin S$, then there exits $\varepsilon > 0$ such that $p_t(p) \in W$ for all $t < \varepsilon$, where $p_t(p) = \lim_{\Delta \to 0} p_t^\Delta$ is the selected solution to (4) with $p_0 = p$.

Requirement (i) guarantees that the sender's strategy gives rise to a well defined belief at all $t > 0$ for all prior beliefs regardless of the receiver's strategy. Requirement (ii) ensures that for any belief $p \in W$, the belief evolution is such that absent jumps the belief remains in the waiting region.

One may wonder why we do not simply require that the stopping region is a closed set. This is stronger than requirement (ii) and it turns out that in some cases it can lead to non-existence of an equilibrium.[35]

Payoffs and Equilibrium. Let $\sigma = (\sigma^S, \sigma^R)$ be a profile of strategies. If σ is not admissible, then both players receive $-\infty$ from playing the strategy profile. If σ is admissible, then for each prior belief p_0, both players' expected payoffs are well-defined:

$$V^\sigma(p_0) = v\,\mathbb{P}\left[\sigma^R(p_\tau) = r \big| p_0\right] - c\mathbb{E}\left[\int_0^\tau \mathbf{1}_{\{\sum \alpha_i(p_t) \neq 0\}} dt \Big| p_0\right]$$

for the sender, and

$$U^\sigma(p_0) = \mathbb{E}\left[U_{\sigma^R(p_\tau)}(p_\tau) - c\int_0^\tau \mathbf{1}_{\{\sum \alpha_i(p_t) \neq 0\}} dt \Big| p_0\right]$$

[35]For example, in the equilibrium characterized in Proposition 2, we have $S = [0, p_*) \cup [p^*, 1]$. If we require S to be closed and set $S = [0, p_*] \cup [p^*, 1]$ instead, the sender does not have a best response for $p \in (\pi_0, p_*)$ since $v(p; q_i)$ fails to be upper semi-continuous in q_i at $q_i = p_*$.

for the receiver, where τ is the stopping time defined by the strategy profile and p_τ is the belief when the receiver stops.

Definition 4 (Markov Perfect Equilibrium)**.** An admissible strategy profile $\sigma = (\sigma^S, \sigma^R)$ is a *Markov perfect equilibrium* (MPE), if

(i) for any $p_0 \in [0,1]$ and any admissible strategy profile $\hat{\sigma} = (\hat{\sigma}^S, \sigma^R)$, $V^{\hat{\sigma}}(p_0) \leq V^{\sigma}(p_0)$,

(ii) for any $p_0 \in [0,1]$ and any admissible strategy profile $\hat{\sigma} = (\sigma^S, \hat{\sigma}^R)$, $U^{\hat{\sigma}}(p_0) \leq U^{\sigma}(p_0)$, and

(iii) for any $p \in S$: (refinement)

$$\sigma^S(p) \in \arg \max_{(\alpha_i, q_i) \in \mathcal{I}} \sum_{i: q_i \neq p} \alpha_i \frac{\lambda p(1-p)}{|q_i - p|} \left(V^\sigma(q_i) - \mathbf{1}_{\{\sigma^R(p) = r\}} v \right) - \mathbf{1}_{\{\sum \alpha_i \neq 0\}} c.$$

Parts (i) and (ii) in this definition require that no player have a profitable deviation to a Markov strategy that, together with the opponent's strategy, forms an admissible strategy profile. Part (iii) formalizes our refinement. We do not explicitly require that deviations to non-Markov strategies should not be profitable. This requirement is in fact hard to formulate since we do not define a game that allows for non-Markov strategies. However, given the opponent's strategy, each player faces a Markov decision problem. Therefore, if there is a policy in this decision problem that yields a higher payoff than the candidate equilibrium strategy, then there is also a profitable deviation that is Markov.

A.2 Belief-free formulation of feasible information structures in continuous time.

In this section, we formulate the class of feasible information in continuous time without reference to beliefs. Denote by $\tilde{\mathcal{I}}$ the set of information structures available to the sender. We have

$$\tilde{\mathcal{I}} = \left\{ (\alpha_i, \gamma_i, \omega_i)_{i \in \mathbb{N}} \;\middle|\; \alpha_i \geq 0; \sum_{i=1}^{\infty} \alpha_i \leq 1; \gamma_i \in [\lambda, \infty); \omega_i \in \{L, R\} \right\}.$$

Let $\tilde{I} = (\alpha_i, \gamma_i, \omega_i)_{i \in \mathbb{N}} \in \tilde{\mathcal{I}}$ be a typical information structure. As discussed in section 2, α_i is the share of resources allocated to experiment i. An information structure with $\sum_{i=1}^{\infty} \alpha_i = 0$ corresponds to "passing". For an experiment with $\alpha_i > 0$, the parameter γ_i controls the arrival rate of jumps that the experiment generates, and the experiment i generates Poisson jumps at rate $\alpha_i \gamma_i$ if the true state is $\omega = \omega_i$, and at rate $\alpha_i(\gamma_i - \lambda)$ if the true state is $\omega \neq \omega_i$.

This specification implies that if the current belief is p_t, a jump from an experiment with $\omega_i = R$ leads to a posterior

$$q(p_t, \gamma_i, R) = \frac{p_t \gamma_i}{p_t \gamma_i + (1 - p_t)(\gamma_i - \lambda)}.$$

Hence the sender can choose any jump target $q > p_t$ by setting (where we solved the previous formula for γ_i):

$$\gamma_i = \frac{\lambda(1 - p_t)q}{q - p_t}.$$

Note that this expression for γ_i (multiplied by α_i) is precisely the conditional arrival rate in state R of the experiment (α_i, q_i) for $q_i > p$ in Section 2. Similarly the arrival rate in state L is

$$\gamma_i - \lambda = \frac{\lambda(1 - p_t)q - \lambda(q - p_t)}{q - p_t} = \frac{\lambda p_t(1 - q)}{q - p_t}.$$

Hence, the experiments $(\alpha_i, \gamma_i, R) \in [0, 1] \times [\lambda, \infty) \times \{R\}$ specify the Poisson experiments with upward jumps that were described with reference to current beliefs and posteriors in Section 2. Similarly, experiments $(\alpha_i, \gamma_i, L) \in [0, 1] \times [\lambda, \infty) \times \{L\}$ specify experiments with downward jumps. Therefore, the set of information structures $\tilde{\mathcal{I}}$ describes precisely the feasible information structures available to the sender as defined in the main text (or the set $\mathcal{I}$ in Appendix A.1).

B Proofs of Propositions 2 and 3

This appendix provides formal proofs for Propositions 2 and 3, from which Theorem 2 and Corollary 1 are also straightforward to obtain. In Section B.1, we derive the general value functions that correspond to the three policies, stationary, R-drifting, and L-drifting experiments. In Section B.2, we define the various cutoffs used in Propositions 2 and 3. In Section B.3, we show that the strategy profiles given in Propositions 2 and 3 (and supplemented by the cutoffs defined in Section B.2) indeed constitute equilibria. Finally, in Section B.4, we show that for c sufficiently small, the strategy profiles in Propositions 2 and 3 are a unique SMPE in each case. To avoid breaking the flow in the presentation, proofs of most Lemmas are relegated to Appendix C.

The main challenge in the proofs is to characterize the best response of the sender if the receiver uses a strategy with waiting region (p_*, p^*), or $[p_*, p^*)$. Inside the waiting region, the value function of the sender's best response must be non-negative and satisfy the HJB equation[36]

$$c = \max_{(\alpha_i, q_i)_{i\in\mathbb{N}}\in\mathcal{I}} \sum_{q_i\neq p} \alpha_i v(p; q_i), \tag{HJB}$$

where $\mathcal{I}$ denotes the set of feasible information structures (see Appendix A.1). If the value function satisfies the conditions of Proposition 1.b, then the maximization problem on the right-hand side can be restricted to information structures $(\alpha_i, q_i)_{i\in\mathbb{N}} \in \mathcal{I}_s$, where $\mathcal{I}_s$ is the set of *simple information structures*

$$\mathcal{I}_s = \{(\alpha_i, q_i)_{i\in\mathbb{N}} | q_1 = p^*,\ q_2 = 0,\ \alpha_2 = 1-\alpha_1,\ \text{and } \alpha_i = 0 \text{ for } i > 2\},$$

so that the sender's problem is to choose a single variable $\alpha = \alpha_1$. This simplifies the HJB equation to

$$c = \lambda p(1-p) \max_{\alpha} \left[\alpha \frac{v - V(p)}{p^* - p} - (1-\alpha)\frac{V(p)}{p} - (2\alpha - 1)\, V'(p)\right]. \tag{HJB-S}$$

Characterizing solutions to (HJB-S) involves the three policies discussed in Section B.1 and the cutoffs in Section B.2. In Lemma 15 in Section B.3, we characterize the sender's (unrestricted) best response and show that it satisfies the restriction $(\alpha_i, q_i)_{i\in\mathbb{N}} \in \mathcal{I}_s$. To prove this, we first construct the best response and the corresponding value function under the restriction $(\alpha_i, q_i)_{i\in\mathbb{N}} \in \mathcal{I}_s$. We then show that the value function satisfies the conditions of Proposition 1.b. This allows us to appeal to Lemmas 13 and 14 (also in Subsection B.3), to show that the candidate value function satisfies (HJB), which implies that the sender's best response under the restriction $(\alpha_i, q_i)_{i\in\mathbb{N}} \in \mathcal{I}_s$ is also an unrestricted best response. This indirect way of characterizing the sender's unrestricted best response

[36] If the value function is not everywhere differentiable, it is a *viscosity solution* of the HJB equation.

is necessary since we do not have a direct proof that the value function satisfies the conditions of Proposition 1.b.[37]

The rest of Section B.3 verifies that the strategy profiles given in Propositions 2 and 3 specify a best response for the sender in the stopping region (Lemma 16), and likewise verifies that the receiver plays a best response to the sender's strategy (Sections B.3.4 and B.3.5). In Section B.4, we show uniqueness. Given the general observation that for any waiting region $W = [p_*, p^*)$ or $W = (p_*, p^*)$, the sender's best response uses only experiments in $\mathcal{I}_s$, we can use the characterization of the sender's best response in Lemma 15 in the uniqueness proof.

B.1 Value Functions

In this section, we derive value functions for the stationary strategy, and the R- and L-drifting policies, and prove crucial properties for these that will be used in later proofs. As discussed in the introductory paragraph to this appendix, if the sender is restricted to information structures in $\mathcal{I}_s$, the sender's value function $V(p)$ satisfies (HJB-S). Similarly, the receiver's value function satisfies

$$c = \lambda p(1-p)\left[\alpha \frac{U_r(p^*) - U(p)}{p^* - p} - (1-\alpha)\frac{u_\ell^L - U(p)}{p} - (2\alpha - 1)\,U'(p)\right]. \qquad \text{(HJB-R)}$$

Note that this is not the HJB equation for the receiver's problem, but it can be used to derive the receiver's payoff for any given information acquisition policy of the sender.

B.1.1 Stationary Strategy

Let $V_S(p)$ denote the sender's expected payoff from playing the stationary strategy at belief p (assuming that the receiver waits at belief p). In the stationary strategy, the sender devotes equal attention to jumps to 0 and p^*. Setting $\alpha = 1/2$ in (HJB-S), we obtain[38]

$$V_S(p) = \frac{\lambda p(1-p)v - 2c(p^* - p)}{\lambda p^*(1-p)} = \frac{p}{p^*}v - C_S(p), \qquad (6)$$

where

$$C_S(p) = \frac{2c(p^* - p)}{\lambda p^*(1-p)}. \qquad (7)$$

[37]Consider for example convexity: Standard arguments to not apply since the sender's payoff depends on the receiver's strategy so that the sender's value cannot be written as the envelope of linear functions.

[38]Intuitively, under the stationary strategy, the belief stays constant in the absence of a jump. In addition, if a jump occurs, either to 0 or to p^*, then the game ends immediately. Therefore, the sender's benefit of playing the strategy is equal to the probability of jumping to p^* times v, while the associated cost is equal to the expected time of the first jump, either to 0 or p^*, times c. For the receiver's value function (below), it suffices to take into account that his payoff is equal to u_ℓ^L if p jumps to 0 and equal to $U_r(p^*)$ if p jumps to p^*.

The corresponding value function of the receiver can be similarly obtained by setting $\alpha = 1/2$ in (HJB-R) and rearranging:

$$U_S(p) = \frac{p^* - p}{p^*} u_\ell^L + \frac{p}{p^*} U_r(p^*) - C_S(p). \tag{8}$$

Note that $C_S(p)$ is concave, and thus both $V_S(p)$ and $U_S(p)$ are convex over $(0, p^*)$:

$$V_S''(p) = U_S''(p) = -C''(p) = -\frac{4c(1-p^*)}{\lambda p^*(1-p)^3} > 0. \tag{9}$$

B.1.2 R-drifting Strategies

Generic value functions: If the sender uses an R-drifting experiment with jump-target zero for an interval of beliefs, then the sender's value function satisfies (HJB-S) with $\alpha = 0$ on that interval:

$$c = -\lambda(1-p)V_+(p) + \lambda p(1-p)V_+'(p). \tag{10}$$

Here, $V_+(p)$ denotes a generic function that satisfies (10); "+" signifies the *upward* drift of the belief. If the sender uses the R-drifting strategy until the belief reaches a stopping bound $q(> p)$, and the sender's value at q is given by the boundary condition $V(q) = X$, we can solve the ODE (10) to obtain the particular solution

$$V_+(p; q, X) = \frac{p}{q} X - C_+(p; q),$$

where

$$C_+(p; q) = \left(p \log\left(\frac{q}{1-q} \frac{1-p}{p} \right) + 1 - \frac{p}{q} \right) \frac{c}{\lambda}.$$

Similarly, let $U_+(p)$ denote the receiver's generic value function from the R-drifting strategy with jumps to zero. It satisfies (HJB-R) with $\alpha = 0$:

$$c = \lambda(1-p)\left(u_\ell^L - U_+(p)\right) + \lambda p(1-p)U_+'(p). \tag{11}$$

For a boundary condition $U(q) = X$, we obtain the particular solution

$$U_+(p; q, X) = \frac{q-p}{q} u_\ell^L + \frac{p}{q} X - C_+(p; q).$$

Note that, as for $V_S(p)$ and $U_S(p)$, $V_+(p)$ and $U_-(p)$ are convex:

$$V_+''(p; q, X) = U_+''(p; q, X) = -C_+''(p; q) = \frac{c}{\lambda p(1-p)^2} > 0. \tag{12}$$

Here and in the following, we are slightly abusing notation for the partial derivatives of these functions with respect to p, e.g., $V_+''(p;q,X) = \partial^2 V_+(p;q,X)/\partial p^2$.

R-drifting with stopping bound p^*: In Propositions 2 and 3, for p close to p^* the sender uses the R-drifting strategy with a stopping bound p^*, and at p^* the receiver takes action r immediately. Hence, we have the boundary condition $V(p^*) = v$ for the sender and $U(p^*) = U_r(p^*)$ for the receiver. To avoid cluttering notation, we denote the corresponding value functions by $V_R(p)$ and $U_R(p)$, suppressing the dependence on p^*. Then, we have

$$V_R(p) := V_+(p;p^*,v) = \frac{p}{p^*}v - C_+(p;p^*), \tag{13}$$

$$U_R(p) := U_+(p;p^*,U_r(p^*)) = \frac{p^*-p}{p^*}u_\ell^L + \frac{p}{p^*}U_r(p^*) - C_+(p;p^*). \tag{14}$$

Comparison of $V_R(p)$ and $V_S(p)$: Clearly, the sender will not use R-drifting with stopping bound p^* at beliefs where $V_R(p) < V_S(p)$. We now derive conditions under which $V_R(p) < V_S(p)$ can arise for some beliefs $p \in (0, p^*)$.

From (6) and (8), we have

$$V_S(p^*) = v = V_R(p^*), \tag{15}$$

$$V_S'(p^*) = \frac{v}{p^*} + \frac{2c}{\lambda p^*(1-p^*)} > \frac{v}{p^*} + \frac{c}{\lambda p^*(1-p^*)} = V_R'(p^*), \tag{16}$$

$$V_S(0) = -2\frac{c}{\lambda} < -\frac{c}{\lambda} = \lim_{p\to 0} V_R(p). \tag{17}$$

Hence, $V_S(p)$ is dominated by $V_R(p)$ for p close to 0 and p^*. This implies that if $V_S(p) > V_R(p^*)$ for some $p \in (0,p^*)$, then there must be at least two intersection points.

We begin by characterizing intersections between $V_S(p)$ and $V_+(p)$ (the latter being a generic solution to (10)). For each $p^* \geq 8/9$, define the following two cutoffs:

$$\xi_1(p^*) := \frac{3p^*}{4} - \sqrt{\left(\frac{3p^*}{4}\right)^2 - \frac{p^*}{2}} \in \left(0, \frac{3}{4}p^*\right),$$

$$\xi_2(p^*) := \frac{3p^*}{4} + \sqrt{\left(\frac{3p^*}{4}\right)^2 - \frac{p^*}{2}} \in \left(\frac{3}{4}p^*, p^*\right).$$

ξ in Propositions 2 and 3 corresponds to $\xi_1(p^*)$. In what follows, we will use ξ_1 instead of ξ.

Lemma 1. *Fix $p^* \in (\hat{p}, 1)$, let $V_+(p)$ be a solution to (10), and suppose that $V_+(p) = V_S(p)$ for some $p \in [0, p^*]$.*

(a) If $p^* < 8/9$, then $V_+'(p) < V_S'(p)$.

(b) If $p^* \geq 8/9$, then

$$V'_+(p) \begin{cases} < V'_S(p) & \text{if } p \notin [\xi_1(p^*), \xi_2(p^*)], \\ = V'_S(p) & \text{if } p \in \{\xi_1(p^*), \xi_2(p^*)\}, \\ > V'_S(p) & \text{if } p \in (\xi_1(p^*), \xi_2(p^*)). \end{cases}$$

Lemma 1.a implies that if $p^* < 8/9$ then V_+ can cross V_S only from above. Since this rules out the possibility of at least two intersections, it follows that $V_S(p) < V_R(p)$ for all $p \in (0, p^*)$ if $p^* < 8/9$.

Lemma 1.b, together with (15)–(17), also shows how V_S and V_R can intersect each other. (17) implies that at the lowest intersection, which we call $\underline{\pi}_{SR}$, $V_R(p)$ must cross $V_S(p)$ from above. At the highest intersection which we call $\overline{\pi}_{SR}$, (15)–(16) imply that $V_R(p)$ must cross $V_S(p)$ from below. The only possibility for this pattern of intersections to arise is that $\underline{\pi}_{SR} \in (0, \xi_1)$, $\overline{\pi}_{SR} \in (\xi_1, \xi_2)$, and there are no other intersections. This implies that there are two intersections if $V_S(\xi_1(p^*)) > V_R(\xi_1(p^*))$, and there is no intersection if $V_S(\xi_1(p^*)) < V_R(\xi_1(p^*))$. If $V_S(\xi_1(p^*)) = V_R(\xi_1(p^*))$, then the two value functions do not intersect but touch each other at $\xi_1(p^*)$. We define the constant η as the value for p^* for which this knife-edge case obtains:[39]

$$V_S(\xi_1(\eta)) = V_R(\xi_1(\eta)).$$

An explicit solution for η is not available, but numerically we obtain $\eta = .94325 > 8/9$.

To complete the comparison of $V_S(p)$ and $V_R(p)$, the following lemma shows that the two functions never intersect if $p^* < \eta$, and intersect exactly twice if $p^* > \eta$.

Lemma 2. *If $p^* < \eta$, then $V_R(p) > V_S(p)$ for all $p \in [0, p^*)$. If $p^* = \eta$, then $V_R(p) \geq V_S(p)$ for all $p < p^*$, with equality holding only when $p = \xi_1$. Finally, if $p^* > \eta$, then there are two points of intersection $\underline{\pi}_{SR} \in (0, \xi_1)$ and $\overline{\pi}_{SR} \in (\xi_1, \xi_2)$.*

***R*-drifting followed by stationary strategy:** We have shown that at ξ_1, the sender strictly prefers the stationary strategy to R-drifting with stopping bound p^*, if and only if $p^* > \eta$. In this case, Propositions 2 and 3 prescribe that for some $p < \xi_1$, the sender use the R-drifting experiment until the belief drifts to ξ_1, where she switches to the stationary strategy. Let $V_{RS}(p)$ and $U_{RS}(p)$ denote the value functions that derive from this dynamic

[39] Note that in this condition η enters both through $\xi_1(\eta)$, and as an omitted argument $p^* = \eta$ of V_S and V_R. That is, we have

$$\frac{\xi_1(\eta)}{\eta}v - \frac{2c(\eta - \xi_1(\eta))}{\lambda\eta(1 - \xi_1(\eta))} = \frac{\xi_1(\eta)}{\eta}v - \left(\xi_1(\eta)\log\left(\frac{\eta}{1-\eta}\frac{1 - \xi_1(\eta)}{\xi_1(\eta)}\right) + 1 - \frac{\xi_1(\eta)}{\eta}\right)\frac{c}{\lambda}.$$

strategy. From the generic solutions, we get (for $p < \xi_1$)

$$V_{RS}(p) = V_+(p; \xi_1, V_S(\xi_1)) = \frac{p}{p^*}v - C_+(p, \xi_1) - \frac{p}{\xi_1}C_S(\xi_1), \tag{18}$$

$$U_{RS}(p) = U_+(p; \xi_1, U_S(\xi_1)) = \frac{p^* - p}{p^*}u_\ell^L + \frac{p}{p^*}U_r(p^*) - C_+(p, \xi_1) - \frac{p}{\xi_1}C_S(\xi_1). \tag{19}$$

B.1.3 L-drifting strategies

Generic Value Functions: If the sender uses the L-drifting experiment with jumps to p^*, then her value function satisfies (HJB-S) with $\alpha = 1$:

$$c = \lambda p(1-p)\left(\frac{v - V_-(p)}{p^* - p} - V_-'(p)\right), \tag{20}$$

where we use $V_-(p)$ to denote a generic function that satisfies (20). For a stopping bound $q < p$, and a boundary condition $V(q) = X$, we obtain the particular solution

$$V_-(p; q, X) := \frac{p - q}{p^* - q}v + \frac{p^* - p}{p^* - q}X - C_-(p; q), \tag{21}$$

where

$$C_-(p; q) = -\frac{p^* - p}{p^*(1-p^*)}\left(p^* \log\frac{1-q}{1-p} + (1-p^*)\log\frac{q}{p} - \log\frac{p^* - q}{p^* - p}\right)\frac{c}{\lambda}. \tag{22}$$

Similarly, When $\alpha = 1$, (HJB-R) simplifies to

$$c = \lambda p(1-p)\left(\frac{U_r(p^*) - U_-(p)}{p^* - p} - U_-'(p)\right). \tag{23}$$

For a boundary condition $U(q) = X$, we obtain the following generic solution:

$$U_-(p; q, X) = \frac{p - q}{p^* - q}U_r(p^*) + \frac{p^* - p}{p^* - q}X - C_-(p; q).$$

Again, the value functions $V_-(p; q, X)$ and $U_-(p; q, X)$ are convex in p:

$$V_-''(p; q, X) = U_-''(p; q, X) = -C_-''(p; q) = \frac{(p^* - p)^2 + p^*(1-p^*)}{p^2(1-p)^2(p^* - p)}\frac{c}{\lambda} > 0. \tag{24}$$

L-drifting with stopping bound p_*: In Proposition 3, the sender uses the L-drifting experiment with a stopping bound equal to p_*. At p_*, the receiver takes action ℓ, which yields the boundary conditions $V(p_*) = 0$ for the sender, and $U(p_*) = U_\ell(p_*)$ for the receiver. Denoting the associated value functions by $V_L(p)$ and $U_R(p)$, we get

$$V_L(p) = V_-(p; p_*, 0) = \frac{p - p_*}{p^* - p_*}v - C_-(p; p_*), \tag{25}$$

$$U_L(p) = U_-(p; p_*, U_\ell(p_*)) = \frac{p - p_*}{p^* - p_*} U_r(p^*) + \frac{p^* - p}{p^* - p_*} U_\ell(p_*) - C_-(p; p_*). \quad (26)$$

L-drifting followed by stationary strategy: If $p^* > \eta$, then Propositions 2 and 3 also prescribe another L-drifting strategy: for an interval above ξ_1, the sender uses the L-drifting strategy with jumps to p^* until belief reaches ξ_1, at which point she switches to the stationary strategy. We denote the values of this strategy by $V_{LS}(p)$ and $U_{RS}(p)$, respectively:

$$V_{LS}(p) = V_-(p; \xi_1, V_S(\xi_1)) = \frac{p}{p^*} v - C_-(p; \xi_1) - \frac{p^* - p}{p^* - \xi_1} C_S(\xi_1), \quad (27)$$

$$U_{LS}(p) = U_-(p; \xi_1, U_S(\xi_1)) = \frac{p^* - p}{p^*} u_\ell^L + \frac{p}{p^*} U_r(p^*) - C_-(p; \xi_1) - \frac{p^* - p}{p^* - \xi_1} C_S(\xi_1). \quad (28)$$

B.1.4 The Crossing Lemma

The following lemma provides a crossing condition for intersections of generic functions $V_+(p)$ and $V_-(p)$.

Lemma 3. *(Crossing Lemma) Let $V_+(p)$ be a solution to (10), and $V_-(p)$ a solution to (20). If $V_-(p) = V_+(p)$ for some $p \in (0, p^*)$, then*

$$\mathit{sign}\left(V_+'(p) - V_-'(p)\right) = \mathit{sign}\left(V_-(p) - V_S(p)\right).$$

Lemma 3 suggests that the crossing patterns between $V_+(p)$ and $V_-(p)$ are fully determined by the relationship between $V_+(p) = V_-(p)$ and $V_S(p)$. This leads to the following crucial observations.

Lemma 4. *Suppose $p^* \in [\eta, 1]$. Then*
(a) $V_S'(\xi_1) = V_{LS}'(\xi_1) = V_{RS}'(\xi_1)$, and
(b) $V_{RS}(p) > V_S(p)$ for all $p \in [0, \xi_1)$ and $V_{LS}(p) > V_S(p)$ for all $p \in (\xi_1, \xi_2)$.

Moreover, the Crossing Lemma 3 will be crucial to show that the cutoffs used in Propositions 2 and 3 are well defined (see Section B.2), and that kinks in the value functions of the strategies specified in Propositions 2 and 3 are convex (see the proof of Lemma 15).

B.1.5 Additional Notation for the Value Functions

For some proofs and derivations that follow, it is convenient to have a unified notation for $V_R(p)$ and $V_{RS}(p)$, as many arguments apply to both in the same way. To this end, note that $V_S(p^*) = V_R(p^*) = v$, and thus $V_R(p)$ can be written as

$$V_R(p) = V_+(p; p^*, V_S(p^*)).$$

On the other hand, we have $V_{RS}(p) = V_+(p; \xi_1, V_S(\xi_1))$. In other words, the two functions have an identical structure except for the point at which the value of stationary strategy is used in the boundary condition. We therefore define

$$q_R := \begin{cases} p^* & \text{if } p^* \leq \eta, \\ \xi_1 & \text{if } p^* > \eta, \end{cases}$$

and

$$V_{RS}(p; q_R) := V_+(p; q_R, V_S(q_R)).$$

Note that this implies that $V_{RS}(p; q_R) = V_R(p)$ if $p^* \leq \eta$ and $V_{RS}(p; q_R) = V_{RS}(p)$ if $p^* > \eta$. We will also use a similar notation for the receiver: $U_{RS}(p; q_R) := U_+(p; q_R, U_S(q_R))$.

B.2 Cutoffs

We proceed to formally define various cutoffs used in Propositions 2 and 3. As a general rule for notation, we use $\phi_\bullet$ to denote cutoffs stemming indifference conditions of the receiver, and $\pi_\bullet$ to denote those stemming from indifference conditions of the sender. The subscripts refer to the strategies or actions between which the player is indifferent at the respective cutoff. For example, at $\pi_{\ell L}$ the sender is indifferent between action ℓ and the L-drifting strategy.

B.2.1 $\overline{p}$: Upper Bound of p^*

In Propositions 2 and 3, the sender always plays R-drifting experiments if the belief p is close to p^*. A necessary condition for this to be possible in equilibrium is that the receiver prefers waiting to taking action r, that is, $U_R(p) \geq U_r(p)$ for $p \in (p^* - \varepsilon, p^*)$ for some $\varepsilon > 0$. The following lemma shows that this is satisfied if and only if $p^* \leq \overline{p}$, where $\overline{p}$ is defined as

$$\overline{p} := 1 - \frac{1}{u_\ell^L - u_r^L} \frac{c}{\lambda}.$$

Lemma 5 (Crossing of U_r and U_R). *For any $p^* \in (\hat{p}, 1)$, $U_R'(p^*) \lesseqgtr U_r'(p^*)$, if and only if $p^* \lesseqgtr \overline{p}$.*

B.2.2 p_* in Proposition 2

In Proposition 2, the sender uses R-drifting experiments if p is slightly above p_*. The lower bound p_* is then given by the belief $\phi_{\ell R}$ at which the receiver is indifferent between R-drifting and taking action ℓ:

$$U_\ell(\phi_{\ell R}) = U_{RS}(\phi_{\ell R}; q_R) = \begin{cases} U_R(\phi_{\ell R}) & \text{if } p^* \leq \eta, \\ U_{RS}(\phi_{\ell R}) & \text{if } p^* > \eta. \end{cases} \tag{29}$$

Although $\phi_{\ell R}$ is not available in close form, the following lemma shows that the cutoff $\phi_{\ell R}$ is well defined, and converges to zero as c tends to 0. In the second case ($p^* > \eta$) the cutoff is well defined if $U_S(\xi_1)$ exceeds the stopping payoff, which is the case for c sufficiently small.

Lemma 6. *Let $p^* \in (\hat{p}, 1)$.*

(a) If $p^* \leq \eta$, then there exists a unique belief $\phi_{\ell R} \in (0, \hat{p})$ that satisfies (29), and $\phi_{\ell R} \to 0$ as $c \to 0$.

(b) If $p^* > \eta$, then for c sufficiently small, $U_S(\xi_1) > \max\{U_\ell(\xi_1), U_r(\xi_1)\}$ and there exists a unique belief $\phi_{\ell R} \in (0, \hat{p})$ that satisfies (29). In addition, $\phi_{\ell R} \to 0$ as $c \to 0$.

Let $\pi_{\ell R}$ denote a similar cutoff for the sender. This cutoff does not appear in the equilibrium characterization, but it will be useful when we verify the sender's incentives. $\pi_{\ell R}$ is given by the sender's indifference between action ℓ and the R-drifting strategy:

$$0 = V_{RS}(\pi_{\ell R}; q_R) = \begin{cases} V_R(\phi_{\ell R}) & \text{if } p^* \leq \eta, \\ V_{RS}(\phi_{\ell R}) & \text{if } p^* > \eta. \end{cases} \tag{30}$$

Convexity in p of $V_{RS}(p; q_R)$ and $\lim_{p \to 0} V_{RS}(p; q_R) = -c/\lambda < 0$ imply that $\pi_{\ell R}$ is well defined for c sufficiently small.[40]

The next lemma shows that the relationship between $\phi_{\ell R}$ and $\pi_{\ell R}$ is fully determined by Condition C2. Intuitively, the player who gains more form persuasion has the lower indifference belief. This also explains why in Proposition 2, where Condition C2 holds, we defined $p_* = \phi_{\ell R} > \pi_{\ell R}$.

Lemma 7. *$\pi_{\ell R} \lesseqqgtr \phi_{\ell R}$ if and only if $v \gtreqqless U_r(p^*) - U_\ell(p^*)$.*

B.2.3 p_* in Proposition 3

Proposition 3 specifies that for p slightly above p_*, the sender uses L-drifting experiments with stopping bound p_* given by the belief $\pi_{\ell L}$, at which the sender is indifferent between stopping with action ℓ and playing the L-drifting experiment for one more instant:

$$-c\,dt + \frac{\lambda \pi_{\ell L}(1 - \pi_{\ell L})}{p^* - \pi_{\ell L}} v\,dt = 0 \iff \pi_{\ell L}^2 \lambda v - (\lambda v + c)\pi_{\ell L} + p^* c = 0. \tag{31}$$

This quadratic equation has a unique solution in $(0, p^*)$ given by

$$\pi_{\ell L} = \frac{1}{2} + \frac{c}{2\lambda v} - \sqrt{\left(\frac{1}{2} + \frac{c}{2\lambda v}\right)^2 - \frac{cp^*}{\lambda v}}. \tag{32}$$

[40] If $p^* > \eta$, then for c sufficiently small, $V_{RS}(\xi_1) = V_S(\xi_1) > 0$, so that the intermediate value theorem implies existence of the cutoff.

We show that the sender prefers a short period of L-drifting experiment followed by action ℓ, over immediate stopping with action ℓ, if and only if $p^* \geq \pi_{\ell L}$. Formally, we insert $V(p_*) = 0$ in (20) and use simple algebra to obtain

$$V_L'(p_*) \gtreqless 0 \iff v \gtreqless \frac{p^* - p_*}{p_*(1-p_*)} \frac{c}{\lambda} \iff p^* \gtreqless \pi_{\ell L}. \tag{33}$$

Let $\phi_{\ell L}$ denote a similar cutoff for the receiver, that is, the belief at which the receiver is indifferent between stopping with action ℓ and allowing the L-drifting experiment for one more instant:

$$-c + \frac{\lambda \phi_{\ell L}(1-\phi_{\ell L})}{p^* - \phi_{\ell L}} (U_r(p^*) - U_\ell(\phi_{\ell L})) - \lambda \phi_{\ell L}(1-\phi_{\ell L}) U_\ell'(\phi_{\ell L}) = 0. \tag{34}$$

The unique solution on $(0,1)$ is given by

$$\phi_{\ell L} = \frac{c + \lambda \Delta U(p^*) - \sqrt{(c + \lambda \Delta U(p^*))^2 - 4\lambda c \Delta U(p^*) p^*}}{2\lambda \Delta U(p^*)},$$

where $\Delta U(p^*) = U_r(p^*) - U_\ell(p^*)$. Analogously to $\pi_{\ell L}$, the receiver prefers waiting to taking action ℓ immediately, if and only if $p \geq \phi_{\ell L}$. Formally we have

$$U_L'(p_*) \gtreqless U_\ell'(p_*) \iff p_* \gtreqless \phi_{\ell L}. \tag{35}$$

As for $\pi_{\ell R}$ and $\phi_{\ell R}$, Condition C2 determines the relationship between $\pi_{\ell L}$ and $\phi_{\ell L}$. As the following lemma shows, the player who benefits more from persuasion has the lower cutoff. This gives an intuition why for Proposition 3, where C2 is violated, we set $p_* = \pi_{\ell L} > \phi_{\ell L}$.[41]

Lemma 8. $\pi_{\ell L} \lesseqgtr \phi_{\ell L}$ *if and only if* $v \gtreqless U_r(p^*) - U_\ell(p^*)$.

The following Lemma shows that for any belief $p < p^*$, the R-drifting strategy dominates the L-drifting strategy with stopping bound $p_* = \pi_{\ell L}$ if the cost is sufficiently low.[42]

Lemma 9. *For any* $p \in (0, p^*)$, *there exists* $c(p) > 0$ *such that for all* $c < c(p)$, *we have* $\pi_{\ell L} < p$ *and* $V_L(p) < V_R(p)$ *when the stopping bound for* V_L *is* $p_* = \pi_{\ell L}$.

We will later use the following corollary of Lemma 9, which holds since by Lemma 2, $V_R(\xi_1) < V_S(\xi_1)$ if $p^* > \eta$.

[41] The careful reader will wonder why we do not set $p^* > \pi_{\ell L}$ since the sender's incentives only rule out $p^* < \pi_{\ell L}$. The argument is more subtle and will become clear in Subsection B.3.5 as well as Subsection B.4.

[42] Note however, that $\pi_{\ell L} \to 0$ as $c \to 0$. Therefore Lemma 9 does not imply that the R-drifting strategy dominates the L-drifting strategy on the whole waiting region. In the lemma, for $c < c(p)$, we have $p_* = \pi_{\ell L} < p$, leaving room for an interval where the L-drifting strategy is not dominated by the R-drifting strategy.

Corollary 2. *Suppose $p^* > \eta$. If c is sufficiently small and $p_* = \pi_{\ell L}$, then $V_L(\xi_1) < V_S(\xi_1)$.*

Proof. This is immediate from Lemma 9 and the fact that $V_R(\xi_1) < V_S(\xi_1)$. $\square$

B.2.4 Cutoffs Derived from Indifference between Modes of Learning

So far, we have considered cutoff beliefs at which the players are indifferent between experimenting and stopping with some action. Now we define the cutoffs that arise from the sender's indifference conditions between different experiments.

π_{LR} in Proposition 3: If $p^* \leq \eta$, then we define the cutoff π_{LR} as the point of indifference between the L-drifting strategy with stopping bound p_* and the R-drifting strategy with stopping bound p^*, that is,

$$V_L(\pi_{LR}) = V_R(\pi_{LR}), \quad \text{if } p_* \leq \pi_{\ell R}. \tag{36}$$

If $p_* > \pi_{\ell R}$, then $V_R(p_*) > 0 = V_L(p_*)$, and we set $\pi_{LR} = p_*$. The following lemma shows that the cutoff is well defined.

Lemma 10. *Suppose $p^* \in (\hat{p}, \eta]$.*

(a) If c is sufficiently small and $p_* \leq \pi_{\ell R}$, then there exists a unique $\pi_{LR} \in [p_*, p^*)$ that solves (36). Moreover, $V_L(p) > V_R(p)$ if $p \in [p_*, \pi_{LR})$ and $V_L(p) < V_R(p)$ if $p \in (\pi_{LR}, p^*)$. If $p_* > \pi_{\ell R}$, then $V_L(p) < V_R(p)$ for all $p \in (p_*, p^*)$.

(b) If $p_* = \pi_{\ell L}$, then $\pi_{LR} \to 0$ as $c \to 0$.

$\overline{\pi}_{LR}$ in Propositions 2 and 3: For $p^* > \eta$, we define $\overline{\pi}_{LR}$ as the point of indifference between R-drifting with stopping bound p^* and L-drifting followed by the stationary strategy at ξ_1:

$$V_R(\overline{\pi}_{LR}) = V_{LS}(\overline{\pi}_{LR}). \tag{37}$$

The following lemma shows that the cutoff is well defined.

Lemma 11. *Suppose $p^* \in (\eta, 1)$. Then there is a unique $\overline{\pi}_{LR} \in (\xi_1, p^*)$ that solves (37). Moreover, $V_{LS}(p) > V_R(p)$ if $p \in [\xi_1, \overline{\pi}_{LR})$ and $V_{LS}(p) < V_R(p)$ if $p \in (\overline{\pi}_{LR}, p^*)$.*

$\underline{\pi}_{LR}$ in Proposition 3: Proposition 3 uses the additional cutoff $\underline{\pi}_{LR}$. We define it as the point of indifference between L-drifting with stopping bound p_* and R-drifting followed by the stationary strategy at ξ_1:

$$V_L(\underline{\pi}_{LR}) = V_{RS}(\underline{\pi}_{LR}) \qquad \text{if } p_* \leq \pi_{\ell R}. \tag{38}$$

If $p_* > \pi_{\ell R}$, then $V_{RS}(p_*) > 0$ and we set $\underline{\pi}_{LR} = p_*$. The following Lemma shows that this cutoff is well defined if $p_* < \pi_{\ell R}$ and $V_L(\xi_1) < V_S(\xi_1)$.[43]

Lemma 12. *Suppose that $p^* \in (\eta, 1)$. If $p_* < \pi_{\ell R}$ and $V_L(\xi_1) < V_S(\xi_1)$, then there exists a unique $\underline{\pi}_{LR} \in (p_*, \xi_1)$ that solves (38), $V_L(p) > V_{RS}(p)$ if $p \in [p_*, \underline{\pi}_{LR})$, and $V_L(p) < V_{RS}(p)$ if $p \in (\underline{\pi}_{LR}, \xi_1)$. If $p_* \geq \pi_{\ell R}$, then $V_L(p) < V_{RS}(p)$ for all $p \in (p_*, \xi_1)$.*

B.3 Equilibrium

In this section we show that the strategy profiles specified in Proposition 2 and 3 are indeed equilibria when c is sufficiently small.

B.3.1 Verifying Sender Optimality in the Waiting Region

We first consider the sender's strategies in the waiting region (p_*, p^*). The following lemmas provide conditions that can be used to verify the optimality of the sender's strategy. Note that they apply unchanged even if the waiting region is the half-open interval $[p_*, p^*)$.

Lemma 13. *(Unimprovability) Let $V(p) \geq 0$ be a candidate value function for the sender that is continuous on $[p_*, 1]$, is strictly convex on (p_*, p^*), and satisfies $V(p) = 0$ for $p < p_*$, $V(p) = v$ for $p \geq p^*$, $p_* V'(p_{*+}) \geq V(p_{*+})$,[44] and $V(p) \geq V_S(p)$ for all $p \in (p_*, p^*)$.*

(a) If $V(p)$ is differentiable and satisfies (20) at $p \in (p_*, p^*)$, then $V(p)$ satisfies (HJB) at p. If $V(p) > V_S(p)$, then L-drifting with jumps to p^* is the unique optimal policy at p.

(b) If $V(p)$ is differentiable and satisfies (10) at $p \in (p_*, p^*)$, then $V(p)$ satisfies (HJB) at p. If $V(p) > V_S(p)$, then R-drifting with jumps to 0 is the unique optimal policy at p.

If the sender's value function is differentiable for all $p \in (p_*, p^*)$, optimality of a candidate solution whose value function $V(p)$ satisfies (20) or (10), is shown in two steps. First, if $V(p) \geq V_S(p)$ for all $p \in (p_*, p^*)$, then the strategy is optimal on the restricted set of information structures $\mathcal{I}_s$, i.e., $V(p)$ satisfies (HJB-S). Second, if it also satisfies the conditions of Proposition 1.b, then $V(p)$ satisfies (HJB) and the candidate solution is optimal on the unrestricted set of feasible information structures $\mathcal{I}$. The next lemma extends this to the case where the value function has convex kinks. In this case, we show that the value function is a *viscosity solution* of (HJB). A standard verification argument then implies that the candidate solution is optimal.

Lemma 14. *(Unimprovability with Kinks) Let $V(p)$ be a candidate value function that satisfies the conditions of Lemma 13. Suppose further that for any $p \in (p_*, p^*)$ where*

[43] Corollary 2 shows that $V_L(\xi_1) < V_S(\xi_1)$ holds for c sufficiently small if $p^* \in (\eta, 1)$.

[44] We define $V(p_-) := \lim_{x \nearrow p} V(x)$, $V'(p_-) := \lim_{x \nearrow p} V'(x)$, $V(p_+) := \lim_{x \searrow p} V(x)$, and $V'(p_+) := \lim_{x \searrow p} V'(x)$.

$V(p)$ is not differentiable, we have $V'(p_-) < V'(p_+)$, and there exists $\varepsilon > 0$ such that $V(p)$ satisfies (20) for $p' \in (p-\varepsilon, p)$; $V(p)$ satisfies (10) for $p' \in (p, p+\varepsilon)$. Then $V(p)$ is a viscosity solution of (HJB) on (p_*, p^*). Moreover, if $p_* V'(p_{*+}) > V(p_{*+})$ and $V(p) > V_S(p)$, then the optimal policy is unique at p if $V(p)$ is differentiable; and the optimal policy at p can only be R-drifting with jumps to 0 or L-drifting with jumps to p^* if $V(p)$ is not differentiable at p.[45]

B.3.2 The Sender's Best Response in the Waiting Region

The following lemma characterizes the sender's best response given the waiting region. Note that $p_* > \pi_{\ell L}$ in Proposition 2 and $p_* = \pi_{\ell L}$ in Proposition 3. Therefore, it suffices to consider the case where $p_* \geq \pi_{\ell L}$. To verify that the strategy profiles in the Propositions are equilibria, it suffices to consider $p_* \leq \phi_{\ell R}$ (cases (a)–(d) in the lemma). For the uniqueness proof, however, we need to consider all possible $p_* \in [\pi_{\ell L}, \hat{p})$.

Lemma 15. *Suppose that the receiver plays a strategy with waiting region $W = [p_*, p^*)$ or $W = (p_*, p^*)$, where $\pi_{\ell L} \leq p_* < \hat{p} < p^* \leq \overline{p}$. Then, for c sufficiently small, the sender's best response and the associated value function $V(p)$ have the following properties in W:*

(a) If $p^* \leq \eta$ and $p_* < \pi_{\ell R}$, then the sender's best response is given by

$$\begin{cases} \textit{L-drifting with jump to } p^* & \textit{if } p < \pi_{LR}, \\ \textit{R-drifting with jump to } 0 & \textit{if } p \geq \pi_{LR}. \end{cases}$$

Admissibility requires $W = (p_*, p^*)$ in this case. The best response is unique for all $p < p^*$ if $p^* < \eta$, and for all $p \neq \xi$ if $p = \eta$.[46] *The corresponding value function satisfies*

$$V(p) = \max\{V_L(p), V_R(p)\} \geq V_S(p),$$

with strict inequality for all $p < p^*$ if $p^* < \eta$, and for all $p \neq \xi$ if $p = \eta$.

(b) If $p^* \leq \eta$ and $p_* \geq \pi_{\ell R}$, then the sender's best response is given by "R-drifting with jumps to 0" for all $p \in [p_*, p^*)$. It is unique for all $p < p^*$ if $p^* < \eta$, and for all $p \neq \xi_1$ if $p = \eta$. The value function satisfies $V(p) = V_R(p) \geq V_S(p)$, with strict inequality for all $p < p^*$ (and $p \neq \xi_1$) if $p^* < \eta$ ($p = \eta$).

[45] This means $\alpha \in \{0, 1\}$ and the policy is unique up to tie-breaking.

[46] If $p = \eta$, then it is also a best response that the sender plays the stationary strategy at ξ.

(c) *If* $p^* > \eta$ *and* $p_* < \pi_{\ell R}$*, then the sender's best response is given by*

$$\begin{cases} \textit{L-drifting with jump to } p^* & \textit{if } p \leq \underline{\pi}_{LR}, \\ \textit{R-drifting with jump to } 0 & \textit{if } p \in [\underline{\pi}_{LR}, \xi_1), \\ \textit{stationary} & \textit{if } p = \xi_1, \\ \textit{L-drifting with jump to } p^* & \textit{if } p \in (\xi_1, \overline{\pi}_{LR}], \\ \textit{R-drifting with jump to } 0 & \textit{if } p \geq \overline{\pi}_{LR}. \end{cases}$$

Admissibility requires $W = (p_*, p^*)$ *in this case. The best response is unique up to tie-breaking at* $\underline{\pi}_{LR}$ *and* $\overline{\pi}_{LR}$*. The associated value function satisfies*

$$V(p) = \begin{cases} \max\{V_L(p), V_{RS}(p)\} > V_S(p), & \textit{if } p < \xi_1, \\ V_S(\xi_1) & \textit{if } p = \xi_1, \\ \max\{V_{LS}(p), V_R(p)\} > V_S(p), & \textit{if } p \in (\xi_1, p^*). \end{cases}$$

(d) *If* $p^* > \eta$ *and* $p_* \in [\pi_{\ell R}, \xi_1)$*, then the sender's best response is given by*

$$\begin{cases} \textit{R-drifting with jump to } 0 & \textit{if } p \in [p_*, \xi_1), \\ \textit{stationary} & \textit{if } p = \xi_1, \\ \textit{L-drifting with jump to } p^* & \textit{if } p \in (\xi_1, \overline{\pi}_{LR}), \\ \textit{R-drifting with jump to } 0 & \textit{if } p \in [\overline{\pi}_{LR}, p^*). \end{cases}$$

The best response is unique up to tie-breaking at $\overline{\pi}_{LR}$*. The associated value function satisfies*

$$V(p) = \begin{cases} V_{RS}(p) > V_S(p), & \textit{if } p < \xi_1, \\ V_S(\xi_1) & \textit{if } p = \xi_1, \\ \max\{V_{LS}(p), V_R(p)\} > V_S(p), & \textit{if } p \in (\xi_1, p^*). \end{cases}$$

(e) *If* $p^* > \eta$ *and* $p_* \in [\xi_1, \overline{\pi}_{SR})$*, existence of a best response requires* $W = [p_*, p^*)$*. With* $W = [p_*, p^*)$*, the sender's best response is given by*

$$\begin{cases} \textit{stationary} & \textit{if } p = p^*, \\ \textit{L-drifting with jump to } p^* & \textit{if } p \in (p^*, \overline{\pi}_{LR}), \\ \textit{R-drifting with jump to } 0 & \textit{if } p \in [\overline{\pi}_{LR}, p^*). \end{cases}$$

The best response is unique up to tie-breaking at $\overline{\pi}_{LR}$*. The associated value function* *satisfies*

$$V(p)=\begin{cases}V_S(\xi_1) & \text{if } p=p^*,\\ \max\{V_{LS}(p),V_R(p)\}>V_S(p), & \text{if } p\in(p^*,p^*).\end{cases}$$

(f) If $p^*>\eta$ and $p_*\geq\overline{\pi}_{SR}$, then the sender's best response and value function are as in case (b). The best response is unique and if $p\neq\overline{\pi}_{SR}$. The value function satisfies $V(p)=V_R(p)\geq V_S(p)$, with strict inequality for $p\neq\overline{\pi}_{SR}$.

(g) In all cases, the value function associated with the sender's best response is strictly convex on $[p_*,p^*)$, and satisfies $V(p^*_-)=v$, $V(p)>0$ for $p>p_*$, and $p_*V'(p_{*+})\geq V(p_{*+})$.

B.3.3 The Sender's Best Response outside the Waiting Region

Lemma 16. *Suppose that the receiver uses a strategy with waiting region $W=[p_*,p^*)$ or $W=(p_*,p^*)$, where $\pi_{\ell L}\leq p_*<\hat{p}<p^*\leq\overline{p}$, and suppose that c is sufficiently small so that $\pi_{\ell R}$ is well defined. Then the sender's best response has the following properties for $p<p_*$:*

(a) The sender passes whenever $p<\pi_{\ell L}$.

(b) If $\pi_{\ell L}\leq p_*\leq\pi_{\ell R}$, then the sender chooses the L-drifting experiment with jumps to p^* for all $p\in(\pi_{\ell L},p_*)$.

(c) If $p_*>\pi_{\ell R}$ and $W=[p_*,p^*)$, then the sender chooses the L-drifting experiment with jumps to p^* for all $p\in[\pi_{\ell L},\pi_0)$ and the L-drifting experiment with jumps to p_* for all $p\in(\pi_0,p_*)$, where $\pi_0\in(\pi_{\ell L},\pi_{\ell R})$ is defined by

$$\frac{V(p_*)}{p_*-\pi_0}=\frac{v}{p^*-\pi_0}\qquad\Longleftrightarrow\qquad\pi_0=\frac{p_*v-p^*V(p_*)}{v-V(p_*)}.$$

(d) If $p_*>\pi_{\ell R}$, $W=(p_*,p^*)$, and $V(p_*)>0$, then for $p\in(\pi_0,p_*)$ the sender's best reply does not exists.

The non-existence in part (d) is the reason why we must allow for both $W=(p_*,p^*)$, and $W=[p_*,p^*)$. The former is required by admissibility to obtain equilibria in which the sender uses the L-drifting experiment with stopping bound p_*. The latter is required to ensure the existence of a best response for the sender in equilibria in which $V(p_*)>0$. To prove Lemma 16, we use the following result, which we state as a separate lemma since it will also be used in other proofs.

Lemma 17. *$\phi_{\ell L}<\phi_{\ell R}$ and $\pi_{\ell L}<\pi_{\ell R}$.*

B.3.4 Equilibrium Verification in Proposition 2

Suppose that Condition C2 holds. Lemmas 15 and 16 imply that the sender's strategy is a best response in each case. For the receiver, if $p\geq p^*$, then it is clearly optimal to stop and take action r, because the sender will no longer experiment. We consider the other cases in more detail below. For the following, recall from Section B.2.2 that $p_* = \phi_{\ell R}$ in Proposition 2.

Receiver optimality when $p < p_*$: If $p \in (\pi_0, p_*)$, then the sender plays L-drifting experiments with jumps to p_*. After observing the signals form this experiment, taking action ℓ is the receiver's best response whether a jump to p_* has occurred or not. Therefore, it is optimal for the receiver to take action ℓ immediately, without incurring more listening costs. If $p < \pi_{\ell L}$, then the sender simply passes and provides no additional information. Therefore, it is obviously optimal for the receiver to stop and take action ℓ.

It remains to consider beliefs $p \in [\pi_{\ell L}, \pi_0]$. Here, the sender uses L-drifting experiments with jumps to p^*. By (35), given the sender's strategy, the receiver's best response is to stop if $p < \phi_{\ell L}$. We complete the proof by showing that for c sufficiently small, $\phi_{\ell L} > \pi_0$.

Lemma 18. *Suppose that Condition C2 holds. Then there exists $c(p^*)$ such that $\phi_{\ell L} > \pi_0$ if $c < c(p^*)$.*

Receiver optimality when $p \in [p_*, p^*)$: First, consider the case where $p^* \leq \eta$ (so that the sender plays only R-drifting experiments over (p_*, p^*)). Waiting yields a payoff of $U_R(p)$. Lemma 5 implies that $U_R(p) > U_r(p)$ for all $p < p^*$, as long as $p^* < \overline{p}$. Since $\overline{p}$ approaches 1 as c tends to 0, $p^* < \overline{p}$ is guaranteed for c sufficiently small. In addition, uniqueness of $\phi_{\ell R}$ (see Lemma 6) implies that $U_R(p) > U_\ell(p)$ for all $p > p_*$. It follows that $U_R(p) \geq \max\{U_r(p), U_\ell(p)\}$ for all $p \in [p_*, p^*]$, with strict inequalities for $p \in (p_*, p^*)$. This implies that waiting is a best response for $p \in [p_*, p^*]$.

Now consider the case where $p^* > \eta$. First, suppose that $p \geq \xi_1$. As for the case where $p^* \leq \eta$, $U_R(p) > U_r(p)$ for $p < p^*$. Note that $U_R(p) \to \frac{p^*-p_*}{p^*}u_\ell^L + \frac{p_*}{p^*}U_r(p^*) > U_\ell(p)$ as $c \to 0$. This implies that $U_R(p) \geq \max\{U_\ell(p), U_r(p)\}$ for all $p \geq \xi_1$ if c is sufficiently small. Next, note that $U_{LS}(p) > U_R(p)$ if and only if $V_{LS}(p) > V_R(p)$ since the sender and the receiver incur the same cost. Therefore, whenever the LS-strategy is used on $[\xi_1, p^*]$, it increases the receiver's value compared to $U_R(p)$. Hence the receiver's value is greater or equal $U_R(p) \geq \max\{U_\ell(p), U_r(p)\}$ for $p \geq \xi_1$, if c is sufficiently small. Turning to $p < \xi_1$, we note that $U_{RS}(p) > \max\{U_\ell(p), U_r(p)\}$ for $p \in [p_*, \xi_1]$: $U_{RS}(p) > U_\ell(p)$ is proven in the same way as $U_R(p) > U_\ell(p)$ in the case where $p^* \leq \eta$; and $U_{RS}(p) > U_r(p)$ follows from $U_{RS}(p) > U_R(p) > U_r(p)$ (which holds since $U_{RS}(\xi_1) = U_S(\xi_1) > U_R(\xi_1)$ by Lemma 2). Hence the receiver's value exceeds $\max\{U_\ell(p), U_r(p)\}$ also for $p \in [p_*, \xi_1]$ if c is sufficiently small. This completes the verification of the receiver's incentives in the equilibrium specified by Proposition 2 when $p^* > \eta$.

B.3.5 Equilibrium Verification in Proposition 3

Suppose Condition C2 fails. As for Proposition 2, Lemmas 15 and 16 imply that the sender's strategy is a best response in each case. The optimality of the receiver's strategy outside the waiting region is straightforward, because the sender passes for all $p \notin (p_*, p^*)$. For the following, recall from Section B.2.3 that $p_* = \pi_{\ell L}$ in Proposition 3.

Receiver optimality when $p \in (p_*, p^*)$ and $p^* \leq \eta$: First, consider $p > \pi_{LR}$. If c is sufficiently small, then $p^* < \overline{p}$, and thus $U_R(p) > U_r(p)$ for all $p < p^*$ (see Lemma 5). Moreover, by Lemma 7, $\phi_{\ell R} < \pi_{\ell R}$ because Condition C2 does not hold. Since $V_R(p) < 0$ for $p < \pi_{\ell R}$ we have $\pi_{LR} \geq \pi_{\ell R} > \phi_{\ell R}$. This implies that $U_R(p) > U_\ell(p)$ for $p > \pi_{LR}$. (This follows from convexity of $U_R(p)$ since $U_R(\phi_{\ell R}) = U_\ell(\phi_{\ell R})$ and $U_R(p^*) > U_\ell(p^*)$.) Hence we have $U(p) = U_R(p) \geq \max\{U_r(p), U_\ell(p)\}$ for all $p \in [\pi_{LR}, p^*]$.

For $p \in [p_*, \pi_{LR}]$, note that $p_* = \pi_{\ell L} > \phi_{\ell L}$, where the inequality follows from Lemma 8 if Condition C2 is violated. Condition (35) and convexity of $U_L(p)$ then imply that $U_L(p) > U_\ell(p)$ for all $p > p_* = \pi_{\ell L}$. To show that $U_L(p) > U_r(p)$ for $p \in [p_*, \pi_{LR}]$ we show that for c sufficiently small, $\pi_{LR} < \hat{p}$ so that $U_r(p) < U_\ell(p)$. To see this note that for $p_* = \pi_{\ell L}$, Lemma 10.b implies that $\pi_{LR} \to 0$ as $c \to 0$ which implies $\pi_{LR} < \hat{p}$ for c sufficiently small. Hence we have $U_L(p) > \max\{U_r(p), U_\ell(p)\}$ for all $p \in [p_*, \pi_{LR}]$. This completes the proof that waiting is a best response of the receiver for all $p \in (p_*, p^*)$.

Receiver optimality when $p \in (p_*, p^*)$ and $p^* > \eta$: For the case $p^* \leq \eta$, we showed that $U(p) = U_R(p) \geq \max\{U_r(p), U_\ell(p)\}$ for all $p \in [\pi_{LR}, p^*]$. The proof works virtually unchanged for the current case and shows that $U(p) = U_R(p) \geq \max\{U_r(p), U_\ell(p)\}$ for all $p \in [\phi_{\ell R}, p^*]$, and hence for all $p \in [\overline{\pi}_{LR}, p^*]$. Moreover by Lemma 6.b, for c sufficiently low, $U_{RS}(\xi_1) = U_S(\xi_1) \geq \max\{U_\ell(\xi_1), U_r(\xi_1)\}$. Noting that $U_{RS}(p) > U_R(p)$ for all $p \leq \xi_1$, this implies that $U_{RS}(p) > U_r(p)$ for all $p \leq \xi_1$. Moreover, since Condition C2 is violated, we have $\underline{\pi}_{LR} > \pi_{\ell R} > \phi_{\ell R}$ which implies that $U_{RS}(p) > U_\ell(p)$ for all $p \in [\underline{\pi}_{LR}, \xi_1]$. This shows that $U(p) = U_R(p) \geq \max\{U_r(p), U_\ell(p)\}$ for all $p \in [\underline{\pi}_{LR}, \xi_1]$.

Next consider the beliefs where the sender uses an L-drifting strategy. For $p \in [\xi_1, \overline{\pi}_{LR}]$, we have to show $U_{LS}(p) > \max\{U_r(p), U_\ell(p)\}$. Note that

$$U_{LS}(p) - U_R(p) = C_+(p; p^*) - C_-(p; \xi_1) - \frac{p^* - p}{p^* - \xi_1} C_S(\xi_1) = V_{LS}(p) - V_R(p).$$

Since $V_{LS}(p) > V_R(p)$ for $p \in [\xi_1, \overline{\pi}_{LR}]$, we therefore have $U_{LS}(p) > U_R(p) > \max\{U_r(p), U_\ell(p)\}$ for $p \in [\xi_1, \overline{\pi}_{LR}]$. We have already shown that $U_R(p) > \max\{U_r(p), U_\ell(p)\}$ for $p \in (\phi_{\ell R}, p^*)$ and since $\phi_{\ell R} < \xi_1$ for c sufficiently small we therefore have $U(p) = U_{LS}(p) > \max\{U_r(p), U_\ell(p)\}$ for $p \in [\xi_1, \overline{\pi}_{LR}]$.

It remains to consider $p \in [p_*, \underline{\pi}_{LR}]$. As in the case where $p^* \leq \eta$, we have $U_L(p) > U_\ell(p)$ for $p \in [p_*, \underline{\pi}_{LR}]$. Again we show that $U_L(p) > U_r(p)$ for $p \in [p_*, \underline{\pi}_{LR}]$ if c is sufficiently small. To show this, note that $\underline{\pi}_{LR} < \pi_{LR}$ since $V_{RS}(p) > V_R(p)$ for all $p < \xi_1$. Hence $\underline{\pi}_{LR} \to 0$ since $\pi_{LR} \to 0$ by Lemma 10.b. This completes the proof that $U(p) = U_L(p) > \max\{U_r(p), U_\ell(p)\}$ for $p \in [p_*, \underline{\pi}_{LR}]$.

B.4 Uniqueness

We complete the proofs of Propositions 2 and 3 by proving that the given strategy profiles are the unique SMPE in each case.

B.4.1 Lower Bound of p_*

We begin with two useful observations, formally reported in the following lemma.

Lemma 19. *Fix $p^* \in (\hat{p}, \overline{p})$. In any SMPE, $p_* \geq \max\{\phi_{\ell L}, \pi_{\ell L}\}$. In addition, if $p_* > \max\{\phi_{\ell L}, \pi_{\ell L}\}$, then $V(p_*) > 0$ and $U_\ell(p_*) = U(p_*)$.*

In the proof, the crucial step is to show that for *any* strategy profile, the sender's payoff is negative if $p < \pi_{\ell L}$, and the receiver's payoff is less than $U_\ell(p)$ if $p < \phi_{\ell L}$. To show this, we consider a hypothetical strategy in which the sender can choose $\alpha_1 = \alpha_2 = 1$, violating the constraint $\alpha_1 + \alpha_2 \leq 1$. The value of this strategy, which is an upper bound for any feasible strategy, is negative for the sender if $p < \pi_{\ell L}$, and below $U_\ell(p)$ for the receiver if $p < \phi_{\ell L}$. Therefore, the lower bound of the waiting region must be greater than or equal to $\max\{\phi_{\ell L}, \pi_{\ell L}\}$.

To understand the second result of Lemma 19, notice first that if $V(p_*) = 0$, then the optimal jump-target for $p < p_*$ is p^*. This implies that for $p_* > \max\{\phi_{\ell L}, \pi_{\ell L}\}$, the receiver would prefer to wait when $p \in (\phi_{\ell L}, p^*)$, in contrast to the conjectured equilibrium. Therefore $p_* > \max\{\phi_{\ell L}, \pi_{\ell L}\}$ requires $V(p_*) > 0$. If $V(p_*) > 0$, however, the sender will target p_* for beliefs in the stopping region close to p_*. $U_\ell(p_*) = U(p_*)$ is needed to guarantee that such jumps do not give the receiver an incentive to wait.

Lemma 19 implies that there are only two cases to consider: (i) $p_* = \max\{\phi_{\ell L}, \pi_{\ell L}\}$; or (ii) $p_* > \max\{\phi_{\ell L}, \pi_{\ell L}\}$ and p_* is determined by the receiver's incentives (i.e., $U_\ell(p_*) = U(p_*)$). Propositions 2 and 3 correspond to each of these two cases.

B.4.2 Proof of Uniqueness in Proposition 2

In Proposition 2, Condition C2 holds, that is, $v > U_r(p^*) - U_\ell(p^*)$. In this case, by Lemma 8, $\phi_{\ell L} > \pi_{\ell L} \Leftrightarrow \max\{\phi_{\ell L}, \pi_{\ell L}\} = \phi_{\ell L}$: the receiver is less willing to continue and, therefore, stops at a higher belief than the sender. In addition, we have the following important observation:

Lemma 20. *If $v > U_r(p^*) - U_\ell(p^*)$ and c is sufficiently small, then $p_* > \phi_{\ell L} = \max\{\phi_{\ell L}, \pi_{\ell L}\}$ in any SMPE.*

The crucial step in the proof is to show that for c sufficiently small, $V_{RS}(p_*; q_R) > 0$ if $p_* = \phi_{\ell L}$. This implies that the sender uses the R-drifting strategy close to p_* which yields negative utility for the receiver for $p \in [\phi_{\ell L}, \phi_{\ell R})$. Hence, $p_* = \phi_{\ell L}$ implies that the receiver's incentives are violated. Therefore, we must have $p_* > \phi_{\ell L}$ in equilibrium.

Since $p_* > \phi_{\ell L} = \max\{\phi_{\ell L}, \pi_{\ell L}\}$ by Lemma 20, Lemma 19 requires that the receiver's value at p_* is equal to $U_\ell(p_*)$. To determine the receiver's value at p_*, note that Lemma 15 implies that the sender either uses R-drifting experiments at p_* (Parts (b), (d) or (f) of the lemma) or she uses the the stationary strategy (Part (e) of the lemma). For $c \to 0$ we have $U_S(p), U_{RS}(p; q_R) \to ((p^* - p)/p^*)U_\ell(0) + (p/p^*)U_r(p^*)$. Since the receiver's value at p_* must be equal to $U_\ell(p_*)$, we must therefore have $p_* \to 0$ for $c \to 0$. Therefore, for c sufficiently small Lemma 15.(e) never applies, and the sender uses R-drifting experiments at p_*. Hence, the receiver's indifference condition implies $p_* = \phi_{\ell R}$. It then follows from Lemma 15 that the strategy profile stated in Proposition 2 is the only one that can arise in equilibrium, establishing uniqueness of the equilibrium in Proposition 2.

B.4.3 Proof of Uniqueness in Proposition 3

In Proposition 3, $v \leq U_r(p^*) - U_\ell(p^*)$, and thus $\pi_{\ell L} = \max\{\phi_{\ell L}, \pi_{\ell L}\}$. Using Lemma 19, we show that $p_* = \pi_{\ell L}$ if Condition C2 fails:

Lemma 21. *If $v \leq U_r(p^*) - U_\ell(p^*)$, then $p_* = \pi_{\ell L}$ in any SMPE.*

With $p_* = \pi_{\ell L}$, uniqueness of the equilibrium in Proposition 3 then follows immediately from the characterization of the sender's best response in Lemma 15.

C Remaining Proofs

C.1 Proof of Proposition 1

Proof of Proposition 1. Recall that $v(p)$ is defined as follows:

$$v(p) := \max_{(\alpha_i,q_i)_i} \sum_{q_i \neq p} \alpha_i v(p;q_i) \text{ subject to } \sum_{i\in\mathbb{N}} \alpha_i \leq 1,$$

where

$$v(p;q_i) := \lambda \frac{p(1-p)}{|q_i-p|}(V(q_i)-V(p)) - \text{sgn}(q_i-p)\lambda p(1-p)V'(p).$$

As explained in Section 5.1, the first result of Proposition 1 (namely that for each $p \in (0,1)$, the sender uses at most two experiments, (α_1,q_1) and (α_2,q_2), where $0 \leq q_2 < p < q_1 \leq p^*$) directly follows from the fact that both $\sum_{q_i \neq p} \alpha_i v(p;q_i)$ and $\sum_{i\in\mathbb{N}} \alpha_i \leq 1$ are linear in α_i.

For the second result, write $v(p)$ as

$$v(p) = \max_{(\alpha_1,q_1),(\alpha_2,q_2)} \lambda p(1-p)\left[\alpha_1 \frac{V(q_1)-V(p)}{q_1-p} + \alpha_2 \frac{V(q_2)-V(p)}{p-q_1} - (\alpha_1-\alpha_2)V'(p)\right],$$

where $\alpha_1+\alpha_2 \leq 1$, and assume that $V(\cdot)$ is non-negative, increasing and convex over $(p_*,p^*]$ and $V(p_*)/p_* < V'(p_*)$. Clearly, the optimal q_1 maximizes $(V(q_1)-V(p))/(q_1-p)$, while the optimal q_2 maximizes $(V(q_2)-V(p))/(p-q_1)$.

Suppose that $p \in (p_*,p^*)$. The optimality of $q_1 = p^*$ follows immediately from convexity of $V(\cdot)$. A similar argument shows that the optimal q_2 is either 0 or p_*. In order to show that 0 is optimal, notice that $V'(p_*) > V(p_*)/p_*$ implies that

$$\frac{V(p)-V(p_*)}{p-p_*} = \frac{\int_{p_*}^{p} V'(x)dx}{p-p_*} \geq \frac{\int_{p_*}^{p} V'(p_*)dx}{p-p_*} = V'(p_*) > \frac{V(p_*)}{p_*},$$

which is equivalent to

$$\frac{V(p)}{p} > \frac{V(p_*)}{p_*} \Leftrightarrow \frac{V(p)-V(p_*)}{p-p_*} > \frac{V(p)}{p} = \frac{V(p)-V(0)}{p} \Leftrightarrow \frac{V(p_*)-V(p)}{p-p_*} < \frac{V(0)-V(p)}{p-0}.$$

Now suppose that $p < p_*$. Again, the optimality of $q_1 = p_*$ or $q_1 = p^*$ follows from convexity of $V(\cdot)$ over $(p_*,p^*]$. The optimality of $q_2 = 0$ simply comes from the fact that $V(p) = 0$ for all $p < p_*$. ☐

C.2 Proof of Lemma 1

Proof of Lemma 1. Substituting $V_+(p) = V_S(p)$ in (10) we obtain $V_+'(p)$, and differentiating (6) we obtain $V_S'(p)$:

$$V_+'(p) = \frac{v}{p^*} + \frac{2p - p^*}{p^* p(1-p)} \frac{c}{\lambda} \quad \text{and} \quad V_S'(p) = \frac{v}{p^*} + \frac{1-p^*}{(1-p)^2} \frac{2c}{\lambda p^*}.$$

Straightforward algebra yields

$$V_+'(p) \lesseqqgtr V_S'(p) \quad \Longleftrightarrow \quad 3pp^* - p^* - 2p^2 \lesseqqgtr 0.$$

The quadratic expression $3pp^* - p^* - 2p^2$ is negative for all $p \in [0, p^*]$ if $p^* < 8/9$. Hence, if $p^* < 8/9$, $V_+'(p) < V_S'(p)$ for all $p \in (0, p^*)$. This proves part (a).

If $p^* \geq 8/9$, the quadratic expression $3pp^* - p^* - 2p^2$ has the real roots $\xi_1(p^*)$ and $\xi_2(p^*)$, which are distinct if $p^* > 8/9$. Hence $V_+'(q) = V_S'(p)$ if $p \in \{\xi_1(p^*), \xi_2(p^*)\}$. Since the quadratic expression is concave in p, $V_+'(q) > V_S'(p)$ if $p \in (\xi_1(p^*), \xi_2(p^*))$, and $V_+'(q) < V_S'(p)$ if $p \notin [\xi_1(p^*), \xi_2(p^*)]$. This proves part (b). □

C.3 Proof of Lemma 2

Proof of Lemma 2. We have already argued that for $p^* < 8/9$ $V_S(p) < V_R(p)$ for all $p \in (0, p^*)$. Now suppose that $p^* \geq 8/9$, so that $\xi_1(p^*)$ and $\xi_2(p^*)$ are well defined.

Our notation for $V_S(p)$ does not explicitly note the dependence on p^*. In this proof, to avoid confusion, we explicitly note this dependence and write $V_S(p; p^*)$. The equation defining η is therefore given by

$$V_S(\xi_1(\eta); \eta) = V_+(\xi_1(\eta); \eta, v),$$

where the right-hand side makes explicit the dependence of $V_R(p)$ on $p^* = \eta$. We have already argued in the text before the statement of the lemma that that $V_S(p; p^*) < V_+(p; p^*, v)$ for all $p \in (0, p^*)$ if and only if $V_S(\xi_1(p^*); p^*) < V_+(\xi_1(p^*); p^*, v)$. We have to show that this is equivalent to $p^* < \eta$. To do so we show that $V_S(\xi_1(p^*); p^*) = V_+(\xi_1(p^*); p^*, v)$ implies that V_S intersects V_+ from below (as functions of p^*).

If $V_S(\xi_1(p^*); p^*) = V_+(\xi_1(p^*); p^*, v)$, then Lemma 1.(b) implies that

$$\frac{\partial V_S(\xi_1(p^*); p^*)}{\partial p} - \frac{\partial V_+(\xi_1(p^*); p^*, v)}{\partial p} = 0.$$

This implies

$$\frac{d\left(V_S(\xi_1(p^*); p^*) - V_+(\xi_1(p^*); p^*, v)\right)}{dp^*} = \frac{\partial\left(V_S(\xi_1(p^*); p^*) - V_+(\xi_1(p^*); p^*, v)\right)}{\partial p^*},$$

where the partial derivative on the RHS is the derivative with respect to the second argument of V_S and V_+, respectively. We show that this derivative is positive:

$$
\begin{aligned}
&\frac{\partial\left(V_S(\xi_1(p^*);p^*)-V_+(\xi_1(p^*);p^*,v)\right)}{\partial p^*}\\
=&\frac{\partial}{\partial p^*}\left(\left(\xi_1(p^*)\log\left(\frac{p^*}{1-p^*}\frac{1-\xi_1(p^*)}{\xi_1(p^*)}\right)+1-\frac{\xi_1(p^*)}{p^*}\right)\frac{c}{\lambda}-\frac{2c(p^*-\xi_1(p^*))}{\lambda p^*(1-\xi_1(p^*))}\right)\\
=&\frac{2p^*-1-\xi_1(p^*)}{(p^*)^2\left(1-p^*\right)\left(1-\xi_1(p^*)\right)}\xi_1(p^*)\frac{c}{\lambda}\\
>&\frac{\frac{5}{4}p^*-1}{(p^*)^2\left(1-p^*\right)\left(1-\xi_1(p^*)\right)}\xi_1(p^*)\frac{c}{\lambda}>0,
\end{aligned}
$$

where the first inequality holds since $\xi_1(p^*)\leq 3p^*/4$, and the second since $p^*\geq 8/9$. This completes the proof. □

C.4 Proof of Lemma 3

Proof of Lemma 3. From (10) we get

$$V_+'(p)=\frac{V_+(p)}{p}+\frac{c}{\lambda p(1-p)},$$

and from (20) we get

$$V_-'(p)=\frac{v-V_-(p)}{p^*-p}-\frac{c}{\lambda p(1-p)}.$$

Given $V_+(p)=V_-(p)$, the difference is equal to

$$V_+'(p)-V_-'(p)=\frac{p^*}{p\left(p^*-p\right)}V_-(p)+\frac{2c}{\lambda p(1-p)}-\frac{v}{p^*-p}=\frac{p^*}{p(p^*-p)}\left(V_-(p)-V_S(p)\right).$$

Since $p^*/(p(p^*-p))$ is positive for all $p<p^*$, this proves the Lemma. □

C.5 Proof of Lemma 4

Proof of Lemma 4. By construction, $V_S(\xi_1)=V_+(\xi_1)=V_{RS}(\xi_1)$ and $V_S'(\xi_1)=V_+'(\xi_1)=V_{RS}'(\xi_1)$. In addition, $V_{LS}(\xi_1)=V_{RS}(\xi_1)$. Then, by Lemma 3, $V_{LS}'(\xi_1)=V_{RS}'(\xi_1)=V_S'(\xi_1)$.

For $p\neq\xi_1$, consider $V_{RS}(p)$ first. We will show that $V_{RS}''(\xi_1)>V_S''(\xi_1)$. Since $V_{RS}'(\xi_1)=V_S'(\xi_1)$ by Lemma 4.(a), this implies that $V_{RS}(p)>V_S(p)$ for $p\in(\xi_1-\varepsilon,\xi_1)$ for some $\varepsilon>0$. Lemma 1 then implies that $V_{RS}(p)>V_S(p)$ for all $p\in[0,\xi_1)$.

To complete the proof it remains to show $V_{RS}''(\xi_1)>V_S''(\xi_1)$. Direct calculation yields

$$V_{RS}''(\xi_1)>V_S''(\xi_1)\Leftrightarrow-\frac{1}{(1-\xi_1)^2\xi_1}>-\frac{4(1-p^*)}{(1-\xi_1)^3p^*}\Leftrightarrow\xi_1<\frac{p^*}{4-3p^*}.$$

Since $\xi_1 \leq 3p^*/4$ and $p^* > 8/9$, the last inequality is satisfied.

Next consider $V_{LS}(p)$. The proof works virtually identically except that we have:

$$V''_{LS}(\xi_1) > V''_S(\xi_1) \Leftrightarrow \frac{p^* - 2pp^* + p}{(\xi_1)^2(p^* - \xi_1)} > -\frac{4(1-p^*)}{(1-\xi_1)p^*}$$
$$\Leftrightarrow \quad (5p^* - 4)(\xi_1)^3 + (3 - 6p^*)p^*(\xi_1)^2 + 3(p^*)^2\xi_1 - (p^*)^2 < 0.$$

Since $\xi_1 \leq 3p^*/4$ and $p^* > 8/9$, the last inequality is satisfied. This implies that $V_{LS}(p) > V_S(p)$ for $p \in (\xi_1, \xi_1+\varepsilon)$ for some $\xi_1 > 0$. Part (a) implies that Lemma 1 can also be applied to $V_-(p)$. Therefore, if $V_{LS}(p)$ and $V_S(p)$ intersect at $p \in (\xi_1, \xi_2)$, then $V_{LS}(p)$ crosses $V_S(p)$ from below. Since $V_{LS}(p) > V_S(p)$ for $p \in (\xi_1, \xi_1 + \varepsilon)$, this implies $V_{LS}(p) > V_S(p)$ for all $p \in (\xi_1, \xi_2)$. □

C.6 Proof of Lemma 5

Proof of Lemma 5. Substituting $U_R(p^*) = U_r(p^*)$ in (11), we get

$$U'_R(p^*) = -\frac{1}{p^*}\left(u_\ell^L - p^* u_r^R - (1-p^*)u_r^L\right) + \frac{c}{\lambda p^*(1-p^*)} = U'_r(p^*) - \frac{u_\ell^L - u_r^L}{p^*} + \frac{c}{\lambda p^*(1-p^*)}.$$

Simple algebra then shows that $U'_R(p^*) \lesseqgtr U'_r(p^*)$ is equivalent to $p^* \lesseqgtr \overline{p}$. □

C.7 Proof of Lemma 6

Proof of Lemma 6. (a) If $p^* \leq \eta$, $\phi_{\ell R}$ is defined by $U_\ell(\phi_{\ell R}) = U_R(\phi_{\ell R})$. Since $p^* > \hat{p}$, $U_R(p^*) = U_r(p^*) > U_\ell(p^*)$, and from (14) we have $\lim_{p \to 0} U_R(p) = U_\ell(0) - c/\lambda$. Therefore an intersection $\phi_{\ell R}$ exists and since $U_\ell(p)$ is linear and $U_R(p)$ is convex, the intersection is unique. Finally, if $p^* \leq \overline{p}$, $U'_R(p^*) \leq U'_r(p^*)$ and, since $U_r(p)$ is linear and $U_R(p)$ is convex, $U_R(p) > U_r(p)$ for all $p < p^*$. In particular this implies that $U_\ell(\phi_{\ell R}) = U_R(\phi_{\ell R}) > U_r(\phi_{\ell R})$ which shows that $\phi_{\ell R} < \hat{p}$.

(b) We first show that $U_S(\xi_1) > \max\{U_\ell(\xi_1), U_r(\xi_1)\}$ for c sufficiently small. This follows directly from $U_S(\xi_1) \to \frac{p^*-\xi_1}{p^*} u_\ell^L + \frac{\xi_1}{p^*} U_r(p^*) > \max\{U_\ell(\xi_1), U_r(\xi_1)\}$, where the strict inequality follows from $p^* > \hat{p}$ and the fact that $\max\{U_\ell(\xi_1), U_r(\xi_1)\}$ is piecewise linear and V-shaped.

Hence, $U_{RS}(\xi_1) = U_S(\xi_1) > \max\{U_\ell(\xi_1), U_r(\xi_1)\}$ for c sufficiently small. From (19) we have $\lim_{p \to 0} U_{RS}(p) = U_\ell(0) - c/\lambda$. Therefore an intersection $\phi_{\ell R} \in (0, \xi_1)$ exists and since $U_\ell(p)$ is linear and by (12), $U_R(p)$ is convex, the intersection is unique. Finally, $U_{RS}(p) > U_R(p)$ for all $p < \xi_1$ and therefore $U_\ell(\phi_{\ell R}) = U_{RS}(\phi_{\ell R}) > U_R(\phi_{\ell R}) > U_r(\phi_{\ell R})$, where we have used that $U_R(p) > U_r(p)$ for all $p < p^*$ as in the case $p^* \leq \eta$. $U_\ell(\phi_{\ell R}) > U_r(\phi_{\ell R})$ shows that $\phi_{\ell R} < \hat{p}$.

For both (a) and (b), the convergence $\phi_{\ell R} \to 0$ follows from $U_R(p), U_{RS}(p) \to ((p^* - p)/p^*)\, u_\ell^L +$

$(p/p^*)\ U_r(p^*) > U_\ell(p)$ for all $p > 0$. ☐

C.8 Proof of Lemma 7

Proof of Lemma 7. In this proof we use the simplified notation $U_{RS}(p; q_R)$ that was introduced in section B.1.5. With this notation (29) and (30) can be written as follows:

$$U_\ell(\phi_{\ell R}) = U_{RS}(\phi_{\ell R}, q_R) \text{ and } 0 = V_{RS}(\pi_{\ell R}, q_R).$$

Substituting $U_\ell(\phi_{\ell R})$ and $U_{RS}(\phi_{\ell R}, q_R)$ in the first condition we obtain

$$\phi_{\ell R} u_\ell^R + (1 - \phi_{\ell R}) u_\ell^L = \frac{p^* - \phi_{\ell R}}{p^*} u_\ell^L + \frac{\phi_{\ell R}}{p^*} U_r(p^*) - C_+(\phi_{\ell R}; q_R) - \frac{\phi_{\ell R}}{q_R} C_S(q_R),$$

which reduces to

$$\frac{1}{p^*}(U_r(p^*) - U_\ell(p^*)) = \frac{1}{\phi_{\ell R}} C_+(\phi_{\ell R}; q_R) + \frac{1}{q_R} C_S(q_R).$$

Similarly, from the second condition we obtain

$$\frac{1}{p^*} v = \frac{1}{\pi_{\ell R}} C_+(\pi_{\ell R}; q_R) + \frac{1}{q_R} C_S(q_R).$$

This implies that

$$\begin{aligned} v > U_r(p^*) - U_\ell(p^*) \quad &\Leftrightarrow \quad \frac{1}{\pi_{\ell R}} C_+(\pi_{\ell R}; q_R) > \frac{1}{\phi_{\ell R}} C_+(\phi_{\ell R}; q_R) \\ &\Leftrightarrow \quad \pi_{\ell R} < \phi_{\ell R}, \end{aligned}$$

where the last equivalence is due to the fact that

$$\begin{aligned} &\frac{d}{dp}\left(\frac{1}{p} C_+(p; q_R)\right) = -\frac{1}{p^2} C_+(p; q_R) + \frac{1}{p} C_+'(p; q_R) \\ &= -\frac{1}{p^2}\left(p \log\left(\frac{q_R}{1-q_R} \frac{1-p}{p}\right) + 1 - \frac{p}{q_R} - p\left(\log\left(\frac{q_R}{1-q_R} \frac{1-p}{p}\right) - \frac{1}{1-p} - \frac{1}{q_R}\right)\right) \frac{c}{\lambda} \\ &= -\frac{1}{p^2(1-p)} \frac{c}{\lambda} < 0. \end{aligned}$$

☐

C.9 Proof of Lemma 8

Proof of Lemma 8. Equation (31) is equivalent to

$$v = \frac{p^* - \pi_{\ell L}}{\pi_{\ell L}(1 - \pi_{\ell L})} \frac{c}{\lambda}$$

and (34) is equivalent to

$$U_r(p^*) - U_\ell(p^*) = \frac{p^* - \phi_{\ell L}}{\phi_{\ell L}(1-\phi_{\ell L})}\frac{c}{\lambda}.$$

The desired result follows from the fact that $(d/dp)\left((p^*-p)/(p(1-p))\right) < 0$. □

C.10 Proof of Lemma 9

Proof of Lemma 9. We first show that

$$\lim_{c\to 0} V_L(p) = \frac{p}{p^*}v = \lim_{c\to 0} V_R(p).$$

For $V_L(p)$ we derive a lower bound that converges to pv/p^*. Since $V_L'(\pi_{\ell L}) = 0$ and $V_L(p)$ is convex, we have $V_L(p) > 0$ for $p > \pi_{\ell L}$. Therefore $V_L(p) > V_-(p;q,0)$ for any $q \in (\pi_{\ell L}, p)$. From (21) and (22), we obtain

$$\lim_{c\to 0} V_-(p;q,0) = \frac{p-q}{p^*-q}v.$$

Since $\pi_{\ell L} \to 0$ we have

$$\lim_{c\to 0} V_L(p) \geq \lim_{q\to 0}\lim_{c\to 0} V_-(p;q,0) = \frac{p}{p^*}v.$$

Moreover, since $C_-(p;\pi_{\ell L}) > 0$, $V_L(p) < pv/p_*$ and therefore $\lim_{c\to 0} V_L(p) = pv/p_*$. For $V_R(p)$ the limit follows from $\lim_{c\to 0} C_+(p;p^*) = 0$.

To show that $V_L(p) < V_R(p)$ for c sufficiently small, we compare the derivatives with respect to c of both functions in a neighborhood of $c = 0$. For $V_R(p)$, we have

$$\frac{dV_R(p)}{dc} = -\frac{\partial C_+(p;p^*)}{\partial c} = -\left(p\log\left(\frac{p^*}{1-p^*}\frac{1-p}{p}\right) + \frac{p^*-p}{p^*}\right) < 0.$$

This is equal to zero for $p = 0$ and $p = p^*$ and continuous in p. Therefore, $\partial V_R(p)/\partial c$ is uniformly bounded in (p, c) on $[0, p^*] \times \mathbb{R}_+$.

For $V_L(p)$ we have

$$\frac{dV_L(p)}{dc} = \frac{\partial V_L(p)}{\partial \pi_{\ell L}}\frac{\partial \pi_{\ell L}}{\partial c} + \frac{\partial V_L(p)}{\partial c}.$$

The first term on the RHS is zero since

$$\frac{\partial V_L(p)}{\partial \pi_{\ell L}} = \frac{(p^*-p)\left[c(p^*-\pi_{\ell L}) - (1-\pi_{\ell L})\pi_{\ell L}v\lambda\right]}{(1-\pi_{\ell L})\pi_{\ell L}(p^*-\pi_{\ell L})^2\lambda} = 0.$$

where the term in the square brackets is zero from (31). The second term on the RHS is

$$\frac{\partial V_L(p)}{\partial c} = -\frac{C_-(p;\pi_{\ell L})}{c} = \frac{p^*-p}{p^*(1-p^*)}\left(p^*\log\frac{1-\pi_{\ell L}}{1-p} + (1-p^*)\log\frac{\pi_{\ell L}}{p} - \log\frac{p^*-\pi_{\ell L}}{p^*-p}\right)\frac{1}{\lambda}.$$

Taking the limit and using $\pi_{\ell L} \to 0$, we get

$$\lim_{c\to 0}\frac{dV_L(p)}{dc} = \lim_{c\to 0}\frac{\partial V_L(p)}{\partial c} = -\infty.$$

for all $p \in (0, p^*)$. This shows that for any $p \in (0, p^*)$ there exists $c(p) > 0$ such that $V_L(p) < V_R(p)$ for $c \in (0, c(p))$. □

C.11 Proof of Lemma 10

Proof of Lemma 10. (a) For c sufficiently small, Lemma 9 implies that $V_L(p) < V_R(p)$ for all $p \in [\xi_1, p^*)$, so that any intersection must be at $p < \xi_1$. Therefore, by Lemma 2, $V_L(p) = V_R(p) > V_S(p)$ at any intersection. Then, by the Crossing Lemma 3, $V_R(p)$ can intersect $V_L(p)$ only from below which implies that there is at most one intersection of $V_R(p)$ and $V_L(p)$.

If $p_* = \pi_{\ell R}$, then an intersection exists at $\pi_{LR} = p_*$ since $V_R(p_*) = 0 = V_L(p_*)$. If $p_* < \pi_{\ell R}$, then $V_R(p_*) < 0 = V_L(p_*)$ and since $V_L(\xi_1) < V_R(\xi_1)$ for c sufficiently small, the intermediate value theorem implies that there exists an intersection of $V_R(p)$ and $V_L(p)$ in (p_*, p^*). In both cases the intersection is unique by the same argument as above. The remaining claim of the Lemma holds since $V_R(p)$ crosses $V_L(p)$ from below.

If $p_* > \pi_{\ell R}$, then $V_R(p_*) > 0 = V_L(p_*)$ so that the number of intersections must be even which implies that there is no intersection and $V_L(p) < V_R(p)$ for all $p \in (p_*, p^*)$.

(b) From Lemma 9 if follows directly that $\pi_{LR} \to 0$ as $c \to 0$ if $p_* = \pi_{\ell L}$ (this means if we adjust p_* to maintain $p_* = \pi_{\ell L}$ as $c \to 0$). □

C.12 Proof of Lemma 11

Proof of Lemma 11. By Lemma 2, $V_{LS}(\xi_1) = V_S(\xi_1) > V_R(\xi_1)$ if $p^* > \eta$. From (22), (13) and (27), we obtain

$$C'_-(p;q) = -\frac{C_-(p;q)}{p^*-p} - \frac{p^*-p}{p^*(1-p^*)}\left(\frac{p^*}{1-p} - \frac{1-p^*}{p} - \frac{1}{p^*-p}\right)\frac{c}{\lambda} \text{ and}$$
$$\lim_{p\to p^*} V'_{LS}(p^*) = -\lim_{p\to p^*} C'_-(p;q) = \infty.$$

Therefore, $V_{LS}(p) < V_R(p)$ for $p \in (p^* - \varepsilon, p^*)$ for some $\varepsilon > 0$. This shows that there is at least one intersection of $V_{LS}(p)$ and $V_R(p)$ in the interval (ξ_1, p^*).

Next we show that the intersection is unique. We have $V_{LS}(\xi_1) = V_S(\xi_1)$ and $V_{LS}(p^*) =$

$V_S(p^*)$. By the Crossing Lemma 3, the conditions characterizing intersections of V_S and V_+ in Lemma 1 also apply to V_-. These crossing conditions imply $V_{LS}(p) > V_S(p)$ for all $p \in (\xi_1, p^*)$. This implies that we can apply the Crossing Lemma 3 to intersections of V_{LS} and V_R. Since $V_{LS}(p) > V_S(p)$, the Crossing Lemma 3 implies that $V'_{LS}(p) < V'_R(p)$ at any intersection $p \in (\xi_1, p^*)$. Therefore there is a unique intersection, and this defines the cutoff $\overline{\pi}_{LR}$. Clearly, for $p \in [\xi_1, \overline{\pi}_{LR})$, $V_{LS}(p) > V_R(p)$, and for $p \in (\overline{\pi}_{LR}, p^*)$, $V_{LS}(p) < V_R(p)$. □

C.13 Proof of Lemma 12

Proof of Lemma 12. We have $V_{RS}(\xi_1) = V_S(\xi_1)$ and from (6) and (18) we have $V_{RS}(0) > V_S(0)$. Lemma 1 implies that if $V_{RS}(p) = V_S(p)$ for $p < \xi$ then $V'_{RS}(p) < V'_S(p)$. Therefore, there cannot be any intersection for $p < \xi$ and we have $V_{RS}(p) > V_S(p)$ for $p < \xi_1$.

Since $V_{RS}(p) > V_S(p)$ for all $p < \xi_1$, the Crossing Lemma 3 implies that there can be at most one intersection of $V_{RS}(p)$ and $V_L(p)$ for $p \in (p_*, \xi_1)$ and $V_L(p)$ must cross $V_{RS}(p)$ from above. If $p_* \geq \pi_{\ell R}$, then $V_L(p_*) = 0 \leq V_{RS}(p_*)$, hence there cannot be any intersection which implies that $V_L(p) < V_{RS}(p)$ for all $p \in (p_*, \xi_1)$. Next suppose that $p_* < \pi_{\ell R}$. This implies that $V_{RS}(p_*) < 0 = V_L(p_*)$, therefore by the intermediate value theorem, an intersection exists if $V_L(\xi_1) < V_S(\xi_1)$, and we have already shown that the intersection is unique. Clearly, $V_L(p) > V_{RS}(p)$ if $p \in [p_*, \underline{\pi}_{LR})$ and $V_L(p) < V_{RS}(p)$ if $p \in (\underline{\pi}_{LR}, \xi_1)$. □

C.14 Proof of Lemma 13

Proof of Lemma 13. Let $p \in (p_*, p^*)$. Strict convexity of $V(p)$ on (p_*, p^*) and $p_* V'(p_{*+}) \geq V(p_*)$ imply that it suffices to consider jumps to zero and p^* in (HJB). If $p_* V'(p_{*+}) > V(p_*)$, then 0 or p^* is the unique optimal jump target. Hence (HJB) simplifies to (HJB-S) which can be written as

$$\frac{c}{\lambda p(1-p)} = -\frac{V(p)}{p} + V'(p) + \max_{\alpha} \left[\frac{V(p)}{p} + \frac{v - V(p)}{p^* - p} - 2V'(p) \right] \alpha.$$

For part (a) we substitute $V'(p)$ using (20) in the coefficient of α and rearrange as follows:

$$\begin{aligned}
\frac{V(p)}{p} + \frac{v - V(p)}{p^* - p} - 2V'(p) &= \frac{2c}{\lambda p(1-p)} + \frac{V(p)}{p} - \frac{v - V(p)}{p^* - p} \\
&= \frac{2c}{\lambda p(1-p)} - \frac{v}{p^* - p} + \frac{V(p)}{p^* - p} + \frac{V(p)}{p} \\
&= \frac{p^*}{p(p^* - p)} V(p) - \frac{\lambda p(1-p)v - 2c(p^* - p)}{\lambda p(1-p)(p^* - p)} \\
&= \frac{p^*}{p(p^* - p)} \left(V(p) - V_S(p) \right).
\end{aligned}$$

Hence $\alpha = 1$ is a maximizer in (HJB-S) if $V(p) \geq V_S(p)$ and (HJB-S) holds if

$$\frac{c}{\lambda p(1-p)} = \frac{v - V(p)}{p^* - p} - V'(p),$$

which is equivalent to (20). If $V(p) > V_S(p)$, $\alpha = 1$ is the unique maximizer.

For part (b), we substitute $V'(p)$ from (10) instead to obtain

$$\begin{aligned}\frac{V(p)}{p} + \frac{v - V(p)}{p^* - p} - 2V'(p) &= \frac{v - V(p)}{p^* - p} - \frac{V(p)}{p} - \frac{2c}{\lambda p(1-p)} \\ &= -\frac{p^*}{p(p^* - p)}\left(V(p) - V_S(p)\right)\end{aligned}$$

Hence $\alpha = 0$ is a maximizer in the HJB equation if $V(p) \geq V_S(p)$ and (HJB-S) holds if

$$\frac{c}{\lambda p(1-p)} = -\frac{V(p)}{p} + V'(p),$$

which is equivalent to (10). If $V(p) > V_S(p)$, $\alpha = 0$ is the unique maximizer. □

C.15 Proof of Lemma 14

Proof of Lemma 14. By Lemma 13, $V(p)$ satisfies (HJB) for all points $p \in (p_*, p^*)$ where it is differentiable. To show that it is a viscosity solution, we have to show that for all points $p' \in (p_*, p^*)$ where $V(p)$ is not differentiable,

$$\max_{\alpha \in [0,1]} \lambda p'(1-p')\left[\alpha \frac{w - V(p')}{p^* - p'} - (1-\alpha)\frac{V(p')}{p'} - (2\alpha - 1)\, z\right] \leq c$$

for all $z \in [V'(p'_-), V'(p'_+)]$. We have simplified the condition for a viscosity solution using the fact that $V(p) \geq 0$, and $V(p)$ satisfies (20) below the kink and (10) above the kink, the kink is convex (i.e., $V'(p'_-) < V'(p'_+)$), and ~~that~~ $V(p)$ is strictly convex and satisfies $p_* V'(p_{*+}) \geq V(p_{*+})$.

Since the term in the square bracket is linear in α, it suffices to check this condition for $\alpha \in \{0, 1\}$. For $\alpha = 1$ we have $-(2\alpha - 1)\, z < -(2\alpha - 1)\, V'(p'_+)$ and for $\alpha = 0$ we have $-(2\alpha - 1)\, z < -(2\alpha - 1)\, V'(p'_-)$. Hence it is sufficient to check

$$\lambda p'(1-p') \max\left\{\frac{w - V(p')}{p^* - p'} - V'(p'_+), -\frac{V(p')}{p'} + V'(p'_-)\right\} \leq c.$$

Since $V(p)$ is continuous and satisfies (20) for $p'' \in (p' - \varepsilon, p')$ and ~~satisfies~~ (10) for $p'' \in (p', p' + \varepsilon)$, the last condition holds with equality. Therefore, $V(p)$ is a viscosity solution of (HJB). Uniqueness is shown as in the proof of Lemma 13. □

C.16 Proof of Lemma 15

Proof of Lemma 15. For the proof of this Lemma, we verify for each case (a)–(f) that the stated value functions verify the conditions of Lemmas 13 and 14. This implies that the candidate value function is a viscosity solution of (HJB). The value function must necessarily be a viscosity solution of (HJB) (see, e.g., Theorem 10.8 in Oksendal and Sulem, 2009). While we are not aware of a statement of sufficiency that covers precisely our model, the arguments in Soner (1986) can be easily extended to show uniqueness of the viscosity solution to (HJB). This proves that the candidate value function $V(p)$ is the value function of the sender's problem in the waiting region. Uniqueness of the best response follows from Lemmas 13 and 14. At the end of the proof we also address the existence issue for part (e).

We now verify the conditions of Lemmas 13 and 14. Outside the waiting region the sender's value function satisfies $V(p) = 0$ for $p < p_*$ and $V(p) = v$ for $p > p^*$. The other properties are verified one by one:

$V(p) > 0$. Note first that for c sufficiently small, $\pi_{\ell R} < \xi_1$, which we use in some of the cases.

(a) Since $p_* \geq \pi_{\ell L}$, we have $V_L(p) > 0$ for $p \in (p_*, p^*]$ and therefore $V(p) > 0$.

(b) Since $p_* \geq \pi_{\ell R}$, we have $V_R(p) > 0$ for $p \in (p_*, p^*]$ and therefore $V(p) > 0$.

(c) The argument for case (a) implies that $V(p) > 0$ for $p \in (p_*, \xi_1)$, and the argument for case (b) implies that $V(p) > 0$ for $p \in (\xi_1, p^*]$ since $\pi_{\ell R} < \xi_1$. Finally, for c sufficiently small, $V_S(\xi_1) > 0$.

(d) Since $p_* \geq \pi_{\ell R}$, we have $V_{RS}(p) > 0$ for $p \in (p_*, \xi_1]$, and therefore $V(p) > 0$ for $p \in (p_*, \xi_1]$. The argument for case (b) implies that $V(p) > 0$ for $p \in (\xi_1, p^*]$ since $\pi_{\ell R} < \xi_1$. Finally, for c sufficiently small, $V_S(\xi_1) > 0$.

(e) Since $p_* \geq \xi_1 > \pi_{\ell R}$, as in (b), we have $V_R(p) > 0$ for $p \in (p_*, p^*]$ and therefore $V(p) \geq 0$. For c sufficiently small $V_S(\xi_1) > 0$.

(f) Since $p_* \overline{\pi}_{SR} > \xi_1 > \pi_{\ell R}$, as in (b) we have $V_R(p) > 0$ for $p \in (p_*, p^*]$ and therefore $V(p) \geq 0$.

Continuity. We only need to verify continuity at the cutoffs since elsewhere the candidate value functions are solutions to the ODEs (10) and (20):

- at p_* if the waiting region is $[p_*, p^*)$. By admissibility, this rules out cases (a) and (c).
  - (e) Since $V_{LS}(p_*) = V_S(p_*)$ is used as the boundary condition for $V_{LS}(p)$, the candidate value function is continuous at p_*.
  - (b,d,f) In all other cases $V_R(p)$ is continuous at p_*.
- at ξ_1: Since $V_{RS}(\xi_1) = V_{LS}(\xi_1) = V_S(\xi_1)$ from the boundary condition used to define V_{LS} and V_{RS}, the candidate value functions in (c)–(d) are continuous at ξ_1.

- All other cutoffs $(\pi_{LR}, \underline{\pi}_{LR}, \overline{\pi}_{LR})$ are given by indifference conditions between the value functions in the adjacent regions of beliefs. Therefore, the candidate value function is continuous at theses cutoffs.

$V(p) \geq V_S(p)$.

(a,b) By Lemma 2, $V_R(p) \geq V_S(p)$ for all $p < p^*$ since $p^* \leq \eta$. The inequality is strict if $p^* < \eta$ or $p^* = \eta$ and $p \neq \eta$. In (a), when $V(p) = V_L(p)$ we have $V_L(p) \geq V_R(p)$ so that $V_L(p) \geq V_S(p)$ (or $V_L(p) > V_S(p)$) as well.

(c,d) By Lemma 12, $V(p) \geq V_{RS}(p)$ for $p \in (p_*, \xi_1)$, and by Lemma 4.(b) $V_{RS}(p) > V_S(p)$ for $p \in (p_*, \xi_1)$. Hence $V(p) > V_S(p)$ for $p \in (p_*, \xi_1)$. By Lemma 2, $V_R(p) > V_S(p)$ for $p \in [\xi_2, p^*)$ and by Lemma 4.(b) $V_{LS}(p) > V_S(p)$ for $p \in (\xi_1, \xi_2)$. Hence $V(p) > V_S(p)$ for $p \in (\xi_1, p^*)$.

(e) By Lemma 2, $V_R(p) > V_S(p)$ for $p \in [\xi_2, p^*)$. Since $p_* < \overline{\pi}_{RS}$ in case (e), Lemma 2 implies that $p_* \in (\xi_1, \xi_2)$. The proof of Lemma 4.(b) can be extended to show that $V_{LS}(p) > V_S(p)$ for $p \in (\xi_1, \xi_2)$ when $V_{LS}(p) = V_-(p; p_*, V_S(p_*))$ for $p_* \in (\xi_1, \xi_2)$. Hence $V(p) > V_S(p)$ for $p \in (p_*, p^*)$.

(f) Since $p_* \geq \overline{\pi}_{RS}$, Lemma 2 implies that $V_R(p) > V_S(p)$ for all $p \in (p^*, p^*)$.

Strict convexity. The candidate value functions are defined piece-wise using the functions V_R, V_{RS}, V_L, and V_{LS}, which are all strictly convex (see (12) and (24), respectively). Lemma 4 implies that $V'_{LS}(\xi_1) = V'_{RS}(\xi_1)$ in (c) and (d). Therefore, it only remains to verify that at the remaining cutoffs the value function has convex kinks. To do this we employ the Crossing Lemma 3. Note that at any cutoff $x \in \{\pi_{LR}, \underline{\pi}_{LR}, \overline{\pi}_{LR}\}$, $V(p)$ satisfies (10) for $p \in (x, x+\varepsilon)$, and (20) for $p \in (x-\varepsilon, x)$ for some $\varepsilon > 0$. Since $V(p) \geq V_S(p)$ (see above), the Crossing Lemma 3 imples that $V'(x_-) \leq V'(x_+)$—that is $V(p)$ is either continuously differentiable at x or has a convex kink. Therefore $V(p)$ is strictly convex.

$p_* V'(p_{*+}) \geq V(p_*)$.

(a,c) In these two cases we have $V(p_{*+}) = V(p_*) = 0$ and by (33), $V'(p_{*+}) \geq 0$.

(b,d,f) We define

$$\tilde{V}(p) = \begin{cases} V(p) & \text{if } p > p_*, \\ V_R(p) & \text{if } p \leq p_* \text{ in cases (b) and (f),} \\ V_{RS}(p) & \text{if } p \leq p_* \text{ in case (d).} \end{cases}$$

Since $V_R(p)$ is strictly convex on $[0, p^*]$ and $V_{RS}(p)$ strictly convex on $[0, \xi_1]$, $\tilde{V}(p)$ is a strictly convex function on $[0, p^*]$. Moreover $\lim_{p \to 0} \tilde{V}(p) = \lim_{p \to 0} V_R(p) = \lim_{p \to 0} V_{RS}(p) = -c/\lambda < 0$. This implies that $p_* \tilde{V}'(p_*) > \tilde{V}(p_*) - \lim_{p \to 0} \tilde{V}(p) > \tilde{V}(p_*)$ and therefore $p_* V'(p_{*+}) \geq V(p_*)$.

(e) In this case we follow a similar argument as in cases (b,d,f) and define

$$\tilde{V}(p) = \begin{cases} V(p) & \text{if } p \geq p_*, \\ V_+(p; p_*, V_S(p_*)) & \text{if } p < p_*. \end{cases}$$

By the Crossing Lemma (3), we have $V_L'(p_*) = V_+'(p; p_*, V_S(p_*))$ so that $\tilde{V}(p)$ is convex on $[0, p^*)$ and as in cases (b,d,f), $\lim_{p\to 0} \tilde{V}(p) = -c/\lambda < 0$. This implies that $p_*\tilde{V}'(p_*) \geq \tilde{V}(p_*)$ and therefore $p_*V'(p_{*+}) \geq V(p_*)$.

Existence in case (e). Suppose $p_* \in [\xi_1, \overline{\pi}_{SR})$. Let $V_{[p_*,p^*)}(p)$ be the value function of the best response if $W = [p_*, p^*)$ (stated in part (e)). Clearly, if $W = (p_*, p^*)$ the sender cannot achieve $V_{[p_*,p^*)}(p)$ for $p < \overline{\pi}_{LR}$, since this would require using $V_{LS}(p)$, but the sender cannot switch to the stationary strategy at p_* if the receiver stops at p_*. However, the sender can achieve a value arbitrarily close to $V_{[p_*,p^*)}(p)$ by using the strategy

$$\begin{cases} R\text{-drifting with jump to } 0 & \text{if } p \in (p_*, p_* + \varepsilon), \\ \text{stationary} & \text{if } p = p_* + \varepsilon, \\ L\text{-drifting with jump to } p^* & \text{if } p \in (p_* + \varepsilon, \overline{\pi}_{LR}), \\ R\text{-drifting with jump to } 0 & \text{if } p \in [\overline{\pi}_{LR}, p^*). \end{cases}$$

For fixed $\varepsilon > 0$ this yields the value function

$$V_\varepsilon(p) = \begin{cases} V_+(p; p^* + \varepsilon, V_S(p^* + \varepsilon)) & \text{if } p \in (p_*, p_* + \varepsilon), \\ V_S(p^* + \varepsilon) & \text{if } p = p_* + \varepsilon, \\ V_-(p; p^* + \varepsilon, V_S(p^* + \varepsilon)) & \text{if } p \in (p_* + \varepsilon, \overline{\pi}_{LR}), \\ V_R(p) & \text{if } p \in [\overline{\pi}_{LR}, p^*). \end{cases}$$

Since the ODE (20) is Lipschitz continuous, $\lim_{\varepsilon\to 0} V_-(p; p^* + \varepsilon, V_S(p^* + \varepsilon)) = V_{LS}(p)$ for all $p \in (p_*, \overline{\pi}_{LR})$. Hence, $\lim_{\varepsilon\to 0} V_\varepsilon(p) = V_{[p_*,p^*)}(p)$ for all $p \in W$. Since for $p < \overline{\pi}_{LR}$, the limit value cannot be achieved by any strategy if $W = (p_*, p^*)$, the sender has no best response. □

C.17 Proof of Lemma 16

Proof of Lemma 16. Let $V(p)$ denote the value function associated with the sender's best response. For $p < p_*$ we have $V(p) = 0$ since the receiver stops immediately and for $p \in W$, $V(p)$ is as characterized in Lemma 15.

For $p < p_*$, our refinement requires that the sender chooses the experiment that yields the highest flow payoff if this flow payoff is positive (see Appendix (A.1)). The flow payoff is given by

$$-c + \max_{0 \le q_2 \le p \le q_1 \le p^*,\, \alpha \in [0,1]} \lambda p(1-p)\left[\alpha \frac{V(q_1) - V(p)}{q_1 - p} + (1-\alpha)\frac{V(q_2) - V(p)}{p - q_2}\right].$$

Since $V(x) = 0$ for all $x < p_*$ and $V(q_1) > 0$ for $q_1 > p_*$, we must have $\alpha = 1$. The optimal value for q_1 maximizes

$$\frac{q_1 - V(p)}{q_1 - p}.$$

Since $V(p)$ is convex for $p \in [p_*, p^*]$ by Lemma 15.e, we must have $q_1 \in \{p_*, p^*\}$.

Consider first the case that $p_* \le \pi_{\ell R}$. In this case, Lemma 15 implies $V(p_*) = 0$. Therefore the optimal jump-target is $q_1 = p^*$. By (33), the flow payoff from the L-drifting experiment with jump-target p^* is negative for $p < \pi_{\ell L}$. This proves part (b) and part (a) for the case $p_* \le \pi_{\ell R}$.

Next consider the case that $p_* > \pi_{\ell R}$. In this case, Lemma 15 implies $V(p_*) > 0$. Therefore, the optimal jump-target is $q_1 = p_*$ if

$$\frac{V(p_*)}{p_* - p} > \frac{v}{p^* - p} \qquad \Longleftrightarrow \qquad p > \pi_0,$$

and $q_1 = p^*$ otherwise, with indifference at $p = \pi_0$. Hence $q_1 = p^*$ if and only if $p \ge \pi_0$. Convexity of $V(p)$ on $[p_*, p^*]$, together with convexity of $V_{RS}(p; q_R)$, implies that $\pi_{\ell R} < \pi_0$ and Lemma 17 implies $\pi_{\ell L} < \pi_{\ell R} < \pi_0$. Hence for $p < \pi_{\ell L}$ the optimal target is $q_2 = p^*$ which yields a negative flow payoff (by (33)) so that passing is optimal.

For $p \in [\pi_{\ell L}, \pi_0)$ the optimal target is $q_2 = p^*$, which yields a positive flow payoff (also by (33)). For $p \in [\pi_0, p_*]$ the optimal target is $q_2 = p_*$. Here, the flow payoff is positive since it is greater than the flow payoff if the target is $q_2 = p^*$, and the latter leads to a positive flow payoff (again by (33)). This shows part (c) and part (a) for $p_* > \pi_{\ell R}$.

Finally, for part (d), note that the flow payoff fails upper semi-continuity in q_2 at $q_2 = p_*$. Hence there exists no best response in $p \in (\pi_0, p_*)$. For $p < \pi_{\ell L}$, the argument in (c) remains valid which proves the statement of part (a) under the assumptions of part (d). □

C.18 Proof of Lemma 17

Proof of Lemma 17. Since $\pi_{\ell R}$ is the lowest value of p such that $V_{RS}(p; q_R) \ge 0$,[47] it suffices to show that $V_{RS}(\pi_{\ell L}; q_R) < 0$. From (31) we have

$$c = \frac{\lambda \pi_{\ell L}(1 - \pi_{\ell L})}{p^* - \pi_{\ell L}} v. \tag{39}$$

[47] See Section B.1.5 for the definition of $V_{RS}(p; q_R)$.

Similarly, for $\phi_{\ell L} < \phi$, it suffices to show that $U_\ell(\phi_{\ell L}) > U_{RS}(\phi_{\ell L}; q_R)$. Rearranging (34) we have.

$$c = \frac{\lambda \phi_{\ell L}(1-\phi_{\ell L})}{p^* - \phi_{\ell L}}(U_r(p^*) - U_\ell(p^*)). \tag{40}$$

Recall that

$$\begin{aligned} V_{RS}(p; q_R) &= \frac{p}{q_R}V_S(q_R) - \left(p \log\left(\frac{q_R}{1-q_R}\frac{1-p}{p}\right) + 1 - \frac{p}{q_R}\right)\frac{c}{\lambda} \\ &= \frac{p}{p^*}v - \left(\frac{2p(p^*-q_R)}{p^* q_R(1-q_R)} + p\log\left(\frac{q_R}{1-q_R}\frac{1-p}{p}\right) + 1 - \frac{p}{q_R}\right)\frac{c}{\lambda}, \end{aligned}$$

and

$$U_{RS}(p; q_R) = \frac{p}{p^*}U_r(p^*) + \frac{p^*-p}{p^*}u_\ell^L - \left(\frac{2p(p^*-q_R)}{p^* q_R(1-q_R)} + p\log\left(\frac{q_R}{1-q_R}\frac{1-p}{p}\right) + 1 - \frac{p}{q_R}\right)\frac{c}{\lambda}.$$

Rearranging the terms, and using (39) and (40), we get that both $V_R(\pi_{\ell L}; q_R) < 0$ and $U_\ell(\phi_{\ell L}) > U_{RS}(\phi_{\ell L}; q_R)$ reduce to the following inequality:

$$\frac{2p(p^*-q_R)}{p^* q_R(1-q_R)} + p\log\left(\frac{q_R}{1-q_R}\frac{1-p}{p}\right) + 1 - \frac{p}{q_R} > \frac{p^*-p}{p^*(1-p)}.$$

First, consider the case where $q_R = p^*$ (i.e., $p^* \leq \eta$). In this case, the necessary inequality simplifies to

$$p\log\left(\frac{p^*}{1-p^*}\frac{1-p}{p}\right) + 1 - \frac{p}{p^*} > \frac{p^*-p}{p^*(1-p)} \Leftrightarrow \log\left(\frac{p^*}{1-p^*}\frac{1-p}{p}\right) > \frac{p^*-p}{p^*(1-p)}.$$

This holds for any $p \in (0, p^*)$, because the two sides are identical if $p = p^*$, and

$$\frac{d}{dp}\left(\log\left(\frac{p^*}{1-p^*}\frac{1-p}{p}\right) - \frac{p^*-p}{p^*(1-p)}\right) = -\frac{1}{(1-p)^2}\left(\frac{1-p}{p} - \frac{1-p^*}{p^*}\right) < 0.$$

Now consider the case where $q_R = \xi_1$ (i.e., $p^* > \eta$). For this case, we show that the following function is always positive:

$$f(p) = \frac{2p(p^*-\xi_1)}{p^*\xi_1(1-\xi_1)} + p\log\left(\frac{\xi_1}{1-\xi_1}\frac{1-p}{p}\right) + 1 - \frac{p}{\xi_1} - \frac{p^*-p}{p^*(1-p)}.$$

It is straightforward that $f(0) = 0$ and $f(\xi_1) = (p^* - \xi_1)/(p^*(1-\xi_1)) > 0$. We obtain the desired result by showing that f is strictly concave over $[0, \xi_1]$. To that end, observe that

$$f'(p) = \frac{2(p^*-\xi_1)}{p^*\xi_1(1-\xi_1)} + \log\left(\frac{\xi_1}{1-\xi_1}\frac{1-p}{p}\right) - \frac{1}{1-p} - \frac{1}{\xi_1} + \frac{1-p^*}{p^*(1-p)^2},$$

and thus

$$f''(p) = -\frac{1}{p(1-p)^2} + \frac{2(1-p^*)}{p^*(1-p)^3} = -\frac{1}{(1-p)^3}\left(\frac{1-p}{p} - \frac{2(1-p^*)}{p^*}\right).$$

For any $p \in (0, \xi_1]$, $f''(p) < 0$, because

$$\frac{1-p}{p} \geq \frac{1-\xi_1}{\xi_1} = \frac{4-3p^* + \sqrt{p^*(9p^*-8)}}{3p^* - \sqrt{p^*(9p^*-8)}} > \frac{2(1-p^*)}{p^*} \text{ whenever } p^* > \eta.$$

□

C.19 Proof of Lemma 18

Proof of Lemma 18. Since $\phi_{\ell L}$ is the lowest value of p such that (cf. (35))

$$\frac{p(1-p)}{p^*-p}\lambda\left(U_r(p^*) - U_\ell(p^*)\right) \geq c,$$

it suffices to show that

$$\frac{\pi_0(1-\pi_0)}{p^*-\pi_0}\lambda\left(U_r(p^*) - U_\ell(p^*)\right) < c. \tag{41}$$

We show that (41) holds if c is sufficiently small and Condition C2 holds.

Step 1: $\phi_{\ell R}$ *as a function of* c, *and* $\lim_{c\to 0}\phi_{\ell R} = 0$.

Adopting the notation introduced in in Section B.1.5, the receiver's value function associated with R-drifting experiment below q_R can be written as

$$U_{RS}(p; q_R) = \frac{p}{p^*}U_r(p^*) + \left(1 - \frac{p}{p^*}\right)u_\ell^L - C_{RS}(p; q_R)$$

where

$$C_{RS}(p; q_R) = C_R(p; q_R) + \frac{p}{q_R}C_S(q_R) = \left(p\log\left(\frac{q_R}{1-q_R}\frac{1-p}{p}\right) + 1 - \frac{p}{q_R} + \frac{p}{q_R}\frac{2(p^*-q_R)}{p^*(1-q_R)}\right)\frac{c}{\lambda}.$$

Recall that $\phi_{\ell R}$ is defined to be the value such that

$$U_\ell(\phi_{\ell R}) = U_{RS}(\phi_{\ell R}; q_R) \Leftrightarrow C_{RS}(\phi_{\ell R}; q_R) = \frac{\phi_{\ell R}}{p^*}(U_r(p^*) - U_\ell(p^*)). \tag{42}$$

As c tends to 0, $C_{RS}(p; q_R)$ approaches 0 for any $p > 0$. Therefore, the right-hand side of this equation must also converge to 0, which implies that $\phi_{\ell R}$ converges 0.

Step 2: π_0 *as a function of* $\phi_{\ell R}$, *and* $\lim_{c\to 0}\pi_0 = 0$.

Recall that π_0 is defined to be the value such that

$$\frac{V_{RS}(\phi_{\ell R};q_R)}{\phi_{\ell R}-\pi_0}=\frac{v}{p^*-\pi_0}\Leftrightarrow \pi_0=\frac{\phi_{\ell R}v-p^*V_{RS}(\phi_{\ell R};q_R)}{v-V_{RS}(\phi_{\ell R};q_R)},$$

where

$$V_{RS}(\phi_{\ell R};q_R)=\frac{\phi_{\ell R}}{p^*}v-C_{RS}(\phi_{\ell R};q_R).$$

Replacing $C_{RS}(\phi_{\ell R};q_R)$ with equation (42), we get

$$V_{RS}(\phi_{\ell R};q_R)=\frac{\phi_{\ell R}}{p^*}v-\frac{\phi_{\ell R}}{p^*}(U_r(p^*)-U_\ell(p^*)).$$

Plugging this into the equation for π_0, we obtain

$$\pi_0=\frac{p^*\phi_{\ell R}(U_r(p^*)-U_\ell(p^*))}{(p^*-\phi_{\ell R})v+\phi_{\ell R}(U_r(p^*)-U_\ell(p^*))}. \tag{43}$$

Since $\lim_{c\to 0}\phi_{\ell R}=0$, we also have $\lim_{c\to 0}\pi_0=0$.
Step 3: *deriving an equivalent inequality to* (41).

Since $\lim_{c\to 0}\pi_0=0$, inequality (41) holds with equality in the limit as c tends to 0. We show that it holds with strictly inequality when c is sufficiently small (but strictly positive) by showing that the derivative of the left-hand side at 0 is strictly less than 1 (which is the derivative of the right-hand side), that is,

$$\lim_{c\to 0}\frac{d}{dc}\left(\frac{\pi_0(1-\pi_0)}{p^*-\pi_0}\lambda\left(U_r(p^*)-U_\ell(p^*)\right)\right)<1.$$

Notice that $\phi_{\ell R}$ is a function of c (via equation (42)), and π_0 can be expressed as a function of $\phi_{\ell R}$ (via equation (43)). Therefore, the above inequality is equivalent to

$$\lim_{c\to 0}\frac{d}{d\pi_0}\left(\frac{\pi_0(1-\pi_0)}{p^*-\pi_0}\lambda\left(U_r(p^*)-U_\ell(p^*)\right)\right)\frac{d\pi_0}{d\phi_{\ell R}}\frac{d\phi_{\ell R}}{dc}<1.$$

Since $\lim_{c\to 0}\pi_0=0$ and

$$\frac{d}{d\pi_0}\left(\frac{\pi_0(1-\pi_0)}{p^*-\pi_0}\lambda\left(U_r(p^*)-U_\ell(p^*)\right)\right)=\frac{p^*-2p^*\pi_0+\pi_0^2}{(p^*-\pi_0)^2}\lambda\left(U_r(p^*)-U_\ell(p^*)\right),$$

we have

$$\lim_{c\to 0}\frac{d}{d\pi_0}\left(\frac{\pi_0(1-\pi_0)}{p^*-\pi_0}\lambda\left(U_r(p^*)-U_\ell(p^*)\right)\right)=\frac{\lambda\left(U_r(p^*)-U_\ell(p^*)\right)}{p^*}.$$

Therefore, the following condition is sufficient for (41) to hold for c sufficiently small:

$$\lim_{c\to 0} \frac{d\pi_0}{d\phi_{\ell R}} \frac{d\phi_{\ell R}}{dc} < \frac{p^*}{\lambda\left(U_r(p^*) - U_\ell(p^*)\right)}. \tag{44}$$

Step 4: *Proving* (44).

We complete the proof by showing that

$$\lim_{c\to 0} \frac{d\pi_0}{d\phi_{\ell R}} = \frac{U_r(p^*) - U_\ell(p^*)}{v} \text{ and } \lim_{c\to 0} \frac{d\phi_{\ell R}}{dc} = \frac{p^*}{\lambda\left(U_r(p^*) - U_\ell(p^*)\right)}.$$

Since Condition C2 ensures that $\lim_{c\to 0} d\pi_0/d\phi_{\ell R} < 1$, these are (exactly) sufficient for the desired inequality.

From equation (43), we get

$$\frac{d\pi_0}{d\phi_{\ell R}} = \frac{p^* v}{((p^* - \phi_{\ell R})v + \phi_{\ell R}(U_r(p^*) - U_\ell(p^*)))^2} p^*(U_r(p^*) - U_\ell(p^*)).$$

Combining this with $\lim_{c\to 0} \phi_{\ell R} = 0$ (from Step 1) immediately yields

$$\lim_{c\to 0} \frac{d\pi_0}{d\phi_{\ell R}} = \frac{U_r(p^*) - U_\ell(p^*)}{v}.$$

For $\lim_{c\to 0} d\phi_{\ell R}/dc$, define

$$\begin{aligned} F(\phi_{\ell R}, c) &:= \frac{\phi_{\ell R}}{p^*}(U_r(p^*) - U_\ell(p^*)) - C_{RS}(\phi_{\ell R}; q_R) \\ &= \frac{\phi_{\ell R}}{p^*}(U_r(p^*) - U_\ell(p^*)) - \left(\phi_{\ell R} \log\left(\frac{q_R}{1-q_R}\frac{1-\phi_{\ell R}}{\phi_{\ell R}}\right) + 1 - \frac{\phi_{\ell R}}{q_R} + \frac{\phi_{\ell R}}{q_R}\frac{2(p^*-q_R)}{p^*(1-q_R)}\right)\frac{c}{\lambda}. \end{aligned}$$

Note that $\phi_{\ell R}$ is implicitly defined by $F(\phi_{\ell R}, c) = 0$. We have

$$\begin{aligned} \frac{\partial F}{\partial c} &= -\left(\phi_{\ell R} \log\left(\frac{q_R}{1-q_R}\frac{1-\phi_{\ell R}}{\phi_{\ell R}}\right) + 1 - \frac{\phi_{\ell R}}{q_R} + \frac{\phi_{\ell R}}{q_R}\frac{2(p^*-q_R)}{p^*(1-q_R)}\right)\frac{1}{\lambda}, \\ &= -\frac{C_{RS}(\phi_{\ell R}; q_R)}{c} = -\frac{1}{c}\left(\frac{\phi_{\ell R}}{p^*}(U_r(p^*) - U_\ell(p^*)) - F(\phi_{\ell R}, c)\right), \\ &= -\frac{1}{c}\frac{\phi_{\ell R}}{p^*}(U_r(p^*) - U_\ell(p^*)), \end{aligned}$$

where we have used (42) in the second line. Next, we have

$$\begin{aligned} \frac{\partial F}{\partial \phi_{\ell R}} &= \frac{U_r(p^*) - U_\ell(p^*)}{p^*} - \left(\log\left(\frac{q_R}{1-q_R}\frac{1-\phi_{\ell R}}{\phi_{\ell R}}\right) - \frac{1}{1-\phi_{\ell R}} - \frac{1}{q_R} + \frac{2(p^*-q)}{p^* q_R(1-q_R)}\right)\frac{c}{\lambda} \\ &= \frac{U_r(p^*) - U_\ell(p^*)}{p^*} - \frac{1}{\phi_{\ell R}} C_{RS}(\phi_{\ell R}; q_R) + \frac{1}{\phi_{\ell R}(1-\phi_{\ell R})}\frac{c}{\lambda} \\ &= \frac{1}{\phi_{\ell R}(1-\phi_{\ell R})}\frac{c}{\lambda} \end{aligned}$$

where we have used (42) to obtain the third line. Therefore, by the implicit function theorem we have[48]

$$\frac{d\phi_{\ell R}}{dc}=\frac{\phi_{\ell R}^2(1-\phi_{\ell R})}{p^*c^2}\lambda(U_r(p^*)-U_\ell(p^*)).$$

Recall that it suffices to show that

$$\lim_{c\to 0}\frac{d\phi_{\ell R}}{dc}=\frac{p^*}{\lambda\left(U_r(p^*)-U_\ell(p^*)\right)}.$$

To obtain this result, notice that the differential equation for $\phi_{\ell R}(c)$ can be expressed as follows:

$$\frac{d\phi_{\ell R}}{dc}\frac{1}{\phi_{\ell R}^2(1-\phi_{\ell R})}=-\frac{d}{dc}\left(\frac{1}{\phi_{\ell R}}+\log\left(\frac{1-\phi_{\ell R}}{\phi_{\ell R}}\right)\right)=K\frac{1}{c^2}=-K\left(\frac{1}{c}\right)'.$$

where $K:=\lambda(U_r(p^*)-U_\ell(p^*))/p^*$. Therefore, we have

$$\frac{1}{\phi_{\ell R}(c)}+\log\left(\frac{1-\phi_{\ell R}(c)}{\phi_{\ell R}(c)}\right)=\frac{K}{c}+\chi, \tag{45}$$

where χ is the constant of integration. Multiplying both sides by $\phi_{\ell R}(c)$ and letting c tend to 0, we have

$$1+\lim_{c\to 0}\phi_{\ell R}(c)\log\left(\frac{1-\phi_{\ell R}(c)}{\phi_{\ell R}(c)}\right)=K\lim_{c\to 0}\frac{\phi_{\ell R}(c)}{c}+\lim_{c\to 0}\phi_{\ell R}(c)\chi.$$

Using that $\lim_{c\to 0}\phi_{\ell R}(c)=0$ (by Step 1) and $\lim_{x\to 0}x\log x=0$, it follows that $\lim_{c\to 0}\phi_{\ell R}(c)/c=1/K$. Applying these results to the original differential equation, we conclude that

$$\lim_{c\to 0}\phi_{\ell R}'(c)=K\left(\lim_{c\to 0}\frac{\phi_{\ell R}(c)}{c}\right)^2(1-\lim_{c\to 0}\phi_{\ell R}(c))=K\frac{1}{K^2}=\frac{1}{K}=\frac{p^*}{\lambda(U_r(p^*)-U_\ell(p^*))}.$$

□

C.20 Proof of Lemma 19

Proof of Lemma 19. We first show that in any SMPE, $p_*\geq\max\{\phi_{\ell L},\pi_{\ell L}\}$. To this end, we consider the following "hypothetical" environment in which the sender is less constrained than in our model: she may choose two experiments, one generating upward jumps to $q_1(>p)$ and the other generating downward jumps to $q_2(<p)$. However, the sender is not constrained to split her attention between the two experiments. Instead, she can devote "full attention" to both. Specifically, she now has access to information structures in which $\alpha_1=\alpha_2=1$, $q_1<p<q_2$, and $\alpha_i=0$ for $i>2$. Assuming that the

[48]Note that we cannot apply the implicit function theorem to determine $d\phi_{\ell R}(0)/dc$ since $F(\phi_{\ell R},c)$ is not continuously differentiable at $(\phi_{\ell R},c)=(0,0)$.

receiver still takes action r whenever $p \geq p^*$, the sender's flow value of experimentation is given by

$$\tilde{v}(p) = \lambda p(1-p) \max_{0 \leq q_2 < p < q_1 \leq 1} \left[\frac{V(q_1) - V(p)}{q_1 - p} + \frac{V(q_2) - V(p)}{p - q_2} \right].$$

Now consider the specific strategy that the sender always targets $q_2 = 0$ and $q_1 = p^*$. Under the strategy, the sender's value function, denoted by $V_{FA}(p)$ (FA for "full attention"), satisfies the following equation whenever she continues to experiment:

$$c = \tilde{p} = \lambda p(1-p) \left(\frac{v - V_{FA}(p)}{p^* - p} - \frac{V_{FA}(p)}{p} \right) \Leftrightarrow V_{FA}(p) = \frac{p}{p^*} v - C_{FA}(p),$$

where

$$C_{FA}(p) = \frac{p^* - p}{p^*(1-p)} \frac{c}{\lambda}.$$

Notice that

$$V'_{FA}(p) = \frac{v}{p^*} + \frac{1}{(1-p)^2} \frac{1-p^*}{p^*} \frac{c}{\lambda} > 0 \text{ and } V''_{FA}(p) = \frac{2}{(1-p)^3} \frac{1-p^*}{p^*} \frac{c}{\lambda} > 0.$$

Therefore, if we set $V_{FA}(p) = 0$ whenever the value $V_{FA}(p)$ derived above is negative, then the function $V_{FA}(p)$ is convex on $[0, p^*]$.

For the same reason as in Section 5.1, the optimal q_1 maximizes the slope $V(q_1) - V(p))/(q_1 - p)$, while the optimal q_2 minimizes the slope $(V(p) - V(q_2))/(p - q_2)$. Convexity of $V_{FA}(p)$ over $[0, p^*]$ implies that if $V(p) = V_{FA}(p)$ then $q_1 = p^*$ and $q_2 = 0$ are optimal for the sender. Conversely, since $V_{FA}(p)$ is the value function that arises with the optimal jump targets $q_1 = p^*$ and $q_2 = 0$, it must be indeed the optimal value function. Finally, since $V_{FA}(p)$ is the value function for a problem in which the sender is less constrained, it forms an upper bound of the sender's value function in the original problem (given the waiting region $[p_*, p^*]$).

The corresponding value function of the receiver can be derived in a similar fashion:

$$U_{FA}(p) = \frac{p^* - p}{p^*} u_\ell^L + \frac{p}{p^*} U_r(p^*) - C_{FA}(p).$$

Specifically, this can be interpreted as the solution to the problem in which the receiver chooses two jump targets in $[0, p^*]$ and can devote full attention to each target.

In order to conclude that $p_* \geq \max\{\phi_{\ell L}, \pi_{\ell L}\}$, first observe that $V_{FA}(\pi_{\ell L}) = 0$ and $U_{FA}(\phi_{\ell L}) = U_\ell(\phi_{\ell L})$, which can be directly shown by plugging the values of $\pi_{\ell L}$ and $\phi_{\ell L}$ in Section B.2 into each function. Combining this with the fact that $V(p) \leq V_{FA}(p)$ and $U(p) \leq U_{FA}(p)$ for all $p \leq p^*$, but $V(\pi_{\ell L}) = V_{FA}(\pi_{\ell L})$ and $U(\phi_{\ell L}) \leq U_{FA}(\phi_{\ell L})$ leads to the desired result.

For the second result, suppose by contradiction that $p_* > \max\{\phi_{\ell L}, \pi_{\ell L}\}$ and $V(p_*) =$

0. If $p^* > \pi_{\ell L}$, then for all $p \in (\pi_{\ell L}, p_*)$, the sender uses the L-drifting experiment with jumps to p^*. This follows from the same arguments as in the proof of Lemma 16.(b). The receiver's strategy in the candidate equilibrium prescribes to stop for $p < p_*$. However, this is not a best response for $p \in (\phi_{\ell L}, p_*)$, which is a non-empty interval if $\phi_{\ell L} < p_*$. Therefore we must have $V(p_*) > 0$ if $p_* > \max\{\phi_{\ell L}, \pi_{\ell L}\}$.

Now suppose by contradiction that $U(p_*) > U_\ell(p^*)$. We show that this implies that there exists $\varepsilon > 0$, such that it is optimal for the receiver to wait if $p \in [p_* - \varepsilon, p_*)$, contradicting the conjectured equilibrium. If $V(p_*) > 0$, the same argument as in the proof of Lemma 16.(c) implies that the optimal experiment for the sender is "L-drifting with jumps to p_*" for $p \in [\pi_0, p_*)$. The (flow) benefit of this experiment for the receiver is

$$\frac{\lambda p(1-p)}{p_* - p}\left(U(p_*) - U_\ell(p)\right) - \lambda p(1-p)U_\ell'(p) = \frac{\lambda p(1-p)}{p_* - p}\left(U(p_*) - U_\ell(p_*)\right).$$

If $U(p_*) > U_\ell(p_*)$, this exceeds the cost c for p sufficiently close to p_*. Hence, the receiver is willing to stop for all $p < p_*$ only if $U(p_*) = U_\ell(p_*)$. □

C.21 Proof of Lemma 20

Proof of Lemma 20. Note first that both $\phi_{\ell L}$ and $\pi_{\ell L}$ approach 0 as c tends to 0. Moreover, both $U_S(p)$ and $U_{RS}(p; q_R)$ (whether $q_R = p^*$ or $q_R = \xi_1$) converge to $pU_r(p^*)/p^* + (p^* - p)u_\ell^L/p^*$. Therefore, the characterization of p_* from Lemma 19 implies that $p_* \to 0$ as $c \to 0$ since either p_* is given by the indifference condition for the receiver which implies $p_* = \phi_{\ell R} \to 0$, or $p_* = \max\{\phi_{\ell L}, \pi_{\ell L}\} \to 0$. Hence, $p_* < \xi_1$ if c is sufficiently low. We therefore assume that c is small enough so that $p_* < \xi_1$.

Suppose $p^* \leq \eta$. If $p_* = \max\{\phi_{\ell L}, \pi_{\ell L}\}$, Lemma 8 and Condition C2 imply that $p_* = \phi_{\ell L}$. Lemma 18 shows that there exits $c(p^*) > 0$ such that for $c < c(p^*)$, $\pi_0 < \phi_{\ell L}$. Note that convexity of $V_R(p)$ implies that $V_R(p) > 0$ for $p > \pi_0$. But this implies that $p_* = \phi_{\ell L}$ cannot be the lower bound in an equilibrium. If it was, the sender would prefer to use R-drifting for p close to $\phi_{\ell L}$ which yields $V_R(\phi_{\ell L}) > 0 = V_-(\phi_{\ell L}; \phi_{\ell L}, 0)$. By Lemma 17, $\phi_{\ell L} < \phi_{\ell R}$ which implies that $U_R(p) < U_\ell(p)$ for $p \in (\phi_{\ell L}, \phi_{\ell R})$. Therefore, it is not a best response for the receiver to wait for p close to $\phi_{\ell L}$. This contradicts the fact that the lower bound of the waiting region is $p_* = \phi_{\ell L}$.

The proof for the case $p^* > \eta$ is similar. The only necessary modification is that we have to show that $V_{RS}(p) > 0$ for $p > \pi_0$, which again follows from convexity of the sender's value function. □

C.22 Proof of Lemma 21

Proof of Lemma 21. Suppose by contradiction that $p_* > \pi_{\ell L}$. Then, by Lemma 19, $U(p_*) = U_\ell(p_*)$ and $V(p_*) > 0$. According to Lemma 15, $V(p_*) > 0$ arises only when the sender does not play L-drifting at p_*, and the game ends only when p reaches 0 or p^*. This implies that the players' expected values at p_* can be written as

$$V(p_*) = \frac{p_*}{p^*}v - C(p_*) \text{ and } U(p_*) = \frac{p_*}{p^*}U_r(p^*) + \frac{p^* - p_*}{p^*}u_\ell^L - C(p_*),$$

where $C(p)$ represents the total cost of delay common for both players. By straightforward algebra,

$$U(p_*) - U_\ell(p_*) = \frac{p_*}{p^*}(U_r(p^*) - U_\ell(p^*)) - C(p_*).$$

If $U(p_*) = U_\ell(p_*)$ and $V(p_*) > 0$, but Condition C2 fails, then we arrive at the following contradiction:

$$C(p_*) < \frac{p_*}{p^*}v \leq \frac{p_*}{p^*}(U_r(p^*) - U_\ell(p^*)) = C(p_*).$$

□

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