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Modelling contacts and transitions in the SIR epidemics model[1]

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Since the outbreak of the Covid-19 pandemic economists have turned to the SIR model and its subsequent variants for the study of the pandemic's economic impact. But the SIR model is lacking the optimising behaviour of economic models, in which agents can inuence future transitions with their present actions. We borrow ideas and modelling techniques from the Mortensen-Pissarides (1994) search and matching model and show that there is a well-defined solution in line with the original claims of Kermack and McKendrick (1927) but in which incentives play a role in determining the transitions. There are also externalities that justify government intervention in the form of imposing more restrictions on actions outside the home than a decentralised equilibrium would yield.

Covid Economics 5, 16 April 2020: 1-20

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1 Introduction

The disruption to the global economy caused by the covid-19 pandemic has led many economists to turn to Kermack's and McKendrick's (1927) SIR model and its subsequent variants for the study of its economic impact.[1] The SIR model is one in which agents inhabit different states and transition according to some process, so it is eminently suitable for economic analysis, being similar to models already in use, for example in the study of labour market dynamics.[2] But it is lacking the optimizing behaviour of economic models, in which agents can influence future transitions with their present actions. Transitions in the SIR model are determined by aggregates without a foundation in individual decision-making, in contrast to economic models, in which transitions are influenced by optimizing behaviour that evaluates the costs and returns of doing something now against the expected future payoffs. In this paper we introduce individual decision making in the SIR model, following established techniques from the economics literature.

To give an example of a process that plays a critical role in our paper, consider the "social distancing" decision of a "susceptible" agent, one that belongs to the state S of the SIR model and who is healthy but could catch the disease by coming into contact with an infected individual. In normal circumstances, without the disease, this person takes various actions that bring her into contact with others, such as working in an office environment, shopping in person or spending her leisure time socializing or attending sports events. When there is a possibility of an infection as a result of such actions, the agent may decide to restrict her social interactions by foregoing some of these actions, e.g., by buying groceries online for home delivery. Such restrictions reduce the payoffs of the agent but they also reduce the probability that the agent will transition to a state of infection (the I in the SIR model). The decision of how much to restrict present action (social distancing) is an optimizing one and it influences the later transitions. Policy makers talk regularly about the need to restrict social contact but individual responses to the covid-19 pandemic and why there is need for policy-makers to impose more social distancing than that chosen by agents are absent from the SIR model or any of its variants.

Our approach is to use the simple three-state model SIR, with state S consisting of individuals who are susceptible to the disease, state I consisting of individuals who are infected and state R consisting of the recovered individuals who have immunity.

[1] See for example Atkeson (2020), Stock (2020), Toda (2020) and Berger et al. (2020). All these papers offer extensions of the SIR model to account for the economic cost of the disease. Eichenbaum et al (2020) also extend the SIR model by endogenizing the infection rate but through working hours and consumption, not contact technologies.

[2] A introduction to the mathematics of the SIR model is in Weiss (2013) Useful summaries of the history of the SIR model and the basic mathematical formulation can be found in *https://en.wikipedia.org/wiki/Mathematical_modelling_of_infectious_disease* and *https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology*

Covid Economics 5, 16 April 2020: 1-20

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We assume that there are no natural births or deaths because of the difference in the time dimension of demographics and covid-19 transitions.

We borrow ideas and modelling techniques from the Mortensen-Pissarides (1994) search and matching model (Pissarides, 2000) and show that there is a well-defined solution in line with the original claims of Kermack and McKendrick (1927) but in which incentives play a role in determining the transitions. There are also externalities that justify government intervention in the form of imposing more restrictions on actions outside the home than a decentralized equilibrium will yield. We show that in an epidemic free agents will restrict their social contacts in order to reduce the probability of a future infection but they will not restrict them enough for two reasons. First, they will ignore the costs they cause when they transmit the disease to others and second they ignore any possible congestion externalities on health services.[3] These externalities justify government action that imposes more social distancing than people will choose.

But in a forward-looking economy restricting social action may delay reaching herd immunity, when the disease is no longer active, and this dynamic externality works against the planner's social distancing policy.[4] We show with simulations that when the transition rates are determined by the optimizing decisions of our model herd immunity is indeed delayed, sometimes substantially, but interestingly the number of people who get infected before it is reached is much lower than the number reached in the standard SIR model. To be more specific, in all our simulations we find that the fraction of susceptible people in the economy converges to the highest possible number consistent with herd immunity. We conjecture that this important finding will hold for a wide set of parameter values.

Section 2 describes the model in more detail and derives the individual maximizing choices. Section 3 shows the divergencies between the decentralized solutions and the choices of a central planner. Section 4 shows with simulations the impact of individual choices on the aggregate flows between states.

[3]These externalities are the two main reasons that the British government is giving for imposing strict social distancing. The slogan is "stay at home, protect the NHS and save lives."

[4]Once again the British context is revealing here. At the onset of the disease the government was emphasizing more the need to keep in good health to withstand the disease but warned about the prospect of many deaths as necessary to get rid of the disease and return to normal. But very quickly once the first deaths appeared the policy was changed to strict social distancing, never again mentioning herd immunity. It seems that faced with the imminent prospect of disease and death people (and their representatives) emphasize much more the short term need for survival than long-term social outcomes (act with a much higher rate of time preference).

Covid Economics 5, 16 April 2020: 1-20

CEPR PRESS

2 A Simple Covid-19 model: Decentralized equilibrium

In this section we develop a model of transitions with individual decision making that restricts functional forms to approximate the features of covid-19 as we know them today. In particular, transitions of susceptible individuals from state S to I depend on contacts, which arise in a variety of situations, such as work, shopping and leisure activities. Transitions for individuals in the infected group I to recovery R depend only on medical conditions related to the disease that the individual cannot influence.

We work in discrete time and define the period to be short; for simplicity we assume that infected individuals spend one period in that state. In terms of covid-19 the period is therefore a minimum of two weeks and a maximum of about five. We ignore deaths, as is usually done in the SIR model, being a small fraction of the infected population, in order to make use of the convenient assumption that population is constant.

Before we move on to describe the transitions in the susceptible state we write the simple value functions implied by these assumptions for individuals in states R and I, working in that order.

We assume that individuals who recover from an infection become immune to further infections. Given infinite horizons we can then write a constant V^R for the value of recovery. In the infected state individuals receive medical care. Although the total medical facilities available to covid-19 patients are not a constant, as even new hospitals have been put in place in some countries, our assumption is that they change much more slowly than the total number of infections. It follows that as total infections rise the facilities available to a patient fall, creating a medical congestion externality. In this state the individual receives care without making her own choices. We assume that the utility from being in this state is v_t, which could be either positive or negative. We assume that it is an increasing function of the per-capita medical facilities available, as in that case the patient is getting better quality care. To simplify the notation we make explicit only the dependence on the number of patients under treatment, I_t, the members of set I in period t, and assume, $v_t = v(I_t)$ with $v'(I_t) \leq 0$. The value of being in state I in period t is therefore,

$$V_t^I = \frac{v(I_t)}{1+r} + \frac{V^R}{1+r}, \tag{1}$$

where r is the rate of discount (making use of end of period discounting). If in turn we make the plausible assumption that the cost of being sick (e.g., hospitalization) depends on the value attached by the individual to the state of recovery (for example, earning capacity is a determinant of V^R and it is lost when the person is sick), (1) further simplifies to

$$V_t^R = \frac{1-\delta(I_t)}{1+r} V^R, \tag{2}$$

Covid Economics 5, 16 April 2020: 1-20

Covid Economics 5, 16 April 2020: 1-20

where δ is the fraction of V^R that corresponds to the cost of the disease to the individual, i.e., $\delta(I_t) \equiv -v(I_t)/V^R$ and so $\delta'(I_t) \geq 0$.

Susceptible individuals enjoy utility from their activities during the period. There are two types of activities in which the person can engage, activities in the home, such as work at home, home production, online shopping and home leisure activities, such as watching TV, and activities in society and the marketplace, such as going to the office, visiting shops and spending leisure time with friends. Social contact results only from the second set of activities. We denote the first set of activities by x_h and the second by x_s and write the per-period utility function as,

$$u_t = u(x_{ht}, x_{st}). \tag{3}$$

This function is assumed to satisfy the standard restrictions of a two-good utility function, with the additional assumption that $u(x_{ht}, 0) \geq 0$, i.e., survival does not require a person to leave the home. The choice of x_{ht} and x_{st} is constrained by a cost function which we assume for simplicity that it is a convex utility cost $c(x_{ht}, x_{st})$. We define net utility from all activities by,

$$\phi_t = \phi(x_{ht}, x_{st}) = u(x_{ht}, x_{st}) - c(x_{ht}, x_{st}), \tag{4}$$

assumed to be single peaked with $\phi(x_{ht}, 0) \geq 0$. The latter defines the value of net utility in the state of complete social distancing.

In state S individuals enjoy net utility as in (4) but run also the risk of infection through social contacts. Social contacts increase in x_{st}, in a way that we specify below, and depend also on the number of people in each of the three states. We assume in addition that not all social contacts lead to infection and let $k \in [0, 1]$ denote the probability that a contact leads to infection.[5] If $k = 0$ the disease is not infectious whereas $k = 1$ makes it extremely infectious, with every single contact between a person in state S and one in state I leading to infection. In general, we write the transition probability of a single agent from S to I as,

$$p_t = p(x_{st}, \bar{x}_{st}, k, S_t, I_t, R_t), \tag{5}$$

where $\bar{x}_{st}$ are the choices of social activities of other agents and S_t, I_t and R_t are the numbers of people in states S, I and R respectively and satisfy the normalization $S_t + I_t + R_t = 1 \ \forall t$. We assume,

$$\begin{aligned} \frac{\partial p(x_{st}, .)}{\partial x_{st}} &\geq 0, \\ p(0, .) &= 0, \end{aligned} \tag{6}$$

[5]Weiss (2013) defines a parameter τ as the fraction of her contacts that an infected individual actually infects and refers to it as the "transmissibility" of the disease. Our k is related to this parameter.

where $p(0,.)$ is the transition to infection in the state of complete social distancing. We will make explicit the dependence of individual transitions on the social actions of other agents and the number of agents in each state later in this section.

The value function of a single individual in state S is,

$$V_t^S = \max_{x_{ht},x_{st}} \left\{ \frac{\phi(x_{ht},x_{st})}{1+r} + p_t \frac{V_{t+1}^I}{1+r} + (1-p_t) \frac{V_{t+1}^S}{1+r} \right\}, \tag{7}$$

with the transition probability p_t given by the function in (5)-(6). The maximization conditions with respect to x_{ht} and x_{st} are

$$\frac{\partial \phi(x_{ht},x_{st})}{\partial x_{ht}} = 0 \tag{8}$$

$$\frac{\partial \phi(x_{ht},x_{st})}{\partial x_{st}} + \frac{\partial p(x_{st},.)}{\partial x_{st}} \left(V_{t+1}^I - V_{t+1}^S \right) = 0. \tag{9}$$

We impose the restriction $\left(V_{t+1}^I - V_{t+1}^S \right) < 0$, which is intuitive as it represents the difference in values from being infected and not being infected. It is clear from the first order conditions that in the case of an infectious disease healthy agents restrict their activities outside the home to avoid infection. Without an infectious disease the first order condition for activities outside the home would be $\partial \phi(x_{ht},x_{st})/\partial x_{st} = 0$, yielding a higher x_{st} than the solution in (8)-(9).

We now specify the contact technology that yields the infection probability $p(x_{st},.)$. This parallels the matching function of labour economics (Petrongolo and Pissarides, 2001) but with some important differences. In the matching function of the labour literature, more workers looking for jobs reduces the success probability of a single worker because of congestion externalities in the application process. Here more individuals coming out in the marketplace increases the chances of infection because a single exposed individual can infect many people; the infectious disease is "non-exhaustible," in the sense that many people could acquire it from a single person at the same time.

To provide an intuitive derivation of our contact function suppose x_s stands for the number of trips outside the house (omitting time subscripts for convenience). In each trip the person comes into contact with some individuals. How many these contacts are depends on how many times on average other people circulate outside their home. Let $\bar{x}_s$ be the number of times that people on average come out each period and assume that each person experiences, again on average, m contacts per period, defined by $m = m(\bar{x}_s)$, with $m'(\bar{x}_s) \geq 0$. The function $m(.)$ is similar to the matching function of labour economics in the sense that it depends on the structure of the marketplace, including density of population, transportation facilities, types of establishments etc. For example, consider two cities that are identical in all respects, except that one has more coffee bars than the other. If a resident goes out for a coffee, she will come across more people in the city with the fewer coffee bars, because each one in that city will be selling more coffee. So if $\bar{x}_s$ is the same in the two cities, $m(\bar{x}_s)$ will be larger in

Covid Economics 5, 16 April 2020: 1-20

the city with the fewer coffee bars. In this paper we assume that the function $m(.)$ is fixed, at least in the short to medium run, although it is likely to be different across locations like cities or countries.[6]

Consider now the choices made by the individual who does not influence market outcomes, where as before x_s without the bar is the chosen activity level of the person. Here we follow the method used in search theory to choose the optimal search intensity (Pissarides, 2000, chapter 5). With $m(\bar{x}_s)$ representing the total number of contacts for $\bar{x}_s$ outings, each outing on average generates $m(\bar{x}_s)/\bar{x}_s$ contacts. So if the individual chooses to go out of the home x_s times, her contacts are on average $x_s m(\bar{x}_s)/\bar{x}_s$. These are total contacts. We are interested in the contacts that can potentially lead to an infection, and these are contacts that involve a person from set I. Here we make a simplifying assumption that is common in the SIR literature, that the susceptible person cannot distinguish a priori who is in which state. We assume that on average the fraction of contacts that are infected is equal to the fraction of persons in set I in the population. With the normalization of the population size to unity, we obtain that the probability that a contact in period t is with an infected person is simply I_t. Given that the probability that a contact with an infected person leads to an infection is the constant k, we write as an approximation the transition from the susceptible to the infected state for the person who chooses x_{st} outside activities as,[7]

$$p_t = k\frac{x_{st}m(\bar{x}_{st})}{\bar{x}_{st}}I_t. \tag{10}$$

This expression satisfies the extreme properties that for a non-infectious disease ($k = 0$) or complete social isolation ($x_{st} = 0$), $p_t = 0$.

It follows from (10) that p_t now depends on a smaller set of variables than in the general expression (5) and its partial derivative satisfies,

$$\frac{\partial p_t}{\partial x_{st}} = k\frac{m(\bar{x}_{st})}{\bar{x}_{st}}I_t = \frac{p_t}{x_{st}}. \tag{11}$$

In moving from individual transitions to the average for a market where all agents optimize we assume a symmetric Nash equilibrium in which all agents choose the same policy, so $x_{st} = \bar{x}_{st}$. For notational simplicity we drop the bar from $\bar{x}_{st}$ and write the equilibrium p_t as,

$$p_t = km(x_{st})I_t, \tag{12}$$

[6]For example, reports in the media warn that it would be very difficult to reduce social contacts in very dense cities like Mumbai, whereas there has been success in such reductions in less dense cities like London.

[7]Another derivation of the probability of meeting at least one infected individual is to reason as follows. Since for each contact there is a probability $(1 - I_t)$ that the person does not meet an infected person, there is a probability $(1 - I_t)^{x_s m(\bar{x}_s)/\bar{x}_s}$ that the person does not meet any infected persons in her x_s outings. If I is a small fraction of the population, this is approximately equal to $\exp\{I x_s m(\bar{x}_s)/\bar{x}_s\}$, so the probability of meeting an infected person is $1 - \exp\{.\}$ and for small transition probability this is approximately equal to the expression in the text.

Covid Economics 5, 16 April 2020: 1-20

Covid Economics 5, 16 April 2020: 1-20

with x_{st} obtained as the solution to (8), (9) and (11), under the restriction $\bar{x}_{st} = x_{st}$ and given all the value equations previously derived.

This completes our specification and derivation of the solution equations for the agents in the model. It is noteworthy that when comparing with the epidemiological SIR model, our innovation is the insertion of x_{st} in the transition probability p_t, which picks up the disincentives that the susceptible individuals have when they go out of their homes. Some obvious properties of this choice, given our strong functional assumptions, can easily be derived. There is social distancing (lower x_{st}), for higher k and higher I_t (more infectiousness of the disease or more infected people) and for higher unpleasantness from treatment (higher difference between the value of avoiding infection V_t^S and getting infected, V_t^I).

We now complete the description of the decentralized equilibrium by deriving the transitions implied by our individual models. With transition probability from state S to state I given by (12), the number of people in the S state falls each period by the fraction in (12). This is also the number of people who join the I state, whereas a period later every infected individual joins the recovery state R. The implied transitions are,

$$\Delta S_{t+1} = -km(x_{st})I_tS_t \tag{13}$$

$$I_{t+1} = km(x_{st})I_tS_t \tag{14}$$

$$\Delta R_{t+1} = I_t, \tag{15}$$

with Δ denoting the first difference operator. We note that in the standard SIR model the parameter β that gives the transition from S to I plays a critical role and is usually assumed to be a constant; here β ban be expressed as,

$$\beta = \beta(x_{st}) = km(x_{st}). \tag{16}$$

In addition, since we assume that infected people recover in one period, our model implies that $R_0 = \beta(x_{st})$, where R_0 is the key parameter referred to as the "basic reproductive number" of the disease and it is critical in determining the future path of the disease. It has also featured prominently in the policy debate around Covid-19.

We are now in a position to define our decentralized equilibrium.

Definition 1 *A decentralized epidemic equilibrium is a set of sequences of state variables* $\{S_t, I_t, R_t\}_{t=0}^{\infty}$*, a set of value functions* $\{V_t^S, V_t^I, V_t^R\}_{t=0}^{\infty}$*, and a set of sequence of probabilities and social contacts* $\{p_t, x_{ht}, x_{st},\}_{t=0}^{\infty}$ *such that, for given initial conditions* $S(0) = 1 - \epsilon$*,* $I(0) = \epsilon$*,* $R(0) = 0$

1. S_t, I_t, R_t *solve equations (13-15)*

2. V_t^S, V_t^I, V_t^R *solve equations (7), (1) and (2)*

3. x_{ht} *and* x_{st} *solve the first order conditions (8) and 9)*

4. p_t solves equation (12)

The next step is to ask whether the social distancing obtained from this equilibrium is the optimal one in a decentralized society or whether stricter government restrictions are needed.

3 Externalities and deviations from social efficiency

As in other models of pairwise interaction, we would expect the decision strategies derived in the preceding section to be subject to externalities and inefficient outcomes. We address this question in the following simple manner. Take equation (7), which describes the value of being in the initial state S and is forward-looking with an infinite horizon. If a social planner was making the choices that the individual was making, would she choose the same level of x_{ht} and x_{st} as the individual? The social planner is aware that the equilibrium is a symmetric Nash equilibrium and that contacts involve at least two people, so when one person meets another the other person is also involved in a meeting. The social planner is also aware that there is a medical congestion externality due to limited medical resources and welfare depends on the quality of medical services, and also has foresight and is aware that with her actions she can influence the size of the states S and I in future periods.

With these assumptions the relevant transition probability for the social planner is (12), in which $x_{st} = \bar{x}_{st,}$ and the choice variable is the average for all persons in S, x_{st}. We do not allow the planner to use "mixed strategies" and allow different individuals to choose different activity levels in the same period. The social planner takes the stocks in period t as predetermined and solves the problem,

$$\hat{V}_t^S(S_t, I_t) = \max_{x_{ht}, x_{st}} \left\{ \frac{\phi(x_{ht}, x_{st})}{1+r} + p_t \frac{\hat{V}_{t+1}^I}{1+r} + (1-p_t) \frac{\hat{V}_{t+1}^S(S_{t+1}, I_{t+1})}{1+r} \right\}, \tag{17}$$

with $\hat{V}_{t+1}^I$ given by (14). The first-order conditions are,

$$\frac{\partial \phi(x_{ht}, x_{st})}{\partial x_{ht}} = 0 \tag{18}$$

$$\begin{aligned} & \frac{\partial \phi(x_{ht}, x_{st})}{\partial x_{st}} + \frac{\partial p(x_{st}, .)}{\partial x_{st}} \left(\hat{V}_{t+1}^I - \hat{V}_{t+1}^S \right) \\ & + p_t \frac{\partial \hat{V}_{t+1}^I}{\partial x_{st}} + (1-p_t) \left(\frac{\partial \hat{V}_{t+1}^S}{\partial I_{t+1}} \frac{\partial I_{t+1}}{\partial x_{st}} + \frac{\partial \hat{V}_{t+1}^S}{\partial S_{t+1}} \frac{\partial S_{t+1}}{\partial x_{st}} \right) \\ = & \ 0 \end{aligned} \tag{19}$$

The first choice in equation (18) corresponds exactly to the one in decentralized equilibrium, (8), so conditional on the choice of x_{st}, the choice of x_{ht} in the decentralized equilibrium is efficient.

Covid Economics 5, 16 April 2020: 1-20

The two first terms in equation (19) in the first line of the equation capture the utility gain from the social activity and the transition cost of the activity associated with more people being infected, respectively. Analogous terms can be found in the first order conditions for the decentralized maximization problem (9). We refer to these as the *static* welfare effects of increasing x_{st}. The choice of the efficient number of social contacts depends in addition on two terms that are not in (9). The first of these captures the medical congestion externalities, in that decisions made today about contacts influence the number of sick people tomorrow and hence the cost of treating them. Recall the definition of V_t^I in (1) and the transition (14), which clearly show this dependence. Finally, the activity level in period t determines the stocks of susceptible and infected people in period $t+1$, and hence the continuation value of V^S. This is captured by the term in (19) in the second line of the equation, and we refer to it as the immunity externality. We refer to these as the *dynamic aspects* of the planner's maximization problem.

The inefficiencies of the static maximization problem

Suppose for the moment that we zero the dynamic effects, so the efficient outcome for x_{st} is given by

$$\frac{\partial \phi(x_{ht}, x_{st})}{\partial x_{st}} + \frac{\partial p(x_{st}, .)}{\partial x_{st}} \left(\hat{V}_{t+1}^I - \hat{V}_{t+1}^S \right) = 0. \tag{20}$$

The solution from this equation for x_{st} coincides with the solution from (9) if the partial derivative $\partial p(x_{st}, .)/\partial x_{st}$ coincides with the solution for the partial in the decentralized problem. The latter is given in (11) whereas the former can easily be calculated from (12), and it is

$$\frac{\partial p(x_{st}, .)}{\partial x_{st}} = km'(x_{st})I_t = \frac{p_t m'(x_{st})}{m(x_{st})}. \tag{21}$$

Comparison with (11) immediately gives that in the absence of the medical externality efficiency of the decentralized decision requires,

$$\frac{x_{st} m'(x_{st})}{m(x_{st})} = 1. \tag{22}$$

This requirement parallels the familiar elasticity condition from matching theory, often referred to as the Hosios (1990) condition, which applies to situations of pairwise matching (see Pissarides, 2000, chapter 8). What does it mean in our context?

Unit elasticity in matching, or linear matching technology, is a restriction that can be justified when the agent has full control over the number of people she meets when going out. For example, suppose an agent decides beforehand to go out to meet exactly x' people and does not come into contact with any other. If she goes a second time with the same plan then she meets $2x'$ people - constant returns. If she goes out to get a coffee and no one crosses her in the street or comes close to her in the coffee bar,

Covid Economics 5, 16 April 2020: 1-20

Covid Economics 5, 16 April 2020: 1-20

she meets exactly one person, the barrista. If she goes out a second time for a coffee, the same happens, has a second meeting with a barrista. If one believes that this is an accurate description of a meeting process then private social distancing is the same as a benevolent social planner would choose.

But in practice we come into contact with many people who are going about their business in social space. These contacts are unintended and on average they will be more the more people choose social activities. It is more likely that the contact process will be exhibiting increasing returns to scale, because as circulation increases in given space the number of random contacts increases by more than in proportion. Consider again the coffee example. Suppose that on the way to getting a coffee the person crosses at random two other people and everyone in this economy goes out of the home twice a day. Then each time she goes out she comes into contact with three people, the barrista and the two street contacts, and so the total meetings of this person during the day are 6. But now if everyone doubles their activity, instead of two random meetings she will have four, so each time she goes out she will meet five people. With double her social activity she will go out four times, so the total meetings during the day are $5 * 4 = 20$. Doubling x_{st} from 2 to 4 led to an increase in contacts from 6 to 20.

The justification for increasing returns is similar to the one used by Peter Diamond in his famous "coconut" paper (Diamond, 1982). In that paper islanders posses a coconut which they acquire by climbing a tree but they cannot consume their own coconut. They have to find another islander with a coconut and swap nuts. Diamond's claim was that if the number of islanders climbing trees doubled, a passive islander was more likely to come out and climb a tree because the probability of finding a trade would be higher. Subsequent work did not find support for this claim because as both buyers and sellers double in number they create congestion for each other and so many swaps are crowded out (Petrongolo and Pissarides, 2001). In the context of an epidemic it is precisely this congestion that justifies the increasing returns, because of the non-exhaustive nature of the disease. I can pass a disease to a very large number of people but I can only give my coconut to one person. Diamond's intuition for increasing returns applies to this model much more than in a model of exchange.

Suppose then for the sake of illustration of the impact of the externality that $m(x_{st}) = x_{st}^{\alpha}$, with $\alpha \geq 1$. Then (21) implies

$$\frac{\partial p(x_{st}, .)}{\partial x_{st}} = km'(x_{st})I_t = \alpha \frac{p_t}{x_{st}}, \tag{23}$$

and so comparison of (9) with (20) immediately yields that the social planner will choose a higher marginal effect $\partial \phi(x_{ht}, x_{st})/\partial x_{st}$, or lower social activity. If individuals choose their own social activity they will go out too much because they ignore the infectious impact that their social activities have on others.

Covid Economics 5, 16 April 2020: 1-20

The inefficiencies of the dynamic maximization problem

In order to examine the role of dynamic externalities we turn off the static externality by working with a linear contact technology. Suppose also for now that the cost of illness is independent of the number of people infected, so that there are no medical externalities. Dynamic externalities still imply that the equilibrium allocation is generically inefficient.

Combining equations (14) and (19) gives that

$$\frac{\partial \hat{V}^S_{t+1}}{\partial I_{t+1}}\frac{\partial I_{t+1}}{\partial x_{st}} + \frac{\partial \hat{V}^S_{t+1}}{\partial S_{t+1}}\frac{\partial S_{t+1}}{\partial x_{st}} = \left(\frac{\partial \hat{V}^S_{t+1}}{\partial I_{t+1}} - \frac{\partial \hat{V}^S_{t+1}}{\partial S_{t+1}}\right) km'(x_{st})S_t I_t \tag{24}$$

Consider first the derivative $\partial \hat{V}^S_{t+1}/\partial I_{t+1}$. This is a contagion externality: if more people are infected in this period, more people are around to infect susceptible people in the next period. As long as the planner wants to keep the number of infected individuals down, this effect is negative.

Consider then the derivative $\partial \hat{V}^S_{t+1}/\partial S_{t+1}$. This is the effect of having fewer susceptible people around, or, since $R_t = 1 - S_t - I_t$, the effect (for a given I_t) of having more recovered people around. This is a positive effect, as it moves the society closer to herd immunity. We refer to this as the immunity externality.

From an *a priori* perspective, it is not clear if the planner would like to implement a higher or a lower activity level than the level realized in the decentralized solution. Clearly, the internalization of the contagion externality may easily lead the planner to reduce the activity level, but the immunity externality may give a strong push-back. Each individual has an incentive to reduce her activity level in order to avoid being among those who get ill before herd immunity is obtained. However, this is similar to a rat race, and introduces a positive externality from activity (a negative externality from passivity) that the planner internalizes.

Consider finally the impact of medical congestion. In a static perspective, this leads to a negative externality associated with activity that the planner might internalize by imposing more social distancing. To show this we note, from (19) and (1), that,

$$p_t \frac{\partial \hat{V}^I_{t+1}}{\partial x_{st}} = p_t \frac{v'[km'(x_{st})I_{t+1}]}{1+r} < 0, \tag{25}$$

and so again the social planner will choose lower social activity than the decentralized equilibrium. This effect works through the number of people in the infected state next period, and so the intuition behind it is that by lowering the transition rate, the planner reduces the medical congestion externalities and improves the medical facilities available to patients. However, in a dynamic equilibrium this is less clear. If the medical externalities are expected to be bigger in the more distant future, the planner on the margin may prefer more people being ill early on (when there is spare capacity in the health sector) rather than later on (when the capacity constraint binds).

Clearly, the planner will aim at reaching herd immunity with the highest possible share of people remaining susceptible. As will be clear below, the decentralized solution reaches herd immunity with the highest possible number remaining susceptible consistent with herd immunity, given that x is privately optimal in the new steady state. We conjecture that the optimal path will converge to the same steady state level of S, with x converging to its pre-infection level, albeit at a slower speed than the decentralized equilibrium.

We close this section by briefly considering the impact of vaccination, had one being made available. The probability that the vaccine arrives between two consecutive periods is denoted λ. If a vaccine arrives, a susceptible individual obtains the same lifetime value as a recovered individual, V^R, without having to go through a costly period of illness. It follows that the Bellman equation of a susceptible individual adjusts to

$$V_t^S = \max_{x_{ht}, x_{st}} \left\{ \frac{\phi(x_{ht}, x_{st})}{1+r} + (1-\lambda)\left(p_t \frac{V_{t+1}^I}{1+r} + (1-p_t)\frac{V_{t+1}^S}{1+r} \right) + \lambda \frac{V^R}{1+r} \right\} \tag{26}$$

People become more cautious to avoid the disease in the hope that a new vaccine will be discovered. We know that V^R is greater than V^S and V^I. Therefore, V^S is increasing in λ while V^I stays constant. It follows that an increase in λ will increase the utility loss associated with getting the disease, and hence reduce the privately optimal x_{st}.

In addition, the possibility of obtaining a vaccine in the future reduces the value of obtaining herd immunity from infections, and hence reduces the positive externality associated with a higher number of recovered individuals. As a result, we conjecture that the possibility of a discovery of a vaccine will reduce the planner's optimal activity level more than the activity level in the decentralized equilibrium.

4 Simulations

Parameterization

We make the following parameterization assumptions: The (indirect) utility function can be written as a function of the control x_{st} only. We suppress the subscript s, and write $\phi(x_t) = x_t - x_t^2/(2c)$, $c \leq 2$. In the simulations below we set $c = 1$. The contact function is $m(x) = kx^\alpha$, $k \leq 4$, $\alpha \geq 1$. In the simulation below $\alpha = 1$ and $k = 2.2$.[8] The interest rate is $r = 1/0.998 - 1 = 0.002$ (if a period is two weeks this gives a r of 0.05 on annual basis.

After recovery, the agents set x_t so as to maximize per period utility $\phi(x_t)$. Hence the agent sets $x = c$, and obtains per period utility $c/2$. The latter implies that $V^R =$

[8]Here k comprises of the product of the contamination probability per contact and the constant in the meeting function, and hence can be greater than 1. The value of $k = 2.2$ is in the range of the parameter R_0 in the SIR model used for simulating the diffusion of Covid-19 (Wu et al., 2020).

Covid Economics 5, 16 April 2020: 1-20

$\frac{c(1+r)}{2r}$. For the cost of hospitalization, we assume an exponential function such that $\delta(I) = \bar{g}e^{g_1 I}$. For assigning values to this function, we simply suppose that the cost of being ill is doubled if 1 percent of the population is infected. Then $g_1 = \ln 2/0.01 \approx 70$. The other parameter in the function is $\bar{g} = .6$

The key 4 difference equations in the simulation thus read (we now use beginning of period discounting)

$$S_{t+1} = S_t - kx_t^\alpha I_t S_t \tag{27}$$

$$I_{t+1} = kx_t^\alpha I_t S_t \tag{28}$$

$$V_t^S = x_t - x_t^2/(2c) + \frac{1}{1+r}\left(x_t^\alpha kI_t(1-\bar{g}e^{g_1 I_{t+1}})\bar{V}_R\right) + \frac{1}{1+r}\left(1 - x_t^\alpha kE_t)V_{t+1}^S\right) \tag{29}$$

$$x_t = c\left(1 - kx_t^\alpha I_t(V_{t+1}^S - (1-\bar{g}e^{g_1 I_{t+1}})V^R)\right) \tag{30}$$

The model features 3 terminal conditions for the sequences x_t, I_t and V_t^S, so that

$$I_\infty = 0; \quad x_\infty = c; \quad V_\infty^S = V^R \tag{31}$$

The model's solution is obtained with shooting algorithm- a standard solution algorithm for system of difference equations that are highly non linear and feature both initial and terminal conditions (Sargent and Stuchurski, 2020) .

Dynamic Path

We perform two simple quantitative exercises. The first simulation plots the dynamics of the states S_t and I_t along a decentralized epidemic equilibrium (Figure 1). The top panel in the figure refers to the dynamics of the susceptible individuals. As patient 0 is exogenously imposed to the system, more and more people are infected as time goes by. The stock of susceptible people, initially normalized to 1, converges to a steady state size of .45, suggesting that approximately 55 percent of the population gets infected before the virus dies out and $I(\infty) = 0$. If one period of time corresponds to two weeks, Figure (1) implies that full herd immunity is reached in more than 10 years. While the full convergence appears very slow, one should also note that after 5 years since the outbreak of the 0-patient, more than 35 percent of the population are infected. This pattern is entirely driven by the optimal fall in activity x, that clearly follows a u-shaped behaviour. Interesting enough, the fall in activity reaches the minimum in the 6th period, or 4 months after the spread of the disease. Thereafter activity rises until the steady state. In percentage terms, the maximum fall in activity corresponds to 55 percent of its steady state value.

The second simulation compares the forward looking epidemic equilibrium with that of a traditional SIR model (Figure 2). The latter simulation applies a constant x throughout the epidemic. As a benchmark case, the level of x is set so as to match the

Covid Economics 5, 16 April 2020: 1-20

transversality condition in the optimizing SIR. The rest of the parameters are identical. The differences between the two paths are striking. In Figure (2), the traditional SIR simulation is the dotted line, while the continuous line refers to the optimizing SIR. The steady state level of susceptible individuals in the standard SIR model is .08, suggesting that 92 percent of the population gets infected.

Clearly, herd immunity is reached much faster in the traditional SIR. After approximately 4 months, 80 percent of the people get the disease, and herd immunity is largely on its way. Nevertheless, the longer time to reach herd immunity with endogenous behaviour comes with a large gain. The precautions of the forward looking individuals save 35 percent of the population from the illness.

Herd Immunity

As discussed in the numerical simulation, an important variable is S^{∞}, the number of susceptible individuals in the new steady state equilibrium after herd immunity is obtained. Since in steady state $I = 0$, we have that $R^{\infty} = N - S^{\infty}$, where N is the total population.[9]

Let $\overline{x}$ denote the activity level in a steady state. It follows that $\overline{x}$ is the activity level in the new steady state, and is obtained by plugging in $I = 0$ in the behavioural equations above. Hence $\bar{x}$ maximizes the current period utility $\phi(x_h, x_s)$ (for the optimal value of x_h). As suggested in the previous section, this is the activity level in period 0 of our model, and is the activity level in the reference model with fixed activity level.

Define $R_0 = km(\bar{x})$. This is the (basic) reproduction number in our model. In steady state, the *effective* reproduction number $S^{\infty}R_0/N$ has to be less than or equal to 1. Hence a lower bound for S^{∞}, $S^{\min}$, is given by

$$S^{\min} = R_0^{-1}N \tag{32}$$

In the standard SIR model, which is similar to our model with constant x, the maximum number of infected individuals is obtained when $I_t = I_{t+1}$. Plugging $I_t = I_{t+1}$ into (14), gives that $S = NR_0^{-1}$ $(= S^{\min})$. At that point, the disease is on retreat, as the effective reproduction number falls below 1. However, it takes time before the disease "burns out", and along the path many people get infected. It can be shown (Weiss 2013) that the equilibrium value of S in the continuous time SIR model, denoted S^{SIR}, is given by the solution to the equation $\ln S^{SIR}/N = R_0(S^{SIR}/N - 1)$. This equation can be solved numerically, and for $R_0 > 1$ it gives that S^{SIR} is substantially lower than $S^{\min}$.

When x is set by forward-looking individuals, this is no longer the case. At the point at which I reaches its maximum level, the probability of obtaining the illness is at its highest, and the agents reduce their activity level relative to the steady-state level. If we denote by x^I the equilibrium value of x at the point at which I reaches its maximum level, it follows that the stock of infected people at this point is given by

[9] In Section 2 total population is normalized to 1.

Covid Economics 5, 16 April 2020: 1-20

$S^{\min} x^I / \bar{x} < S^{\min}$. From that point S still increases (as does x) until the disease burns out, but from a lower level.

In our simulations, $S^{\infty} = S^{\min}$, meaning that the stock of susceptible people converges to the highest level consistent with herd immunity. Hence, when the model is extended to allow for forward-looking agents, our simulations illustrate that herd immunity may be obtained with the lowest possible number of people becoming infected.

5 Conclusions

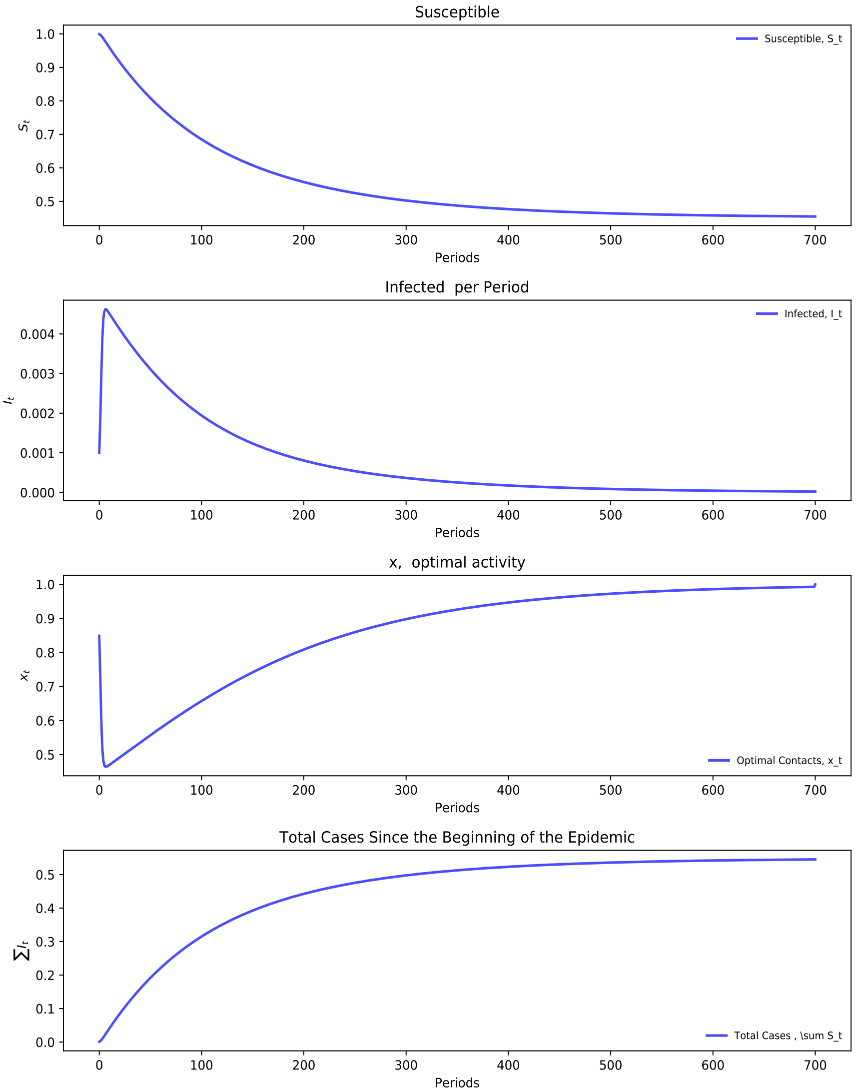
With the outbreak of Covid-19, the SIR epidemics model has entered mainstream economics. The dynamic properties of the SIR models (Kermack and McKendrick, 1927) naturally feature a herd immunity at the end of the epidemic. Yet, the coefficients that describe the transitions across the three main states of the model are independent of private decision-making. This paper has borrowed concepts from the search and matching model (Pissarides, 2000) to endogenize the key transition from the susceptible state to the infected one. Forward looking agents now choose the intensity of their contacts to maximize utility, but are fully aware that higher social contacts lead to a higher probability of infection. A first contribution of this paper is the introduction of the contact function and the forward looking decisions of the susceptible agents to the simple SIR model, in a way that will be familiar to economists and easily extendable to other more complex situations.

Our theoretical perspective has also welfare implications. The decentralized epidemic equilibrium is likely to be suboptimal. The paper uncovered four types of externalities, referring to static or dynamic situations. The externalities in a static, short-horizon context refer to the transition probability from the susceptible to the infected state and how it relates to the social distance between agents and the hospitalization congestion effect when large numbers become infected. In a dynamic context the externalities arise from changes in the stocks of susceptible and infected persons as they affect contagion and herd immunity. We argue that when comparing the private and social equilibrium, only the herd immunity externality provides incentives to the central planner to speed up the spread of the epidemic. We believe that the latter two externalities would survive to a broader class of model, and are not specific to the search and matching approach.

Much remains to be done. The model certainly needs to be taken to the data. We argue that the contact function features increasing returns to scale, but the actual size of the parameters is an empirical question.

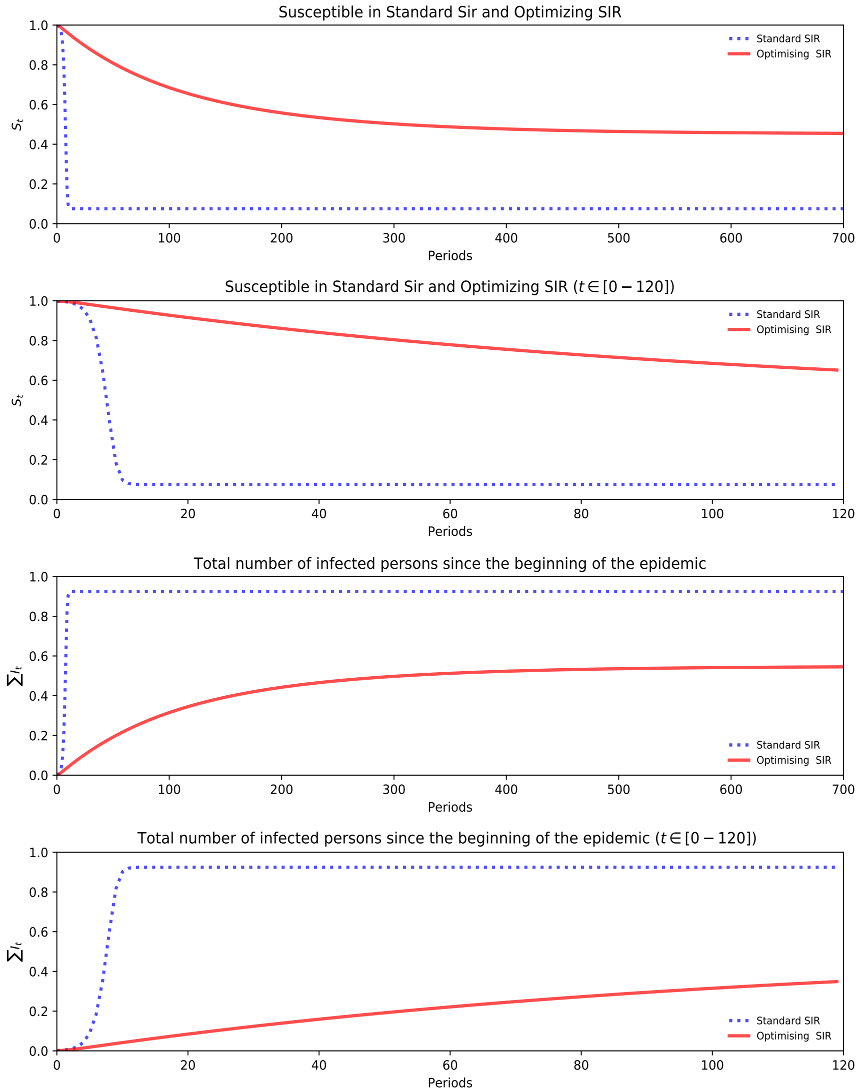
Covid Economics 5, 16 April 2020: 1-20

Figure 1: Dynamics of the Epidemic in Optimizing SIR



Covid Economics 5, 16 April 2020: 1-20

Figure 2: Epidemic in Optimizing SIR and Standard SIR



Covid Economics 5, 16 April 2020: 1-20

Covid Economics 5, 16 April 2020: 1-20

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Annex: Shooting Algorithm for the Simulation

1. Chose initial values $I_0 = \epsilon$, $S_0 = 1 - \epsilon$.
2. Choose a number of periods, $t = 0, ..., T$.
3. Choose a vector of activity levels $x_0, ...x_T$, with x_T given by the transversality condition.
4. Set $\bar{x}_t = x_t \forall t$.
5. Calculate $I_0, ...I_T$ and $S_0, ...S_T$ using (27) and (28)
6. Calculate V_T^S using the transversality (endpoint) conditions
7. Calculate backward $V_{T-1}^S, V_{T-2}^S, ..., V_0^S$ using (29)
8. Calculate the optimal $x_0^o, x_1^o, ...x_T^0$ using (30)
9. Update choosing $x_t' = \lambda x_t + (1 - \lambda) x_t^o$ for $t = 0, ..., T - 1$, $\lambda \in (0, 1)$
10. Repeat the procedure from step 5 until $|x_t' - x_t| \approx 0$

Covid Economics 5, 16 April 2020: 1-20