Bond Risk Premia: The Information in Really Long-Maturity Forward Rates

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This version: February 2, 2021

Abstract

Long term forward rates contain information that greatly improves the precision with which expectations of future short rates can be distinguished from risk premia in the term structure. Indeed, in affine models, the slope of the term structure of risk premia for long maturities is very closely approximated by the sum of (i) the slope of the forward rate curve and (ii) a term that depends only of yield volatility and maturity ("convexity"), two quantities that can easily be estimated independently of the details of model specification. Key to extracting the risk premium information in long-term forward rates is capturing the dynamics of convexity, which requires a model with time-varying volatility. Using a four-factor ATSM, we find that risk premia on long term bonds are almost entirely driven by volatility. Short rate expectations in our model account for a much larger fraction of the volatility of yields than is typically reported, a result that we show is mainly the result of econometric bias in some estimates of Gaussian models. We also show – using Monte Carlo simulation – that including data on long-term yields in the estimation of term structure models greatly improves the precision of estimated values of short rate expectations, term premia and risk premia. Compared with benchmark estimates that use yield data up to 25 years, excluding data on yields longer than 10 years results in the standard errors of both estimated means and volatilities roughly tripling in size.

Keywords: Risk Premia, Forward Rates, Yield Volatility, Convexity, Expectations Hypothesis
JEL Classification: E43, G12

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1 Introduction

Disentangling expectations of future interest rates from risk premia in the term structure of interest rates has long been a challenge, not only for financial economists but for policymakers. Speaking to the Economic Club of New York in 2006, Federal Reserve Chairman Ben Bernanke made the following much quoted observation:

"*What does the historically unusual behavior of long-term yields imply for the conduct of monetary policy? The answer, it turns out, depends critically on the source of that behavior. To the extent that the decline in forward rates can be traced to a decline in the term premium, ... the effect is financially stimulative and argues for greater monetary policy restraint, all else being equal. ... However, if the behavior of long-term yields reflects current or prospective economic conditions, the implications for policy may be quite different - indeed, quite the opposite*".

Estimates of expected rates and term premia that appear on the Federal Reserve website are derived from term structure models (see Kim and Wright (2005) and Adrian et al. (2013)) but the risk premium on government debt may also be estimated from ex-post returns. Using 28 years of data (1963-1991), Fama and French (1993) report that:

"*the average excess returns on the [US] government and corporate bond portfolios ... are puny. All the average excess bond returns are less than 0.15% per month, and only one of seven is more than 1.5 standard errors from 0*".

As with all risk premium estimates derived from ex-post returns, those for government bonds will also depend a great deal on the sample period. Using 120 years of data (1900-2019), Dimson et al. (2020) report the average risk premium on US Treasuries to be 1.2% while, over the last 50 years of this period (1970-2019), it has been a far from "puny" 3.4%. And over the last 20 years it has been a positively "macho" 6.9%.[1] The historical average of excess returns provides an inevitably noisy estimate of the ex-ante risk premium and the precision of term premia derived from term structure models is far from clear. At the least, it seems fair to say that there is a good deal of uncertainty about the ex-ante risk premium on government debt.

[1] Source: Barclays 20^{+} year US Treasury index and T-bill return from Ken French website.

The central idea of this paper is very easily explained. In virtually all models of the term structure, the current value of the instantaneous forward rate for some future date is equal to the expected value of the short rate for that date plus other terms that will typically include a risk premium. Since the term structure of short rate expectations may reasonably be supposed to become flat for long maturities, differences in long-term forward rates will be largely independent of short rate expectations. The slope of the long end of the term structure therefore provides information about the term structure of risk premia that, again, will be little affected by short rate expectations. Indeed, as we show later, in affine models the average slope of the term structure of risk premia is very closely approximated by the sum of the slope of the forward curve and the difference in "convexity", a term that we define in detail below and which is a simple (known) function of yield volatility. Since both forward rates and convexity can be easily estimated independently of any model, this provides a very simple "model-free" method of estimating the slope of the term structure of risk premia as well as a powerful diagnostic for model mis-specification and/or mis-estimation.

We exploit this property of long-term rates using a decomposition of the forward rate that is common to all (time-homogeneous) affine term structure models (hereafter, ATSMs).[2] In these models, the forward rate at time t for maturity τ, $f_t(\tau)$, may be expressed as the sum of four terms:[3]

$$f_t(\tau) = \mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right] + brp_t(\tau) + cvx_t(\tau) + dur_t(\tau), \tag{1}$$

where $\mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right]$ is the time t expected value, under the $\mathbb{P}$ measure, of the short rate at time $t+\tau$, $brp_t(\tau)$ is the instantaneous risk premium on a zero-coupon bond with time-to-maturity τ, i.e., $brp_t(\tau) = \mathrm{E}\left[\frac{dP_t(\tau)}{P_t(\tau)} - r_t dt\right]$, with $P_t(\tau)$ the price at time t of a τ-maturity zero coupon bond (ZCB) and r_t the instantaneous short rate, and $cvx_t(\tau)$ is a "convexity term" that depends on the volatility at time t of the yield $Y_t(\tau)$ on the same ZCB: $cvx_t(\tau) = -\frac{1}{2}\tau^2 Var(Y_t(\tau))$. The convexity term, which is negative, arises as the joint result of the volatility of yields and the convex relation between the bond price and future interest rates. Finally, $dur_t(\tau)$ is a "duration adjustment" term that we

[2] We use ATSM to refer to a *time-homogeneous* ATSM since effectively all ATSMs used in the academic (as distinct from practitioner) literature are time-homogeneous.

[3] In Appendix A we provide some intuition on the decomposition of the forward rate and, in particular, on the effect of including stochastic volatility, using the simple case of the affine single-factor models of Vasicek (1977) and Cox et al. (1985).

describe in detail later.

From equation (1), the *difference* between the forward rates for maturities τ_2 and τ_1 ($\tau_2 > \tau_1$), i.e., the "forward rate spread" $\Delta f_t(\tau) = f_t(\tau_2) - f_t(\tau_1)$, is given by:

$$\Delta f_t(\tau) = \Delta \mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right] + \Delta brp_t(\tau) + \Delta cvx_t(\tau) + \Delta dur_t(\tau), \tag{2}$$

where Δ is used to indicate the time-t difference between the value of the variable in question for the two maturities τ_2 and τ_1. As suggested above, when τ_2 and τ_1 are both large, it is intuitively reasonable to suppose that the term structure of expected short rates becomes flat, i.e., that $\Delta \mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right]$ becomes small. This conjecture is strongly supported by our estimates from which we find that the estimated value of $\Delta dur_t(\tau)$ is also generally small apart from a period around the Great Financial Crisis. In this case, the *difference* between the risk premia for maturities τ_2 and τ_1 is approximately equal to the difference between the two forward rates minus the difference in the convexity terms, i.e.,

$$\Delta brp_t(\tau) \approx \Delta f_t(\tau) - \Delta cvx_t(\tau). \tag{3}$$

How significant is the convexity component of the forward rate spread at long maturities? Consider the case of the spread between forward rates at 15 and 25 years. Since the average variance of the 15- and 25-year yields in our sample is 0.84 bps and 0.75 bps per year, respectively, the average value of $\Delta cvx_t(\tau)$ predicted by the expression above is -139 bps.[4] The average value of the observed forward spread, by comparison, is 82 bps (see Appendix C) and so the convexity term at long maturities is quantitatively very significant.

The other important feature of the convexity term, *cvx*, apart from its size, is that, because volatility is time varying, *cvx* is also time varying. Using thirty-two years of data on the US Treasury curve, we find that the standard deviation of the convexity term for the 15- to 25-year spread is 179 bps. Much of the recent literature on the term structure employs Gaussian models under which yields have constant volatility and, later in the paper, we document the impact that an assumption of constant volatility has on estimates of expected rates and risk premia. In the model that we present below the volatility of yields is stochastic.

[4] $\Delta cvx_t(\tau) = -\frac{1}{2}\left(\tau_2^2 \times Var(Y_t(\tau_2)) - \tau_1^2 \times Var(Y_t(\tau_1))\right) = -\frac{1}{2}\left(25^2 \times 0.000075 - 15^2 \times 0.000084\right) = -0.0139.$

Taking the time-average of the quantities in equation (3) provides an immediate estimate of the average difference in risk premia. Again, for the 25- minus 15 year forward spread and using the average values given above, the implied average difference in the instantaneous risk premia between 25- and 15-year zero-coupon bonds is:

$$\overline{\Delta brp_t(\tau)} \approx \overline{\Delta f_t(\tau)} - \overline{\Delta cvx_t(\tau)} = -82 \text{ bps} + 139 \text{ bps} = 57 \text{ bps}. \tag{4}$$

This simple result is useful in a number of ways and, as already suggested, as a diagnostic for model mis-specification or mis-estimation. The forward rate decomposition (1) above is true for all ATSMs, and this means that there are only two reasons that a model in this class would give a value for $\overline{\Delta brp_t(\tau)}$ that was significantly different from 57 bps. The first is that the average value of at least one of $\Delta \mathrm{E}^{\mathbb{P}}[r_t(\tau)]$ or $\Delta dur_t(\tau)$ is not small. The second is that the model fails to fit at least one of (a) the average forward spread or (b) the average convexity term. We find that a well-known Gaussian model fails this test for each of a number of estimation methods, but only for the second reason: an inability to fit both the slope of the forward curve and the volatility of the long-term yields. By the same token, the reason that the implied average difference in risk premia in our stochastic volatility model (62 bps) *is* very close to the value 57 bps implied by equation (4) is that it *does* fit both the forward curve and convexity.[5] Thus, the implied average difference in risk premia obtained from equation (4) is an extremely useful diagnostic of model mis-specification and mis-estimation.

Because the forward spread separates bond risk premia from expectations of the short rate, it is an object of interest in its own right and in Appendix C we provide long-run estimates of the forward spread for a number of markets. For most of our analysis we use data on US Treasury STRIPs but we also provide estimates of the forward spread for nominal bonds in the UK, Germany and Canada as well as inflation linked bonds in both the US and the UK. In all these cases we find that the forward spread is almost always negative (a "downward tilt"), in other words that the sum of $\Delta brp_t(\tau)$ and $\Delta cvx_t(\tau)$ is negative. If convexity were small, this would mean that the term structure of risk premia for long maturities was downward sloping. In fact, we find that

[5]The difference in 5 bps is the result of the estimated average value of $\Delta dur_t(\tau)$ being 5 bps rather than precisely zero.

the convexity term is typically larger than the forward spread, implying an upward sloping term structure of risk premia.[6]

We implement a four factor "essentially affine" model in which the dynamics of the term structure are driven by three latent state variables and by a variance factor that drives the conditional volatility of the other three. The inclusion of stochastic volatility is important in order to accommodate the time-varying nature of the convexity term. The model is estimated using data on US Treasury STRIPs for the period from January 1988 to December 2019. The model provides a good fit to the forward rate spread as well as to yields and forward rates.

Two of the three latent variables are highly correlated with the level and slope of the yield curve but the third is highly correlated with the second principal component of the term structure of yield volatility. Our variance factor captures the level of volatility and so, together, our state variables capture the dynamics of the term structure of both yields and their variance.

As a robustness check, we also investigate the deviations from the Expectations Hypothesis documented by Fama and Bliss (1987) and Campbell and Shiller (1991) (CS). Dai and Singleton (2002) and Joslin and Le (2013) have previously found that, while it was possible to capture the CS effect in a Gaussian model, this was not true for stochastic volatility models. In contrast, we find that our model – that includes stochastic volatility – is consistent with the deviations from the pure Expectations Hypothesis observed in the data.

Our model estimates of the time series of the four components of the forward spread ($\Delta \mathrm{E}^{\mathbb{P}}[r_t(\tau)]$, $\Delta brp_t(\tau)$, $\Delta cvx_t(\tau)$ and $\Delta dur_t(\tau)$) show that, as with the average values reported above, the risk premium and convexity terms dominate and that the differences in both the expected short rate and the duration adjustment term are small.

The first main result of the paper is in identifying the two variance related state variables as the main determinants of risk premia on long-term bonds. The dynamics of risk premia are almost entirely driven by the variance factor and the latent variable that is correlated with the second principle component of the term structure of yield volatility. In contrast the state variables

[6] In an early paper, Brown and Schaefer (2000) explain the downward tilt in long-term forward rates purely in terms of a convexity effect and provide an empirical investigation for both US Treasuries and UK Gilts based on a Gaussian two-factor framework. Since then, however, the convexity effect has attracted little attention in the literature, perhaps because – as observed earlier – few studies pay much attention to the behavior of long-term rates.

corresponding to the level and slope of the yield curve explain only a small fraction of the variation in risk premia.

In the term structure literature there is less focus on the bond risk premium than on the term premium, the difference between the yield on a bond and the average expected short rate up to the bond's maturity. The main difference between these quantities is the convexity term and we show that, as a result of the relation between, on one hand, convexity and, on the other, maturity and volatility, the properties of term premia and risk premia can be quite different. For example, the fact that convexity increases strongly with maturity leads to a term structure of term premia that is typically much flatter than risk premia. More significantly, since (i) we find that risk premia are increasing in volatility and (ii) the term premium includes a (negative) convexity term that increases in size with volatility, the term premium displays much lower sensitivity to volatility than risk premia. Overall, the inclusion of convexity in the standard definition means that interpreting the term premium as a risk premium will be often unreliable.

A major difference between our results and those in much of the literature is that, under our model, short rate expectations account for a much larger fraction of the variability of yields than in well known Gaussian models (e.g., those utilized by the Fed cited above). In the latter, the fraction of the variance of the 10-year yield accounted for by expected short rates is around 40%; in our model it is over 70%. Because this is such a significant difference and because this issue speaks directly to the question of disentangling expectations from term premia in yield curve changes, we carry out an extensive investigation to identify the source of the difference.

This is the second main result of the paper. We find that the difference between our results and others in the literature on the volatility of expectations versus term premia turns out to have two components. First, in comparing the results of our stochastic volatility model to those from a Gaussian model, model specification clearly plays a role. But we find that the more important determinant is the econometric method used to derive the estimates of the Gaussian model. Using a Monte Carlo approach, we find that the two-step methods used in estimating the Gaussian model in Joslin et al. (2011) ($KWJSZ$) or Adrian et al. (2013) (ACM) substantially under-estimate the volatility of short rate expectations and over-estimate the volatility of term premia and risk premia relative to the "correct" values generated by a *Gaussian* model. This result is in line with the findings already obtained by Bauer et al. (2014) and arises from the small sample bias in the estimated mean reversion coefficients when the latter are estimated using a Vector AutoRegression (VAR). Since our view is that stochastic volatility plays an important role in determining the term structure, we repeat the same Monte Carlo exercise when the "correct" model is our stochastic volatility model as estimated below. The results are similar. We find that $KWJSZ$ and ACM continue to underestimate the volatility of short rate expectations (by around 40% at 10 years and by 60% at 20 years). Somewhat paradoxically, we also find that, while this method continues to overestimate the volatility of term premia (by about 80% at 10 years and 140% at 20 years) and hugely overestimates the volatility of risk premia (by a factor of about five at 10 years and nine at 20 years), the degree of overestimation is actually slightly lower for our stochastic volatility model than when the correct model is Gaussian. Thus a significant part of the difference between our results and those in the literature is the result of econometric method.

The third main finding goes back to our claim that long-term rates provide information that greatly facilitates the separation between expectations and risk premia. In most of the current term structure literature the maximum bond maturity that is used in estimation is 10 years. If our claim is correct, then including long-term yields should improve the precision with which expectations, term premia and risk premia are estimated. We again use Monte Carlo and, to avoid conflating the effect of including data on long-term yields with the effects of stochastic volatility, we address this question using a Gaussian model (our claim about the importance of long-term rates applies irrespective of whether volatility is stochastic). For each Monte Carlo trial we estimate the model twice, first using maturities up to 25 years and second, in line with much of the literature, including maturities only up to 10 years. The results are striking and remarkably consistent, both across the various components of yields and across maturity: when the maximum maturity of yields used to estimate the model is reduced from 25 years to 10 years, the standard errors of the average expected short rate, the term premium and the risk premium all almost triple. This is a huge difference and strongly suggests that the estimation of term structure models would be significantly improved by including data on long-term rates.

The paper is related to a number of different areas of the term structure literature. As mentioned above, the only previous study focusing on the downward tilt in long-term forward rates is Brown and Schaefer (2000), although it is also explicitly considered in other studies, such as Sack (2000) and Gurkaynak et al. (2007). The interrelation between volatility, convexity and long-term yields has been examined by Brown and Schaefer (1994) and, more recently, by Cieslak and Povala (2016), Giglio and Kelly (2018), Rebonato and Putyatin (2018) and Carr and Wu (2019).

Our term structure model belongs to the family of stochastic volatility affine term structure models. Pioneered by Longstaff and Schwartz (1992) and Chen and Scott (1992), this literature has grown rapidly and includes, among the others, Jacobs and Karoui (2009), Bansal and Shaliastovich (2012) and Feldhütter et al. (2018) as well as tests for unspanned volatility in fixed income markets by Collin-Dufresne and Goldstein (2002), Bikbov and Chernov (2009), Andersen and Benzoni (2010), Creal and Wu (2015) and Joslin (2017). Haubrich et al. (2012), Cieslak and Povala (2016) and Berardi and Plazzi (2019) develop stochastic volatility term structure models for the estimation of time-varying term premia, which differ from other studies in this area based on Gaussian type models, such as Duffee (2002), Dai and Singleton (2002) and Kim and Wright (2005) and, even more recently, Wright (2011), Adrian et al. (2013), Abrahams et al. (2016) and Malik and Meldrum (2016).

2 The Model

In this section, we outline our term structure model and derive the analytical expression for the forward rate spread, and the term premium.

2.1 Term Structure

We assume that the dynamics of the term structure are determined by three latent factors x_1, x_2 and x_3, and by a variance factor, v, which drives the conditional volatility of the other variables. In addition to being jointly related in the diffusion component through v, the factors are potentially mutually dependent via their conditional means.

We collect the latent factors in the vector $X = (v\ x_1\ x_2\ x_3)'$. In the notation of Dai and

Singleton (2000), under the physical probability measure $\mathbb{P}$ the dynamics of X at time t evolve according to the following $\mathrm{A}_1(4)$ specification:

$$dX_t = K(\Theta - X_t)dt + \Sigma\sqrt{S_t}dz_t, \tag{5}$$

where K is a (4×4) full matrix of mean-reversion coefficients, Θ is a (4×1) vector of long-run means, Σ is a (4×4) diagonal matrix, and S_t is a diagonal (4×4) matrix whose element (i,i) is given by $S_{t(ii)} = \beta_i' X_t$, with β_i denoting the i-th column of the (4×4) matrix β which has ones in the first row and zeros elsewhere.[7] dz_t denotes a vector of independent Brownian motions. Our specification thus implies that v follows a non-negative square-root process which drives the conditional volatility of the other, conditionally gaussian, state variables.

We assume that the functional form of the market price of risk is given by an "essentially affine" specification (see Duffee (2002) and Duarte (2004)) of this type:

$$\Lambda_t = \sqrt{S_t^-}\,\left(\Lambda_0 + \Lambda_1\,X_t\right), \tag{6}$$

where S_t^- denotes the inverse of S_t, Λ_0 is a (4×1) vector and Λ_1 a (4×4) full matrix. This formulation allows risk premia to vary over time and change sign. The stochastic process for X under the risk-adjusted probability measure $\mathbb{Q}$ is then $dX_t = (\widetilde{K}\widetilde{\Theta} - \widetilde{K}X_t)dt + \Sigma\sqrt{S_t}d\widetilde{z}_t$, where $\widetilde{K} = K + \Sigma\Lambda_1$, $\widetilde{K}\widetilde{\Theta} = K\Theta - \Sigma\Lambda_0$, and $d\widetilde{z}_t$ is the risk-neutral Brownian motion.

The time t instantaneous nominal interest rate r is an affine function of the state vector:

$$r_t = \delta_0 + \delta' X_t, \tag{7}$$

where δ_0 is a scalar and δ a (4×1) vector.

The equilibrium price of a nominal unit zero coupon bond with time to maturity τ at time t has an exponentially affine closed-form solution:

$$P_t(\tau) = \exp[A(\tau) - B'(\tau)X_t], \tag{8}$$

[7] What this means is that, in our case, matrix S_t takes a very simple form, namely a diagonal matrix with all elements equal to v_t.

where the coefficients $A(\tau)$ and $B(\tau)$ depend on the underlying model parameters and solve the Dai and Singleton (2000) system of ordinary differential equations.

The term structure of nominal zero coupon yields is therefore affine in the state vector:

$$Y_t(\tau) = a(\tau) + b'(\tau)X_t, \tag{9}$$

where $a(\tau) \equiv -A(\tau)/\tau$ and $b(\tau) \equiv B(\tau)/\tau$.

The diffusion term in the dynamics of X is a function of v, which implies that yield volatilities are time-varying and are driven by that factor. In particular, the time t term structure of the instantaneous variance of yield changes is proportional to v and is given by:

$$V_t(\tau) = b'(\tau) \left(\Sigma\, S_t \Sigma'\right) b(\tau). \tag{10}$$

2.2 Forward Rate Spread

The partial differential equation whose solution provides the time t equilibrium price of the unit zero coupon bond in equation (8) is:

$$\frac{1}{2}\frac{\partial^2 P_t(\tau)}{\partial X_t' \partial X_t}(\Sigma S_t \Sigma') + \frac{\partial P_t(\tau)}{\partial X_t'}\left[K(\Theta - X_t) - \Sigma(\Lambda_0 + \Lambda_1\, X_t)\right] + \frac{\partial P_t(\tau)}{\partial t} - r_t P_t(\tau) = 0. \tag{11}$$

Since the model is time homogeneous, the time t instantaneous forward rate for date $t+\tau$ is given by $f_t(\tau) = \frac{1}{P_t(\tau)}\frac{\partial P_t(\tau)}{\partial t}$. Furthermore, using equation (8), we have that:

$$f_t(\tau) = r_t + B'(\tau)K(\Theta - X_t) - B'(\tau)\Sigma(\Lambda_0 + \Lambda_1\, X_t) - \frac{1}{2}B'(\tau)\left(\Sigma\, S_t \Sigma'\right) B(\tau). \tag{12}$$

The second term on the right hand side can be decomposed as:

$$B'(\tau)K(\Theta - X_t) = \left[B_G'(\tau) + (B'(\tau) - B_G'(\tau))\right] K(\Theta - X_t). \tag{13}$$

The first term in the square brackets, $B_G'(\tau) \equiv \delta' K^{-1}(I - e^{-K\tau})$, is the expression for the $B(\tau)$ coefficient in a Gaussian model that shares the same mean reversion matrix K, but has a constant market price of risk (for example, as in Vasicek (1977) and Langetieg (1980)). (This is the term which enters the expression for the expected short rate). The second term, $(B'(\tau) - B'_G(\tau))$, can be interpreted as the adjustment in the $B(\tau)$ coefficient, the "duration" coefficient, due both to stochastic volatility and the essentially affine specification. This term goes to zero if $S_t = I$ and $\Lambda_1 = 0$.

As noted earlier, the instantaneous forward rate can be written as the sum of four components (see equation 1 for a definition of each variable):

$$f_t(\tau) = \mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right] + brp_t(\tau) + cvx_t(\tau) + dur_t(\tau), \tag{14}$$

and the analytical expressions for these components can now be derived from equations (12) and (13) above and are given, respectively, by:

$$\mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right] = r_t + B'_G(\tau)K(\Theta - X_t), \tag{15}$$

$$brp_t(\tau) = -B'(\tau)\Sigma(\Lambda_0 + \Lambda_1 X_t), \tag{16}$$

$$cvx_t(\tau) = -\frac{1}{2}B'(\tau)\left(\Sigma\, S_t\Sigma'\right)B(\tau), \tag{17}$$

$$dur_t(\tau) = \left[B'(\tau) - B'_G(\tau)\right]K(\Theta - X_t). \tag{18}$$

Computing equation (14) at maturities τ_2 and τ_1 ($\tau_2 > \tau_1$) and taking the difference, we obtain an expression for the *forward rate spread*:

$$\begin{aligned} f_t(\tau_2) - f_t(\tau_1) = &\left[\mathrm{E}^{\mathbb{P}}\left[r_t(\tau_2)\right] - \mathrm{E}^{\mathbb{P}}\left[r_t(\tau_1)\right]\right] + \left[brp_t(\tau_2) - brp_t(\tau_1)\right] \\ &+ \left[cvx_t(\tau_2) - cvx_t(\tau_1)\right] + \left[dur_t(\tau_2) - dur_t(\tau_1)\right], \end{aligned} \tag{19}$$

which implies that the slope of the term structure of forward rates depends on the slope of its four components. As each of these terms is affine in the variance factor v and the latent factors x_1, x_2 and x_3, the forward rate spread is also affine in the state variables.

2.3 Term Premium

Dai and Singleton (2002) and Kim and Wright (2005) refer to the difference between the instantaneous forward rate and the expected short rate in equation (14) as the "forward term premium",

$$FTP_t(\tau) = f_t(\tau) - \mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right] = brp_t(\tau) + cvx_t(\tau) + dur_t(\tau), \tag{20}$$

while the average of the forward term premium between time t and $t+\tau$ is defined as the (yield) "term premium",

$$TP_t(\tau) = \frac{1}{\tau}\int_0^{\tau} FTP_t(s)ds. \tag{21}$$

The definitions of both FTP and TP include not only the bond risk premium but also the convexity and duration adjustment terms. In our case, consistent with other stochastic volatility models, such as for example Cieslak and Povala (2016) and Carr and Wu (2019), we separate out the convexity term, $cvx_t(\tau)$. In addition, we consider the duration adjustment term, $dur_t(\tau)$, as a distinct element from the risk premium.

The yield on a τ maturity zero coupon bond can therefore be written as the sum of the average expected short rate and term premium, where the term premium includes the average risk premium, convexity and duration adjustment from t to $t+\tau$. We refer to the average risk premium as the "adjusted term premium", i.e., the conventional term premium net of the convexity and duration components. Therefore, we have:

$$Y_t(\tau) = ESR_t(\tau) + TP_t(\tau) = ESR_t(\tau) + ATP_t(\tau) + CX_t(\tau) + DA_t(\tau), \tag{22}$$

where the τ maturity yield of equation (9) can be written as $Y_t(\tau) = \frac{1}{\tau}\int_0^{\tau} f_t(s)ds$, the average expected short rate is defined as $ESR_t(\tau) = \frac{1}{\tau}\int_0^{\tau} \mathrm{E}^{\mathbb{P}}\left[r_t(s)ds\right]$, the "adjusted term premium" as $ATP_t(\tau) = \frac{1}{\tau}\int_0^{\tau} brp_t(s)ds$, the average convexity term as $CX_t(\tau) = \frac{1}{\tau}\int_0^{\tau} cvx_t(s)ds$, and the average duration adjustment term as $DA_t(\tau) = \frac{1}{\tau}\int_0^{\tau} dur_t(s)ds$.

Allowing for stochastic volatility, the model implies that the convexity term may vary over time and may have a large effect on long-term yields in periods of high yield volatility. This is in contrast with most Gaussian models for the modeling of term premia in which the convexity term is a constant and is not explicitly separated from the term premium. We investigate the implications of this assumption in section 3.6 below.

3 Estimation Results

3.1 Data and Estimation Method

The model is estimated using the STRIPs data described in Appendix B. We use end-of-month observations on annualized zero coupon yields and forward rates with maturities ranging from 2 to 25 years.

Estimates of realized yield variance for a given end-month date are computed as the annualized variance of daily changes in yields during that month. We build an estimator for the τ-maturity yield variance at time t by regressing the τ-maturity realized variance at time t against the realized variance at time $t-1$ and the $t-1$ implied variance calculated from the 10-year U.S. Treasury Note Volatility Index (TYVIX).[8] The fitted values of the regression are our estimates of the time series of the τ-maturity yield variance. In the estimation, we use these estimates of yield variance for maturities of 10, 15, 20 and 25 years.

The estimation period runs from January 1988 through December 2019. **Table 1** reports summary statistics on yields, forward rates and yield variances. We notice that, on average, the term structure of yields is upward sloping and declines very slightly from the 23-year maturity. In contrast, the average term structure of forward rates is upward sloping only up to 13 years and then declines markedly. The difference between forward rates at 13 and 25 years is about 90 bps. Average realized yield variance declines from 0.89 bps at 10 years to 0.75 bps at 25 years. The corresponding series for yield volatility decrease from an average value of 89 bps at 10-year maturity to 82 bps at 25-year maturity.

The parameters of the state-space model are estimated by maximum likelihood, with an approximate Kalman filter algorithm being used to calculate the values of the unobserved state variables. An approximate linear filtering method is needed as the state vector has affine dynamics but is not

[8]The TYVIX is calculated from CBOT's options on 10-year Treasury futures using the same methodology as VIX.

Gaussian. In this scenario, an approximate transition equation can be obtained by exploiting the existence of an analytical expression for the first two conditional moments of the state vector (see, for example, Christoffersen et al. (2014)).

The system of observation equations includes series on both yields and forward rates with maturities between 2 and 25 years and yield variance at 10, 15, 20 and 25 years, for a total of fifty two equations. The observation equations are obtained by adding to the variables' model-implied expression an observation error, which is assumed to be normally distributed and homoskedastic. The four state equations are the discrete time (monthly) equivalents of the continuous-time model in (5). The transition equation is obtained by exploiting the solution to the stochastic differential equations that describe the dynamics of the variables, as in Christoffersen et al. (2014).

3.2 Goodness of Fit and Specification Tests

Table 2 reports statistics on the goodness of fit of the model. Errors are defined as the difference between the actual and model values of yields, forward rates and yield variances. The average standard deviation of errors, across maturities from 2 to 25 years, is 5.5 bps for yields and 22.3 bps for forward rates. The model fits medium-long term yields more accurately than short term yields. In fact, if we exclude the maturities below 5 years, the standard deviation of yield errors is only 3.9 bps. The average difference between monthly ex-post yield variance and our ex-ante estimate, for the maturities 10, 15, 20 and 25 years, is around zero and the standard deviation of the differences is on average 0.14 bps. These errors imply an average standard deviation of the differences in yield volatilities of 7.7 bps.

As a specification test, we calculate the maximal attainable Sharpe Ratio (see Duffee (2010)), computed as $\sqrt{(1/\tau)(\exp\{\Lambda_t'\Lambda_t\,\tau\}-1)}$, with τ equal to one month. The maximal Sharpe ratio can be interpreted as a diagnostic on the specification of the stochastic discount factor. We obtain very reasonable values, with a maximum of 0.70 and a full-sample average of 0.39.

We also test if the model is consistent with the deviations from the expectations hypothesis that are observed in the data. Fama and Bliss (1987) show that, contrary to the implications of the traditional expectations hypothesis, bond risk premia are time-varying and positively correlated with the slope of the yield curve. According to the Campbell and Shiller (1991) test, if the traditional expectations hypothesis is true, then the $\phi(\tau)$ coefficient in the following regression should be equal to 1:

$$Y_{t+\Delta}(\tau-\Delta) - Y_t(\tau) = \phi_0(\tau) + \phi(\tau)\left[\frac{Y_t(\tau) - Y_t(\Delta)}{\tau/\Delta - 1}\right] + u_{t+\Delta}(\tau), \tag{23}$$

where in the following we consider Δ equal either to one month or one year. However, as shown by Campbell and Shiller (1991) and subsequently by Backus et al. (2001) and Joslin and Le (2013), among the others, this coefficient is usually below one and largely negative, thus contradicting the assumptions of the expectations hypothesis.

This issue has been taken up in the literature on ATSMs where Duffee (2002), Dai and Singleton (2002) and Joslin and Le (2013) claim that stochastic volatility models, i.e., models of the $A_m(n)$-type, with $m > 0$, $m \leq n$, in Dai and Singleton (2000) notation, fail to match the negative and declining pattern of the $\phi(\tau)$ coefficient observed in the data.

As a statistical test, Dai and Singleton (2002) propose using the estimated model parameters to calculate the model-implied population coefficient $\phi(\tau)$:

$$\hat{\phi}(\tau) = (\tau/\Delta - 1)\left[\frac{Cov\{Y_{t+\Delta}(\tau-\Delta) - Y_t(\tau), Y_t(\tau) - Y_t(\Delta)\}}{Var\{Y_t(\tau) - Y_t(\Delta)\}}\right], \tag{24}$$

and then comparing this with the sample coefficients estimated by regression (23). Following Dai and Singleton (2002), we also run a Monte Carlo experiment to control for the small-sample bias in the calculated projection coefficient in equation (24). Using the estimated model parameters, we simulate one hundred samples of length 250,000 (a thousand years) and, for each sample, we calculate $\phi(\tau)$. The average estimated coefficients match those calculated in equation (24) almost perfectly and we use one standard deviation of these estimates to compute the upper and lower confidence intervals of the $\hat{\phi}(\tau)$ coefficients.

Figure 1 shows the estimates of the $\phi(\tau)$ coefficients for both monthly and annual yield changes. As the model fits actual yields very well, the coefficients computed by running the Campbell-Shiller regression with fitted yields are very close to those obtained using actual data. More interestingly, we observe that the model-implied projection coefficients are negative, decreasing with maturity, statistically significantly different from one and, therefore, consistent with the deviations from the expectations hypothesis observed in the data.

This result is in sharp contrast with the evidence on the failure of stochastic volatility ATSMs presented by Dai and Singleton (2002) and Joslin and Le (2013), where an $A_1(3)$ model is shown to provide slightly negative coefficients, which increase towards zero as the maturity lengthens.

3.3 Estimated State Variables and Parameters

Figure 2 plots the time series of the Kalman filter estimates for the variance factor v and the latent state variables x_1, x_2 and x_3, while **Table 3** reports the estimated model parameters. For comparison, the figure also includes the time series of the first two principal components of yields and realized volatilities, both in levels and with maturities between two and 25 years.

Monthly changes in the variance factor v and the latent factors x_1 and x_2 are almost uncorrelated, whereas changes in the latent factor x_3 are correlated with the other three factors, with a correlation of around 0.3.

The latent factor x_1 is very persistent and its monthly changes are correlated with the first principal component of monthly changes in yields (correlation coefficient 0.51), i.e., the level of the curve, whereas the factor x_2 has mean reversion coefficient 0.25 and is significantly correlated (0.69) with the second principal component of yields, i.e., the slope of the curve.

Monthly changes in the variance factor v are highly correlated (0.76) with the first principal component of monthly changes in realized volatilities. The factor exhibits relatively low mean reversion, with coefficient 0.13. Interestingly, monthly changes in the variance factor exhibit a non-negligible correlation (0.50) only with the third principal component of yield changes, i.e., the convexity of the curve. The latent factor x_3 has low mean reversion (0.07) and appears to be a second volatility-related factor, as it is significantly correlated (0.41) with the second principal component of realized yield volatilities.

The coefficients in matrix Σ show that innovations in x_1 and x_2 are more sensitive to v than to innovations in x_3. The estimated elements of matrix Λ_1 are all negative along the main diagonal and are all statistically significant with the exception of $\lambda_{4,4}$.

3.4 Downward Tilt in Forward Rates

Figure 3 reports the time series behaviour of the four components of forward rates for maturities of 15 and 25 years (equation (14)). Panel A contains the expected value of the instantaneous short rate and shows that the difference is very small. In contrast, Panel B shows that the term structure of risk premia at these maturities is positively sloped apart – interestingly – from a short period leading up to the GFC. Panel C shows the dynamics of the convexity term and, consistent with equation (17), we find that, in absolute terms, this increases strongly with maturity. Panel D shows that the duration adjustment term fluctuates around zero for most of the sample period and becomes positive after the 2008 financial crisis. However, as with short rate expectations, the slope of the duration adjustment term for these maturities is close to zero.

Panel A of **Figure 4** shows that the model spread between the 25- and 15-year maturity instantaneous forward rates corresponds quite well to the estimates obtained from raw data (see Appendix C). The average value of the model spread is -82 bps, which corresponds exactly to the average observed in raw data. The estimated series shows a lower volatility, as the standard deviation of monthly changes is 55 bps vs 81 bps for changes in the actual spread. As we suggest later, this seems to be the result of noise in the direct estimates of the forward rate.

Panel B shows the decomposition of the spread between the 25- and 15-year maturity instantaneous forward rates into its four components (equation (19)). We observe that the difference between the values at 25 and 15 years of both short rate expectations and the duration adjustment term are both close to zero. What this means is that the downward tilt in forward rates is almost entirely explained by the substantial negative value of the convexity term, which is partially offset by the positive difference in risk premia. The average difference in the convexity term is -139 bps, while the average difference in bond risk premia is $+62$ bps. As regards the difference in expected short rates and duration adjustment, the average values are equal to 0 and -5 bps, respectively.

Panel A of **Figure 5** shows that, holding the difference in maturity constant at 10 years, the term structure of the average long-term forward rate spread decreases almost linearly with maturity from -23 bps for the 10- to 20-year interval to -104 bps for the 17- to 27-year interval.

As the figure shows, the size of the forward rate spread increases even though the difference in risk premia remains approximately constant at around 60 bps, and is due almost entirely to the increase in the size of the convexity term with maturity.

A central conclusion we obtain from our model estimates is that for long maturities the forward spread is determined almost entirely by the sum of only two terms: a negative convexity term and a partially offsetting positive risk premium. Since volatility (and hence the convexity effect) can be measured straightforwardly and with reasonable precision, we can calculate a "model-free" estimate of the difference in average long-term bond risk premia directly from the difference in forward rates.

The average value of the convexity term for maturity τ can be estimated directly from the realized variance of yields, $RV_t(\tau)$,

$$\overline{cvx_t(\tau)} = -\frac{\tau^2}{2}\overline{RV_t(\tau)}, \tag{25}$$

where the overbar indicates the average value. Hence a model-free estimate of the difference in average long-term bond risk premia can be computed as:

$$\overline{brp_t(\tau_2) - brp_t(\tau_1)} = \overline{f_t(\tau_2) - f_t(\tau_1)} - \overline{cvx_t(\tau_2) - cvx_t(\tau_1)}. \tag{26}$$

Panel B of **Figure 5** shows that, holding the difference in maturity constant at 10 years, the model free estimate of the difference in average risk premia is very close to our model estimates with a value of about 60 bps, although the model free estimates are slightly downward sloping rather than constant. The differences in the convexity term decrease with maturity and closely match those estimated by the model. To summarise, estimates of differences in average bond risk premia at long maturities can be obtained directly and relatively reliably from the average slope of the forward curve by simply subtracting the (negative) difference in the estimated convexity terms. Because expected future short rates are very similar for long maturities, these risk premium estimates have little dependence on how short rate expectations are characterized.

The reason that our estimates of the difference in risk premia match the model free estimates so well is because (i) our model fits both long-term average forward rates and long-term average volatility quite well and (ii) the term structures of expected short rates and the duration adjustment term for long maturities are flat. However, as already noted, the decomposition of the forward rate into four components holds for all ASTMs, including those with stochastic risk premia. So, if the second condition holds (flat terms structures of short rate expectations and the duration adjustment term), a different ATSM would give a different estimate of the average difference in long-term risk premia only because it failed to fit long-term forward rates and/or long-term yield volatility. In section 3.6 we use this fact to illustrate some of the limitations of Gaussian models.

3.5 Risk Premia

Panel A of **Figure 6** shows that the average term structure of estimated bond risk premia is upward sloping, with a value of 93 bps at 5 years, increasing to 143 bps at 10 years and 211 bps at 20 years.

To better understand their determinants, we decompose the risk premia to show the contribution of each of the four state variables. Since the risk premia are affine in the state variables, the risk premium at time t is simply equal to a constant plus the sum of four terms, each of which is the component attributable to one of the state variables and is calculated as (see, for example, Adrian et al. (2013)): $brp_{t(i)}(\tau) = -B'_{(i)}\Sigma_{(ii)}\Lambda_{0(i)} - \sum_{j=1}^{4} B'_{(j)}\Sigma_{(jj)}\Lambda_{1(ji)}X_{t(i)}, i = 1, 2, 3, 4$. Panels B and C of **Figure 6** show the evolution of these four terms over time for the 10- and 20-year maturity risk premium, respectively.

For both the 10- and 20-year maturities, v and x_3 account for the majority of both the average level and the variation. A variance decomposition of the 10-year risk premium shows that x_3 accounts for 79% and v for 12% of the total variability. In the case of the 20-year risk premium, v and x_3 explain, respectively, 49% and 39% of the variability, while x_1 accounts for the remaining 12%.

Figure 3 provided a decomposition of forward rates into their four components: the expected short rate, risk premium, convexity and the duration adjustment. Since much of the discussion of term premia focusses on yields, we provide a corresponding decomposition for yields.

Table 4 decomposes the term structure of average yields into the estimated average expected short rate, adjusted term premium, convexity and duration adjustment. We observe that the term structures of both average short rate expectations and adjusted term premia are upward sloping. In particular, the adjusted term premium increases, on average, from 53 bps at the 5-year maturity to 86 bps at the 10-year maturity and to 132 bps for a maturity of 20 years. Consistent with the theoretical assumptions, the term structure of average convexity is also strongly upward sloping, in absolute terms. Average convexity, which is equal to -15 bps for the 10-year maturity, reaches -72 bps at 25 years. The average duration adjustment is almost flat for maturities above 10 years, with values ranging between 10 and 15 bps.

Figure 7 shows the time series estimates of the four components for the 10-year (Panel A) and the 20-year (Panel B) yields. In both cases, short rate expectations follow a declining trend, reaching a minimum of about 1.5% and 2%, respectively, around May 2013,[9] slowly increasing towards 3% between 2014 and 2018, and then declining sharply in 2019. Estimated adjusted term premia are relatively stable in the period preceding the Great Recession, while after they decrease and become somewhat more volatile. Between 1988 and 2007 the 10-year adjusted term premium averages 130 bps with a standard deviation of 36 bps, while between 2008 and 2019 the average declines to 15 bps and the volatility increases to 71 bps. The convexity component reflects the level of yield volatility and is therefore much larger for longer maturity yields. In some periods convexity has a very significant impact on long-term yields. For example, at the peak of the 2008 financial crisis, convexity reduces the 10-year yield by about 50 bps and the 20-year yield by almost 150 bps.

3.6 Definition and Specification of Term Premia

In this section we address two questions. First, how different are estimates of the forward term premium obtained in the conventional way as $f_t(\tau) - \mathrm{E}^{\mathbb{P}}[r_t(\tau)]$ from the conventional risk premium as defined in our equation (16)? Second, how sensitive are estimates of the term premium to model specification and, in particular, to the inclusion of stochastic volatility?

As a benchmark for this analysis, we consider the Gaussian models of Kim and Wright (2005)

[9] On May 22, 2013, the Fed Chairman Ben Bernanke stated in testimony before Congress that the Fed had decided to reduce the size of its quantitative easing program, thus generating expectations for a normalization of short-term interest rates.

(hereafter, KW) and Adrian et al. (2013) (hereafter, ACM) that are used by the Fed and other central banks to provide estimates of term premia and expected short rates.[10] ACM use a Gaussian affine model with the first five principal components of yields as the underlying "observable" state variables. Their procedure uses only yield curve information and is based on a multi-step linear regression approach. KW use maximum likelihood to estimate a Gaussian three-factor affine term structure model with unobservable state variables. Unlike ACM, the data used for estimation in KW include not only Treasury yields, but also survey-based expectations of future short-term interest rates.

3.6.1 Definition of Term Premium

The term premium is typically defined as the difference between the yield on a bond and the average expected short rate up to the bond's maturity (see, for example, Dai and Singleton (2002), Kim and Wright (2005) and many others). There can be no disagreement with the way any variable – in this case the "term premium" – is defined, only with the way that it is interpreted. The following quotation from Cohen et al. (2018) expresses a widely held view about the way that conventionally defined term premia should be understood: "This term premium is normally thought of as the *extra return (a risk premium) that investors demand to compensate them for the risk associated with a long-term bond, particularly for long maturities.*" (Emphasis added)

For a bond with time-to-maturity τ, the time t conventional τ-maturity (yield) term premium, $TP_t(\tau)$, is defined as in equations (20)-(21) and represents the expected value, at time t, of the average log-excess return on the bond from t to $t + \tau$, i.e., over the life of the bond as its maturity goes from τ to zero. The risk premium in our analysis, $brp_t(\tau)$, is the instantaneous risk premium at time t on the same bond, i.e., with time-to-maturity τ.

These two measures differ in a number of ways, but most significantly in that, because it is defined in terms of log-returns, the conventional TP includes convexity which is not a component of the risk premium (see, e.g., Cieslak and Povala (2016)) while convexity is not included in $brp_t(\tau)$.

[10] The estimates of Kim and Wright (2005) and Adrian et al. (2013) are based on the Gurkaynak et al. (2007) data set and are calculated for maturities only up to 10 years. They are produced by staff in the Federal Reserve System and are updated daily at `http://libertystreeteconomics.newyorkfed.org/2014/05/treasury-term-premia-1961-present.html` and `https://www.federalreserve.gov/pubs/feds/2005/200533/200533abs.html`. KW estimates start only in January 1990. In what follows, we refer to the values published as of April 1, 2020.

As we show later, the inclusion of convexity has a substantial impact on the level, term structure and sensitivity to volatility of the conventional term premium.

Panel A of **Figure 8** shows the difference between the 20- and 10-year (i) forward term premium (FTP) and (ii) risk premium (brp), both evaluated using the estimates of our stochastic volatility model. The figure shows that for the FTP, which includes convexity, there are many examples in our data where the term structure of forward term premia at long maturities is downward sloping while the term structure of risk premia is upward sloping. Indeed, we find that the difference between the 20-year and the 10-year maturity FTPs is negative on 73% of the occasions, while according to our model the difference in brps is negative on only 9% of cases.

Panel B reports the difference between the 20- and 10-year (yield) term premium (TP) and adjusted term premium (ATP) and, although almost always positive, clearly shows that the slope of the long-maturity term premium is much lower than that of the adjusted term premium, which does not include the effect of convexity and the duration adjustment term. We emphasize that the estimates of TP and ATP (and FTP and brp above) in **Figure 8** are derived from the same model estimates and the differences between the estimates are therefore due entirely to the difference in definition.

The inclusion of convexity in the definition of forward term premium may also lead to misleading conclusions about the impact of volatility on risk premia on long-term bonds. **Table 5** shows the average change in 10-year and 20-year maturity forward term premia (Panel A) and term premia (Panel B) – and in their components – in response to one standard deviation shock in the variance factor v. We find that, although brp and convexity (cvx) are highly sensitive to changes in yield volatility, the FTP may appear insensitive because changes in yield volatility impact brp and cvx in opposite directions with the result that the net effect on the FTP can be small. This effect is also evident when we consider yield term premia and is especially significant at longer maturities (20 years). Therefore, when long-term yields move in response to an increase (or decrease) in interest rate volatility, the impact on the risk premium component of yields may be significantly under-estimated by the impact on the conventional term premium.

3.6.2 Volatility Specification

Next, we examine the impact of assuming constant rather than time-varying yield volatility on the estimated dynamics of the term premium.

As a benchmark for the constant volatility case, we use a four-factor version of KW (a Gaussian model) estimated using the same technique that we employ for our model. Specifically, we fit the model to both yields and forward rates (from STRIPs) for maturities from two to 25 years and also to the variance of yields at maturities of 10, 15, 20 and 25 years (calculated as explained in section 3.1). As the model is Gaussian, yield volatility for a given maturity is constant and so "fitting to volatility" means fitting to average yield volatility. We assume that the state variables are latent factors and use a maximum likelihood Kalman filter methodology to obtain estimates of the factors and the model parameters. For brevity, in what follows we refer to the KW model estimated in this way as $KWML$.

$KWML$ fits yields and forward rates well, with an average (across maturities) standard deviation of errors of 5.6 bps and 24.0 bps, respectively. The average error in fitting average yield variance increases with maturity. In particular, the average error is close to zero at 15 years but becomes equal to -0.10 bps at 25 years. In terms of yield volatility, this is equivalent to under-estimating the 25-year yield volatility by almost 5 bps (74.4 bps for the $KWML$ vs 79.1 bps in actual data).

For maturities of 10 and 20 years, Panel A of **Table 6** reports the ratio of the mean and the standard deviation of (i) the expected short rate, (ii) the term premium and (iii) the "adjusted" term premium for $KWML$, to the corresponding means and standard deviations in our model. We find that both the average level and the volatility of short rate expectations in $KWML$ are close to the average level and the volatility estimated by our stochastic volatility model (i.e., that the ratio in the table is close to one). However, the constant volatility feature of the Gaussian model leads to significant under-estimation of the volatility of the term premium. In fact, the standard deviation of term premia in $KWML$ is only about one third of that of our model, both for the 10-year and 20-year maturity. Panel A also contains the estimated standard errors of the ratios and they show clearly that the ratio of the volatility of short rate expectations is not significantly different from unity while the ratio of term premia volatilities is strongly different from unity.

3.7 The Impact of the Estimation Method

In this section we address two further questions, which are mainly related to the choice of the estimation method and the data used to make inference on short rate expectations and term premia: (i) does the technique used in estimation have a significant impact on the estimated expected short rate and term premium? and (ii) does the inclusion of long-term yields provide information that is useful in better identifying the expected short rate and term premium?

3.7.1 Estimation Procedure

Panel A of **Table 6** showed that the volatility of the term premium in $KWML$ (a Gaussian model) is much lower than in our stochastic volatility model. However, as we now show, very similar Gaussian models may give very different estimates of the volatility of term premia depending on the estimation method that is used. Thus, while some of the difference between the estimates of the volatility of term premia for our stochastic volatility model and those from Gaussian models that appear in the literature (and in Panel A of **Table 6**) may be due to model specification (stochastic versus constant volatility), some differences may be the result of the estimation method. This is the issue we address next.

For this analysis, we use $KWML$ and also estimate the four-factor version of KW with a technique based on the two-step approach proposed by Joslin et al. (2011) and that we implement as in Wright (2011). Specifically, we first run a principal components analysis on yields and take the first four principal components as state variables. We then use OLS to estimate a vector autoregression (VAR) for the dynamics of the state variables, which provides the coefficients of the factor processes under the physical measure. Finally, keeping these coefficients fixed, we estimate the remaining parameters, including those determining the market prices of risk, while imposing cross-sectional no-arbitrage restrictions. This method uses maximum likelihood, assuming that the difference between the observed yields and the corresponding model-implied values are i.i.d. measurement errors. For brevity, in what follows we refer to the KW model estimated using this approach as $KWJSZ$.

We also estimate a four-factor version of ACM. Here, we also use the first four principal components of yields as state variables and again apply a two-step procedure. The first step is the same as $KWJSZ$, i.e., the dynamics of the factors are estimated by applying OLS to a VAR specification of the state variables under the physical measure. The second step gives estimates for the coefficients of the market prices of risk by regressing monthly excess returns on lagged state variables and contemporaneous innovations in the VAR system.

As before, we fit the models to our STRIP yield curves using maturities between two and 25 years. Both models fit yields relatively well with an average (across maturities) standard deviation of errors of 4.4 bps for $KWJSZ$ and 4.3 bps for ACM. The implied fit of forward rates is very similar for the two models (and also relatively accurate when compared to our model and $KWML$), with an average standard deviation of the errors of 24.4 for $KWJSZ$ and 25.6 bps for ACM. The average error for the average yield variance increases with maturity in $KWJSZ$ (from -0.02 bps at 15 years to -0.08 bps at 25 years), while it is almost constant at -0.05 bps in ACM. In terms of yield volatility, these errors imply an under-estimation of the 25-year yield volatility of 3.4 bps in $KWJSZ$ and 1.6 bps in ACM (again, these are accurate estimates when compared to $KWML$).

The use of OLS to estimate the VAR means that both $KWJSZ$ and ACM are subject to the small-sample bias critique of Bauer et al. (2014), i.e., that this method tends to over-estimate the speed of mean reversion and, therefore, both under-estimates the volatility of short rate expectations and over-estimates the volatility of the term premium.

As Panel B of **Table 6** shows, this is indeed what we find and, as expected, this effect is stronger at longer maturities. The Panel reports the ratio of the mean and the standard deviation of the 10-year and 20-year maturity expected short rate and term premium in $KWJSZ$ and ACM to the corresponding values in $KWML$. Since the model in each case is Gaussian, the difference in the results is due entirely to the estimation method. For the 10-year maturity, we find that, relative to $KWML$, the volatility of short rate expectations is underestimated by 32% ($KWJSZ$) and 43% (ACM) while the volatility of the term premium is overestimated by over 200% in both cases. For the 20-year maturity, the biases are somewhat larger. These effects are even more larger in the case of the "adjusted" term premium, i.e., the term premium net of the convexity and duration adjustment components. In both $KWJSZ$ and ACM, although the duration adjustment term is on average zero,[11] it is both highly volatile and significantly negatively correlated with the term premium. What this means is that when we subtract this term from the term premium (to obtain the adjusted term premium), the volatility of the resulting variable is considerably higher than that of the term premium.

Panels A and B of **Table 6** are in-sample and so, to judge whether the results are specific to the sample period in our analysis or the result of model specification and/or estimation technique, we conduct two Monte Carlo experiments. In the first, we use the parameter estimates provided by $KWML$ to simulate 250 paths for the four state variables with the same length as our data, i.e., 384 months. For each month and trial in the simulation, we use the $KWML$ parameters to compute the term structure of yields. Then, for each trial, we re-estimate $KWML$, $KWJSZ$ and ACM from the simulated yield curves and, in each case, compute the mean and the standard deviation of short rate expectations, term premium and adjusted term premium.

In the second experiment, we repeat the same procedure but generate the yield curves for each trial using the parameter estimates of our stochastic volatility model.

Thus the first experiment asks whether, when yield curves are generated by a Gaussian model, the biases that we find in **Table 6** are generic, or specific to our sample period. Since the models are Gaussian and the "data" are generated by a Gaussian model, this question is entirely about estimation methodology as distinct from model specification and has already been substantially answered by Bauer et al. (2014). However, the second experiment asks a different question, namely, when a Gaussian model is estimated from yield curves that are generated by a model with stochastic volatility, what is the bias in estimates of the volatility of short rate expectations and the term premium?

Panel A of **Table 7** gives the results of the first "methodology only" experiment. Although the table is similar to **Table 6**, there are two important differences. First, rather than a single in-sample estimate of a ratio, the table reports the average value across the 250 Monte Carlo trials.

[11] The *average* duration adjustment term in $KWJSZ$ and ACM is zero by construction, as the state variables X (the principal components) have zero mean and the model is estimated imposing that the long-run mean Θ is zero.

Second, the denominator in each case is the "correct" value in the sense that, for each trial, it is computed using the same parameters that are used to generate the state variables and, from these, the yield curves.

Panel A gives results for $KWML$ as well as for $KWJSZ$ and ACM. Although the data are generated using the in-sample estimates from $KWML$, the results in the first column for $KWML$ are meaningful since the model is re-estimated for each trail. The results are striking in that, for maturities of both 10 years and 20 years, $KWML$ gives essentially unbiased estimates of both the mean (Panel A(i)) and standard deviation (Panel A(ii)) of short rate expectations as well as the two definitions of term premium. Panel A(i) also shows that $KWJSZ$ and ACM give estimates of the mean values of short rate expectations and term premia that are either unbiased or, in the case of ACM's estimates of term premia, modestly upward biased.

In contrast, the results for $KWJSZ$ and ACM in Panel A(ii) are qualitatively very similar to the in-sample results in **Table 6**. Both methods significantly under-estimate the volatility of the expected short rate (by 26% and 13% respectively at 10 years and 44% and 36% at 20 years) and hugely over-estimate the volatility of the term premium (by over 100% for both methods at 10 years and over 300% for both methods at 20 years). The Panel also reports the standard error of the mean in each case and these show that all the biases in volatility for $KWJSZ$ and ACM are highly significant.

The difference in the precision of the estimates is noteworthy. For example, for $KWJSZ$ the mean value of the ratio of the estimated volatility of the term premium at 10 years to the correct value is 2.33 and the standard error of the mean is 0.044. Since the sample size is 250, the standard error of the ratio is 0.70 while the corresponding value for $KWML$ is 0.06. Overall, the estimates produced by $KWJSZ$ and ACM are not only biased but also highly variable while those from $KWML$ are both unbiased and much less variable.[12]

Finally, Panel B of **Table 7** reports the corresponding results where yield curves are generated using the parameter estimates from our stochastic volatility model. The results for mean values

[12] Ironically, the fact that the $KWML$ method is so much more precise means that, even though $KWML$ produces estimates of both mean and volatility that are close to unbiased – the ratios in Panel A of **Table 7** for $KWML$ range from 0.93 to 1.08 compared with 0.56 to 12.51 for $KWJSZ$ and ACM – these quite modest deviations from unity in the case of $KWML$ are also statistically significant.

in Panel B(i) are broadly similar to those in Panel A(i) with one important exception. While the biases in the mean value of the (conventional) term premium for $KWML$, $KWJSZ$ and ACM are very similar to those in Panel A(i), the biases in the case of the adjusted term premium are much larger. This can be traced to underestimation of the convexity term by Gaussian models.

The results for the volatility of the expected short rate in Panel B(ii) are broadly similar to those in Panel A(ii), although the underestimation in the case of $KWJSZ$ and ACM is much greater. However, it is interesting that, even in the case of a stochastic volatility model, the estimate of the volatility of the expected short rate produced by $KWML$ is unbiased and relatively precise. However, it is not surprising that the results for the volatility of the term premium are quite different from those in Panel A(ii) since in our model risk premia are strongly dependent on time-varying volatility which is constant in Gaussian models.

For $KWML$ the volatility of the term premium is underestimated by around 60% while, for $KWJSZ$ and ACM it is overestimated by around 80% at 10 years and 140% at 20 years. The somewhat lower level of overestimation for $KWJSZ$ and ACM in this case can be understood as the result of a combination of the overestimation found in Panel A combined with a higher true level of the volatility of the risk premium.

The simulation exercise confirms the in-sample finding (Panel B of **Table 6**) that both $KWJSZ$ and ACM tend to under-estimate the volatility of short rate expectations and significantly over-estimate the volatility of the term premium. In **Table 7**, comparing Panel B(ii), where estimates from our stochastic volatility model are used as the benchmark, to Panel A(ii), where $KWML$ is the benchmark, the underestimation of the volatility of short rate expectations is significantly more marked, while the overestimation of the volatility of term premia is less marked only because term premia in the stochastic volatility model are significantly more volatile than in $KWML$. The biases produced by $KWJSZ$ and ACM are also significantly greater at the 20-year maturity than at 10 years.

Finally, and significantly, the first column in Panels B(i) and B(ii) show that the estimation procedure we use for our model produces unbiased estimates for the average level and the volatility of short rate expectations at both 10 and 20 years. The average term premium is also unbiased as well as the volatility of the term premium at 20 years while at 10 years the volatility of the term premium is slightly under-estimated (the ratio in this case is 0.91 with a standard error of 0.013).

To sum up, we find that: (i) conventional yield or forward term premia may give a misleading view of both the level and term structure of the compensation that bondholders require to hold bonds of a particular maturity, i.e., the bond risk premium; (ii) under Gaussian models (i.e., models with constant volatility), the dynamics of the convexity component of yields are mis-specified and this leads in turn to mis-estimation of risk premia (and term premia) and, finally, (iii) common estimation techniques based on OLS estimation of a VAR specification for the underlying factors tend to generate expected short rates that are too stable and term premia that are too volatile. This last result applies regardless of whether the underlying model is Gaussian or, as for our model, has stochastic volatility.

3.7.2 Additional Information in Long-Maturity Yields

Most of the empirical work on the term structure is based on yield data that has a maximum maturity of 10 years. In general, these estimates provide a very good fit for yields and apparently reasonable estimates for the expected short rate and the term premium, at least for maturities up to 10 years.

The central idea in this paper is that, because the term structure of expectations at long maturities is relatively flat, long-term rates facilitate the separation of expectations and risk premia much better than short rates. From this it seems likely that excluding data on long rates in the estimation of term structure models will reduce the precision with which both expectations and risk premia are estimated.

To test this proposition, we again use Monte Carlo simulation and employ a Gaussian model to generate the yield curves from which the parameter estimates are obtained. Our proposition on the importance of long-term rates applies irrespective of whether or not volatility is stochastic and using a Gaussian model allows us to keep separate the effect of including or excluding data on long-term yields from the effect of model specification (including stochastic volatility). As before, we generate 250 paths of length 384 months of the term structure of yields simulated using the in-sample parameter estimates provided by $KWML$ (see section 3.7.1 above). Then, for each trial, we fit $KWML$ twice, once using yield data only up to 10 years ($KWML10$) and a second time using yields up to 25 years ($KWML25$).[13]

Panel A of **Table 7** showed that, for maturities of both 10 and 20 years and using yield data for maturities up to 25 years, the $KWML$ estimation method produced estimates of the mean level and volatility of the expected short rate (ESR), term premium (TP) and adjusted term premium (ATP) that were relatively unbiased. **Table 8** contains the results from the same set of estimates (using yield data up to 25 years maturities) along with corresponding results when the yield data used in estimation is limited to 10 years maturity. Since our claim is that including data on long-term rates improves the precision in general, and not simply at long maturities, we report results for maturities of 5 and 10 years rather than 10 and 20 years.

The results given in **Table 8** partially overlap those already discussed in **Table 7**. Specifically, **Table 7** used yield data up to 25 years and included results for maturities of 10 and 20 years. **Table 8** gives results both using data up to 25 years (as in **Table 7**) and up to 10 years and reports results for both the 5- and 10-year maturities.

We find that biases in the estimates of the average level and the volatility of ESR, TP and ATP are similar to those in **Table 7**, irrespective of whether estimates are based on yield data up to 10-years or 25 years. However, the question we address here is not about bias but about precision and in **Table 8** we therefore report only the standard errors of the estimates (along with the "true" average value of the quantity in question in order to be able to judge whether scale of any improvement in precision is "worthwhile").

Panel A gives the standard errors of the estimated values of the average ESR, TP and ATP and Panel (ii) gives the standard errors of their estimated standard deviations. For example, the first row of Panel A gives results for the ESR at 5 years (left-hand sub-panel) and at 10 years (right-hand sub-panel). The "true" average value of ESR at 5 years is 429 bps, which is computed as follows: for each of the 250 trials, the average value of the ESR is computed using the realization of the state variables for that trial and the "true" parameters (used to generate each of the MC

[13]In the latter case ($KWML25$), the results are the same as in Panel A of Table 7 (for $KWML$).

trials). The value of 429 bps is then the average of these values across the 250 trials. To compute the standard error of the estimated average ESR we proceed as follows: for each trial, we compute estimate error in the average ESR as the difference between the value using (i) the estimated parameters for that trial and (ii) the true parameters. The standard error of an individual estimate is then the standard error of these estimate errors across the 250 trials.

Using yield data only up to 10 years ($KWML10$), the standard error of the estimated average ESR at a 5-year horizon is 32.8 bps but, when yield data up to 25 years is used, this falls to 11.6 bps. In other words, failing to use long-term yield data roughly triples the standard error in this case. At 10 years, the picture is much the same: the standard error is 35.5 bps for $KWML10$ and 12.8 bps for $KWML25$. Indeed, the pattern of improvement is remarkably similar across the estimated average values of all three variables: in each case the standard error falls by around 2.8 times.

Panel B gives the corresponding results for volatilities. For estimates of the volatility of ESR, the standard error falls by a factor of around two (2.1 for 5 years and 1.7 for 10 years). The improvement in the precision with which the volatility of TP and ATP is estimated is even greater: the standard errors of the estimated volatility of the 5-year TP is 3.4 times smaller using data up to 25 years and 3.5 times for the 10-year TP. Finally, the improvement in ATP is greater still: for the 5-year ATP, it is reduced from 9.3 bps using $KWML10$ years to 2.5 bps using $KWML25$ and, for the 10-year ATP, from 12.0 bps to 3.2 bps. In this last case the standard error of the estimate is reduced by a factor of almost four.

Why are long-term rates so helpful in improving precision? The answer would appear to be that the difference in average long-term risk premia is effectively "pinned down" by the slope of the corresponding long-term forward rate curve and this, combined with the fact that the risk premium goes to zero at zero maturity, greatly constrains the term structure of risk premia. Overall, these results argues strongly for the inclusion of data on long-term rates when estimating term structure models.

3.8 Implied Term Structure of Long-Term Risk Premia

As noted earlier (see equation 2), the forward rate spread between the maturities τ_1 and τ_2 for both our model and the Gaussian model can the decomposed into four components:

$$f_t(\tau_2) - f_t(\tau_1) = \Delta f_t(\tau) = \Delta \mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right] + \Delta brp_t(\tau) + \Delta cvx_t(\tau) + \Delta dur_t(\tau). \tag{27}$$

Table 9 reports the average value (Panel A) and volatility (Panel B) of the actual 25- minus 15-year spread calculated from STRIP prices (row 1) along with the model values of the spread and its four components for our model, $KWML$, $KWJSZ$ and ACM (rows 2-5). For these long maturities, our model estimates of the average values of $\Delta \mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right]$ and $\Delta dur_t(\tau)$ are small (0 bps and -5 bps, respectively) and so the average value of $\Delta brp_t(\tau)$ is well approximated by:

$$\overline{\Delta brp_t(\tau)} \approx \overline{\Delta f_t(\tau)} - \overline{\Delta cvx_t(\tau)}, \tag{28}$$

where an overbar indicates the average value. Interestingly, as Panel A of **Table 9** shows, the sum of $\overline{\Delta \mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right]}$ and $\overline{\Delta dur_t(\tau)}$ is also small in the case of the three estimates of the Gaussian model: -3 bps for $KWML$ and zero for the other two (for $KWJSZ$ and ACM both $\overline{\Delta \mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right]}$ and $\overline{\Delta dur_t(\tau)}$ are zero individually). Thus, across our model and the three estimates of the Gaussian model, the only reason for a significant difference in the estimate of $\overline{\Delta brp_t(\tau)}$ is a difference in either the estimate of $\overline{\Delta f_t(\tau)}$ or the estimate of $\overline{\Delta cvx_t(\tau)}$. The top Panel of **Table 9** shows that three of the four estimates of $\overline{\Delta f_t(\tau)}$ (our model, $KWJSZ$ and ACM) are close to each other and to the direct estimate from STRIPs while the value estimated in $KWML$ is 18 bps larger (more negative) than the direct estimate.

The average convexity term, $\overline{\Delta cvx_t(\tau)}$, depends simply on the average variance of yields and computing this directly from the zero-coupon yields calculated from STRIP prices, we obtain 139 bps. The estimate from our model matches this perfectly (-139 bps) but the other three estimates are significantly lower. This results mainly from the fact that the (negative) slope of the term structure of yield volatility is overestimated in the Gaussian model. In the case of $KWML$, $\overline{\Delta cvx_t(\tau)}$ is underestimated by 48 bps (relative to the estimate of -139 bps obtained directly from

STRIP prices) and in the other two cases by 40 bps ($KWJSZ$) and 28 bps (ACM). As a result of underestimating the convexity term, the estimated values of $\overline{\Delta brp_t(\tau)}$ are much lower than the 62 bps that we find in our model; the value in the case of $KWML$ is -6 bps ($KWML$) and for $KWJSZ$ and ACM the values are 23 bps and 27 bps, respectively.

Because the convexity term is proportional to the variance of yields rather than the standard deviation, it is highly sensitive to the estimated value of yield volatility. For example, the average yield volatility in the data at long maturities is around 86 bps.[14] Underestimating yield volatility by, say, 10 bps, will reduce the value of $\overline{\Delta cvx_t(\tau)}$ by around 30 bps and if, as is the case for the three estimates of the Gaussian model, (i) the estimates in question fit the forward curve relatively well and (ii) $\overline{\Delta \mathrm{E}^{\mathbb{P}}\left[r_t(\tau)\right]}$ and $\overline{\Delta dur_t(\tau)}$ are small, then this underestimation will feed directly into the estimate of $\overline{\Delta brp_t(\tau)}$.

Panel B of **Table 9** gives the annualised volatilities of monthly changes in each of the quantities in Panel A and reveals some significant differences not evident from the mean values. The first point to note, as already mentioned above, is that the volatility of changes in the forward spread calculated directly from STRIPs (81 bps) is significantly higher than the value computed in our model (55 bps). This is largely the result of noise in the direct estimates of the forward rate spread: monthly changes in the direct estimates are significantly negatively serially correlated (with a correlation coefficient of -0.21). The volatility of annual changes in direct estimates of the change in the forward spread (43 bps) is much more in line with the volatility of annual changes in the values from our model (50 bps).

For our stochastic volatility model, the most volatile components of the forward spread are the convexity term and the risk premium. These are negatively correlated (correlation coefficient -0.80) because the convexity term is both negative and proportional to the variance of yields while our estimates of the stochastic volatility model give bond risk premia (and $\Delta brp_t(\tau)$) that are positively related to variance.

Although the three Gaussian models fit the average level of the spread quite well, each of them predicts a much lower volatility for the spread than exhibited in the data. In all three cases the

[14]Computed from the average value of the variance of long term yields, which is equal to about 0.75 bps for the 25-year maturity.

convexity term has zero volatility since the volatility of yields in the Gaussian model is constant. The volatility of $\Delta brp_t(\tau)$ and $\Delta dur_t(\tau)$ for $KWJSZ$ (159 bps and 156 bps, respectively) and ACM (129 bps and 124 bps, respectively) are strikingly large since the overall volatility of the spread is only 19 bps ($KWJSZ$) and 5 bps (ACM). This apparently anomalous result comes about because in both $KWJSZ$ and ACM, estimates of $\Delta brp_t(\tau)$ and $\Delta dur_t(\tau)$ are almost perfectly negatively correlated.

The results in **Tables 6 – 9** suggest that both estimation method and model specification (inclusion or exclusion of stochastic volatility) can have a major impact on estimates of the importance of expectations versus term premia in the variation of long-term yields. Following Ang and Piazzesi (2003) and Moench (2019), we carry out a variance decomposition analysis and **Table 10** shows the decomposition of the variance of monthly changes in the 10-year yield estimated using: (i) our model, (ii) and (iii) our estimates of $KWJSZ$ and ACM and (iv) and (v) the estimates of KW and ACM published by the Federal Reserve ($KWFed$ and $ACMFed$, respectively). The first point to note is that the results for our estimates of $KWJSZ$ and ACM are very similar to those reported by the Fed. Second, and more significantly, we see that for each of the four estimates of the Gaussian model, around 60% of the variability of monthly changes in the 10-year yield is explained by the term premium component, while for our model this percentage is less than 30%. The higher level of variation in the term premium for KW and ACM is consistent with the results of the Monte-Carlo experiment in Panel B of **Table 7**. This showed that, when the yield curve is generated by our stochastic volatility model, the average estimated volatility of the 10-year term premium for $KWJSZ$ and ACM is around 80% too high.

Thus, in contrast with the predictions from KW and ACM and others (see, for example, Jotikasthira et al. (2015) and Moench (2019)), who identify fluctuations in the term premium as the major determinant of recent movements in long-term yields, we conclude that, in fact, it is variation in short rate expectations that plays the larger role.

We showed earlier that a "model-free" estimate of the average difference in risk premia at long maturities ($\overline{\Delta brp_t(\tau)}$) can be obtained by applying a convexity adjustment ($\overline{\Delta cvx_t(\tau)}$) to the average difference in forward rates ($\overline{\Delta f_t(\tau)}$). Panel B of **Figure 5** showed that the value of these three quantities ($\overline{\Delta brp_t(\tau)}$, $\overline{\Delta f_t(\tau)}$ and $\overline{\Delta cvx_t(\tau)}$) computed from our model were very close to the "model-free" values.

Figure 9 carries out the same comparison between model-free estimates and the three estimates of the Gaussian model. The results for $KWML$ are in Panel A, for $KWJSZ$ in Panel B and for ACM in Panel C.[15]

In all three cases, the Gaussian model fits the average differences in forward rates relatively well but all three under-estimate the convexity difference ($\overline{\Delta cvx_t(\tau)}$). Since the average difference in the expected short rate and the duration adjustment is small in each case, underestimation of $\overline{\Delta cvx_t(\tau)}$ leads to a lower estimate of the slope of the term structure of risk premia at long maturities than that implied by both the model-free calculations and our stochastic volatility model. In $KWJSZ$ and ACM the slope of the term structure of bond risk premia is about 4 bps per year, compared to 6 bps in the case of model free estimates. In $KWML$ the under-estimation of convexity is more severe and, as a result, the implied term structure of long-term bond risk premia is effectively flat.

4 Conclusion

Disentangling risk premia and interest rate expectations in bond yields is important for both policy makers and portfolio investors and term structure models provide an internally consistent way of addressing the problem. While most term structure research has focused on relatively short maturities (up to 10 years) we pay particular attention to the long end of the yield curve where a flat term structure of interest rate expectations plays a crucial role in allowing expectations to be separated from risk premia.

Using an affine model, we show that the difference between two long-term forward rates for maturities τ_1 and τ_2 ($\tau_2 > \tau_1$) is largely determined by two terms. The first is the difference in the instantaneous risk premia on zero coupon bonds with the same maturities and the second is a convexity term that depends on yield volatility at these maturities. At long maturities, the convexity component is large and negative and is responsible for the persistent downward slope in the term structure of long-term forward rates (the forward rate "tilt"). As volatility varies

[15] In **Figure 9**, the results for the 15- to 25-year interval are those given in Panel A of **Table 9**.

significantly over time, a model with stochastic volatility is necessary in order to account for the dynamics of the convexity component. The model provides a good fit to yields, forward rates, yield volatility and to the dynamics of the forward rate tilt.

We find that yield volatility is the major driver of bond risk premia and that, in contrast with previous empirical evidence, our stochastic volatility model produces estimates which are consistent with deviations from the pure Expectations Hypothesis that are observed in the data.

We compare our estimates of term premia with some others in the literature. Since term premia are conventionally defined as the difference between a yield (or forward rate) and the average expected short rate (or expected short rate at corresponding horizon), these include not only risk premia but also the effects of convexity and the duration adjustment. We show first that, under this definition, it is often the case that the term structure of term premia at long maturities is downward sloping while the term structure of risk premia is upward sloping.

Estimating a Gaussian model on the same data set gives results that, naturally, reflect the difference in model specification but also turn out to depend strongly on the estimation method that is used. A maximum likelihood approach, similar to the method we use to estimate our model, provides very similar estimates of the mean level and volatility of short rate expectations but higher means and lower volatility for both term and risk premia. However, estimating the same model using a two-step approach quite widely used in the literature, results in much lower volatilities for short rate expectations and much higher volatilities for term and risk premia. The results using the two-step approach imply that around 30%-40% of the variance of yields is explained by short rate expectations while the remainder, 60%-70%, is explained by variation in the term premium. These estimates are consistent with the estimates published by the Federal Reserve but are quite different from the results for our model where around 70% of the variance of yields is explained by short rate expectations and 30% by variation in the term premium.

Using a Monte Carlo approach, we confirm previous evidence showing that the two-step methods used in estimating Gaussian models – such as $KWJSZ$ or ACM – suffer from a small sample bias problem (see Bauer et al. (2014)), which induces significant under-estimation of the volatility of the expected short rate and over-estimation of the volatility of term premia. The simulation shows that this result holds both in the case that the "correct" values are generated by a Gaussian model and in the case that data on yields are generated using our stochastic volatility model. The Monte Carlo experiment also shows that estimating the Gaussian model with a maximum likelihood approach, rather than the two-step methods cited above, produces relatively unbiased estimates for the level and the volatility of short rate expectations and term premia, when the generating process is a Gaussian model, whereas when the "correct" values are generated by our stochastic volatility model, the estimates for the expected short rate are still unbiased, but the volatility of the estimated term premium is too low. Thus the difference between our results and those in the "Gaussian" literature certainly depends on model specification (constant vs time-varying volatility), but is to a greater extent the result of differences in the econometric method used in estimation.

We claim that a flat term structure of short rate expectations at long maturities greatly helps to isolate the risk premium information in long-term yields and so facilitates the separation of the expectations and risk/term premia in the yield curve. We put this claim to a very simple test by estimating the same model twice for each realisation of a simulated term structure data set, once using yield data up to 25 years and once limiting the data used – as in much of the literature – to 10 years. When data on yields longer than 10 years are excluded, we find a dramatic increase in the standard errors of estimated means and volatilities of short rate expectations, term premia and risk premia. Significantly, we find that the improvement in precision is just as great at short maturities (5 years) as at long maturities (10 years).

Finally, although it is necessary to estimate the model to obtain the full term structure of risk premia, we can gauge the average difference between risk premia on two long-term bonds quite easily. Since forward rates and yield volatility can be quite easily measured, the difference in risk premia at any time t between two long maturity zero-coupon bonds can be quite well approximated by:

$$brp_t(\tau_2) - brp_t(\tau_1) \simeq [f_t(\tau_2) - f_t(\tau_1)] - \frac{1}{2}\left[\tau_2^2 V_t(\tau_2) - \tau_1^2 V_t(\tau_1)\right], \tag{29}$$

where $brp_t(\tau)$ is the time t instantaneous risk premium on a τ-maturity zero coupon bond, $f_t(\tau)$ is the τ-maturity instantaneous forward rate and $V_t(\tau)$ is the τ-maturity yield variance. Evaluating this expression using average forward rate differences spanning 10 years, from 10–20 years to 17–27 years, shows that, in this region of the term structure, average risk premia are approximately linear in maturity and increase at the rate of around 6 bps per year.

We find that the "model-free" estimates of average risk premia in equation (29) are matched almost exactly by the estimated values in our stochastic volatility model which also matches the directly estimated average values of the forward rate spread and convexity. However, when we apply the same test to estimates derived from a Gaussian model, we find that, although the fit to average forward rates is quite good, the difference is risk premia is underestimated because the size of the convexity component is underestimated. Thus, in addition to providing a simple way to gauge the slope of the term structure of risk premia, equation (29) also provides a valuable diagnostic for prospective models.

The main conclusion from our paper is that long-term yield data, and particularly forward rates, provide information on the slope of the term structure of term premia that greatly improves the precision of estimates. Even when policy makers are concerned to assess the contribution of expectations versus risk premia at relatively short horizons, we find that including data on long-term yields greatly improves results.

Finally, we return to the simple approximation for the slope of the term structure of risk premia in terms of the slope of the forward rate curve, and convexity:

$$\overline{\Delta brp_t(\tau)} \approx \overline{\Delta f_t(\tau)} - \overline{\Delta cvx_t(\tau)}, \tag{30}$$

Multi-factor affine models appear to be quite flexible. Thus, even though we view Gaussian models as mis-specified, and even though different estimation methods given different results, we found in Section 3.7.1 that all the different estimates fitted the term structure quite well, even very well. This means that, as we found in **Table 9**, model values of the difference in forward rates are likely to be close to the observed values. Since model values of the difference in risk premia will be well approximated by equation (,30), this means that the difference in risk premia will be mis-estimated unless the convexity term is correct. As we have seen, relatively small errors in volatility lead to quite large errors in estimated convexity and so this emphasises once again the critical importance of modelling the level and dynamics yield volatility in order to obtain reliable estimates of risk premia.

Appendix

A Forward Rate Decomposition in Affine Models

This appendix shows the decomposition of the forward rate in the case of two well-known, single-factor models of the term structure: Vasicek (1977) (hereafter VAS or V) and Cox et al. (1985) (hereafter CIR or C). In both models the short rate, r, follows a mean reverting stochastic process:

$$dr_t = \kappa(\theta - r_t)dt + \sigma(r_t)dz_t, \tag{A.1}$$

where κ is the mean reversion coefficient, θ is the mean value of r, dz_t is a Brownian motion, and the volatility function $\sigma(r_t)$ is equal to σ, a constant, in VAS and to $\sigma\sqrt{r_t}$ in CIR. For both the VAS and CIR models, the price at time t of a zero coupon bond with a face value of unity and maturity τ can be written as a linear-exponential function of r, $P_t(\tau) = \exp\left[A(\tau) - B(\tau)r_t\right]$, which solves the following partial differential equation:

$$\frac{1}{2}\frac{\partial^2 P_t(\tau)}{\partial r_t^2}(\sigma(r_t))^2 + \frac{\partial P_t(\tau)}{\partial r_t}\left[\kappa(\theta - r_t) - \sigma(r_t)\lambda(r_t)\right] + \frac{\partial P_t(\tau)}{\partial t} - r_t P_t(\tau) = 0. \tag{A.2}$$

In equation (A.2) $\lambda(r_t)$ is the market price of risk, which is constant in VAS, $\lambda(r_t) = \lambda_v$, and time-varying in CIR, $\lambda(r_t) = \lambda_c\sqrt{r_t}$. Since the models are time homogeneous, the time t instantaneous forward rate for date $t+\tau$ is given by $f_t(\tau) = \frac{1}{P_t(\tau)}\frac{\partial P_t(\tau)}{\partial t}$, and in the VAS and the CIR cases can be written as follows (for simplicity, in what follows we suppress the time index t and the maturity index τ):

$$\begin{aligned} f_{\mathrm{V}} &= r + B_{\mathrm{V}}\kappa(\theta - r) - B_{\mathrm{V}}\sigma\lambda - \frac{1}{2}B_{\mathrm{V}}^2\sigma^2, && \text{(A.3)} \\ f_{\mathrm{C}} &= r + B_{\mathrm{C}}\kappa(\theta - r) - B_{\mathrm{C}}\sigma\lambda r - \frac{1}{2}B_{\mathrm{C}}^2\sigma^2 r. && \text{(A.4)} \end{aligned}$$

In both equations (A.3) and (A.4), the third and the fourth terms on the right hand side represent, respectively, the instantaneous bond risk premium, $\frac{1}{dt}[E(\frac{dP}{P})] - r$, and minus one half of the rate of return variance, $-\frac{1}{2}\frac{1}{dt}[Var(\frac{dP}{P})]$, of a τ-maturity zero coupon bond. The latter is the "convexity term".

The value of $B(\tau)$ in the VAS model, B_{V}, is a function only of the mean reversion coefficient κ and time-to-maturity τ, whereas its value in the CIR model, B_{C}, is also a function of σ and λ. In both the VAS and the CIR models the conditional expectation of the short rate under the physical measure $\mathbb{P}$ is equal to: [16]

$$\mathrm{E}^{\mathbb{P}}[r] = r + B_{\mathrm{V}}\kappa(\theta - r). \tag{A.5}$$

Substituting (A.5) in (A.3) and (A.4), the τ-maturity instantaneous forward rate can be written

[16] This result derives from the fact that the short rate in CIR is conditionally gaussian.

as:

$$f_{\mathrm{V}} = \mathrm{E}^{\mathbb{P}}[r] - B_{\mathrm{V}}\sigma\lambda - \frac{1}{2}B_{\mathrm{V}}^2\sigma^2, \tag{A.6}$$

$$f_{\mathrm{C}} = \mathrm{E}^{\mathbb{P}}[r] - B_{\mathrm{C}}\sigma\lambda r - \frac{1}{2}B_{\mathrm{C}}^2\sigma^2 r + (B_{\mathrm{C}} - B_{\mathrm{V}})\,\kappa(\theta - r). \tag{A.7}$$

In the case of CIR, the equation for the forward rate contains an additional term that depends on the difference between the $B(\tau)$ functions in the CIR and VAS models. The $B(\tau)$ function can be interpreted as a measure of duration and therefore we define the difference $B_{\mathrm{C}} - B_{\mathrm{V}}$ as the "duration adjustment". It arises because of the presence in the CIR model of stochastic volatility and a stochastic price of risk.

If we compute the difference between the forward rates calculated at maturities τ_2 and τ_1 ($\tau_2 > \tau_1$) (the forward rate spread) in the case of the VAS model we obtain:

$$f_{\mathrm{V}}(\tau_2) - f_{\mathrm{V}}(\tau_1) = \mathrm{E}^{\mathbb{P}}\left[r_{t+\tau_2} - r_{t+\tau_1}\right] - (B_{\mathrm{V}}(\tau_2) - B_{\mathrm{V}}(\tau_1))\sigma\lambda - \frac{1}{2}\left[\tau_2^2 V_{\mathrm{V}}(\tau_2) - \tau_1^2 V_{\mathrm{V}}(\tau_1)\right], \tag{A.8}$$

where $V_{\mathrm{V}}(\tau) = (1/\tau^2)B_{\mathrm{V}}^2\sigma^2$ is the variance of the τ-period yield in the VAS model. When both τ_2 and τ_1 are "long", it is reasonable to expect that the difference in the expected value of the short rate r will be small and so, in this case, the forward rate spread will be (approximately) equal to the difference in the risk premia between the longer and the shorter bonds minus the difference in the convexity terms. However, in the VAS model, these two terms are both constant and so the forward rate spread is approximately constant.

In the CIR model, the equation for the forward rate spread is:

$$\begin{aligned} f_{\mathrm{C}}(\tau_2) - f_{\mathrm{C}}(\tau_1) = \mathrm{E}^{\mathbb{P}}\left[r_{t+\tau_2} - r_{t+\tau_1}\right] &- (B_{\mathrm{C}}(\tau_2) - B_{\mathrm{C}}(\tau_1))\sigma\lambda r - \frac{1}{2}\left[\tau_2^2 V_{\mathrm{C}}(\tau_2) - \tau_1^2 V_{\mathrm{C}}(\tau_1)\right] \\ &+ \left[(B_{\mathrm{C}}(\tau_2) - B_{\mathrm{C}}(\tau_1)) - (B_{\mathrm{V}}(\tau_2) - B_{\mathrm{V}}(\tau_1))\right]\kappa(\theta - r), \end{aligned} \tag{A.9}$$

where $V_{\mathrm{C}}(\tau) = (1/\tau^2)B_{\mathrm{C}}^2\sigma^2 r$ is the variance of the τ-period yield in the CIR model. The duration adjustment term may be either positive or negative depending on the sign at time t of the drift of the short rate process. In fact, as the difference $(B_{\mathrm{C}} - B_{\mathrm{V}})$ is always negative and increasing in size with maturity, and so increases the downward tilt in long-term forward rates when the short term interest rate is below its long-term mean, and decreases the tilt when it is above the mean. In the CIR model, therefore, even if the difference in expected short rates at long maturities is negligible, the forward spread is potentially affected by volatility and the market price of risk effects that are neither a part of risk premia nor of the convexity effect. We estimate this term along with the other components of the forward rate and find that, for most of our sample, along with the term related to the expected short rate, it is relatively small.

B Data on STRIPs

Our empirical analysis is based on US Treasury STRIP prices (STRIP is an acronym for the Separate Trading of Registered Interest and Principal), which are obtained from Street Software and Bloomberg.

The interest and principal payments made by US Treasury notes and bonds are traded and registered separately and STRIP prices are direct observations of the discount factors that define the zero coupon yield curve. Street Software's data is daily covering the period 1 January 1988 to 31 July 2010. Bloomberg data are also daily and cover the period 1 August 2010 to 31 December 2019. For the years 1988 through to mid-1990 the prices of only the thirty most liquid STRIPs are recorded. From 1 June 1990 onwards all prices are recorded, resulting in around 160 observations of discount factors on a typical day. Both bid and offer quotes are available. We use only data on coupon STRIPs and run a spline interpolation through the observed yields to obtain estimates of the yield curve at annual maturities ranging between 1 and 30 years.

C The Behaviour of the Long-Term Forward Rate Spread

Figure A.1 presents some evidence on the downward tilt in long-term forward rates.

Panel A shows the spread between the US 25-year and 15-year instantaneous forward rates for end-of-month observations over the period January 1988 to December 2019. The data are obtained from (i) the STRIP prices described above and (ii) the zero coupon yield data estimated by Gurkaynak et al. (2007) (GSW).[17]

The average spread is -82 bps for STRIPs and -58 bps for GSW data. In the case of STRIPs, 88% of the observations exhibit a negative value, whereas 72% of the GSW values are negative.

Panel B reports the 25-year minus 15-year forward rate spread for the UK (starting in January 1998), Germany (January 2000) and Canada (January 1995).[18] We find that the spread is on average equal to -54 bps for the UK, -75 bps for Germany and -112 bps for Canada, with 92%, 89% and 97% negative values, respectively.

Panel C contains the corresponding data for US TIPS and UK index-linked Gilts and shows that a similar pattern of long-term forward rates also exists for inflation-indexed bonds.[19]

Thus the forward rate spread at long maturities is almost always negative. This is true, not

[17] It is, of course, not possible to measure instantaneous forward rates directly from STRIP prices. As a proxy for the instantaneous forward rate for maturity τ, we therefore use the average forward rate between maturities $\tau - \Delta\tau$ and $\tau + \Delta\tau$, where $\Delta\tau$ is one year. For example, we approximate the 25-year instantaneous forward rate as: $f_{25} = \frac{1}{2} \ln\left(\frac{P_{24}}{P_{26}}\right)$, where P_{24} (P_{26}) is the price of the 24-year (26-year) STRIP. The GSW data are available on the Federal Reserve website and are derived by the estimation of the Svensson (1994) model which endogenously generates instantaneous forward rates.

[18] Data for the UK are available from the Bank of England website (see Anderson and Sleath (2001)), while those for Germany are calculated using daily prices of Bund STRIPs downloaded from Bloomberg. Data for Canada zero coupon yields are obtained from the Bank of Canada website (see Bolder et al. (2004)).

[19] Data for the US TIPS are from Gurkaynak et al. (2008), while data for UK index-linked Gilts are available from the Bank of England website (see, again, Anderson and Sleath (2001)).

only for the US but for the other countries for which we have estimates of long-term nominal forward rates (the UK, Germany and Canada) and also for inflation-indexed rates in the US and the UK.[20] The negative slope in forward rates is arguably the single most persistent feature of the term structure. Second, we observe that the size of the forward spread is highly variable. We show below that both features of the forward rate spread – a negative average value and its variability – derive mainly from the behaviour of the convexity term.

[20] All our estimates of forward rate spreads, apart from one, are derived from term structure estimates obtained using splines, or some other form of curve fitting technique, to represent the discount function. It is therefore possible that the behaviour of the forward rates we observe is, to a greater or lesser extent, dependent on the estimation methodology used. After all, long term rates will usually be the most difficult to estimate (because they depend on the prices of a relatively small number of the longest bonds in the sample) and the forward rate spread then depends on the difference between these rates. The forward rate spreads we obtain from STRIP prices are therefore particularly significant because they are calculated directly from prices of zero coupon bonds with only simple interpolation to standardized maturities.

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Table 1
Summary Statistics

This table reports summary statistics for zero coupon yields and forward rates with maturities between 2 and 25 years and realized yield variance for maturities from 10 to 25 years. Average values are calculated from the variables in level, while annualised standard deviations are computed from monthly changes in the variables. Data are expressed in basis points. The sample period is January 1988 to December 2019.

Maturity	Yields		Forward Rates		Realized Yield Variance	
	Level	Changes	Level	Changes	Level	Changes
(years)	Average	Ann. St. Dev.	Average	Ann. St. Dev.	Average	Ann. St. Dev.
2	356	88	386	106		
3	382	93	434	111		
4	404	95	472	112		
5	423	95	499	104		
6	441	93	532	106		
7	457	93	554	123		
8	471	93	567	117		
9	483	91	578	97		
10	493	90	583	95	0.89	2.21
11	503	89	597	101	0.88	2.15
12	511	88	603	100	0.86	2.08
13	518	87	603	101	0.85	2.04
14	524	86	601	96	0.85	2.01
15	529	85	596	100	0.84	1.97
16	532	85	587	105	0.83	1.95
17	535	83	580	91	0.81	1.95
18	537	82	575	96	0.79	1.93
19	539	81	566	95	0.78	1.91
20	540	80	559	92	0.78	1.90
21	541	79	555	99	0.77	1.88
22	541	79	549	94	0.76	1.89
23	541	78	540	99	0.75	1.89
24	540	78	526	101	0.75	1.89
25	539	77	514	104	0.75	1.91

Table 2
Estimation Errors

This table reports summary statistics (in basis points) for estimation errors of yields, forward rates and yield variance for the period January 1988 to December 2019.

Maturity (years)	Yields Average	Yields St. Dev.	Forward Rates Average	Forward Rates St. Dev.	Yield Variance Average	Yield Variance St. Dev.
2 - 5	0.60	13.54	6.16	23.44		
6 - 9	2.86	5.18	0.51	22.18		
10	2.41	5.13	3.26	17.10	-0.03	0.15
11 - 14	0.68	4.99	-6.19	18.70		
15	-0.04	3.43	0.11	24.26	-0.06	0.14
16 - 19	0.52	2.85	3.97	25.20		
20	1.04	2.96	5.54	21.55	-0.01	0.12
21 - 24	0.83	3.17	-1.34	21.81		
25	0.53	1.93	-2.17	26.53	0.04	0.14

Table 3
Estimated Parameters

This table reports the maximum likelihood estimates of the model parameters. The coefficients are ordered as $[v; x_1; x_2; x_3]$. Bootstrapped standard errors are calculated: *** indicates statistical significance at 1% level, ** at 5% level, and * at 10% level.

K

0.1325 ***	0.0126 ***	-0.0162 ***	0.1627 ***
-0.0032	0.0198	0.0784 ***	0.0017
-0.0076 *	0.0571 *	0.2533 ***	-0.0030
0.0092 ***	0.0291 ***	-0.0058 **	0.0692 ***

Λ_1

-0.3186 ***	-0.0062 ***	-0.0035 ***	0.0607 ***
-0.1279 ***	-0.5935 ***	-0.0030 ***	0.1467 ***
0.0124 ***	-0.0133 ***	-1.5793 ***	1.0231 ***
0.0564 ***	-0.0520 ***	0.0121 ***	-0.3605

δ_0

0.0376 ***

δ	Θ	Σ_{ii}	Λ_0
0	0.3755 ***	0.0231 ***	-0.0014 ***
1.0000 ***	0	0.0154 ***	-0.0019
1.0000 ***	0	0.0188 **	0.0024
0	-0.1457 ***	0.0073	0.0025 ***

Table 4
Components of Yields

This table reports summary statistics (in basis points) for average short rate expectations, adjusted term premia, convexity and duration adjustment estimated by our model for the period January 1988 to December 2019.

Maturity	Short Rate Exp.		Adjusted Term Premia		Convexity		Duration Adjustment	
(years)	Average	St. Dev.	Average	St. Dev.	Average	St. Dev.	Average	St. Dev.
2	329	244	25	28	-1	1	1	2
5	373	203	53	54	-5	2	4	7
10	410	172	86	76	-15	7	10	16
15	428	160	111	84	-28	12	13	23
20	437	154	132	86	-47	21	15	27
25	443	151	151	85	-72	32	15	30

Table 5
Sensitivity of Term Premia Components to Volatility Shocks

This table shows, in Panel A, the average changes in the τ-maturity (τ equal to 10 or 20 years) forward rate, f, expected short rate, $E^{\mathbb{P}}[r]$, forward term premium, FTP, and its components (brp, cvx and dur in equation (20)), in response to one standard deviation shock in the variance factor v, and, in Panel B, the average changes in the τ-maturity (τ equal to 10 or 20 years) yield, Y, average expected short rate, ESR, term premium, TP, and its components (ATP, CX and DA in equation (21)) in response to one standard deviation shock in the variance factor v.

Panel A: Forward term premium

Maturity	f	$E^{\mathbb{P}}[r]$	brp	+	cvx	+	dur	=	FTP
10-Year	19.8	14.1	13.6		-7.1		-0.8		5.7
20-Year	7.9	8.5	27.5		-25.2		-3.0		-0.6

Panel B: Term premium

Maturity	Y	ESR	ATP	+	CX	+	DA	=	TP
10-Year	17.4	13.3	6.7		-2.8		0.1		4.1
20-Year	15.8	12.1	13.6		-8.9		-1.1		3.7

Table 6
In-Sample Comparison vs Gaussian Models

This table reports the ratio of the mean and the standard deviation of (i) average short rate expectations, (ii) term premium and (iii) "adjusted" term premium for Gaussian models (i.e., $KWML$, $KWJSZ$ or ACM), to the corresponding means and standard deviations in a benchmark model, i.e., our model or $KWML$. Results are for maturities of 10 and 20 years and the sample period from January 1988 to December 2019. Panel A shows the ratio between estimates in $KWML$ and in our model (benchmark), while Panel B shows the same ratios between estimates in $KWJSZ$ and ACM and estimates in $KWML$ (benchmark). Standard errors in parentheses.

Panel A: $KWML$ vs our model (benchmark)

	Ratio between means		Ratio between standard deviations	
	10-Year	20-Year	10-Year	20-Year
Short rate expectations	0.94	0.97	0.97	1.05
	(0.078)	(0.087)	(0.158)	(0.151)
Term premium	1.25	1.11	0.35	0.36
	(0.519)	(0.442)	(0.243)	(0.261)
Adjusted term premium	1.59	1.29	0.46	0.54
	(0.993)	(0.733)	(0.409)	(0.508)

Panel B: $KWJSZ$ and ACM vs $KWML$ (benchmark)

	Ratio between means				Ratio between standard deviations			
	10-Year		20-Year		10-Year		20-Year	
	KWJSZ	*ACM*	*KWJSZ*	*ACM*	*KWJSZ*	*ACM*	*KWJSZ*	*ACM*
Short rate expectations	0.87	0.95	0.79	0.85	0.68	0.57	0.56	0.46
	(0.129)	(0.102)	(0.157)	(0.145)	(0.252)	(0.173)	(0.247)	(0.156)
Term premium	1.48	1.19	1.37	1.54	3.33	3.11	3.67	3.67
	(0.883)	(0.509)	(1.098)	(0.743)	(2.249)	(1.945)	(2.541)	(2.339)
Adjusted term premium	1.26	1.04	1.49	1.33	7.03	9.23	9.89	11.69
	(0.984)	(0.631)	(1.379)	(1.093)	(8.540)	(10.354)	(14.014)	(15.337)

Table 7
Simulation-Based Comparison vs Gaussian Models

This table reports two Monte Carlo experiments. In the first (Panel A), we use the parameter estimates provided by $KWML$ to simulate 250 paths for the four state variables with the same length as our data, i.e., 384 months, and, for each month and trial in the simulation, we compute the term structure of yields. Then, for each trial, we fit $KWML$, $KWJSZ$ and ACM to these curves and, in each case, compute the mean and the standard deviation of short rate expectations, term premium and adjusted term premium. The table shows the average ratio between estimated and simulated values, (i) for the means and (ii) for the standard deviations. In the second experiment (Panel B), we repeat the same procedure but generate the yield curves for each trial using the parameter estimates of our stochastic volatility model. Standard errors in parentheses.

Panel A: Simulations with $KWML$ as benchmark

(i) Average of the ratio between means

	10-Year			20-Year		
	KWML	*KWJSZ*	*ACM*	*KWML*	*KWJSZ*	*ACM*
Short rate expectations	0.98	0.98	0.95	1.00	0.96	0.90
	(0.003)	(0.007)	(0.017)	(0.003)	(0.012)	(0.022)
Term premium	1.07	1.04	1.19	0.98	1.08	1.26
	(0.008)	(0.023)	(0.053)	(0.008)	(0.033)	(0.059)
Adjusted term premium	1.08	0.97	0.93	0.97	1.02	0.99
	(0.009)	(0.019)	(0.040)	(0.007)	(0.026)	(0.041)

(ii) Average of the ratio between standard deviations

	10-Year			20-Year		
	KWML	*KWJSZ*	*ACM*	*KWML*	*KWJSZ*	*ACM*
Short rate expectations	0.98	0.74	0.87	0.98	0.56	0.64
	(0.004)	(0.013)	(0.012)	(0.009)	(0.013)	(0.013)
Term premium	0.97	2.33	2.02	0.93	4.40	4.39
	(0.004)	(0.044)	(0.042)	(0.004)	(0.059)	(0.063)
Adjusted term premium	1.01	7.72	7.64	0.99	12.51	12.28
	(0.007)	(0.243)	(0.231)	(0.007)	(0.347)	(0.361)

Panel B: Simulations with our model as benchmark

(i) Average of the ratio between means

	10-Year				20-Year			
	MODEL	*KWML*	*KWJSZ*	*ACM*	*MODEL*	*KWML*	*KWJSZ*	*ACM*
Short rate expectations	0.99	1.02	1.00	0.97	0.99	1.04	1.02	0.96
	(0.004)	(0.012)	(0.015)	(0.013)	(0.003)	(0.011)	(0.021)	(0.021)
Term premium	1.01	1.06	1.01	1.12	1.03	0.99	1.03	1.26
	(0.006)	(0.040)	(0.059)	(0.056)	(0.009)	(0.038)	(0.073)	(0.071)
Adjusted term premium	1.02	1.63	1.30	1.31	1.02	1.52	1.76	1.79
	(0.012)	(0.079)	(0.079)	(0.079)	(0.007)	(0.071)	(0.107)	(0.118)

(ii) Average of the ratio between standard deviations

	10-Year				20-Year			
	MODEL	*KWML*	*KWJSZ*	*ACM*	*MODEL*	*KWML*	*KWJSZ*	*ACM*
Short rate expectations	0.99	1.06	0.56	0.62	0.99	1.05	0.37	0.40
	(0.003)	(0.013)	(0.015)	(0.015)	(0.004)	(0.011)	(0.014)	(0.015)
Term premium	0.91	0.40	1.81	1.84	0.99	0.38	2.38	2.39
	(0.013)	(0.022)	(0.035)	(0.038)	(0.014)	(0.023)	(0.039)	(0.040)
Adjusted term premium	0.92	0.45	5.80	6.08	1.00	0.48	9.78	10.14
	(0.013)	(0.033)	(0.177)	(0.198)	(0.012)	(0.039)	(0.276)	(0.307)

Table 8
Information in Long-Maturity Yields

This table provides evidence on the additional information contained in long-maturity yields for the estimation of the expected short rate and the term premium. We use the parameter estimates provided by $KWML$ to simulate 250 paths for the four state variables with the same length as our data, i.e., 384 months, and, for each month and trial in the simulation, we compute the term structure of yields. Then, for each trial, we fit $KWML$, in one case using only yields up to 10 years ($KWML10$), in the other case using yields up to 25 years, as in Panel A of Table 7 ($KWML25$). The first and the fourth columns report the average across the 250 trials of the means (Panel A) and the standard deviations (Panel B) of short rate expectations, term premium and adjusted term premium computed in each trial from the simulated data for the 5-year and 10-year maturity, respectively (the benchmark, BMK). We then calculate, for each trial, the difference between these values and the corresponding values estimated by $KWML10$ and $KWML25$. We treat the standard deviation of these estimate errors across the 250 trials as an estimate of the standard error of an individual estimate. These values are reported in the second and third columns (5-year maturity) and in the fifth and sixth columns (10-year maturity). All data are expressed in basis points.

Panel A: Means

	5-Year			10-Year		
	BMK	*KWML10*	*KWML25*	*BMK*	*KWML10*	*KWML25*
Short rate expectations	429	32.8	11.6	432	35.5	12.8
Term premium	76	33.3	12.4	106	35.8	13.4
Adjusted term premium	106	45.7	16.0	145	59.4	20.0

Panel B: Standard deviations

	5-Year			10-Year		
	BMK	*KWML10*	*KWML25*	*BMK*	*KWML10*	*KWML25*
Short rate expectations	129	6.4	2.9	90	10.2	6.0
Term premium	33	5.4	1.6	24	4.8	1.4
Adjusted term premium	23	9.3	2.5	29	12.0	3.2

Table 9
Components of Forward Rate Spread

This table shows the average value (Panel A) and the annualised volatility of monthly changes (Panel B) in differences between average (i) forward rates, Δf, (ii) expected short rates, $\Delta \mathrm{E}^{\mathbb{P}}[r]$, (iii) bond risk premia, Δbrp, (iv) convexity terms, Δcvx, and (v) duration adjustment terms, Δdur, for two long maturities $\tau_1 = 15$ years and $\tau_2 = 25$ years. The table reports values for the forward rate spread calculated directly from STRIPs and for all the five variables estimated from our model and the Gaussian model using either $KWML$, $KWJSZ$ or ACM. Values are expressed in basis points and refer to the period January 1988 to December 2019.

Panel A: Average value

	Δf	$\Delta \mathrm{E}^{\mathbb{P}}[r]$	Δbrp	Δcvx	Δdur
STRIPS	-82				
MODEL	-82	0	62	-139	-5
KWML	-100	-12	-6	-91	9
KWJSZ	-76	0	22	-99	0
ACM	-84	0	27	-111	0

Panel B: Annualised volatility

	Δf	$\Delta \mathrm{E}^{\mathbb{P}}[r]$	Δbrp	Δcvx	Δdur
STRIPS	81				
MODEL	55	19	83	90	11
KWML	17	10	12	0	7
KWJSZ	19	14	159	0	156
ACM	5	11	129	0	124

Table 10
Variance Decomposition of Yields

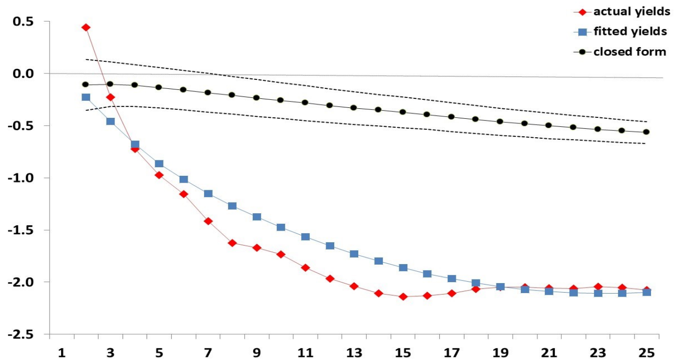
This table shows the variance decomposition of monthly changes in the 10-year yield into short rate expectations and term premia provided by our model, our estimates of $KWJSZ$ and ACM, and the estimates of KW and ACM published by the Federal Reserve ($KWFed$ and $ACMFed$, respectively). Values are in percentage terms and refer to the period January 1988 (January 1990 for $KWFed$) to December 2019.

	Our Model	*KWJSZ*	*ACM*	*KWFed*	*ACMFed*
Short rate expectations	70.9	43.9	33.4	41.7	36.6
Term premium	29.1	56.1	66.6	58.3	63.4

Figure 1
Expectations Hypothesis Test

This figure displays estimates of the Campbell-Shiller regression (23). In particular, it shows the estimates of the sample coefficients estimated by running the regression using actual yields and fitted yields, and the closed form model-implied population coefficient generated by equation (24). The one–standard deviation upper and lower bounds are obtained by running a Monte Carlo simulation in which the estimated model parameters are used to generate one hundred samples of length 240,000 and calculate the regression coefficients for each of them. Panel A plots the coefficients estimated using monthly changes of yields in (23), whereas Panel B contains the coefficients estimated using annual changes in yields. The sample period is January 1988 to December 2019.

Panel A: Monthly yield changes



Panel B: Annual yield changes

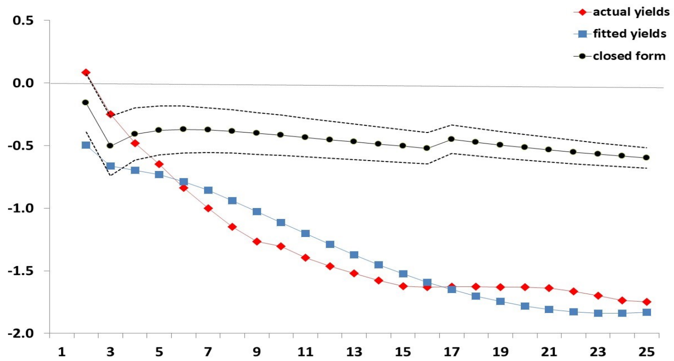
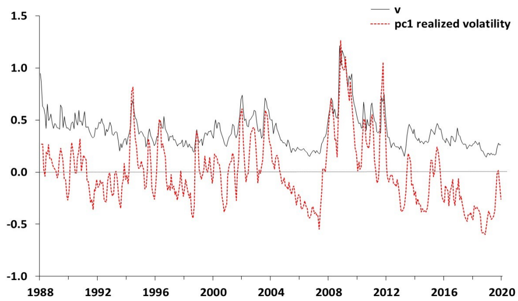


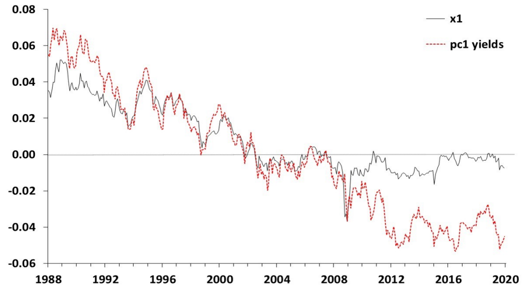
Figure 2
Estimated State Variables

This figure shows time series features of the Kalman Filter estimates of the unobserved state variables for the period January 1988 to December 2019. In particular, it contains estimates for the variance factor v and the three latent factors x_1, x_2 and x_3 and compares them with the first two principal components of realized yield volatilities (from 2- to 25-year maturity) and the first two principal components of yields (from 2- to 25-year maturity).

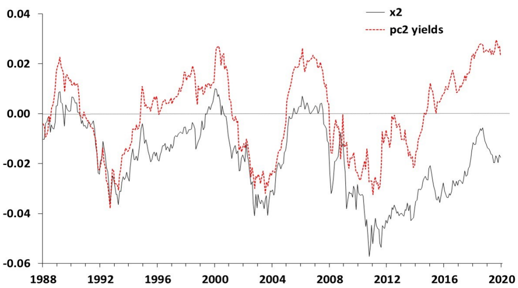
Panel A: Variance factor v



Panel B: Latent factor x_1



Panel C: Latent factor x_2



Panel D: Latent factor x_3

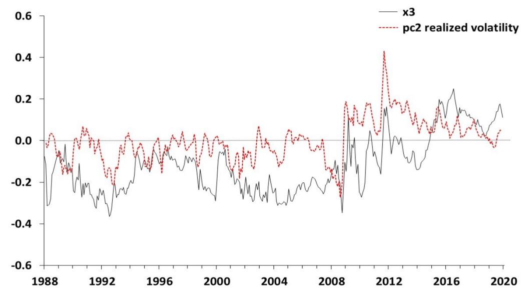
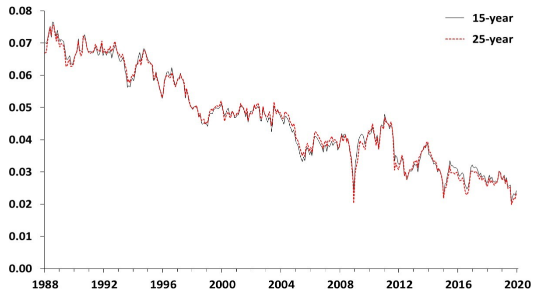


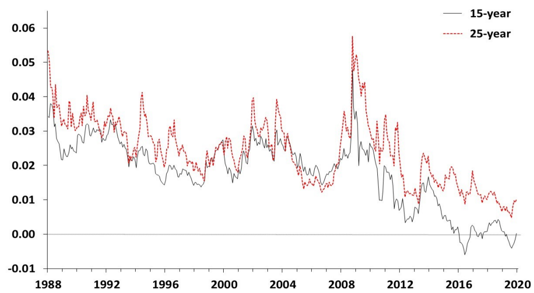
Figure 3
Structure of Long-Term Forward Rates

This figure displays the composition of long-term forward rates (equation (14)) for the period January 1988 to December 2019. Panel A shows the time series of the instantaneous expected short rate for the 15-year and 25-year maturity, Panel B, Panel C and Panel D the time series of the instantaneous bond risk premium, convexity and duration adjustment term, respectively, for the same maturities.

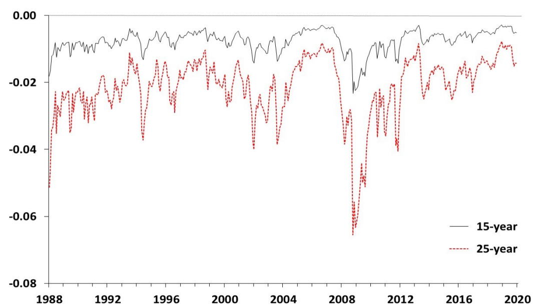
Panel A: Expected short rate



Panel B: Bond risk premium



Panel C: Convexity



Panel D: Duration adjusment

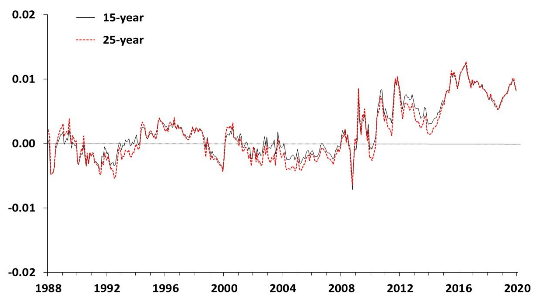
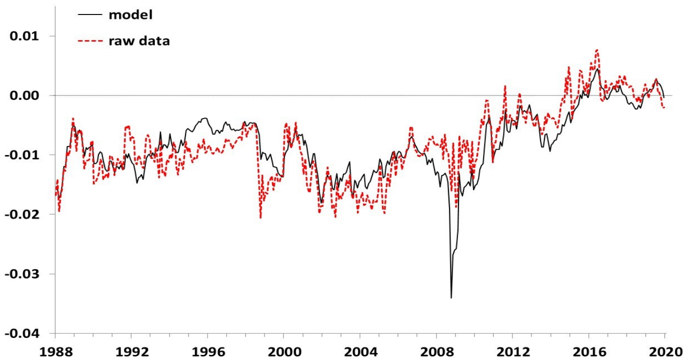


Figure 4
Estimated Downward Tilt

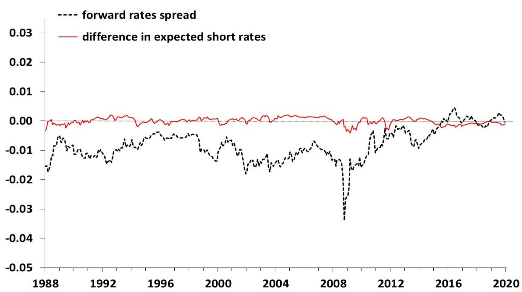
This figure displays estimates of the forward spread between the 25- and 15-year maturity instantaneous forward rates over the period January 1988 to December 2019. Panel A compares the model spread with the corresponding spread obtained from raw data (see Panel A of Figure A.1). Panel B shows the decomposition of the forward rates spread into its four components, i.e., the difference in expected short rates, bond risk premia, convexity, and duration adjustment terms.

Panel A: Forward spread fit

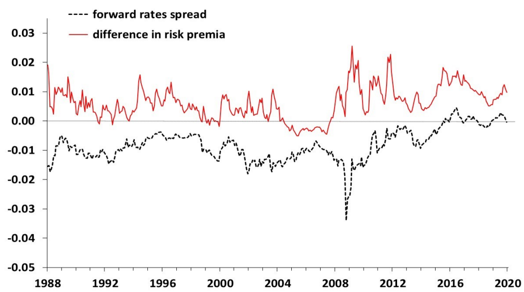


Panel B: Forward spread components

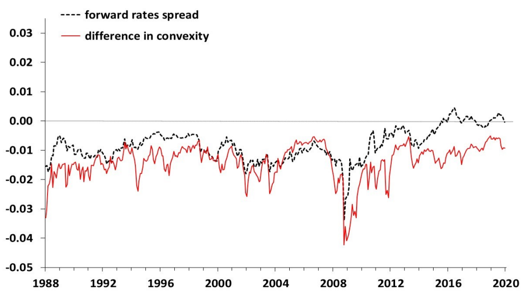
Difference in expected short rates



Difference in risk premia



Difference in convexity



Difference in duration adjustment terms

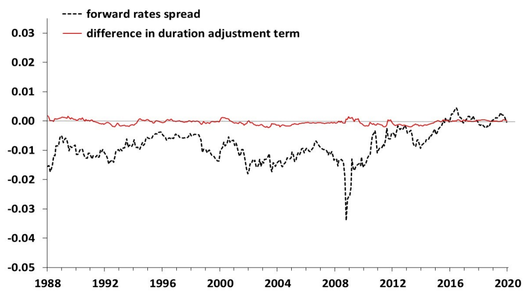
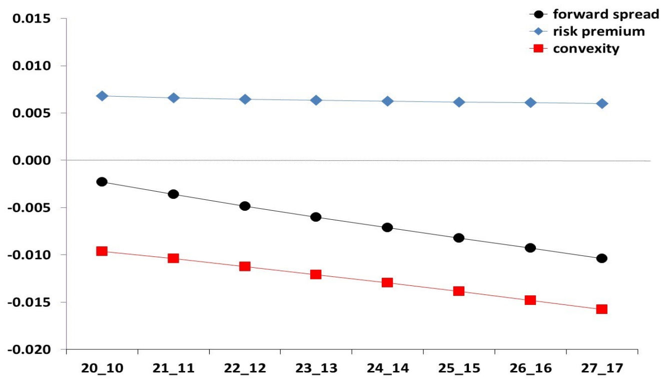


Figure 5
Term Structure of Differences in Risk Premia

This figure reports the term structure of forward rate spreads, differences in risk premia and differences in convexity obtained by holding the difference in maturity constant at 10 years. Panel A contains the term structures estimated by the model. Panel B shows a comparison between these estimates and the “model-free” term structures calculated using measurable proxies for forward rates and convexity terms. The sample period is January 1988 to December 2019.

Panel A: Term structure of differences in forward rates, risk premia and convexity



Panel B: Differences in risk premia: model vs “model-free” estimates

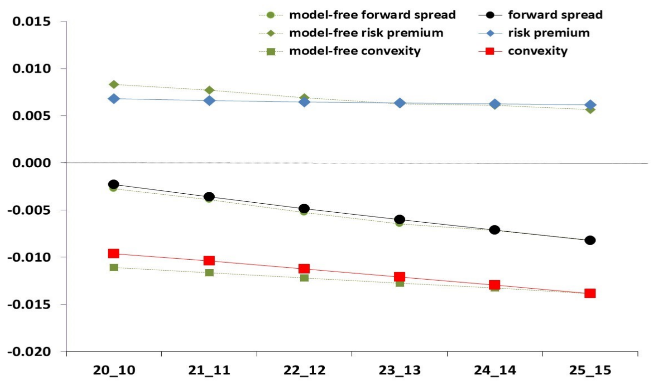
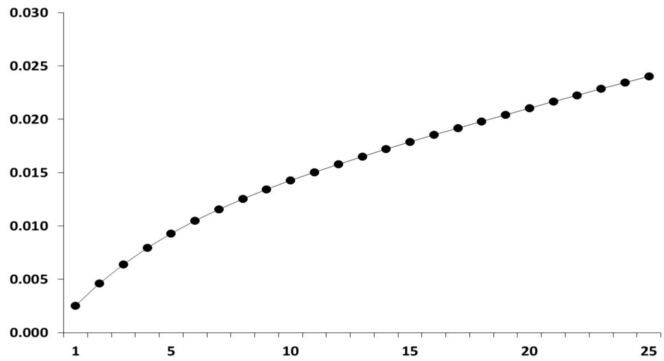


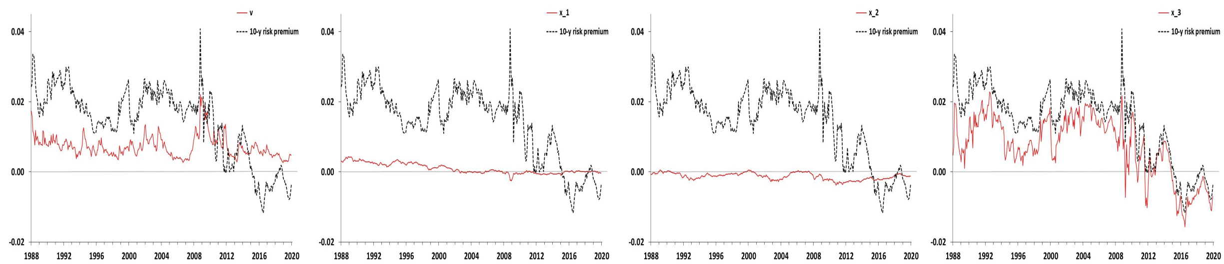
Figure 6
Decomposition of Long-Term Bond Risk Premia

This figure shows the average term structure of risk premia and the decomposition of long-term bond risk premia into the components related to the four state variables, i.e., the variance factor v and the three latent factors x_1, x_2 and x_3, over the period January 1988 to December 2019. Panel A reports the average term structure of risk premia. Panel B and Panel C show the decomposition of the 10- and 20-year risk premia, respectively.

Panel A: Average term structure of risk premia



Panel B: 10-year risk premium



Panel C: 20-year risk premium

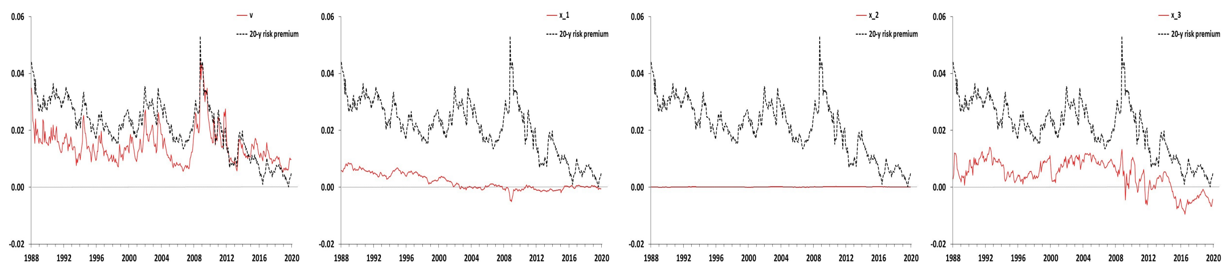
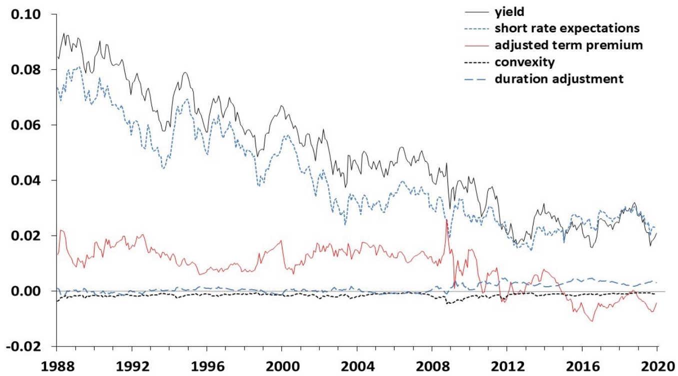


Figure 7
Structure of Long-Term Yields

This figure plots the time series estimates of the components of yields for the period January 1988 to December 2019. Panel A shows the splitting of the estimated 10-year yield into its four components, i.e., average short rate expectations, adjusted term premium, convexity and duration adjustment. Panel B contains the same series for the 20-year maturity.

Panel A: 10-year yield components



Panel B: 20-year yield components

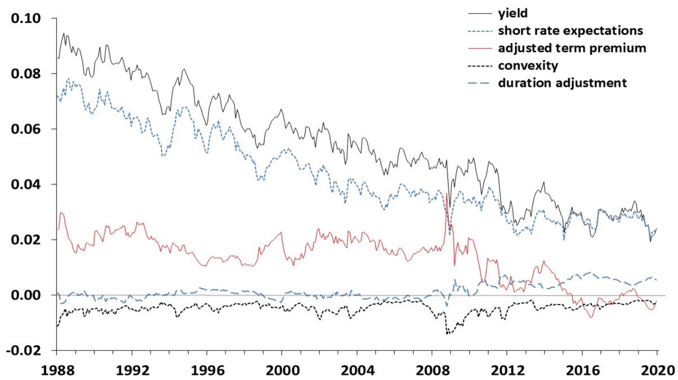
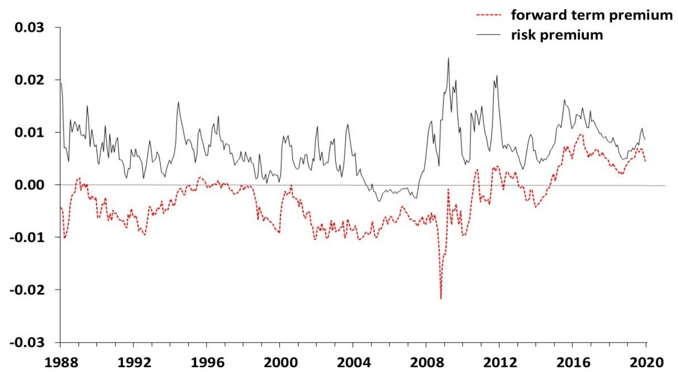


Figure 8
Spread in Long-Term Forward Term Premia

This figure compares the spread between the 20-year and the 10-year forward term premia (Panel A) and term premia (Panel B) implied by our estimates with the spread between the 20-year and the 10-year risk premia and adjusted term premia, respectively. The sample period runs from January 1988 to December 2019.

Panel A: Slope of long-maturity forward term premia vs. risk premia



Panel B: Slope of long-maturity term premia vs. adjusted term premia

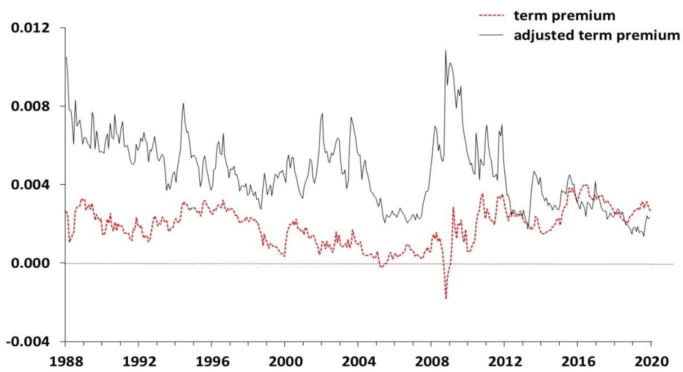
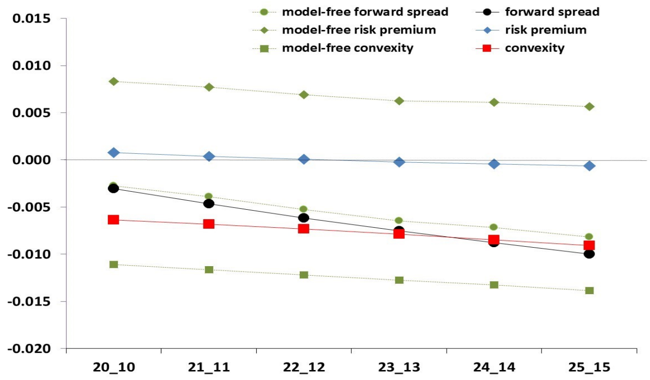


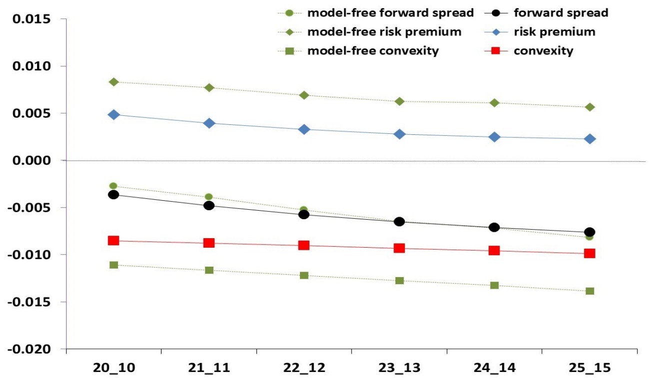
Figure 9
Slope of Long-Term Risk Premia in Gaussian Models

This figure reports the term structure of forward rate spreads, differences in risk premia and differences in convexity obtained by holding the difference in maturity constant at 10 years. The estimated term structures are compared with the "model-free" term structures calculated using measurable proxies for forward rates and convexity terms. Panel A shows estimates provided by $KWML$, Panel B by $KWJSZ$, Panel C by ACM. The sample period is January 1988 to December 2019.

Panel A: Term structure of differences in risk premia in $KWML$



Panel B: Term structure of differences in risk premia in $KWJSZ$



Panel C: Term structure of differences in risk premia in ACM

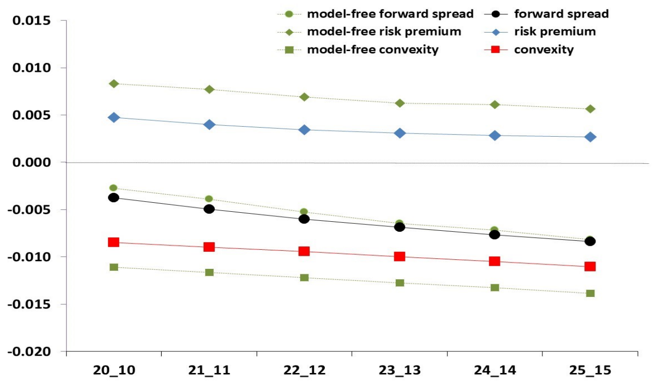
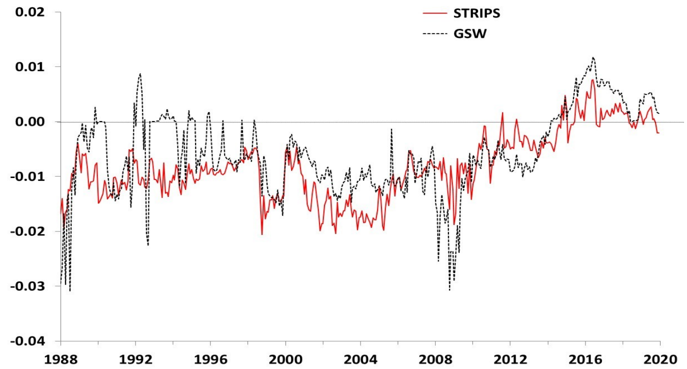


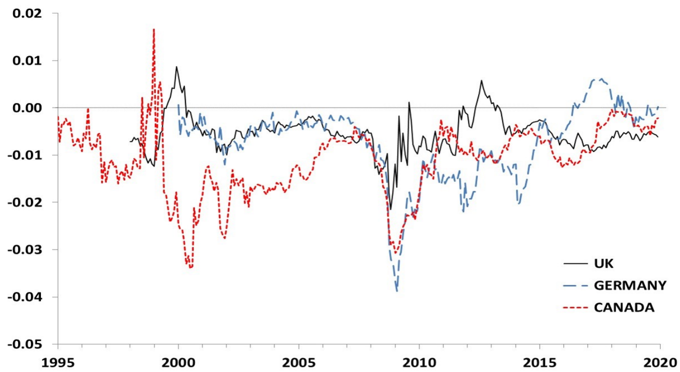
Figure A.1
Downward Tilt

This figure reports the spread between long-term instantaneous forward rates. Panel A shows the 25-year minus 15-year spread for US instantaneous forward rates calculated using end-of-month observations for both STRIPs and GSW data from January 1988 through to December 2019. Panel B contains the same spread for the UK, Germany and Canada over different time periods ranging between January 1995 and December 2019, whereas Panel C for the US TIPS and the UK index-linked Gilts over the period January 2000 (January 2004 for the US) to December 2019.

Panel A: US 25- minus 15-year forward rate spread



Panel B: UK, Germany and Canada 25- minus 15-year forward rate spread



Panel C: US and UK inflation-indexed 25- minus 15-year Forward Rate Spread