Competition in Signaling

Federico Vaccari*

Abstract

I study a multi-sender signaling game between an uninformed decision maker and two senders with common private information and conflicting interests. Senders can misreport information at a cost that is tied to the size of the misrepresentation. The main results concern the amount of information that is transmitted in equilibrium and the language used by senders to convey such information. Fully revealing and pure-strategy equilibria exist but are not plausible. I first identify sufficient conditions under which equilibria are essentially unique, robust, and always exist, and then deliver a complete characterization of these equilibria. As an application, I study the informative value of different judicial procedures.

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*Department of Economics and Management, University of Trento, I-38122 Trento, Italy. E-mail: vaccari.econ@gmail.com. I am grateful to Luciano Andreozzi, Paulo Barelli, Ennio Bilancini, John Duggan, Srihari Govindan, Tasos Kalandrakis, Santiago Oliveros, Marco Ottaviani, and Andriy Zapechelnyuk. All errors are mine. This project has received funding from the European Union's Horizon 2020 Research and Innovation Programme (Marie Skłodowska-Curie grant no. 843315-PEMB).

Contents

1 Introduction

How and how much information is revealed when two equally informed senders with conflicting interests provide advice to a decision maker? When senders are well informed and misreporting is prohibitively expensive, the decision maker can "rely on the information of the interested parties" to always make the right choice.[1] However, there are many situations where information is not fully verifiable and it is possible to misreport it at a reasonable cost.[2] Intuition would suggest that, in these cases, the decision maker might obtain conflicting advice and make wrong choices as a result of being poorly informed.

This type of interaction is at the core of a large number of real-world scenarios: candidates competing for consensus during an electoral campaign may provide voters with different accounts of the same events; newspapers with opposed political leanings may deliver conflicting and inaccurate news; prosecutors and defendants trying to persuade a jury may tamper with evidence; co-workers competing for a promotion may exaggerate their contributions to a team project; advocacy groups trying to influence court cases may use *amicus curiae* briefs; and methods used in lobbying against public health may include "industry-funded research that confuses the evidence and keeps the public in doubt" (Chan, 2013).

I address the above questions with a costly signaling game between an uninformed decision maker and two senders with common information and conflicting interests. The two senders observe the realization of a random variable—the state—and then simultaneously or privately deliver a report to the decision maker. These reports are literal statements about the realized state. Senders can misreport such information, but incur "misreporting costs" that are increasing in the magnitude of misrepresentation. By contrast, reporting truthfully is costless. After observing the reports, the decision maker must select one of two alternatives. At the end of the game, each player obtains a payoff that depends on the realized state and on the alternative selected by the decision maker. I assume that the payoff players receive from the selection of the first alternative is increasing with the state, while the payoff players receive from the selection of the other alternative is normalized to zero in every state. Players prefer the first alternative in relatively higher states, whereas they prefer the second alternative in relatively lower states. Therefore, the state is a valence parameter, and it can be thought of as representing the relative quality of the two alternatives.[3]

[1] See, e.g., Milgrom and Roberts (1986b).

[2] Misreporting information is a costly activity due to the time and effort that is required to misrepresent the information, or due to the expected loss in reputation, credibility, and future influence, and more. Moreover, misreporting is more difficult, and hence more costly, when information is harder.

[3] For example, the state can represent the leadership or competence of politicians, the durability or product quality of commercial goods, and the fitness with the state of the world of policies.

Throughout the paper, I restrict attention to equilibria where the decision maker's posterior beliefs satisfy a first-order stochastic dominance condition with respect to the senders' reports. Under this condition, the decision maker's expected payoff from selecting the first alternative does not decrease when senders announce that the state takes higher values. This condition is natural given that reports are literal, misreporting is costly, and the decision maker's payoff from selecting the first alternative is increasing in the state. It imposes some sort of monotonicity on the senders' reporting strategies, and thus it is akin to restrictions that are widely used in many economic applications, such as in auction theory and in models of communication with lying costs.

The main results of this paper concern the amount of information that can be plausibly transmitted in equilibrium and the "language" used by senders to deliver such information. I first show that misreporting occurs in every equilibrium. Yet, there are "receiver-efficient" equilibria where the decision maker obtains enough information to always select her preferred alternative *as if* she were fully informed. In spite of the senders' misreporting behavior, the decision maker might even end up obtaining more information than she needs. However, these equilibria, while important for our analysis, turn out to be implausible. Specifically, I show that all receiver-efficient—and hence all fully revealing—equilibria rely on an ad-hoc choice of beliefs that have implausible discontinuities to discourage deviations. I identify two well-known refinements that eliminate such equilibria: unprejudiced beliefs (Bagwell & Ramey, 1991) and ε-robsutness (Battaglini, 2002).[4] Similarly, I show that all pure-strategy equilibria are receiver-efficient and hence implausible. This result motivates the search for mixed-strategy equilibria that are robust to such refinements. However, this analysis is difficult in signaling games due to the fact that they typically yield a wealth of equilibria and, in this setting, the difficulty is compounded by the presence of multiple senders and rich state and signal spaces. Canonical refinements based on the notion of strategic stability (Kohlberg & Mertens, 1986) are of little help, as they are developed for settings with a single sender. I overcome this difficulty by imposing reasonable restrictions on the decision maker's posterior beliefs. More specifically, I focus the subsequent analysis on equilibria that satisfy two additional conditions on the posterior beliefs of the decision maker: the first condition is a strong form of first-order stochastic dominance that requires that conflicting reports announcing a strictly higher state must signal that the expected payoff of the first alternative is also strictly higher; the second condition is a dominance argument under which the decision maker excludes the possibility that senders may deliver reports that are equilibrium-dominated.[5] I refer to equilibria satisfying these two conditions as "direct equilibria," as they may feature reports that are

[4] See Section 4 for a formal definition of unprejudiced beliefs and ε-robustness. I show that in this model these two refinements are tightly connected: equilibria that are ε-robust must be supported by unprejudiced beliefs (Lemma 3). This result suggests a novel rationale for the use of ε-robustness in multi-sender communication games.

[5] See Definition 4 in Section 5 for a complete and formal statement of these two conditions.

direct signals of the realized state.

I then provide a complete characterization of direct equilibria, and show that in the game considered here they possess desirable properties: they always exist, they are essentially unique, and they withstand the refinement criteria that break down fully revealing, receiver-efficient, and pure-strategy equilibria. The two conditions imposed on direct equilibria, even though relatively natural and mild, are therefore sufficient to ensure robustness and uniqueness while preserving existence.

In this model, the transmission of information takes place in a qualitatively different way than in comparable models of strategic communication: differently than models of cheap talk, here there are no "babbling" equilibria; differently than models of verifiable disclosure, here misreporting occurs in every equilibrium, and full revelation is not a plausible equilibrium outcome. By contrast, in direct equilibria of this model "revelation" is a probabilistic phenomenon in the sense that the decision maker fully learns almost every state with some positive probability. This probabilistic revelation of the state is more likely to occur in relatively extreme states, where the decision maker obtains a substantially different payoff from selecting the first alternative rather than the second one. In sufficiently extreme states, senders always truthfully reveal the state to the decision maker even though they have conflicting interests.

The senders' equilibrium behavior is mixed, as they report the truth with some positive probability, and they misreport otherwise. Therefore, in every state the two senders may deliver exactly the same truthful report even though they have conflicting interests.[6] They might also end up delivering different reports that nevertheless propose the same recommended action to the decision maker. Whenever one of these two events takes place, the decision maker fully learns the realized state. In the former case, revelation of the state occurs without wasteful signaling expenditures. In the latter case, revelation of the state requires a sender to engage in wasteful misreporting. This result highlights a difference with previous results in multi-sender signaling games, where full revelation is either always inefficient (Emons & Fluet, 2009) or always efficient (Bagwell & Ramey, 1991).[7]

Conditional on misreporting, senders deliver reports in a convex set, and no particular misrepresentation in such a set is delivered with strictly positive probability. The misreporting behavior of each sender is directly determined by the individual characteristics of his opponent, such as the opponent's cost structure and payoff function, and it is determined only indirectly by his own characteristics. Upon observing two conflicting reports recommending different actions, the decision maker understands that "the truth

[6] More precisely, this occurs in *almost* every state.

[7] Signaling games with a single sender typically have inefficient separating equilibria. See, e.g., Spence (1973), Milgrom and Roberts (1982, 1986a), Kartik (2009), Kartik, Ottaviani, and Squintani (2007).

is somewhere in between" and that at least one of the two senders is misreporting. The decision maker cross-validates reports and allocates the burden of proof between the senders by taking into account their characteristics.

The setting studied in the main part of the paper allows for a large number of asymmetries. I also analyze the specific case where senders have a similar payoff and cost structure, and where the distribution of the state is such that no sender has an ex-ante advantage of any kind. In this "symmetric environment," I provide a closed-form solution to direct equilibria and show that they naturally display symmetric strategies. The decision maker equally allocates the burden of proof between the senders by following the recommendation of the sender delivering the most extreme report. The senders' misreporting behavior depends on the shape of the common cost function: with convex costs, senders are more likely to convey large misrepresentations of the state rather than small lies, while the opposite is true for concave misreporting costs.

As a brief application, I use insights from the analysis of direct equilibria to study the informational value of different judicial systems. Shin (1998) shows that, when information is fully verifiable, the adversarial judicial procedure is always superior to the inquisitorial procedure. However, Shin (1998) also conjectures that such a sharp result may crucially depend on the assumption of verifiability. I show that, when information is not fully verifiable, then the inquisitorial procedure may indeed be superior to the adversarial procedure, thus proving Shin's conjecture to be correct.

The remainder of this article is organized as follows. In Section 2, I discuss the related literature. Section 3 introduces the model, which I solve in Sections 4 and 5. In Section 6, I provide an example and an application. Finally, Section 7 concludes. Formal proofs are relegated to Appendix A.

2 Related Literature

This paper contributes to different strands of literature. First, it relates to models of strategic communication with multiple senders. This line of work shows several channels through which full information revelation can be obtained (Battaglini, 2002; Krishna & Morgan, 2001; Milgrom & Roberts, 1986b). Papers in this literature typically assume that misreporting is either costless (cheap talk) or impossible (verifiable disclosure). By contrast, in this article misreporting is possible at a cost that depends on the magnitude of misrepresentation. Under this modeling specification, I show that fully revealing equilibria exist but are not plausible.

Therefore, this paper relates to models of strategic communication with misreporting costs (Chen, 2011; Chen, Kartik, & Sobel, 2008; Kartik, 2009; Kartik et al., 2007; Ottaviani

& Squintani, 2006). All these papers are concerned with the single-sender case, while I consider a multi-sender setting. An exception is Dziuda and Salas (2018), who study a communication game with endogenous lying costs and consider a case with two senders.

The introduction of misreporting costs makes this a costly signaling model. Therefore, this paper relates to extant models of multi-sender signaling with perfectly correlated types, but it differs from these models in a number of ways. First, in my model the messages or signals of senders are used only to transmit information, and thus do not directly affect how players value each alternative. This is not the case, e.g., in related models of limit entry (Bagwell & Ramey, 1991; Schultz, 1996), price competition (Bester & Demuth, 2015; Fluet & Garella, 2002; Hertzendorf & Overgaard, 2001; Yehezkel, 2008), and public good provision (Schultz, 1996).[8] Second, I model a setting where the signals of senders are fully observable.[9] By contrast, in the entry deterrence models of Harrington (1987) and Orzach and Tauman (1996), incumbent firms simultaneously select their own pre-entry output, but the entrant can observe only the resulting market price.

A key feature of the model analyzed in this paper is that both senders pay their own signaling costs independently of the decision maker's choice. This *all-pay* feature is missing in related multi-sender signaling models of electoral competition (Banks, 1990; Callander & Wilkie, 2007), where only the elected candidate incurs the signaling cost.[10]

The type of strategic interaction and competition analyzed in this model is closely related to that analyzed in all-pay contest models, where contestants compete for a prize by simultaneously submitting costly bids (Baye, Kovenock, & De Vries, 1996; Siegel, 2009). In these papers, the mapping from signals or bids to outcomes is exogenously determined by a contest success function. For example, Skaperdas and Vaidya (2012) study persuasion by contending parties as an all-pay contest. The model studied here differs from these models in that the decision maker is a strategic actor whose choice is endogenously determined as part of the equilibrium. Similarly, Gul and Pesendorfer (2012) study political contests where two parties with conflicting interests provide costly payoff-relevant signals to a strategic voter. However, in their model only one party incurs a cost at each moment, and parties cannot distort information.

Finally, this paper contributes to the literature on adversarial judicial procedures (Dewatripont & Tirole, 1999; Shin, 1998). However, it differs from this line of work by considering a model where information is not fully verifiable. In this regard, Emons and

[8] For example, in these models firms may signal quality through prices, which affect market demand and hence profits. Some of these papers also study signaling via a combination of pricing and advertising.

[9] Signals are not fully observable if, e.g., they are aggregated into a single score and the receiver can observe only that score, but cannot observe each individual signal.

[10] These models also differ from my model in that they consider settings where senders do not have common information. Similarly, Mailath (1989) and Daughety and Reinganum (2007) study price signaling and Honryo (2018) studies risk shifts in settings with imperfectly correlated types. My model should be seen as complementary to this line of work.

Fluet (2009) constitute an exception. However, they consider a setting with a continuum of types, signals, and receiver's actions, which yields only fully revealing equilibria.

3 The Model

Setup and timeline. There are three players: two informed senders (1 and 2) and one uninformed decision maker (dm). Let $\theta \in \Theta \subseteq \mathbb{R}$ be the underlying state, distributed according to the full support probability density function f. After observing the realized state θ, two senders simultaneously or privately deliver to the decision maker a report $r_j \in R_j$, where r_j is the report of sender j and R_j is the report space of sender j (he). The decision maker (she), after observing the pair of reports (r_1, r_2) but not the state θ, selects an alternative $a \in \{\oplus, \odot\}$.

Payoffs. Player $i \in \{1, 2, dm\}$ obtains a payoff of $u_i(a, \theta)$ if the decision maker selects alternative a in state θ. I normalize $u_i(\odot, \theta) = 0$ for all $\theta \in \Theta$ and let $u_i(\theta) \equiv u_i(\oplus, \theta)$, where $u_i(\theta)$ is weakly increasing in θ. Thus, the state θ is a valence or vertical differentiation score over which players share a common preference, and it is interpreted as the relative quality of alternative $\oplus$ with respect to alternative $\odot$. The decision maker's expected utility from selecting $\oplus$ given the senders' reports is $U_{dm}(r_1, r_2)$.

Misreporting costs. Sender j bears a cost $k_j C_j(r_j, \theta)$ for delivering report r_j when the state is θ. The cost function $C_j(r_j, \theta) \geq 0$ is continuous and such that, for every $\theta \in \Theta$ and $j \in \{1, 2\}$, we have that $C_j(\theta, \theta) = 0$ and

$$\text{if } \; r_j \gtrless \theta, \; \text{ then } \; \frac{dC_j(r_j, \theta)}{dr_j} \gtrless 0 \gtrless \frac{dC_j(r_j, \theta)}{d\theta}.$$

The scalar $k_j > 0$ is a finite parameter measuring the intensity of misreporting costs. Therefore, misreporting is increasingly costly in the magnitude of misrepresentation, while truthful reporting is always costless. Sender j's total utility is

$$w_j(r_j, a, \theta) = \mathbb{1}\{a = \oplus\} u_j(\theta) - k_j C_j(r_j, \theta),$$

where $\mathbb{1}\{\cdot\}$ is the indicator function. It follows that, conditional on the decision maker's eventual choice, both senders prefer to deliver reports that are closer to the truth.

Definitions and assumptions. I assume that the state space and the report spaces are the same, i.e., $R_1 = R_2 = \Theta$. Thus, a generic report r has the literal or exogenous meaning of "The state is $\theta = r$." I say that sender j reports truthfully when $r_j = \theta$, and misreports otherwise. I sometimes use $-j$ to denote the sender who is not sender j.

I define the "threshold" τ_i as the state in which player i is indifferent between the two alternatives. Formally, $\tau_i := \{\theta \in \Theta | u_i(\theta) = 0\}$. I assume that utilities $u_i(\theta)$ are such that τ_i exists and is unique[11] for every $i \in \{1, 2, dm\}$. The threshold τ_i tells us that player i prefers $\oplus$ to $\ominus$ when the state θ is greater than τ_i. Throughout the paper, I consider the case where senders have opposing biases, i.e., $\tau_1 < \tau_{dm} < \tau_2$. To make the problem non-trivial I let $\tau_{dm} \in \Theta$, and I normalize $\tau_{dm} = 0$. Therefore, the decision maker prefers to select the positive alternative $\oplus$ when the state θ takes positive values, and prefers to select the negative alternative $\ominus$ when the state is negative. I assume that when the decision maker is indifferent between the two alternatives at given beliefs, she selects $\oplus$.

I define the "reach" of sender j in state θ as the report whose associated misreporting costs offset j's gains from having his own preferred alternative eventually selected. Formally, the "upper reach" $\bar{r}_j(\theta) \geq \theta$ of sender j in state θ is defined as

$$\bar{r}_j(\theta) := \max\left\{r \in \mathbb{R} \mid (-1)^{\mathbb{1}\{\theta < \tau_j\}} u_j\left(\theta\right) = k_j C_j\left(r, \theta\right)\right\}. \tag{1}$$

Similarly, the "lower reach" $\underline{r}_j(\theta) \leq \theta$ of sender j in state θ is defined as

$$\underline{r}_j(\theta) := \min\left\{r \in \mathbb{R} \mid (-1)^{\mathbb{1}\{\theta < \tau_j\}} u_j\left(\theta\right) = k_j C_j\left(r, \theta\right)\right\}. \tag{2}$$

I will sometimes use the "inverse reaches" $\bar{r}_1^{-1}(r_1)$ and $\underline{r}_2^{-1}(r_2)$, where $\bar{r}_j^{-1}(\cdot)$ and $\underline{r}_j^{-1}(\cdot)$ map from R_j to Θ and are defined as the inverse functions of $\bar{r}_j(\theta)$ and $\underline{r}_j(\theta)$, respectively.

I assume that the state and report spaces are large enough, that is,

$$\Theta \supseteq \hat{R} := \left[\underline{r}_2(0), \bar{r}_1(0)\right].$$

This assumption ensures that the information senders can transmit is not artificially bounded by restrictions in the reports that they can deliver.[12]

Strategies. A pure strategy for sender j is a function $\rho_j : \Theta \to R_j$ such that $\rho_j(\theta)$ is the report delivered by sender j in state θ. A mixed strategy for sender j is a mixed probability measure $\phi_j : \Theta \to \Delta(R_j)$, where $\phi_j(r_j, \theta)$ is the mixed probability density that $\phi_j(\theta)$ assigns to a report $r_j \in R_j$. I denote by $S_j(\theta)$ the support of sender j's strategy in state θ. Section 5 introduces additional notation that is required to study equilibria in mixed strategies.

I say that a pair of reports (r_1, r_2) is off path if, given the senders' strategies, (r_1, r_2) will never be observed by the decision maker. Otherwise, I say that the pair (r_1, r_2) is on path. A posterior belief function for the decision maker is a mapping $p : R_1 \times R_2 \to \Delta(\Theta)$

[11] These assumptions are for notational convenience. The model can accommodate for senders that always strictly prefer one alternative over the other and for utility functions such that $u_i(\theta) \neq 0$ for every $\theta \in \Theta$, including step utility functions.

[12] For example, if $\max \Theta < \bar{r}_1(0)$, then sender 1 cannot deliver reports that are strictly dominated by truthful reporting when the realized state is strictly negative.

that, given any pair of reports (r_1, r_2), generates posterior beliefs $p(\theta|r_1, r_2)$ with CDF $P(\theta|r_1, r_2)$. Given a pair of reports (r_1, r_2) and posterior beliefs $p(\theta|r_1, r_2)$, the decision maker selects an alternative in the sequentially rational set $\beta(r_1, r_2)$, where

$$\beta(r_1, r_2) = \arg\max_{a \in \{\oplus, \ominus\}} \mathbb{E}_p \left[u_{dm}(a, \theta) | r_1, r_2 \right].$$

As mentioned above, if $p(\theta|r_1, r_2)$ is such that $U_{dm}(r_1, r_2) = 0$, then $\beta(r_1, r_2) = \oplus$.

Solution concept. The solution concept used in this paper is perfect Bayesian equilibrium (PBE).[13] Throughout the paper, I restrict attention to equilibria where posterior beliefs p satisfy the following first-order stochastic dominance condition: for every $r_j \geq r'_j$ and $j \in \{1, 2\}$,

$$U_{dm}(r_1, r_2) \geq U_{dm}(r'_1, r'_2). \tag{FOSD}$$

The (FOSD) condition[14] says that a higher report cannot signal to the decision maker a lower expected utility from selecting alternative $\oplus$. Focusing on these equilibria is natural given that the value of $\oplus$ is increasing in the state, reports are literal, and misreporting is costly.

Since in equilibrium the decision maker has correct beliefs, imposing conditions on p has consequences on the senders' equilibrium reporting behavior. An immediate implication of (FOSD) is that senders play strategies that satisfy some sort of monotonicity condition: in every equilibrium, a sender that prefers alternative $\oplus$ to $\ominus$ is never going to deliver a report that is strictly lower than the actual realized value of θ. The next lemma formalizes this result.

Lemma 1. *In every perfect Bayesian equilibrium satisfying* (FOSD) *we have that, for* $j \in \{1, 2\}$, $\min S_j(\theta) \geq \theta$ *for* $\theta \geq \tau_j$ *and* $\max S_j(\theta) \leq \theta$ *otherwise.*

Lemma 1 shows how (FOSD) is akin to assumptions that are widely used in many economic applications, such as monotone bidding strategies in auction theory (e.g., Wilson, 1977), monotone likelihood ratio property in signal distributions (e.g., Milgrom, 1981), and message monotonicity in related communication games (e.g., Kartik, 2009). To study mixed-strategy equilibria, I will use a stronger version of (FOSD) coupled with an additional condition that draws on a dominance argument. Section 5 introduces these conditions together with additional notation that is required in order to describe mixed strategies. Hereafter, I refer to perfect Bayesian equilibria of the game described in this section where the condition (FOSD) is satisfied simply as "equilibria."

[13] For a textbook definition of a perfect Bayesian equilibrium, see Fudenberg and Tirole (1991).

[14] Posterior beliefs $p(\theta|r_1, r_2)$ first-order stochastically dominate $p(\theta|r'_1, r'_2)$ for $r_j \geq r'_j$, $j \in \{1, 2\}$, if and only if $\int u(\theta)p(\theta|r_1, r_2)d\theta \geq \int u(\theta)p(\theta|r'_1, r'_2)d\theta$ for every weakly increasing utility function $u(\theta)$. Thus, the (FOSD) condition is even weaker than that in this definition as it applies only to $u(\theta) \equiv u_{dm}(\theta)$.

3.1 Benchmark

Before solving for the equilibria of the model, I briefly consider a number of benchmark cases that are useful for interpreting the results in the next sections.

Full information. Under full information about the state θ, the decision maker selects $\oplus$ when $\theta \geq 0$ and selects $\ominus$ otherwise. Both senders always report truthfully. The ex-ante full information welfare obtained by the decision maker in this scenario is

$$W_{fi} = \int_0^{\max \Theta} f(\theta) u_{dm}(\theta) d\theta. \tag{3}$$

Perfect alignment. Sender j is perfectly aligned with the decision maker when $\tau_j = \tau_{dm}$. There is an equilibrium where the perfectly aligned sender j always reports truthfully and the decision maker blindly trusts j's reports. The other sender, even if not perfectly aligned, can do no better than reporting truthfully as well. In this case, the decision maker gets her full information welfare W_{fi}, and no misreporting takes place.

Verifiable information. Consider the case where information about the state is fully verifiable, that is, $k_j = \infty$ for $j \in \{1, 2\}$. Senders cannot profitably withhold information, but even if they could we would obtain an equilibrium where in every state at least one of the two senders discloses truthfully (Milgrom & Roberts, 1986b).[15] As before, the decision maker gets the full information welfare W_{fi}.

Cheap talk. Suppose that $k_1 = k_2 = 0$; then there exists a babbling equilibrium where the decision maker adjudicates according to her prior f only, while the senders deliver uninformative messages. There is no equilibrium where the decision maker obtains enough information to always select her preferred alternative.[16] In an informative equilibrium, the decision maker can only learn that the state is between the senders' thresholds τ_j. Therefore, when misreporting is "cheap," the decision maker obtains an ex-ante welfare that is strictly lower than W_{fi}.

Competition in signaling. Consider the model described in Section 3, where the decision maker is uninformed, senders are not perfectly aligned, and misreporting is possible but costly. The following observation points out that, differently than cheap talk games, the game considered here does not have babbling equilibria.

Observation 1. *There is no babbling equilibrium.*

[15] If withholding information is not possible or is prohibitively costly, then this result holds even when only one of the two senders has verifiable information, i.e., $0 \leq k_j < k_i = \infty$ for $i \neq j$: in equilibrium, the decision maker pays attention only to sender i and disregards every report delivered by sender j, who cannot do better than reporting truthfully as well.

[16] This is because, in this model with a binary action space, the decision maker cannot take extreme actions that punish both senders after observing conflicting reports (cfr. Battaglini, 2002).

To see why, suppose that there is a babbling equilibrium in which there is no communication. The decision maker's choice is unaffected by the senders' reports. Given that misreporting is useless, the senders always report truthfully to economize on misreporting costs. Since the senders' reports are fully revealing, it is not sequentially rational for the decision maker to ignore them, contradicting that this is an equilibrium. In the next section, I show that in this case there are equilibria where the decision maker gets the full information welfare W_{fi}, but also that these equilibria are not plausible.

4 Receiver-efficient and Pure-strategy Equilibria

The goal of this section is to analyze the existence and plausibility of equilibria where the decision maker always obtains the information she needs to select her preferred alternative. This class of equilibria is important because it is believed that competition in "the marketplace of ideas" may result in the truth becoming known (Gentzkow & Shapiro, 2008). Competing forces may indeed yield full information revelation in cheap talk settings (Battaglini, 2002) as well as in models of verifiable disclosure (Milgrom & Roberts, 1986b).

In the setting studied here, the combination of a rich state space together with a binary action space implies that the decision maker does not need to know the realized state θ in order to find her favorite alternative. All she needs to know is whether the state is positive or negative. For the purposes of this section, an analysis of fully revealing equilibria would therefore be too restrictive. The following definition gives a weaker notion of revelation that will prove useful for the analysis that follows.

Definition 1. *A "fully revealing equilibrium" (FRE) is an equilibrium where for every $\theta' \in \Theta$, $r_j \in S_j(\theta')$, and $j \in \{1,2\}$, we have $P(\theta|r_1,r_2) = 1$ if and only if $\theta \geq \theta'$. A "receiver-efficient equilibrium" (REE) is an equilibrium where for every $\theta \in \Theta$, $r_j \in S_j(\theta)$, and $j \in \{1,2\}$, we have $\beta(r_1,r_2) = \oplus$ if $\theta \geq 0$, and $\beta(r_1,r_2) = \odot$ otherwise.*

A fully revealing equilibrium is also receiver-efficient, but a receiver-efficient equilibrium is not necessarily fully revealing. If competing forces could discipline senders into always reporting truthfully their private information about the state, then full revelation would naturally occur. However, the following observation points out that, in the game considered here and described in Section 3, misreporting occurs in every equilibrium.

Observation 2. *Misreporting occurs in every equilibrium.*

To see why, suppose by way of contradiction that there exists an equilibrium[17] where

[17]Observations 1 and 2 apply to all perfect Bayesian equilibria of the game described in Section 3, and not only to those satisfying (FOSD).

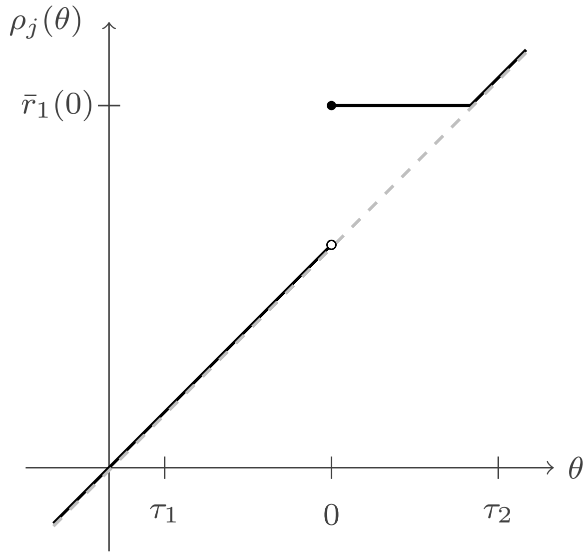

Figure 1: Senders' strategies in a receiver-efficient and fully revealing equilibrium. The reporting rules of senders 1 and 2 are indicated by black and dashed gray lines, respectively.

misreporting never occurs, that is, where $\rho_1(\theta) = \rho_2(\theta) = \theta$ for every $\theta \in \Theta$. Consider such a truthful equilibrium and a state $\theta = \epsilon > 0$, where ϵ is small enough. To discourage deviations, off path beliefs must be such that $\beta(\epsilon, -\epsilon) = \oplus$. However, there always exists an $\epsilon > 0$ such that, when the state is $\theta = -\epsilon$, sender 1 can profitably deviate from the prescribed truthful strategy by reporting $r_1 = \epsilon$, as $u_1(-\epsilon) > k_1 C_1(\epsilon, -\epsilon)$. This contradicts our supposition that there exists an equilibrium where misreporting never occurs.

The question is: if senders misreport in every equilibrium, do receiver-efficient equilibria exist? Figure 1 provides a positive graphical answer by showing reporting strategies that not only constitute a receiver-efficent equilibrium, but are also fully revealing.[18] To verify that Figure 1 depicts an equilibrium, consider the following strategies: sender 1 delivers $\rho_1(\theta) = \bar{r}_1(0)$ for every $\theta \in [0, \bar{r}_1(0)]$, where for simplicity we assume that $\bar{r}_1(0) < \tau_2$. Otherwise, sender 1 reports truthfully. By contrast, sender 2 always reports truthfully, i.e., $\rho_2(\theta) = \theta$ for all $\theta \in \Theta$. Given any on path pair of reports, posterior beliefs are such that $P(\theta|r_1, r_2) = 0$ for every $\theta < r_2$ and $P(\theta|r_1, r_2) = 1$ otherwise, which is consistent with sender 2 playing a separating strategy. off path beliefs are such that $U_{dm}(r_1, r_2) < 0$ if $r_1 < \bar{r}_1(0)$, and $P(\theta|r_1, r_2) = 1$ if and only if $\theta \geq r_1 \geq \bar{r}_1(0)$. By the definition of reach, sender 1 would never find it profitable to deliver a report $r_1 \geq \bar{r}_1(0)$ when $\theta < 0$. Sender 2 cannot deviate from his truthful strategy by delivering a negative report when the state is positive: since $\rho_1(\theta) \geq \bar{r}_1(0)$ for every $\theta \geq 0$, such a deviation would induce $\beta(\cdot) = \oplus$. No sender has a profitable individual deviation from the prescribed equilibrium strategies. Therefore, there exist equilibria where senders always fully reveal the state to the decision

[18] Kartik et al. (2007) study a fully revealing equilibrium where misreporting occurs in every state in a single-sender setting with an unbounded state space.

maker, even though full revelation involves misreporting.[19]

Notably, Figure 1 also illustrates the existence of equilibria in pure strategies. Intuitively, when senders play pure strategies and in each state at least one of the two senders plays a separating strategy, then the decision maker can always "reverse" their reports to recover the underlying truth. This argument suggests that all pure-strategy equilibria are receiver-efficient. The next lemma shows that such an intuition is correct and, in addition, that all receiver-efficient equilibria are in pure strategies.

Lemma 2. *An equilibrium is receiver-efficient if and only if it is in pure strategies.*

The receiver-efficient and fully revealing equilibrium strategies discussed above are, however, problematic. To see what the problem is, consider again the strategies depicted in Figure 1 and a state $\theta' \in (0, \bar{r}_1(0))$. Suppose that in state θ' sender 1 deviates from the prescribed equilibrium by reporting the truth instead of $\rho_1(\theta') = \bar{r}_1(0)$, whereas sender 2 sticks to his separating reporting rule. Notice that, in the equilibrium under consideration, sender 1 never delivers $r_1 = \theta'$. Once the decision maker receives the off path pair of reports (θ', θ'), her posterior beliefs p induce an expected payoff of $U_{dm}(\theta', \theta') < 0$, and hence lead to $\beta(\theta', \theta') = \circleddash$. These off path beliefs require the decision maker to conjecture that the state is likely to be negative. However, this means that the decision maker must entertain the possibility that (i) both senders simultaneously deviated from the prescribed equilibrium strategies, and (ii) sender 2 delivered a strictly dominated report.

In addition, the receiver-efficient equilibrium in Figure 1 is supported by posterior beliefs that are discontinuous in the senders' reports: for every on path pair of reports $(\bar{r}_1(0), r_2)$ such that $r_2 \in (0, \bar{r}_1(0))$, posterior beliefs are such that $U_{dm}(\bar{r}_1(0), r_2) = u_{dm}(r_2) > 0$; by contrast, $U_{dm}(\bar{r}_1(0) - \epsilon, r_2) < 0$ for every arbitrarily small $\epsilon > 0$. This discontinuity is crucial to discourage deviations, but it does not seem plausible especially in light of its problematic implications discussed above. In the remaining part of this section, I test for receiver-efficient equilibria using two well-known refinements that apply to games with multiple senders: unprejudiced beliefs (Bagwell & Ramey, 1991) and ε-robustness (Battaglini, 2002).

Unprejudiced beliefs. Consider again a deviation from the equilibrium depicted in Figure 1 where both senders report truthfully in some state $\theta' \in (0, \bar{r}_1(0))$. If, whenever possible, the decision maker conjectures deviations as individual and thus as originating from one sender only, then she should infer that sender 1 performed the deviation: sender 1 never reports $r_1 = \theta'$ on the equilibrium path, whereas sender 2 truthfully reports $r_2 = \theta'$ only when the state is indeed θ'. Since sender 2 is following his separating strategy, the

[19] Battaglini (2002) uses an argument similar to the revelation principle to show that, in a multi-sender cheap talk model, if there exists a fully revealing equilibrium then there exists a fully revealing equilibrium in truthful strategies. Such an argument cannot be applied here because of the presence of misreporting costs. In this model there are fully revealing equilibria, but there are no equilibria in truthful strategies.

decision maker should infer that the state is $\theta' > 0$. According to this line of reasoning, off path beliefs must be such that $P(\theta|\theta', \theta') = 1$ if and only if $\theta \geq \theta'$, and thus $\beta(\theta', \theta') = \oplus$. Therefore, such a deviation becomes profitable for sender 1 because it saves on misreporting costs without affecting the outcome.

Bagwell and Ramey (1991) introduce the concept of "unprejudiced beliefs," which formalize the idea that the decision maker should rule out the possibility that multiple senders are deviating at the same time whenever it is possible that only a single sender is deviating. Vida and Honryo (2019) show that, in generic multi-sender signaling games, strategic stability (Kohlberg & Mertens, 1986) implies unprejudiced beliefs. Apart from their association with the notion of strategic stability, unprejudiced beliefs are intuitive, easily applicable, and consistent with the notion of Nash equilibrium and, as such, constitute a sensible way to refine equilibria in multi-sender signaling games when other criteria fail to do so. The following definition formalizes unprejudiced beliefs.[20]

Definition 2 (Vida & Honryo, 2019)**.** *Given senders' strategies ρ_j, the decision maker's posterior beliefs p are unprejudiced if, for every pair of reports (r_1, r_2) such that $\rho_j(\theta') = r_j$ for some $\theta' \in \Theta$ and $j \in \{1, 2\}$, we have that $p(\theta''|r_1, r_2) > 0$ only if there is a sender $i \in \{1, 2\}$ such that $\rho_i(\theta'') = r_i$.*

We have seen how the above "informational free-riding" argument breaks down the receiver-efficient equilibrium depicted in Figure 1. A natural question is whether such an argument applies only in that particular case or if instead it rules out other equilibria. The next proposition tells us that in fact there is no receiver-efficient equilibrium that supports unprejudiced beliefs.

Proposition 1. *There are no receiver-efficient equilibria with unprejudiced beliefs.*

ε-robustness. In the model described in Section 3, senders are perfectly informed and the receiver can perfectly observe the senders' reports. In other words, there are no perturbations, or "noise," in the senders' report or the decision maker's observations. This modeling strategy allows me to isolate the effects of strategic interactions from the effects of statistical information aggregation. At the same time, however, it allows for excessive freedom to pick ad-hoc beliefs that would not survive the presence of even arbitrarily small perturbations in the transmission of information.

I follow Battaglini (2002) and define an ε-perturbed game as the game described in Section 3, in which the decision maker perfectly observes the report of sender j with probability $1 - \varepsilon_j$ and with probability ε_j observes a random report $\tilde{r}_j$, where $\tilde{r}_j$ is a random variable with continuous distribution G_j, density g_j, and support in Θ. This

[20]Definition 2 is weaker than the definition originally introduced by Bagwell and Ramey (1991), and therefore it is useful to test for equilibria that do not support unprejudiced beliefs.

may correspond to a situation where with some probability the decision maker misreads reports; or, alternatively, where with some probability senders deliver a wrong report by mistake.[21] As before, senders incur misreporting costs that depend only on the realized state θ and on their "intended" report r_j, and not on the wrongly observed or delivered $\tilde{r}_j$. The introduction of noise makes any pair of reports possible on the equilibrium path. The decision maker's posterior beliefs depend on $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $G = (G_1, G_2)$, and the senders' reporting strategies $\rho_j(\theta)$.

Definition 3 (Battaglini, 2002)**.** *An equilibrium is ε-robust if there exist a pair of distributions $G = (G_1, G_2)$ and a sequence $\varepsilon^n = (\varepsilon_1^n, \varepsilon_2^n)$ converging to zero such that the off path beliefs of the equilibrium are the limit of the beliefs that the equilibrium strategies would induce in an ε-perturbed game as $\varepsilon^n \to 0^+$.*

Intuitively, as the noise ε fades away, the event in which the decision maker misreads both reports becomes negligible. At the limit as $\varepsilon \to 0^+$, the decision maker infers that she is correctly observing at least one of the two reports. Therefore, once she observes an off path pair of reports in an ε-robust equilibrium, the decision maker conjectures—whenever possible—that one sender is following his prescribed reporting strategy while the other is not. This last implication of ε-robustness suggests that there might be a tight connection between the refinement criteria of ε-robustness and unprejudiced beliefs. The next lemma confirms the existence of such a relationship.[22]

Lemma 3. *If an equilibrium is ε-robust, then it has unprejudiced beliefs.*

A straightforward implication of Lemma 3 and Proposition 1 is that no receiver-efficient or fully revealing equilibrium is ε-robust. By Lemma 2, we obtain that also pure-strategy equilibria are not supported by unprejudiced beliefs and are not ε-robust. These results suggest that mixed-strategy equilibria are qualitatively important, whereas they are unexplored in related work.[23] The next section is dedicated to finding equilibria that are robust in the sense that they are ε-robust and supported by unprejudiced beliefs.

5 Direct Equilibria

Findings in the previous section show that pure-strategy equilibria exist and are receiver-efficient, but are supported by off path beliefs that are not plausible. Such results motivate

[21] Battaglini (2002) perturbs the senders' observation of the realized state, whereas I perturb the decision maker's observed reports. My perturbation is qualitatively similar to Battaglini's.

[22] Lemma 3 applies to perfect Bayesian equilibria of the game described in Section 3 with $n \geq 2$ senders.

[23] Most research on single-sender signaling focuses only on pure-strategy equilibria. For example, Kartik et al. (2007), Kartik (2009), and Chen (2011) consider pure strategies only. Most research on multi-sender signaling also focuses only on pure-strategy equilibria (see Section 2).

the quest for “robust” equilibria in mixed strategies. The two main goals of this section are to provide sufficient conditions under which equilibria are robust and to characterize such robust equilibria.

Since (FOSD) is not enough to rule out implausible equilibria, I impose a different set of restrictions to obtain robust mixed-strategy equilibria. However, classic refinements for signaling games such as the “intuitive criterion” (Cho & Kreps, 1987) and the “universal divinity criterion” (Banks & Sobel, 1987) have little bite here, as they are developed for single-sender settings. To date, there is no large consensus on how to extend these criteria to multi-sender settings. By contrast, the ε-robustness and unprejudiced beliefs criteria proved to be useful in testing pure-strategy equilibria of multi-sender signaling games, but cannot be easily applied to finding non-separating equilibria in mixed strategies.

Given the implausibility of receiver-efficient equilibria, I impose two conditions on how the decision maker interprets the senders' reports. I refer to equilibria satisfying these conditions as “direct equilibria.”

Definition 4. *A “direct equilibrium” (DE) is a perfect Bayesian equilibrium of the game described in Section 3 where the decision maker's posterior beliefs p satisfy the following conditions:*

i) The (FOSD) *condition holds, and for every pair of reports* (r_1, r_2) *such that* $\underline{r}_2(0) < r_2 \leq 0 \leq r_1 < \bar{r}_1(0)$*, and for* $j \in \{1, 2\}$*, we have*

$$\frac{dU_{dm}(r_1, r_2)}{dr_j} > 0; \tag{D}$$

ii) Once the decision maker observes the pairs of reports $(\bar{r}_1(0), \underline{r}_2(0))$ *and* $(0, 0)$*, her posterior beliefs p are such that she is indifferent between the two alternatives, i.e.,*

$$U_{dm}\left(\bar{r}_1(0), \underline{r}_2(0)\right) = U_{dm}\left(0, 0\right) = 0. \tag{C}$$

The first condition, (D), imposes a “strict” first-order stochastic dominance condition on posterior beliefs p, but only for pairs of reports consisting of conflicting recommendations. Otherwise, (FOSD) applies. Since (D) implies (FOSD), Lemma 1 applies also to direct equilibria. Intuitively, (D) means that strictly higher conflicting reports inform the decision maker that the expected value of selecting alternative $\oplus$ is strictly higher. Similarly to (FOSD), condition (D) is natural and consistent with the idea that reports are literal statements about the state and that misreporting is costly.

Condition (C) draws on a simple argument of equilibrium dominance. To see how, consider a report $r_j \in \hat{R}$, and denote by $Q_j(r_j)$ the set of states for which delivering report r_j is potentially profitable for sender j given that posterior beliefs p satisfy (D).

By Lemma 1 and the definition of inverse reach, we obtain that $Q_1(r_1) = [\bar{r}_1^{-1}(r_1), r_1] \cap \Theta$ and $Q_2(r_2) = [r_2, \underline{r}_2^{-1}(r_2)] \cap \Theta$. Denote the intersection of these two sets by $Q(r_1, r_2) = Q_1(r_1) \cap Q_2(r_2)$. If $Q(r_1, r_2) \neq \varnothing$, then the decision maker will rule out the possibility that the realized state lies outside $Q(r_1, r_2)$, i.e., $p(\theta|r_1, r_2) = 0$ for all $\theta \notin Q(r_1, r_2)$. Since $Q(\bar{r}_1(0), \underline{r}_2(0)) = Q(0, 0) = \{0\}$, upon receiving the pairs of reports $(\bar{r}_1(0), \underline{r}_2(0))$ or $(0, 0)$, the decision maker will believe that the realized state[24] is for sure $\theta = 0$. Otherwise, the decision maker will believe that at least one of the two senders delivered a report that is equilibrium-dominated. Condition (C) is even less stringent than this argument suggests, as it does not require that the decision maker's posterior beliefs be degenerate at 0, and does not impose conditions over pairs of reports[25] other than $(\bar{r}_1(0), \underline{r}_2(0))$ and $(0, 0)$.

As an immediate application of direct equilibria, let's revisit the fully revealing and voter-efficient equilibrium in Figure 1 discussed in Section 4. To prevent a deviation by sender 1, the decision maker's posterior beliefs p are such that $U_{dm}(\theta', \theta') < 0$ for any $\theta' \in (0, \bar{r}_1(0))$, and thus $\beta(\theta', \theta') = \ominus$. This cannot be a direct equilibrium: by (C) we have that $U_{dm}(0, 0) = 0$, and by (D) we have that $U_{dm}(\theta', \theta') \geq 0$, leading to $\beta(\theta', \theta') = \oplus$ and thus to a profitable deviation by sender 1. Therefore, conditions (C) and (D) rule out at least some equilibria that are not plausible.

In the rest of this section I will show that direct equilibria have a number of remarkable properties: they always exist, they are essentially unique, and there are direct equilibria that are ε-robust and hence with unprejudiced beliefs.

5.1 Notation for Mixed Strategies

Before analyzing direct equilibria, I first introduce some useful notation. To describe mixed strategies, I use a "mixed" probability distribution $\phi_j(r_j, \theta)$ that, for every state θ, assigns a mixed probability density to report r_j by sender j. This specification allows me to describe the senders' reporting strategies as mixed random variables whose distribution can be partly continuous and partly discrete.[26]

Formally, I partition the support $S_j(\theta)$ of each sender in two subsets, $C_j(\theta)$ and $D_j(\theta)$. To represent atoms in $\phi_j(\theta)$, I define a partial probability density function $\alpha_j(\cdot, \theta)$ on $D_j(\theta)$ such that $0 \leq \alpha_j(r_j, \theta) \leq 1$ for all $r_j \in D_j(\theta)$, and $\hat{\alpha}_j(\theta) = \sum_{r_j \in D_j(\theta)} \alpha_j(r_j, \theta)$. By contrast, the continuous part of the distribution $\phi_j(\theta)$ is described by a partial probability density

[24]From $P(\theta|\bar{r}_1(0), \underline{r}_2(0)) = P(\theta|0, 0) = 1$ iff $\theta \geq 0$ we get $U_{dm}\left(\bar{r}_1(0), \underline{r}_2(0)\right) = U_{dm}\left(0, 0\right) = u_{dm}(0) = 0$.

[25]As we shall see, in every direct equilibrium the pair $(\bar{r}_1(0), \underline{r}_2(0))$ is on path only for $\theta = 0$, and thus it fully reveals that the state is indeed zero. By contrast, no sender ever delivers $r_j = 0$, and thus the pair of reports $(0, 0)$ is not only off path, but it must constitute a double deviation.

[26]Mixed-type distributions that have both a continuous and a discrete component to their probability distributions are widely used to model zero-inflated data such as queuing times. For example, the "rectified Gaussian" is a mixed discrete-continuous distribution.

function $\psi_j(\cdot, \theta)$ on $C_j(\theta)$ such that $\int_{r_j \in C_j(\theta)} \psi_j(r_j, \theta) d\theta = 1 - \hat{\alpha}_j(\theta)$. I set $\alpha_j(r', \theta) = 0$ for all $r' \notin D_j(\theta)$ and $\psi_j(r'', \theta) = 0$ for all $r'' \notin C_j(\theta)$.

As we shall see (Lemma 7 and Proposition 4), in every direct equilibrium $D_j(\theta) = \{\theta\}$ for all $\theta \in \Theta$ and $j \in \{1, 2\}$. Therefore, for ease of notation, I set $\alpha_j(\theta) \equiv \alpha_j(\theta, \theta) = \hat{\alpha}_j(\theta)$. The score $\alpha_j(\theta)$ thus represents the probability that sender j reports truthfully in state $\theta \in \Theta$. The partial probability density functions[27] $\alpha_j(\theta)$ and $\psi_j(\cdot, \theta)$ determine the "generalized" density function $\phi_j(\theta)$ through the well-defined mixed distribution

$$\phi_j(x, \theta) = \delta(x - \theta)\alpha_j(\theta) + \psi_j(x, \theta),$$

where $\delta(\cdot)$ is the Dirac delta "generalized" function.[28]

A mixed-strategy for sender j is a mixed probability measure $\phi_j(\theta) : \Theta \to \Delta(R_j)$ with support $S_j(\theta)$. I indicate with $\phi_j(r_j, \theta)$ the mixed probability assigned by $\phi_j(\theta)$ to a report r_j in state θ that satisfies

$$\int_{r_j \in S_j(\theta)} \phi_j(r_j, \theta) dr_j = \alpha_j(\theta) + \int_{r_j \in C_j(\theta)} \psi_j(r_j, \theta) dr_j = 1.$$

I denote by $\Phi_j(r_j, \theta)$ and $\Psi_j(r_j, \theta)$ the CDFs of ϕ_j and ψ_j, respectively. Sender j's expected utility from delivering r_j when the state is θ in a direct equilibrium ω is $W_j^\omega(r_j, \theta)$.

5.2 Solving for Direct Equilibria

In the remaining parts of this section, I characterize direct equilibria and show their properties. All proofs and a number of intermediate results are relegated to Appendix A.2.

Given a pair of reports (r_1, r_2) the decision maker forms posterior beliefs $p(\theta | r_1, r_2)$, which determine whether she selects $\oplus$ or $\odot$. Consider a direct equilibrium and a pair of reports (r_1, r_2) such that $r_2 < 0$ and $U_{dm}(r_1, r_2) < 0$, and suppose that there exists a report $r_1' \in R_1$ such that $U_{dm}(r_1', r_2) > 0$. By conditions (C) and (D), it must be[29] that there exists a report $r_1'' \in (r_1, r_1')$ such that $U_{dm}(r_1'', r_2) = 0$. In this case, r_1'' "swings" the decision maker's choice as $\beta(r, r_2) = \oplus$ for all $r \geq r_1''$ and $\beta(r, r_2) = \odot$ otherwise. Let r_1'' denote the "swing report" of r_2. The notion of swing report is key for the analysis of direct equilibria, and the following definition formalizes the concept.

[27] Under this specification, even the "mass" $\alpha_j(\cdot)$ is a partial probability "density."

[28] The Dirac delta $\delta(x)$ is a generalized function such that $\delta(x) = 0$ for all $x \neq 0$, $\delta(0) = \infty$, and $\int_{-\epsilon}^{\epsilon} \delta(x) dx = 1$ for all $\epsilon > 0$.

[29] By (C) we have $U_{dm}(0, 0) = 0$, and by (D) we have $U_{dm}(0, r_2) < 0$ and $r_1' > r_1$. Since the differentiability of U_{dm} for conflicting reports implies its continuity, and since $U_{dm}(r_1', r_2) > 0$, it follows from the intermediate value theorem that there must be $r_1'' \in (0, r_1')$ such that $U_{dm}(r_1'', r_2) = 0$.

Definition 5. *Given a report r, the "swing report" $s(r)$ is defined as*

$$s(r) = \begin{cases} \{r_2 \in R_2 \mid U_{dm}(r, r_2) = 0\} & \text{if } r \geq 0 \\ \{r_1 \in R_1 \mid U_{dm}(r_1, r) = 0\} & \text{otherwise.} \end{cases}$$

If $s(r) = \varnothing$, then I set $s(r) = -\infty$ for $r \geq 0$, and $s(r) = \infty$ otherwise.

With a slight abuse of language, I hereafter say that sender j "swings" the report of his opponent $-j$ whenever the pair of reports (r_1, r_2) induce the selection of sender j's preferred alternative. When there is a conflict of interest between senders, i.e., for some $\theta \in (\tau_1, \tau_2)$, sender 1 swings the report of sender 2 when $r_1 \geq s(r_2)$. Similarly, sender 2 swings the report of sender 1 when $r_2 < s(r_1)$.

In a direct equilibrium, the swing report $s(r)$ has a number of intuitive properties: first, condition (D) ensures that the swing report, if it exists, is unique; second, condition (C) pins down the swing report for $s(\bar{r}_1(0)) = \underline{r}_2(0)$, $s(\underline{r}_2(0)) = \bar{r}_1(0)$, and $s(0) = 0$. From the interaction of conditions (C) and (D), it follows that every report $r \in \hat{R} = [\underline{r}_2(0), \bar{r}_1(0)]$ has a unique swing report $s(r) \in \hat{R}$ such that if $r > 0$ then $s(r) < 0$. Moreover, for all $r \in \hat{R}$, the swing report of a swing report is the report itself, i.e., $s(s(r)) = r$, and strictly higher reports have strictly lower swing reports. Importantly, $s(r)$ is endogenously determined in equilibrium through the posterior beliefs p. The following lemma formalizes these equilibrium features of the swing report function.

Lemma 4. *In a direct equilibrium, every report $r \in \hat{R}$ has a swing report $s(r) \in \hat{R}$ such that: (i) if $r \gtreqless 0$ then $s(r) \lesseqgtr 0$ and $s(0) = 0$; (ii) $s(s(r)) = r$; (iii) for every $r \in \hat{R}$, $\frac{ds(r)}{dr} < 0$; (iv) $s(\bar{r}_1(0)) = \underline{r}_2(0)$.*

Therefore, $s(r)$ is effectively a strictly decreasing function of r in the set $[\underline{r}_2(0), \bar{r}_1(0)]$, and in such a domain I refer to $s(r)$ as the "swing report function." When the state takes extreme values, a sender may not be able to profitably swing the report of his opponent even when the opponent reports truthfully. This happens when $s(\theta)$ is beyond a sender's reach. In such cases, we should expect both senders to always report truthfully, and therefore to deliver matching reports that reveal the state. It is therefore useful to define cutoffs in the state space that help determine when truthful reporting always occurs in direct equilibria.

Definition 6. *The "truthful cutoffs" are defined as*

$$\theta_1 := \{\theta \in \Theta \mid s(\theta) = \bar{r}_1(\theta)\},$$

$$\theta_2 := \{\theta \in \Theta \mid s(\theta) = \underline{r}_2(\theta)\}.$$

The truthful cutoffs are also determined in equilibrium, as they depend on $s(r)$. Recall that by condition (C) we have that $s(\underline{r}_2(0)) = \bar{r}_1(0)$ and $s(0) = 0$. Since $\bar{r}_1(\theta)$ is increasing in θ and $\bar{r}_1(\tau_1) = \tau_1 < 0 < s(\tau_1)$, it follows that $0 > \theta_1 > \max\{\tau_1, \underline{r}_2(0)\}$. Similarly, we obtain that $0 < \theta_2 < \min\{\tau_2, \bar{r}_1(0)\}$. Therefore, in any direct equilibrium the set of states that lie within the truthful cutoffs (θ_1, θ_2) is also a strict subset of $[\tau_1, \tau_2]$ and of $[\underline{r}_2(0), \bar{r}_1(0)]$. This equilibrium feature of the truthful cutoffs is convenient because it implies that for every state $\theta \in (\theta_1, \theta_2)$ there is always a conflict of interest between senders, and that the swing report function $s(\theta)$ exists and is well defined in such a set.[30]

In a direct equilibrium, we expect both senders to always report truthfully—and therefore to play pure strategies—when the state lies outside the truthful cutoffs. By contrast, when the state takes values within the truthful cutoffs, we expect senders to play mixed strategies and to engage in some misreporting. As the following lemma shows, these two expectations turn out to be correct in every direct equilibrium.

Lemma 5. *In a direct equilibrium,* $S_j(\theta) = \{\theta\}$ *for all* $\theta \notin (\theta_1, \theta_2)$*, and* $|S_j(\theta)| > 1$ *for every* $\theta \in (\theta_1, \theta_2)$*,* $j \in \{1, 2\}$*.*

This result, together with the previous observation that $(\theta_1, \theta_2) \subset [\tau_1, \tau_2]$, shows an interesting characteristic of direct equilibria: in relatively extreme states, both senders always deliver matching and truthful reports even though they have conflicting interests. Since the senders' reports coincide, it follows from Lemma 1 that in these cases the decision maker learns the underlying state. Therefore, full information revelation always occurs in extreme states that lie outside the truthful cutoffs.

Given the results outlined above, from now on I focus on the senders' behavior when the state takes values within the truthful cutoffs. I proceed by first studying the reporting strategies when senders misreport their private information. Conditional on misreporting in state θ, sender j's strategy $\phi_j(\theta)$ has support in the set $S_j(\theta) \setminus \{\theta\}$. To describe and study equilibrium supports and strategies, it is useful to understand whether such a set is convex or not. The next lemma tells us that $S_j(\theta) \setminus \{\theta\}$ is always convex.

Lemma 6. *In a direct equilibrium,* $S_j(\theta) \setminus \{\theta\}$ *is convex for all* $\theta \in (\theta_1, \theta_2)$ *and* $j \in \{1, 2\}$*.*

The intuition for Lemma 6 is the following: in equilibrium, the presence of a "gap" in the set $S_j(\theta) \setminus \{\theta\}$ must imply that j's opponent never wastes resources on swinging reports that are in such a gap, as they are never delivered. However, this means that in $S_j(\theta) \setminus \{\theta\}$ there are two different reports that yield approximately the same probability of inducing the selection of j's preferred alternative but have different costs. This would not be possible in an equilibrium, and therefore the set $S_j(\theta) \setminus \{\theta\}$ must be convex.

[30]These results are formalized in Lemma A.1 in Appendix A.2.

While senders may misrepresent the same state in a number of ways, the above argument also suggests that, conditional on misreporting, there is no report that they deliver with strictly positive probability. To see why, suppose by way of contradiction that sender j misreports some state θ by delivering $r_j \in S_j(\theta) \setminus \{\theta\}$ with some strictly positive "mass" probability $\alpha_j(r_j, \theta) > 0$. That is, j's strategy $\phi_j(\theta)$ has an atom in r_j. It follows that sender $-j$'s expected payoff is discontinuous around $r_{-j} = s(r_j)$, and therefore $s(r_j)$ cannot be in the interior[31] of $S_{-j}(\theta)$. If $s(r_j) \notin S_{-j}(\theta)$, then j can profitably deviate from $\phi_j(\theta)$ by "moving" mass probability from the atom to some cheaper report that ensures the selection of his own favorite alternative. If instead $s(r_j)$ is on the boundary of $S_{-j}(\theta)$, then one of the two senders would have a profitable deviation: either there are reports outside $S_{-j}(\theta)$ that yield a higher expected payoff than reports inside the support, or there is some report that dominates r_j. In both cases, we obtain a contradiction to the supposition that j's strategy is part of an equilibrium. The following lemma formalizes the idea that the equilibrium reporting strategies are non-atomic when senders misreport their private information.[32]

Lemma 7. *In a direct equilibrium, the strategies $\phi_j(\theta)$ have no atoms in $S_j(\theta) \setminus \{\theta\}$ for every $\theta \in (\theta_1, \theta_2)$ and $j \in \{1, 2\}$.*

5.2.1 Strategies, Supports, and Beliefs

I am now ready to state the main results of this section. Lemmata 6 and 7 tell us that, conditional on misreporting, senders play an atomless reporting strategy with support in a convex set. In equilibrium, all reports in the support of a sender's strategy must yield him the same expected payoff. By setting the derivative of the expected payoff sender $-j$ gets from delivering reports in $S_{-j}(\theta) \setminus \{\theta\}$ to zero, I obtain the partial probability density $\psi_j(r_j, \theta)$. The next proposition establishes senders' misreporting behavior.

Proposition 2. *In a direct equilibrium, for every $\theta \in (\theta_1, \theta_2)$ and $i, j \in \{1, 2\}$ with $i \neq j$, sender j delivers report $r_j \in S_j(\theta) \setminus \{\theta\}$ according to*

$$\psi_j(r_j, \theta) = \frac{k_i}{-u_i(\theta)} \frac{dC_i(s(r_j), \theta)}{dr_j}.$$

Each sender's misreporting behavior depends directly on his opponent's utility and costs, and indirectly on his own characteristics through the swing report function $s(r)$. Whether a sender is more likely to deliver small lies or large misrepresentations depends on the shape of his opponent's misreporting cost function together with the shape of the

[31] Recall that every report in the equilibrium support must yield the same expected payoff.

[32] The intuition for the results provided in this section omits a number of additional steps that are necessary to prove Lemmata 6 and 7. See Lemmata A.2 to A.7 in Appendix A.2.

swing report function, where the latter is determined in equilibrium. In Section 6.1 I discuss in detail the senders' misreporting behavior for the particular case where senders have symmetric features.

Since the sets $S_j(\theta)\setminus\{\theta\}$ are convex and the strategies $\phi_j(\theta)$ are atomless on $S_j(\theta)\setminus\{\theta\}$, I can integrate the partial probability densities ψ_j to pin down the senders' equilibrium supports. This procedure allows me to prove the following proposition.

Proposition 3. *In a direct equilibrium, for every state $\theta \in (\theta_1, \theta_2)$, supports $S_j(\theta)$ are*

$$S_1(\theta) = \{\theta\} \cup [\max\{s(\theta), \theta\}, \min\{\bar{r}_1(\theta), s(\underline{r}_2(\theta))\}],$$

$$S_2(\theta) = \{\theta\} \cup [\max\{\underline{r}_2(\theta), s(\bar{r}_1(\theta))\}, \min\{s(\theta), \theta\}].$$

So far, I have focused the analysis on the senders' misreporting behavior. However, the above proposition shows that "the truth" is always in the support of the senders' equilibrium strategies. Having fully characterized the senders' misreporting strategies $\psi_j(\cdot, \theta)$ and supports $S_j(\theta)$, I am now in a position to establish the senders' truthful reporting behavior.

Proposition 4. *In a direct equilibrium, for every state $\theta \in (\theta_1, \theta_2)$, strategies $\phi_j(\theta)$ have an atom at $r_j = \theta$ of size $\alpha_j(\theta)$, where*

$$\alpha_1(\theta) = \begin{cases} \frac{k_2}{-u_2(\theta)} C_2\left(s(\theta), \theta\right) & \text{if } \theta \in [0, \theta_2) \\ 1 - \frac{k_2}{-u_2(\theta)} C_2\left(s(\bar{r}_1(\theta)), \theta\right) & \text{if } \theta \in (\theta_1, 0], \end{cases}$$

$$\alpha_2(\theta) = \begin{cases} 1 - \frac{k_1}{u_1(\theta)} C_1\left(s(\underline{r}_2(\theta)), \theta\right) & \text{if } \theta \in [0, \theta_2) \\ \frac{k_1}{u_1(\theta)} C_1\left(s(\theta), \theta\right) & \text{if } \theta \in (\theta_1, 0]. \end{cases}$$

Both senders report truthfully with strictly positive probability in almost every state. The only exception is $\theta = 0$, where the truth is never reported as $\alpha_1(0) = \alpha_2(0) = 0$. With probability $\alpha_1(\theta)\alpha_2(\theta)$ both senders deliver the truth, and by Lemma 1 we obtain that whenever this event occurs the decision maker fully learns the realized state. Moreover, by Proposition 3 we get that the decision maker may learn the realized state even when only one of the two senders reports truthfully: if the realized state is positive, then full revelation occurs when sender 2 reports truthfully; if the state is negative, then full revelation occurs when sender 1 reports truthfully. In these cases, the senders deliver different reports that nevertheless recommend to the decision maker to select the same alternative.

The probability of full revelation and the probability of observing matching reports

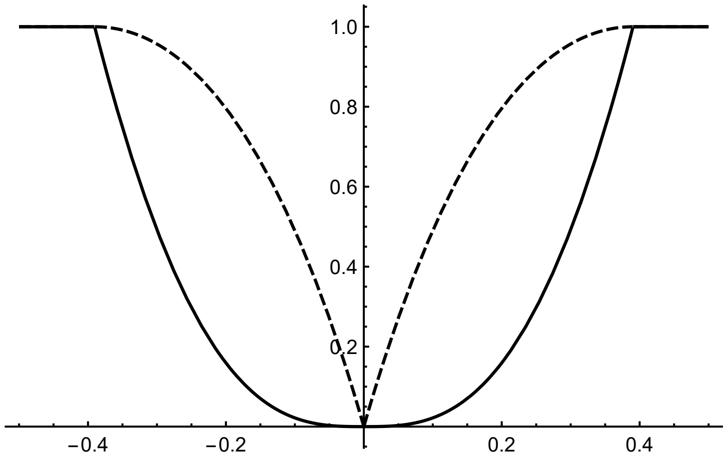

Figure 2: The probability that the decision maker fully learns the state (dashed black line) and the probability that senders deliver matching reports (solid black line) as functions of the realized state in a direct equilibrium of a symmetric environment with linear utilities and quadratic loss misreporting costs.

are both increasing as the realized state moves further away from zero.[33] Therefore, we obtain that in direct equilibria the revelation of the state and the congruence of reports are phenomena that are more likely to occur in extreme states than in intermediate states. To see this, note that

$$\frac{d\alpha_1(\theta)}{d\theta} = \begin{cases} \frac{k_2}{u_2(\theta)^2}\frac{du_2(\theta)}{d\theta}C_2\left(s(\theta),\theta\right) + \frac{k_2}{-u_2(\theta)}\frac{dC_2(s(\theta),\theta)}{d\theta} > 0 & \text{if } \theta \in [0,\theta_2) \\ -\frac{k_2}{u_2(\theta)^2}\frac{du_2(\theta)}{d\theta}C_2\left(s(\bar{r}_1(\theta)),\theta\right) - \frac{k_2}{-u_2(\theta)}\frac{dC_2(s(\bar{r}_1(\theta)),\theta)}{d\theta} < 0 & \text{if } \theta \in (\theta_1,0), \end{cases}$$

$$\frac{d\alpha_2(\theta)}{d\theta} = \begin{cases} \frac{k_1}{u_1(\theta)^2}\frac{du_1(\theta)}{d\theta}C_1\left(s(\underline{r}_2(\theta)),\theta\right) - \frac{k_1}{u_1(\theta)}\frac{dC_1(s(\underline{r}_2(\theta)),\theta)}{d\theta} > 0 & \text{if } \theta \in [0,\theta_2) \\ -\frac{k_1}{u_1(\theta)^2}\frac{du_1(\theta)}{d\theta}C_1\left(s(\theta),\theta\right) + \frac{k_1}{u_1(\theta)}\frac{dC_1(s(\theta),\theta)}{d\theta} < 0 & \text{if } \theta \in (\theta_1,0). \end{cases}$$

Figure 2 depicts both the probability that senders deliver the same report and the probability that the decision maker fully learns the realized state.

After establishing the senders' equilibrium supports and strategies, I can now proceed to study the decision maker's posterior beliefs. It is key to this analysis to understand how posterior beliefs p determine the decision maker's choice given any pair of reports. To this end, it is sufficient to examine how posterior beliefs p shape the swing report function $s(r)$. By Lemma 4, we have that $s(r) \in \hat{R}$ for every $r \in \hat{R}$, with $s(r) < 0$ if $r > 0$, $s(r) > 0$ if $r < 0$, and $s(0) = 0$. Given the supports and the strategies as in Propositions 2, 3, and 4, we obtain that every pair of reports (r_1, r_2) such that $\underline{r}_2(0) \leq r_2 < 0 < r_1 \leq \bar{r}_1(0)$ is on path. By Definition 5 and Lemma 4 we have that, for a pair of reports $(r_1, r_2 = s(r_1))$,

$$U_{dm}(r_1, s(r_1)) = U_{dm}(s(r_2), r_2) = \int_\Theta u_{dm}(\theta)p(\theta|r_1, s(r_1))d\theta = 0.$$

[33]This is because the decision maker's threshold τ_{dm} is normalized to zero.

Therefore, I can use $p(r_1, s(r_1)|\theta) = \phi_1(r_1, \theta) \cdot \phi_2(s(r_1), \theta)$ and previous results to show how posterior beliefs p pin down the swing report function $s(r)$ in a direct equilibrium. The next proposition shows how the swing report depends on the model's parameters.

Proposition 5. *In a direct equilibrium, the swing report function $s(r_i)$ is implicitly defined for $i, j \in \{1, 2\}$, $i \neq j$, and $r_i \in \hat{R}$, as*

$$s(r_i) = \left\{ r_j \in R_j \;\middle|\; \int_{\max\left\{r_2, \bar{r}_1^{-1}(r_1)\right\}}^{\min\left\{r_1, \underline{r}_2^{-1}(r_2)\right\}} f(\theta) \frac{u_{dm}(\theta)}{u_1(\theta)u_2(\theta)} \frac{dC_j(r_j, \theta)}{dr_j} \frac{dC_i(r_i, \theta)}{dr_i} d\theta = 0 \right\}. \tag{4}$$

5.2.2 Uniqueness, Robustness, and Existence

Propositions 2 to 5 complete the characterization of direct equilibria. However, there are three potential issues that must be addressed: first, there may be multiple direct equilibria that yield different solutions; second, direct equilibria may not be robust to the refinements discussed in Section 4, and therefore they may be implausible; third, direct equilibria may not exist. I conclude this section by showing that direct equilibria of the game described in Section 3 are essentially unique, robust, and always exist.

The issue of multiplicity is resolved by the observation that equation (4), which implicitly determines the swing report function $s(r)$, depends only on the primitives of the model. In particular, the swing report function depends on the decision maker's prior beliefs, the players' utilities, and the senders' costs. Given these primitives, the swing report function is the same in every direct equilibrium, and therefore the senders' reporting strategies and supports are the same in all direct equilibria. Conditions (C) and (D) are thus sufficient to ensure that all equilibria are essentially unique in the sense that they are all strategy- and outcome-equivalent.

Corollary 1. *Direct equilibria are essentially unique.*

In Section 4, I find that all pure-strategy and all receiver-efficient equilibria are not plausible for two different reasons: they feature informational free-riding opportunities that generate individual profitable deviations, and they are not robust to the presence of even arbitrarily small noise in communication. Robustness to informational free-riding opportunities and to noise requires equilibria to support unprejudiced beliefs (Bagwell & Ramey, 1991) and to be ε-robust (Battaglini, 2002), respectively. I also show that these two different criteria are tightly connected, as ε-robust equilibria are supported by unprejudiced beliefs. The question is: can direct equilibria be ε-robust?

To study whether there exist direct equilibria with unprejudiced beliefs I apply the following definition, which is adapted from Bagwell and Ramey (1991) to accommodate non-degenerate mixed strategies.[34]

Definition 7. *Given senders' strategies ϕ_j, the decision maker's posterior beliefs p are unprejudiced if, for every off path pair of reports (r_1, r_2) such that $\phi_j(r_j, \theta') > 0$ for some $j \in \{1, 2\}$ and $\theta' \in \Theta$, we have that $p(\theta''|r_1, r_2) > 0$ if and only if there is a sender $i \in \{1, 2\}$ such that $\phi_i(r_i, \theta'') > 0$.*

The next corollary confirms that there exist direct equilibria supported by unprejudiced beliefs (as in both Definition 2 and 7) that are also ε-robust.[35]

Corollary 2. *There are direct equilibria with unprejudiced beliefs that are also ε-robust.*

Even well-behaved signaling games may have no equilibria (Manelli, 1996). However, given posterior beliefs p, the equilibrium reporting strategies and supports in Propositions 2 to 4 are by construction such that no sender has individual profitable deviations. Moreover, given such strategies, the decision maker choice is sequentially rational. Therefore, as long as the assumptions established in Section 3 are satisfied, a direct equilibrium always exists.

Corollary 3. *A direct equilibrium always exists.*

6 An Example and Application

6.1 Example: Symmetric Environments

In what follows, I provide an example where senders have similar characteristics and the state is symmetrically distributed. This environment is an important benchmark because it deals with situations where no sender has an ex-ante advantage. In addition, it gives us a closed-form solution for senders' equilibrium strategies and supports. The following definition formalizes what is meant by a "symmetric environment."

Definition 8. *In a symmetric environment,*

i) the state is symmetrically distributed around zero, i.e., $f(\theta) = f(-\theta)$ for all $\theta \in \Theta$;

ii) $k_j C_j(r, \theta) = kC(r, \theta)$ for $j \in \{1, 2\}$, where $k > 0$ and $C(\cdot)$ satisfies $C(\theta + x, \theta) = C(\theta - x, \theta)$ for every $\theta \in \Theta$ and $x \in \mathbb{R}$; and

[34]Definition 2, introduced by Vida and Honryo (2019) and used in Section 4, is a weaker version of Definition 7. Lemma 3 applies to unprejudiced beliefs as in both definitions.

[35]Since ε-robustness implies unprejudiced beliefs, it would be sufficient to show that there exist direct equilibria that are ε-robust. Corollary 2 simply remarks that the two refinements are different.

iii) payoffs satisfy[36] $u_{dm}(\theta) = -u_{dm}(-\theta)$ *and* $u_1(\theta) = -u_2(-\theta)$ *for all* $\theta \in \Theta$.

Conditions i) to iii) are in addition to the assumptions in Section 3.

In symmetric environments the two senders differ only because they have conflicting interests. In other words, there is no particular reason why the decision maker should give more importance to the report of one sender than to that of the other. Intuition would suggest that, in a symmetric environment, the decision maker should assign the "burden of proof" equally between the senders. The next corollary confirms that this intuition is indeed correct in a direct equilibrium.

Corollary 4. *In a direct equilibrium of a symmetric environment,* $s(r) = -r$ *for every* $r \in \hat{R}$.

In a symmetric environment, the decision maker follows the recommendation of the sender that delivers the most extreme report. The burden of proof is equally distributed between the senders, as Corollary 4 shows. Moreover, the swing report function is linear even though some fundamentals, e.g., the cost functions, may be non-linear. Remarkably, in symmetric environments direct equilibria naturally have symmetric strategies.[37]

With an explicit solution to the swing report function, we obtain a natural closed-form solution to the senders' equilibrium strategies and supports. In applications this is particularly useful because in similar environments, such as in contests, typically little is known about mixed-strategy equilibria except in some special cases (see Levine & Mattozzi, 2019; Siegel, 2009).

I can now use this closed-form solution to examine the determinants and the characteristics of the senders' misreporting behavior. I show that the shape of the cost function, in particular its convexity/concavity, determines whether senders are more likely to deliver small lies or large misrepresentation or the other way around. By Proposition 2 and Corollary 4 we obtain that, in a symmetric environment, misreporting behavior is described by the following partial density, for $j \in \{1, 2\}$ and $j \neq i$,

$$\psi_j(r_j, \theta) = \frac{k}{-u_i(\theta)} \frac{dC(-r_j, \theta)}{dr_j}.$$

Therefore, if $C(\cdot)$ is strictly convex, we have that $d\psi_1(r_1, \theta)/dr_1 > 0$ for all $\theta \in S_1(\theta) \setminus \{\theta\}$ and $d\psi_2(r_2, \theta)/dr_2 < 0$ for all $\theta \in S_2(\theta) \setminus \{\theta\}$. This means that, conditional on misreporting, senders are more likely to deliver large misrepresentations of the state than small lies. By contrast, when senders have concave costs, misreports that are closer to

[36] By definition of threshold τ_j (see Section 3), this last condition implies that $\tau_2 = -\tau_1$.

[37] Corollary 4 is reminiscent of results in all-pay auctions with complete information, where it is shown that with two bidders only symmetric solutions exist (Baye et al., 1996).

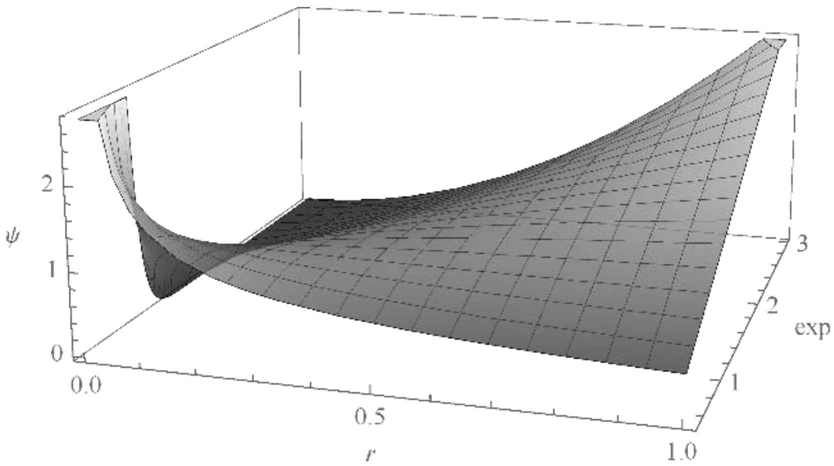

Figure 3: The partial probability density ψ as a function of the shape of misreporting costs and of the extent of misreporting, in a symmetric environment where $C(r,\theta) = |(r-\theta)^{exp}|$ and $\theta = 0$. With quadratic loss costs, $exp = 2$, the density ψ grows linearly as reports get further away from the truth. With absolute value linear costs, $exp = 1$, every misreport in the support has the same partial density. With concave costs, $exp \in (0,1)$, small misrepresentations are more likely than large lies, and when $exp > 1$ the opposite is true.

the truth are more likely to be delivered than large misrepresentations. The type of the senders' misreporting behavior is entirely driven by the shape of the cost function C, and not by k or by utilities u_j. Figure 3 shows senders' misreporting behavior for different concavities of the misreporting cost function.

6.2 Application: Judicial Procedures

In a seminal paper, Shin (1998) compares the informative value of adversarial and inquisitorial procedures. Under the adversarial procedure, two parties with conflicting interests make their case to an uninformed decision maker. By contrast, the inquisitorial procedure requires the decision maker to adjudicate based only on her own acquired information. The question of which procedure allows the decision maker to take more informed decisions is of interest in a host of applications.

To answer this question, Shin (1998) studies a model of verifiable disclosure, where parties can either disclose or withhold information, but they cannot misrepresent it because the information is fully verifiable. In the adversarial procedure, the decision maker cannot rely on the information of the interested parties to secure full revelation because the two parties may be uninformed. In the inquisitorial procedure, the decision maker obtains with some probability an informative signal of the underlying evidence. In choosing between the two procedures, the decision maker must choose whether to obtain two pieces of biased information or one piece of unbiased information. Within this framework, Shin (1998) finds that the adversarial procedure is always superior to the inquisitorial procedure.

This sharp result raises a natural question: why are inquisitorial procedures so are often used in practice? On this point, Shin (1998) argues that the assumption of full verifiability might play a key role in determining the superiority of adversarial procedures, and that "potential violations of the verifiability assumption will be an important limiting factor in qualifying our findings in favor of the adversarial procedure" (Shin, 1998, p. 403).

In this section, I analyze the validity of this conjecture by using results derived in this paper. The framework introduced in Section 3 allows me to model the adversarial procedure for when information is not fully verifiable and parties can thus misrepresent it. The results derived in Section 4 suggest that under this procedure the decision maker cannot plausibly achieve receiver-efficiency and obtain the full information welfare W_{fi}. Moreover, if we accept that conditions (C) and (D) are plausible modeling assumptions, then the results in Section 5 indicate that the ex-ante equilibrium welfare of the decision maker is also strictly lower and bounded away from W_{fi}. To see this, notice that the expected payoff obtained by the decision maker in direct equilibria is bounded above by $\bar{W}_{dm}$, where[38]

$$\bar{W}_{dm} = \underbrace{\int_0^{\max\Theta} f(\theta)u_{dm}(\theta)d\theta}_{=W_{fi}} + \underbrace{\int_{\theta_1}^{0} f(\theta)u_{dm}(\theta)(1-\alpha_1(\theta))\alpha_2(\theta)d\theta}_{<0} < W_{fi}.$$

To model the inquisitorial procedure, I follow Shin (1998) in assuming that the decision maker obtains with probability q a potentially noisy signal σ of the realized state θ. It is straightforward to see that, under the inquisitorial procedure, the decision maker can obtain an expected payoff that, for high q and sufficiently precise σ, is arbitrarily close to W_{fi} and thus higher than $\bar{W}_{dm}$. Therefore, there is always a combination of parameters under which the inquisitorial procedure is superior to the adversarial procedure in that it yields more information to the decision maker. The conjecture of Shin (1998) is thus proved correct for any finite intensity of misreporting costs $k_j > 0$.

It is worth pointing out that, in addition to the verifiability assumption, there are other modeling differences between my setting and that of Shin (1998): first, I assume that the two senders are always perfectly informed about the realized state, while in Shin (1998) they may be uninformed or observe a noisy signal of the realized state; second, I consider a decision maker that is less informed than the two parties, while in Shin (1998) every player is, on average, equally informed.[39] These two differences give in my setting a

[38]The upper bound $\bar{W}_{dm}$ is obtained by assuming that the decision maker makes fewer mistakes than she would in a direct equilibrium: she mistakenly selects $\oplus$ only when $\theta \in (\theta_1, 0)$ and sender 2 reports truthfully while sender 1 misreports. Otherwise, she chooses the correct alternative. Therefore, $\bar{W}_{dm}$ is a strict upper bound of the ex-ante welfare obtained by the decision maker in direct equilibria.

[39]In Shin's model, the decision maker's ability to gather precise information in the inquisitorial system increases in the information possessed in expectation by the contending parties. This is because all players are assumed to be equally informed on average.

relative advantage to the adversarial procedure, and therefore add further force to the potential superiority of the inquisitorial procedure.[40]

7 Concluding Remarks

This article studies a multi-sender signaling model with two informed senders and one uninformed decision maker. Senders have perfectly correlated information, which they can misreport at a cost that is tied to the magnitude of misrepresentation. This setting covers a number of applications in economics and politics, including electoral campaigns, contested takeovers, lobbying, informative advertising, and judicial decision making.

I restrict attention to equilibria of this model where the decision maker's posterior beliefs satisfy a first-order stochastic dominance condition. Fully revealing, receiver-efficient, and pure-strategy equilibria exist, but they are not robust. I identify two natural restrictions on the decision maker's posterior beliefs under which equilibria are essentially unique, robust, and always exist. I dub equilibria that satisfy these two conditions as "direct equilibria."

Therefore, this paper provides a tractable and appealing approach to studying strategic communication from multiple senders with common information that is neither fully verifiable nor totally "cheap." As an application of direct equilibria, I study the informative value of judicial procedures and show that, when information is not fully verifiable, then inquisitorial systems may be superior to adversarial systems.

The transmission of information takes place in a qualitatively different way in direct equilibria of this model than in equilibria of related models of strategic communication. I conclude that the introduction of misreporting costs is not just a technical twist that adds an element of realism to the model; rather, it is an essential component to understanding how and how much information can be plausibly transmitted in this setting.[41]

[40] Moreover, in my model "withholding" information is not possible. In Shin's model, if parties are perfectly informed but cannot withhold information, then the decision maker can obtain full revelation under the adversarial procedure, making it superior to the inquisitorial procedure.

[41] Accounting for misreporting costs also allows one to perform comparative statics on such costs. For example, it allows one to study the effects of "fake news laws" or of technological advancements such as "deepfake videos" that affect senders' misreporting costs. This is left for future research.

A Appendix

Lemma 1. *In every perfect Bayesian equilibrium satisfying* (FOSD) *we have that, for* $j \in \{1, 2\}$, $\min S_j(\theta) \geq \theta$ *for* $\theta \geq \tau_j$ *and* $\max S_j(\theta) \leq \theta$ *otherwise.*

Proof of Lemma 1. Consider a PBE satisfying (FOSD) and consider a state $\theta \geq \tau_1$. For sender 1, every report $r_1 < \theta$ is dominated by truthful reporting because $C_1(r_1, \theta) > 0 = C_1(\theta, \theta)$ and (by (FOSD)) $U_{dm}(\theta, r_2) \geq U_{dm}(r_1, r_2)$ for every $r_2 \in R_2$. Therefore, it must be that $r_1 \notin S_1(\theta)$ for all $r_1 < \theta$ and $\theta \geq \tau_1$. A similar argument applies to sender 2 and to states $\theta \leq \tau_j$, $j \in \{1, 2\}$. □

A.1 Receiver-efficient and Pure-strategy Equilibria

Lemma 2. *An equilibrium is receiver-efficient if and only if it is in pure strategies.*

Proof of Lemma 2. Consider a pure-strategy equilibrium and suppose that it is not receiver-efficient, e.g., because $\beta(\rho_1(\theta'), \rho_2(\theta')) = \ominus$ for some $\theta' \geq 0$. In equilibrium, senders never engage in misreporting to implement their less preferred alternative with certainty, and therefore it must be that $\rho_1(\theta') = \theta'$. Posterior beliefs p must be such that $\beta(r_1, \rho_2(\theta')) = \ominus$ for all $r_1 \in (\underline{r}_1(\theta'), \bar{r}_1(\theta'))$, as otherwise sender 1 would have a profitable deviation. The pair of reports $(\theta', \rho_2(\theta'))$ can induce $\ominus$ only if $(\rho_1(\theta''), \rho_2(\theta'')) = (\theta', \rho_2(\theta'))$ for some $\theta'' < 0$. There is no $\theta \in [\tau_1, 0)$ such that sender 1 would misreport by delivering $r_1 = \theta' \geq 0$ to implement $\ominus$, and therefore it must be that $\theta'' < \tau_1$. Since there is always $r_1' \in (\underline{r}_1(\theta'), \theta')$ such that $C_1(r_1', \theta'') < C_1(\theta', \theta'')$ and $\beta(r_1', \rho_2(\theta'')) = \ominus$, sender 1 has a profitable deviation in state θ'', contradicting that there exists a pure-strategy equilibrium that is not receiver-efficient.

Now consider a REE and suppose that it is not in pure strategies, but that there is a state $\theta' \in \Theta$ and a sender $j \in \{1, 2\}$ such that $S_j(\theta') \supseteq \{r_j', r_j''\}$, with $r_j' \neq r_j''$. Since in a REE we have that $\beta(r_1', r_2') = \beta(r_1'', r_2'')$ for every $r_i', r_i'' \in S_i(\theta)$, $i \in \{1, 2\}$, it must be that $C_j(r_j', \theta') = C_j(r_j'', \theta')$. By Lemma 1, this is possible only if $r_j' = r_j''$, contradicting that there exists a REE that is not in pure strategies. □

Proposition 1. *There are no receiver-efficient equilibria with unprejudiced beliefs.*

Proof of Proposition 1. In a REE, senders play pure strategies (Lemma 2) and the decision maker always selects her preferred alternative *as if* she has complete information, that is, $\beta(\rho_1(\theta), \rho_2(\theta)) = \oplus$ for all $\theta \geq 0$ and $\beta(\rho_1(\theta), \rho_2(\theta)) = \ominus$ otherwise. Since misreporting is costly, senders report truthfully in states where their least preferred alternative is implemented: $\rho_2(\theta) = \theta$ for all $\theta \in [0, \tau_2]$ and $\rho_1(\theta) = \theta$ for all $\theta \in [\tau_1, 0)$. However, there are no REE where $\rho_j(\theta) = \theta$ for all $\theta \in [\tau_1, \tau_2]$ and $j \in \{1, 2\}$; for otherwise, there would always be a state $\theta \in (\tau_1, \tau_2)$ and an off path pair of reports (r_1, r_2), $r_1 \neq r_2$, such that a sender can profitably deviate from truthful reporting (see also Observation 2). Therefore, in every REE either sender 1 misreports in some state $\theta \in [0, \tau_2)$, or sender 2 misreports in some $\theta \in (\tau_1, 0]$, or both.

Consider now a REE where $\rho_1(\theta') \neq \theta'$ for some $\theta' \in [0, \tau_2)$. By Lemma 1, we have that $\rho_1(\theta') > \theta'$. To support the equilibrium, off path beliefs p must be such that $\beta(r_1, \theta') = \odot$ for all $r_1 \in [\theta', \rho_1(\theta'))$ and $\beta(\rho_1(\theta''), r_2) = \oplus$ for all $r_2 \in (\underline{r}_2(\theta''), \theta'']$ and $\theta'' \in [\theta', \tau_2)$. This implies that there must be an open set S of non-negative states such that $\rho_1(\theta''') \geq \rho_1(\theta') > \theta''' = \rho_2(\theta''')$ for all $\theta''' \in S$. It follows that, for every $\theta''' \in S$, the pair of reports (θ''', θ''') is off path. By Lemma 1, and since $\rho_2(\theta) = \theta$ for all $\theta \in [0, \tau_2]$ and $\rho_1(\theta) = \theta$ for all $\theta \in [\tau_1, 0)$, we have that posterior beliefs p are unprejudiced (Definition 2) only if $p(\theta|\theta''', \theta''') = 0$ for all $\theta < 0$. Therefore, unprejudiced beliefs imply that $\beta(\theta''', \theta''') = \oplus$, and therefore sender 1 can profitably deviate by reporting the truth in state $\theta''' \in S$. A similar argument applies to REE where $\rho_2(\theta') \neq \theta'$ for some $\theta' \in (\tau_1, 0]$. Therefore, there are no REE (and, by Lemma 2, no pure-strategy equilibria) with unprejudiced beliefs. □

Lemma 3. *If an equilibrium is ε-robust, then it has unprejudiced beliefs.*

Proof. Consider the posterior beliefs $p_{G,\varepsilon}$ that the strategies ϕ_j of a PBE (see Section 5 for the notation used to describe mixed strategies) induce in an ε-perturbed game for some distribution G and sequence ε^n, i.e.,

$$p_{G,\varepsilon}(\theta|r_1, r_2) = f(\theta)\frac{p_{G,\varepsilon}(r_1, r_2|\theta)}{p_{G,\varepsilon}(r_1, r_2)}$$
$$= \frac{f(\theta)\left[\varepsilon_1\varepsilon_2 g_1(r_1)g_2(r_2) + \varepsilon_1(1-\varepsilon_2)g_1(r_1)\phi_2(r_2, \theta) + (1-\varepsilon_1)\varepsilon_2 g_2(r_2)\phi_1(r_1, \theta)\right]}{\varepsilon_1\varepsilon_2 g_1(r_1)g_2(r_2) + \varepsilon_1(1-\varepsilon_2)g_1(r_1)\int_\Theta f(\theta)\phi_2(r_2, \theta)d\theta + (1-\varepsilon_1)\varepsilon_2 g_2(r_2)\int_\Theta f(\theta)\phi_1(r_1, \theta)d\theta}.$$

As $\varepsilon^n \to 0^+$ the event in which both reports are wrongly delivered or observed becomes negligible, and thus we have that $p_{G,\varepsilon} \to p_{G,0^+}$, where

$$p_{G,0^+}(\theta|r_1, r_2) = \frac{f(\theta)\left[\varepsilon_1 g_1(r_1)\phi_2(r_2, \theta) + \varepsilon_2 g_2(r_2)\phi_1(r_1, \theta)\right]}{\varepsilon_1 g_1(r_1)\int_\Theta f(\theta)\phi_2(r_2, \theta)d\theta + \varepsilon_2 g_2(r_2)\int_\Theta f(\theta)\phi_1(r_1, \theta)d\theta}. \tag{5}$$

By (5) we obtain that, for any distribution G with full support and any sequence $\varepsilon^n \to 0^+$, $p_{G,0^+}(\theta|r_1, r_2) > 0$ if and only if $\phi_j(r_j, \theta) > 0$ for some $j \in \{1, 2\}$. By Definition 7 (and hence even by Definition 2) we get that the limit beliefs $p_{G,0^+}$ are unprejudiced, and therefore every PBE of the game described in Section 3 (and hence every equilibrium) that is ε-robust is supported by unprejudiced beliefs.[42] □

[42]Notice that the proof of Lemma 3 readily extends to an n-sender version of the game, for any finite

A.2 Direct Equilibria

Lemma 4. *In a direct equilibrium, every report $r \in \hat{R}$ has a swing report $s(r) \in \hat{R}$ such that: (i) if $r \gtreqless 0$ then $s(r) \lesseqgtr 0$ and $s(0) = 0$; (ii) $s(s(r)) = r$; (iii) for every $r \in \hat{R}$, $\frac{ds(r)}{dr} < 0$; (iv) $s(\bar{r}_1(0)) = \underline{r}_2(0)$.*

Proof. Consider a report r_1 by sender 1 such that $r_1 \in (0, \bar{r}_1(0)]$. By (C) and (D) we obtain that $U_{dm}(r_1, \underline{r}_2(0)) < 0 < U_{dm}(r_1, 0)$, and therefore there exists $r_2 \in [\underline{r}_2(0), 0)$ such that $U_{dm}(r_1, r_2) = 0$. Thus, $r_2 = s(r_1)$. A similar argument holds for a report $r_2 \in [\underline{r}_2(0), 0)$. It follows that, for every $r \in \hat{R}$, there exists $s(r) \in \hat{R}$ such that if $r > 0$ then $s(r) < 0$, and if $r < 0$ then $s(r) > 0$. From (C) and Definition 5 we obtain that $s(0) = 0$ and $s(\bar{r}_1(0)) = \underline{r}_2(0)$. From Definition 5 and point (i) we get that if $r' = s(r)$ then $r = s(r')$, and therefore $s(s(r)) = r$. By applying the implicit function theorem and (D) to $s(r)$, we obtain that for every $r \in \hat{R}$, $\frac{ds(r)}{dr} < 0$. □

Lemma A.1. *In a direct equilibrium, truthful cutoffs are such that $\theta_1 < 0 < \theta_2$ and $(\theta_1, \theta_2) \subset [\tau_1, \tau_2] \cap \hat{R}$.*

Proof. By Lemma 4 we have that $s(\bar{r}_1(0)) = \underline{r}_2(0) < 0$ and, for every $r \in \hat{R}$, $ds(r)/dr < 0$. Moreover, $d\underline{r}_2(\theta)/d\theta > 0$ and thus $\underline{r}_2(\theta) > \underline{r}_2(0)$ for every $\theta > 0$. Since $s(0) = 0$, there is a state $\theta' \in (0, \bar{r}_1(0))$ such that $s(\theta') = \underline{r}_2(\theta')$. From Definition 5, we obtain that $\theta' = \theta_2 \in (0, \bar{r}_1(0))$. Similarly, we get that $\theta_1 \in (\underline{r}_2(0), 0)$. Since $\bar{r}_1(\tau_1) = \tau_1 < 0$ and $\underline{r}_2(\tau_2) = \tau_2 > 0$, it follows from Definition 6 that $(\theta_1, \theta_2) \subset [\tau_1, \tau_2]$. □

Lemma 5. *In a direct equilibrium, $S_j(\theta) = \{\theta\}$ for all $\theta \notin (\theta_1, \theta_2)$, and $|S_j(\theta)| > 1$ for every $\theta \in (\theta_1, \theta_2)$, $j \in \{1, 2\}$.*

Proof. I begin by proving that $S_j(\theta) = \{\theta\}$ for all $\theta \notin (\theta_1, \theta_2)$. Consider a DE and a state $\theta \geq \theta_2$. Since by Lemma 1 we have that $\min S_1(\theta) \geq \theta \geq \theta_2$, it must be that $S_2(\theta) = \{\theta\}$ as $s(r_1) \leq \underline{r}_2(\theta)$ for every $r_1 \in S_1(\theta)$. Since $\beta(\theta, \theta) = \oplus$, sender 1 best replies to $r_2 = \theta$ with $r_1 = \theta$ and therefore $S_1(\theta) = \{\theta\}$ as well. A similar argument applies to states $\theta \leq \theta_1$, completing the first part of the proof. Note that when $\theta = \theta_1$, sender 1 is actually indifferent between reporting θ_1 and $\bar{r}_1(\theta_1)$. Since this is a measure zero event, which is irrelevant to the analysis that follows, I will consider only the case where $S_1(\theta_1) = \{\theta_1\}$, without any loss of generality.

I turn now to prove that $S_j(\theta)$ contains more than one element for every $\theta \in (\theta_1, \theta_2)$. Suppose by way of contradiction that $S_1(\theta) = \{r_1\}$ for some $\theta \in (\theta_1, \theta_2)$. By Lemma 1, we have that $r_1 \geq \theta$. Consider first the case where $\theta \leq r_1 < 0$. In a DE, sender 2 best replies

$n \geq 2$. In particular, given a profile of reports $(r_1, \ldots, r_n)$ and a set of senders $N = \{1, \ldots, n\}$, we have that $p_{G,0^+}(\theta | r_1, \ldots, r_n) > 0$ if and only if $\phi_j(r_j, \theta) > 0$ for $n - 1$ senders. This is consistent with the idea behind unprejudiced beliefs, where the decision maker conjectures that deviations are individual.

to $r_1 \in [\theta, 0)$ with $r_2 = \theta = S_2(\theta)$ because, by (C) and (D), we get $\beta(r_1, \theta) = \odot$. However, sender 1 can profitably deviate from the prescribed strategy by delivering $r_1' = s(\theta)$, where $0 < s(\theta) < \bar{r}_1(\theta)$ (Lemmata 4 and A.1), contradicting that $S_1(\theta) = \{r_1\}$. Consider next the case where $r_1 \geq 0$ and $r_1 \geq \theta$. If $s(r_1) \leq \underline{r}_2(\theta)$, then it must be that $S_2(\theta) = \{\theta\}$. By Definition 6 and Lemma 4 we have that $\underline{r}_2(\theta) < 0$ and $r_1 \geq s(\underline{r}_2(\theta)) > 0$. Since $\underline{r}_2(\theta) < \theta$, sender 1 can profitably deviate from the prescribed strategy by reporting either $r_1' = s(\theta) \in (0, r_1)$ if $\theta < 0$, or $r_1' = \theta$ if $\theta \geq 0$, as in both cases we get that $\beta(r_1', \theta) = \oplus$ and $C_1(r_1', \theta) < C_1(r_1, \theta)$. If instead $s(r_1) > \underline{r}_2(\theta)$, then sender 2 must be delivering some report $r_2' \in (\underline{r}_2(\theta), s(r_1))$. Therefore, if $r_1 > \theta$, then sender 1 is strictly better off reporting θ rather than r_1 because $\beta(\theta, r_2') = \beta(r_1, r_2') = \odot$ and $C_1(r_1, \theta) > 0 = C_1(\theta, \theta)$. If instead $r_1 = \theta$, then $\theta \geq 0$ and since $\underline{r}_2(\theta) \geq \underline{r}_2(0)$ we have that $s(r_2') \leq \bar{r}_1(\theta)$ (Lemma 4). In this case, sender 1 can profitably deviate from the prescribed strategy by reporting $r_1' = s(r_2')$. Similar arguments apply to $S_2(\theta) = \{r_2\}$, completing the proof. □

Lemma A.2. *In a direct equilibrium, for every $\theta \in (\theta_1, \theta_2)$, supports $S_j(\theta)$ are such that*

$$\max S_1(\theta) \leq \min \{\bar{r}_1(0), \bar{r}_1(\theta), s(\underline{r}_2(\theta))\},$$

$$\min S_2(\theta) \geq \max \{\underline{r}_2(0), \underline{r}_2(\theta), s(\bar{r}_1(\theta))\}.$$

Proof. Consider a DE and a $\theta \in (\theta_1, \theta_2)$. By the definition of reach (equations (1) and (2)) every $r_1 > \bar{r}_1(\theta)$ is strictly dominated by truthful reporting, and thus $\max S_1(\theta) \leq \bar{r}_1(\theta)$. Similarly, we obtain that $\min S_2(\theta) \geq \underline{r}_2(\theta)$ and therefore by (D) and by Definition 5 every $r_1 > s(\underline{r}_2(\theta))$ is dominated by $r_1' = s(\underline{r}_2(\theta))$ and every $r_2 < s(\bar{r}_1(\theta))$ is dominated by $r_2' = s(\bar{r}_1(\theta))$. Thus, $\max S_1(\theta) \leq s(\underline{r}_2(\theta))$ and $\min S_2(\theta) \geq s(\bar{r}_1(\theta))$. For every $\theta \in [0, \theta_2)$ we have $\bar{r}_1(\theta) \geq \bar{r}_1(0)$ and $\underline{r}_2(\theta) \geq \underline{r}_2(0)$, and therefore $\min S_2(\theta) \geq \underline{r}_2(0)$. Since $s(\underline{r}_2(0)) = \bar{r}_1(0)$ (Lemma 4), it follows from (D) and Definition 5 that $s(r_2) \leq \bar{r}_1(0)$ for every $r_2 \in S_2(\theta)$, and therefore $\max S_1(\theta) \leq \bar{r}_1(0)$. Similarly, we obtain that $\min S_2(\theta) \geq s(\bar{r}_1(0))$ for every $\theta \in (\theta_1, 0)$. □

Lemma A.3. *In a direct equilibrium, $r_2 \notin S_2(\theta)$ for every $r_2 \in (s(\min S_1(\theta)), \theta)$ and $\theta > 0$, and $r_1 \notin S_1(\theta)$ for every $r_1 \in (\theta, s(\max S_2(\theta)))$ and $\theta < 0$.*

Proof. Consider $\theta \in (0, \theta_2)$. By Lemmata 1 and 4 we have that $s(\min S_1(\theta)) < 0$, and by Definition 5 we have that $\beta(r_1, r_2) = \oplus$ for every $r_1 \in S_1(\theta)$ and $r_2 \in (s(\min S_1(\theta)), \theta)$. Therefore, for sender 2 every $r_2 \in (s(\min S_1(\theta)), \theta)$ is strictly dominated by truthful reporting, and therefore $r_2 \notin S_2(\theta)$. A similar argument applies to sender 1 for $\theta \in (\theta_1, 0)$ and Lemma 5 proves the case of $\theta \notin (\theta_1, \theta_2)$, completing the proof. □

Lemma A.4. *In a direct equilibrium, for every $\theta \in (\theta_1, \theta_2)$, reports $r_1 \in (\min S_1(\theta), \max S_1(\theta))$ have $s(r_1) > \underline{r}_2(\theta)$, and reports $r_2 \in (\min S_2(\theta), \max S_2(\theta))$ have $s(r_2) < \bar{r}_1(\theta)$.*

Proof. Suppose not, and consider $r_1' \in (\min S_1(\theta), \max S_1(\theta))$ for some $\theta \in (\theta_1, \theta_2)$ such that $s(r_1') < \underline{r}_2(\theta)$. By Definition 6 we have that $\underline{r}_2(\theta) < 0$ and by Lemma 4 we have that $s(\underline{r}_2(\theta)) < r_1'$. This is in contradiction to Lemma A.2, which states that $\max S_1(\theta) \leq s(\underline{r}_2(\theta))$. A similar argument holds for reports $r_2 \in (\min S_2(\theta), \max S_2(\theta))$, completing the proof. □

Lemma A.5. *In a direct equilibrium,* $\alpha_j(r_j, \theta) = 0$ *for all* $r_j \in (\min S_j(\theta), \max S_j(\theta))$, $j \in \{1, 2\}$, *and* $\theta \in (\theta_1, \theta_2)$.

Proof. Consider $\theta \in (\theta_1, \theta_2)$ and suppose that there is a DE ω where sender 1's strategy $\phi_1(\theta)$ has an atom $\alpha_1(r_1', \theta) > 0$ in some report $r_1' \in (\min S_1(\theta), \max S_1(\theta))$. By Lemma A.4 we have that $s(r_1') > \underline{r}_2(\theta)$. The expected payoff of sender 2, $W_2^\omega(\cdot, \theta)$, is discontinuous around $r_2 = s(r_1')$ and therefore it must be that, for some $\epsilon > 0$ small enough, $(s(r_1'), s(r_1') + \epsilon) \cap S_2(\theta) = \varnothing$. Therefore, there exists an $\epsilon' > 0$ small enough such that $W_1^\omega(r_1'', \theta) > W_1^\omega(r_1', \theta)$ for some $r_1'' \in (s\left(s(r_1') + \epsilon'\right), r_1')$, where by Lemma 4 we have that $s\left(s(r_1') + \epsilon'\right) < r_1'$, thus contradicting that this is an equilibrium. A similar argument applies to atoms in sender 2's strategy, completing the proof. □

Lemma A.6. *In a direct equilibrium,* $\min S_1(\theta) = \theta$ *for all* $\theta \geq 0$, *and* $\max S_2(\theta) = \theta$ *for all* $\theta \leq 0$.

Proof. Consider a DE ω and $\theta \geq 0$. By Lemma 1, it must be that $\min S_1(\theta) \geq \theta$. Suppose by way of contradiction that $\min S_1(\theta) > \theta$. By Lemma 5 it has to be that $\theta < \theta_2$ and by Lemma A.3 we obtain that $S_2(\theta) \cap (s(\min S_1(\theta)), \theta) = \varnothing$. Therefore, unless sender 2's strategy has an atom $\alpha_2(s(\min S_1(\theta)), \theta) > 0$, we have that $\Phi_2(s(\min S_1(\theta)), \theta) = \Phi_2(s(\theta), \theta)$. However, since $\beta(r_1, s(\min S_1(\theta))) = \oplus$ for all $r_1 \in S_1(\theta)$ and $C_2(s(\min S_1(\theta)), \theta) > 0$, it must be that $\alpha_2(s(\min S_1(\theta)), \theta) = 0$ as $s(\min S_1(\theta))$ is strictly dominated by $r_2 = \theta$. Hence we have that, for some $\epsilon > 0$, $W_1^\omega(\theta, \theta) > W_1^\omega(r_1, \theta)$ for every $r_1 \in [\min S_1(\theta), \min S_1(\theta) + \epsilon) \cap S_1(\theta)$, contradicting that there can be a DE with $\min S_1(\theta) > \theta$ for a $\theta \geq 0$. A similar argument holds for sender 2 and $\theta \leq 0$, completing the proof. □

Lemma A.7. *In a direct equilibrium,* $|S_j(\theta) \setminus \{\theta\}| > 1$ *for every* $\theta \in (\theta_1, \theta_2)$ *and* $j \in \{1, 2\}$.

Proof. Consider a DE ω and a state $\theta \in [0, \theta_2)$. By Lemma A.6 we have that $\min S_1(\theta) = \theta$, and by Lemma 5 we have that $|S_1(\theta)| > 1$. Suppose by way of contradiction that $S_1(\theta) \setminus \{\theta\} = \{r_1\}$ for some $r_1 > 0$. Since $C_1(r_1, \theta) > 0$, it must be that, in equilibrium, r_1 induces $\oplus$ with strictly higher probability than truthful reporting. This implies that there is some $r_2 \in [s(r_1), s(\theta))$ in the support of sender 2's strategy, $r_2 \in S_2(\theta)$. Since reports that are further away from the realized state are more costly, it must be that

$\alpha_2(r_2', \theta) > 0$ for some $r_2' \in [s(r_1), s(\theta))$, and $\phi_2(r_2, \theta) = 0$ for all $r_2 \in [s(r_1), r_2')$. But then $W_1^\omega(s(r_2'), \theta) > W_1^\omega(r_1, \theta)$, contradicting that this is an equilibrium.

Consider now the case where $\theta \in (\theta_1, 0)$ and suppose again that $S_1(\theta) \setminus \{\theta\} = \{r_1\}$. By Lemma 5, we have that $|S_j(\theta)| > 1$ for $j \in \{1, 2\}$, and therefore $\min S_1(\theta) = \theta$. By Lemmata A.2 and A.3 we have that $r_1 \geq s(\theta) > 0$ and $\max S_2(\theta) = \theta$. If $r_1 = s(\theta)$, then sender 2 can profitably deviate from the prescribed strategy by always reporting $\theta - \epsilon$ for some $\epsilon > 0$ small enough. If instead $r_1 > s(\theta)$, then it must be that $S_2(\theta) \cap [s(r_1), \theta) = \varnothing$ as every $r_2 \in [s(r_1), \theta)$ would be strictly dominated by truthful reporting. Since $|S_2(\theta)| > 1$, there must be some $r_2 < s(r_1)$ such that $r_2 \in S_2(\theta)$. Therefore, $W_1^\omega(s(\theta), \theta) > W_1^\omega(r_1, \theta)$, contradicting that this is an equilibrium. A similar argument applies to $S_2(\theta) \setminus \{\theta\}$, completing the proof. □

Lemma 6. *In a direct equilibrium, $S_j(\theta) \setminus \{\theta\}$ is convex for all $\theta \in (\theta_1, \theta_2)$ and $j \in \{1, 2\}$.*

Proof. Consider a DE ω and a state $\theta \in (\theta_1, \theta_2)$. By Lemma A.7 we have that $|S_j(\theta) \setminus \{\theta\}| > 1$, $j \in \{1, 2\}$. Suppose by way of contradiction that $S_1(\theta) \setminus \{\theta\}$ is not convex, but instead there are two reports $r_1', r_1'' \in S_1(\theta) \setminus \{\theta\}$ with $r_1' < r_1''$, such that $r_1 \notin S_1(\theta) \setminus \{\theta\}$ for every $r_1 \in (r_1', r_1'')$. By Lemmata 1, 4, and A.3 we have that $r_1' > 0$, $r_1' \geq s(\theta)$, and $s(r_1'') < s(r_1') < 0$. Since $C_1(r_1'', \theta) > C_1(r_1', \theta)$ and $\frac{dC_j(r,\theta)}{dr} > 0$ for every $r > \theta$, it must be that every report $r_1 \geq r_1''$ such that $\phi_1(r_1, \theta) > 0$ induces the implementation of alternative $\oplus$ with strictly higher probability than every report $r_1''' \leq r_1'$ such that $\phi_1(r_1''', \theta) > 0$. This is possible only if $r_2 \in S_2(\theta)$ for some $r_2 \in [s(r_1''), s(r_1')]$. Since $\Phi_1(r_1, \theta)$ is constant for all $r_1 \in (r_1', r_1'')$, it must be that sender 2's strategy has an atom $\alpha_2(r_2, \theta) > 0$ in some $r_2 \in (s(r_1''), s(r_1')]$, and $\phi_2(r_2', \theta) = 0$ for all $r_2' \in [s(r_1''), s(r_1')]$ such that $r_2' \neq r_2$. However, for some $\epsilon > 0$ small enough we have that $W_1^\omega(s(r_2), \theta) > W_1^\omega(r_1, \theta)$ for all $r_1 \in [r_1'', r_1'' + \epsilon)$ such that $r_1 \in S_1(\theta)$, where $s(r_2) < r_1''$, contradicting that this is an equilibrium. A similar argument applies to $S_2(\theta) \setminus \{\theta\}$, completing the proof. □

Lemma 7. *In a direct equilibrium, the strategies $\phi_j(\theta)$ have no atoms in $S_j(\theta) \setminus \{\theta\}$ for every $\theta \in (\theta_1, \theta_2)$ and $j \in \{1, 2\}$.*

Proof. Consider a DE ω. Lemma 5 shows that $|S_j(\theta)| > 1$ for all $\theta \in (\theta_1, \theta_2)$ and Lemma A.5 shows that $\phi_j(\theta)$ has no atoms in $(\min S_j(\theta), \max S_j(\theta))$. Consider a state $\theta \in (\theta_1, \theta_2)$, and suppose that $\phi_1(\theta)$ has an atom in $\max S_1(\theta)$, i.e., $\alpha_1(\max S_1(\theta), \theta) > 0$. By Lemma A.2, we have that $\max S_1(\theta) \leq \min\{s(\underline{r}_2(\theta)), \bar{r}_1(\theta)\}$ and $\min S_2(\theta) \geq \max\{\underline{r}_2(\theta), s(\bar{r}_1(\theta))\}$. If $\min S_2(\theta) > s(\max S_1(\theta))$, then $W_1^\omega(r_1, \theta) > W_1^\omega(\max S_1(\theta), \theta)$ for any $r_1 \in [s(\min S_2(\theta)), \max S_1(\theta))$. If $\min S_2(\theta) = s(\max S_1(\theta))$, then, since sender 2's expected payoff $W_2^\omega(\cdot, \theta)$ is discontinuous at $r_2 = s(\max S_1(\theta))$, it must be that $r_2' \notin S_2(\theta)$ for all $r_2' \in (s(\max S_1(\theta)), s(\max S_1(\theta)) + \epsilon)$ and some small $\epsilon > 0$. Otherwise, for some $\epsilon' > 0$, $W_2^\omega(s(\max S_2(\theta)) - \epsilon', \theta) > W_2^\omega(r_2', \theta)$ for any $r_2' \in [s(\max S_1(\theta)), s(\max S_1(\theta)) + \epsilon]$.

However, this would contradict either Lemma A.7 or Lemma 6, and therefore it would not be possible in a DE.

Suppose now that $\phi_1(\theta)$ has an atom in $\min S_1(\theta)$, i.e., $\alpha_1(\min S_1(\theta), \theta) > 0$. By Lemma A.6, if $\theta \geq 0$ then $\min S_1(\theta) = \theta$, and therefore let us suppose that $\theta \in (\theta_1, 0)$ and that $\min S_1(\theta) > \theta$ when $\theta < 0$. By Lemmata 4, A.3, and A.6 we have that $\min S_1(\theta) \geq s(\theta) > 0$. If $\min S_1(\theta) = s(\theta)$, then it must be that $\phi_2(\theta, \theta) = 0$, as $W_2^\omega(\theta - \epsilon, \theta) > W_2^\omega(\theta, \theta)$ for some $\epsilon > 0$ small enough. But then the atom in $\min S_1(\theta)$ would be strictly dominated by truthful reporting as $C_1(s(\theta), \theta) > 0$ and $\beta(s(\theta), r_2) = \odot$ for every $r_2 \in S_2(\theta)$, contradicting that this is an equilibrium. Consider now the case where $\min S_1(\theta) > s(\theta)$. We have that $\Phi_1(r_1, \theta) = 0$ for every $r_1 < \min S_1(\theta)$, and by Lemma 4 we have that $s(\min S_1(\theta)) < \theta$. Therefore, it must be that $\phi_2(r_2, \theta) = 0$ for every $r_2 \in [s(\min S_1(\theta)), \theta)$. However this implies that, for sender 1, $\min S_1(\theta)$ is dominated by $s(\theta)$, contradicting that this can be an equilibrium. Similar arguments hold for atoms $\alpha_2(r_2, \theta)$ for $r_2 \in S_2(\theta) \setminus \{\theta\}$, completing the proof. □

Proposition 2. *In a direct equilibrium, for every $\theta \in (\theta_1, \theta_2)$ and $i, j \in \{1, 2\}$ with $i \neq j$, sender j delivers report $r_j \in S_j(\theta) \setminus \{\theta\}$ according to*

$$\psi_j(r_j, \theta) = \frac{k_i}{-u_i(\theta)} \frac{dC_i(s(r_j), \theta)}{dr_j}.$$

Proof. Consider a DE and a state $\theta \in (\theta_1, \theta_2)$. Given strategy $\phi_1(\theta)$, sender 2 gets an expected utility of $W_2^\omega(r_2, \theta) = (1 - \Phi_1(s(r_2), \theta))u_2(\theta) - k_2 C_2(r_2, \theta)$ from delivering $r_2 \in S_2(\theta) \setminus \{\theta\}$. By Lemmata 6 and 7 we have that $S_j(\theta) \setminus \{\theta\}$ is convex and atomless. By Lemmata 1, A.1, and A.2, we have that $S_j(\theta) \subset \hat{R}$ for all $\theta \in (\theta_1, \theta_2)$, and therefore by Lemma 4 we have that $\frac{ds(r)}{dr} < 0$ for all $r_j \in S_j(\theta)$. Therefore, we can set $\frac{dW_2^\omega(r_2,\theta)}{dr_2}|_{r_2 \in S_2(\theta) \setminus \{\theta\}} = 0$, and since $\phi_j(r_j, \theta) = \psi_j(r_j, \theta)$ for all $r_j \in S_j(\theta) \setminus \{\theta\}$ (Lemma 7), we obtain the partial pdf $\psi_1(s(r_2), \theta) = \frac{k_2}{-u_2(\theta)} \frac{dC_2(r_2,\theta)}{dr_2} \frac{dr_2}{ds(r_2)} = \frac{k_2}{-u_2(\theta)} \frac{dC_2(r_2,\theta)}{ds(r_2)}$. By replacing $r_1 = s(r_2)$ we obtain that $\psi_1(r_1, \theta) = \frac{k_2}{-u_2(\theta)} \frac{dC_2(s(r_1),\theta)}{dr_1}$ for $r_1 \in S_1(\theta) \setminus \{\theta\}$. Similarly, we obtain that for $r_2 \in S_2(\theta) \setminus \{\theta\}$, $\psi_2(r_2, \theta) = \frac{k_1}{-u_1(\theta)} \frac{dC_1(s(r_2),\theta)}{dr_2}$. □

Lemma A.8. *In a direct equilibrium, $S_1(\theta)$ is convex for all $\theta \geq 0$ and $S_2(\theta)$ is convex for all $\theta \leq 0$.*

Proof. Consider a DE ω and suppose by way of contradiction that $S_1(\theta)$ is not convex for some $\theta \in [0, \theta_2)$. By Lemma A.6 we have that $\min S_1(\theta) = \theta$, and by Lemma 6 we have that $S_1(\theta) \setminus \{\theta\}$ is convex. Therefore, it must be that $\min S_1(\theta) \setminus \{\theta\} > \theta$ and $\phi_1(r_1, \theta) = 0$ for every $r_1 \in (\theta, \min S_1(\theta) \setminus \{\theta\})$. In equilibrium, every $r_1 > \min S_1(\theta) \setminus \{\theta\}$ such that $\phi_1(r_1, \theta) > 0$ must yield the implementation of alternative $\oplus$ with strictly higher probability than truthful reporting, as $C_1(r_1, \theta) > 0$. This is possible only if $\phi_2(r_2, \theta) > 0$ for some $r_2 \in [s(\min S_1(\theta) \setminus \{\theta\}), s(\theta))$. However, for some $\epsilon > 0$ small enough, it must be that $\phi_2(r_2', \theta) = 0$ for every $r_2' \in [s(\min S_1(\theta) \setminus \{\theta\}), s(\theta) - \epsilon)$, as every such report r_2' is dominated by reporting $s(\theta) - \epsilon$. Therefore, there exists an $\epsilon' > 0$ such that $W_1^\omega(s(s(\theta)-\epsilon), \theta) > W_1^\omega(r_1', \theta)$ for all $r_1' \in [\min S_1(\theta) \setminus \{\theta\}, \min S_1(\theta) \setminus \{\theta\} + \epsilon')$, contradicting that this is an equilibrium. Lemma 5 considers the case where $\theta \notin (\theta_1, \theta_2)$, and a similar argument applies to states $\theta \leq 0$ and support $S_2(\theta)$. □

Proposition 3. *In a direct equilibrium, for every state $\theta \in (\theta_1, \theta_2)$, supports $S_j(\theta)$ are*

$$S_1(\theta) = \{\theta\} \cup [\max\{s(\theta), \theta\}, \min\{\bar{r}_1(\theta), s(\underline{r}_2(\theta))\}],$$

$$S_2(\theta) = \{\theta\} \cup [\max\{\underline{r}_2(\theta), s(\bar{r}_1(\theta))\}, \min\{s(\theta), \theta\}].$$

Proof. Consider a direct equilibrium and a state $\theta \in [0, \theta_2)$. Since for every $\theta \geq 0$ we have that $\theta \in S_1(\theta)$ (Lemma A.6) and both sets $S_1(\theta)$ and $S_1(\theta) \setminus \{\theta\}$ are convex (Lemmata A.8 and 6), it follows that $S_1(\theta) = [\theta, \max S_1(\theta)]$.

Lemma 6 shows that also $S_2(\theta) \setminus \{\theta\}$ is convex. Since $\min S_1(\theta) = \theta$, Lemma A.3 says that when $\theta > 0$ we have that $\phi_2(r_2, \theta) = 0$ for all $r_2 \in (s(\theta), \theta)$, and therefore $\max S_2(\theta) \setminus \{\theta\} \leq s(\theta)$ for all $\theta \in (0, \theta_2)$. Suppose that $\max S_2(\theta) \setminus \{\theta\} < s(\theta)$. In this case, it must be that $\phi_1(r_1, \theta) = 0$ for every $r_1 \in (\theta, s(\max S_2(\theta) \setminus \{\theta\}))$, as for sender 1 every such a report r_1 would be dominated by truthful reporting. This is in contradiction to Lemma A.8, and therefore it must be that $\max S_2(\theta) \setminus \{\theta\} = s(\theta)$ for every $\theta \in (0, \theta_2)$. When $\theta = 0$, we have that $\max S_2(0) = 0$ (Lemma A.6).

Lemma 7 shows that $\phi_2(r_2, \theta)$ is atomless in $S_2(\theta) \setminus \{\theta\}$. Therefore, for $r_2 \in S_2(\theta) \setminus \{\theta\}$ we have that $\Phi_2(r_2, \theta) = \Psi_2(r_2, \theta)$, and therefore by using Proposition 2 we can write

$$\Phi_2(r_2, \theta)|_{r_2 \in S_2(\theta) \setminus \{\theta\}} = \int_{\min S_2(\theta)}^{r_2} \psi_2(r, \theta) dr = \frac{k_1}{u_1(\theta)} [C_1(s(\min S_2(\theta)), \theta) - C_1(s(r_2), \theta)].$$

The probability that sender 2 misreports information in state $\theta \in (0, \theta_2)$ is therefore

$$\Phi_2(s(\theta), \theta) = \frac{k_1}{u_1(\theta)} C_1(s(\min S_2(\theta)), \theta). \tag{6}$$

Since $\min S_2(\theta) \geq \underline{r}_2(\theta)$ (Lemma A.2), it follows from Lemma 4 that, for every $\theta \in (0, \theta_2)$, $s(\min S_2(\theta)) < \bar{r}_1(\theta)$. Lemma A.7 shows that the set $S_2(\theta) \setminus \{\theta\}$ is not a singleton, and since $\max S_2(\theta) \setminus \{\theta\} \leq s(\theta)$ it must be that $\min S_2(\theta) < s(\theta)$. Therefore by Lemma 4 we have that $s(\min S_2(\theta)) \in (\theta, \bar{r}_1(\theta))$ for $\theta \in (0, \theta_2)$. Finally, by the definition of upper reach we get that $C_1(\bar{r}_1(\theta), \theta) = u_1(\theta)/k_1$, and $C_1(r_1, \theta) < u_1(\theta)/k_1$ for every $r_1 \in [\theta, \bar{r}_1(\theta))$. Therefore, it follows that $\Phi_2(s(\theta), \theta) \in (0, 1)$ for every $\theta \in (0, \theta_2)$. By using $s(s(r)) = r$ and $s(0) = 0$ (Lemma 4), when $\theta = 0$ we obtain that $\Phi_2(s(0), 0) = 1$ only if $\min S_2(0) = \underline{r}_2(0)$.

The above argument shows that $\theta \in S_2(\theta)$ and that $\phi_2(\theta)$ has an atom in $r_2 = \theta$

of size $\alpha_2(\theta) = 1 - \Phi_2(s(\theta), \theta)$. Lemma 1 implies that every pair of on path reports (r_1, r_2) such that $r_j \geq 0$, $j \in \{1, 2\}$, must yield $\beta(r_1, r_2) = \oplus$. Therefore, by reporting truthfully when $\theta \geq 0$, sender 2 obtains a payoff of $W_2^\omega(\theta, \theta) = u_2(\theta)$. It must be that $\max S_1(\theta) \leq s(\min S_2(\theta))$, as otherwise every report $r_1 > s(\min S_2(\theta))$ would be dominated by $s(\min S_2(\theta))$. Since $\phi_1(\theta)$ has no atom in $s(\min S_2(\theta)) > \theta$ (Lemma 7), by reporting $r_2 = \min S_2(\theta)$ sender 2 (almost) always induces the selection of his preferred alternative $\ominus$, and gets an expected payoff of $W_2^\omega(\min S_2(\theta), \theta) = -k_2 C_2(\min S_2(\theta), \theta)$.

In equilibrium, each sender must receive the same expected payoff from delivering any report that is in the support of its own strategy. Since by the definition of lower reach we obtain $C_2(\underline{r}_2(\theta), \theta) = -u_2(\theta)/k_2$, it follows that $W_2^\omega(\min S_2(\theta), \theta) = u_2(\theta) = W_2^\omega(\theta, \theta)$ only if $\min S_2(\theta) = \underline{r}_2(\theta)$. Therefore, for a $\theta \in [0, \theta_2)$, we have that $S_2(\theta) = [\underline{r}_2(\theta), s(\theta)] \cup \{\theta\}$. It also follows that $\max S_1(\theta) = s(\underline{r}_2(\theta))$: if $\max S_1(\theta) < s(\underline{r}_2(\theta))$, then $\underline{r}_2(\theta) < s(\max S_1(\theta))$ and every $r_2 < s(\max S_1(\theta))$ would be strictly dominated by $s(\max S_1(\theta))$. Thus, $S_1(\theta) = [\theta, s(\underline{r}_2(\theta))]$. Similar arguments apply to the case where $\theta \in (\theta_1, 0)$, completing the proof. □

Proposition 4. *In a direct equilibrium, for every state $\theta \in (\theta_1, \theta_2)$, strategies $\phi_j(\theta)$ have an atom at $r_j = \theta$ of size $\alpha_j(\theta)$, where*

$$\alpha_1(\theta) = \begin{cases} \frac{k_2}{-u_2(\theta)} C_2\left(s(\theta), \theta\right) & \text{if } \theta \in [0, \theta_2) \\ 1 - \frac{k_2}{-u_2(\theta)} C_2\left(s(\bar{r}_1(\theta)), \theta\right) & \text{if } \theta \in (\theta_1, 0], \end{cases}$$

$$\alpha_2(\theta) = \begin{cases} 1 - \frac{k_1}{u_1(\theta)} C_1\left(s(\underline{r}_2(\theta)), \theta\right) & \text{if } \theta \in [0, \theta_2) \\ \frac{k_1}{u_1(\theta)} C_1\left(s(\theta), \theta\right) & \text{if } \theta \in (\theta_1, 0]. \end{cases}$$

Proof. Consider a direct equilibrium and a state $\theta \in [0, \theta_2)$. The proof of Proposition 3 shows that $\phi_2(\theta)$ has an atom in $r_2 = \theta$ of size $\alpha_2(\theta) = 1 - \Phi_2(s(\theta), \theta)$. From equation (6) and given $\min S_2(\theta) = \underline{r}_2(\theta)$, we obtain that

$$\alpha_2(\theta) = 1 - \frac{k_1}{u_1(\theta)} C_1(s(\underline{r}_2(\theta)), \theta).$$

By Lemma 7, sender 1's strategy $\phi_1(\theta)$ admits an atom only in $\min S_1(\theta) = \theta$. Therefore, we can use Proposition 2 to write

$$\begin{aligned} \Phi_1(r_1, \theta)|_{r_1 \in S_1(\theta)} &= \alpha_1(\theta) + \int_\theta^{r_1} \psi_1(r, \theta) dr \\ &= \alpha_1(\theta) + \frac{k_2}{-u_2(\theta)} \left[C_2(s(r_1), \theta) - C_2(s(\theta), \theta)\right]. \end{aligned}$$

Since $\max S_1(\theta) = s(\underline{r}_2(\theta))$, it must be that $\Phi_1(s(\underline{r}_2(\theta)), \theta) = 1$. By using $s(s(\underline{r}_2(\theta))) = \underline{r}_2(\theta)$ (Lemma 4) and given that from the definition of lower reach we obtain $C_2(\underline{r}_2(\theta), \theta) =$

$-u_2(\theta)/k_2$, we have that

$$\Phi_1(s(\underline{r}_2(\theta)),\theta) = \alpha_1(\theta) + \frac{k_2}{-u_2(\theta)}\left[C_2(s(s(\underline{r}_2(\theta))),\theta) - C_2(s(\theta),\theta)\right]$$
$$= \alpha_1(\theta) + 1 - \frac{k_2}{-u_2(\theta)}C_2(s(\theta),\theta) = 1,$$

from which we obtain that

$$\alpha_1(\theta) = \frac{k_2}{-u_2(\theta)}C_2(s(\theta),\theta).$$

A similar procedure can be used for $\theta \in (\theta_1, 0)$, completing the proof. □

Lemma A.9. *In a direct equilibrium, for every (on path) pair of reports* (r_1, r_2) *such that* $r_2 = s(r_1)$*, the decision maker's posterior beliefs are*

$$p(\theta|r_1,r_2) > 0 \;\; \textit{if and only if} \;\; \theta \in [\max\{r_2, \bar{r}_1^{-1}(r_1)\}, \min\{r_1, \underline{r}_2^{-1}(r_2)\}].$$

Proof. Consider a DE and a pair of reports (r_1, r_2) such that $\bar{r}_1(0) \geq r_1 > 0 > r_2 \geq \underline{r}_2(0)$. Given equilibrium supports in Proposition 3, all such pairs are on path (e.g., for $\theta = 0$). Upon observing (r_1, r_2), the decision maker forms posterior beliefs $p(\theta|r_1, r_2)$. By Lemma 1, it must be that $p(\theta|r_1, r_2) = 0$ for every $\theta \notin [r_2, r_1]$. By Lemma 5, it must be that $p(\theta|r_1, r_2) = 0$ for every $\theta \notin [\theta_1, \theta_2]$. By Proposition 3 we have that $\min S_2(\theta) \geq \underline{r}_2(\theta)$ and $\max S_1(\theta) \leq \bar{r}_1(\theta)$, and therefore $p(\theta|r_1, r_2) = 0$ for every $\theta \notin \left[\bar{r}_1^{-1}(r_1), \underline{r}_2^{-1}(r_2)\right]$, where from equations (1) and (2) we obtain that

$$\bar{r}_1^{-1}(r_1) = \min\left\{\theta \in \Theta | u_1(\theta) = k_1 C_1(r_1,\theta)\right\},$$

$$\underline{r}_2^{-1}(r_2) = \max\left\{\theta \in \Theta | -u_2(\theta) = k_2 C_2(r_2,\theta)\right\}.$$

From Proposition 3 we also have that, for every $\theta \in [0, \theta_2)$, $\max S_1(\theta) = s(\underline{r}_2(\theta)) \leq r_1(\theta)$. Therefore, given the report $r_1 \in (0, \bar{r}_1(0)]$, it must be that $p(\theta|r_1, r_2) = 0$ for all θ such that $s(\underline{r}_2(\theta)) < r_1$. By Lemma 4 and since $d\underline{r}_2(\theta)/d\theta > 0$, there is a state θ' such that $s(\underline{r}_2(\theta')) = r_1$. Denote such a state by $t_1(r_1) := \{\theta \in \Theta | s(\underline{r}_2(\theta)) = r_1\}$, where $t_1(r_1) > 0$ and $dt_1(r_1)/dr_1 > 0$. Similarly, denote $t_2(r_2) := \{\theta \in \Theta | s(\bar{r}_1(\theta)) = r_2\}$. Given equilibrium supports, it must be that $p(\theta|r_1, r_2) = 0$ for all $\theta \notin [t_2(r_2), t_1(r_1)]$.

By Lemma 4 and since $s(\underline{r}_2(\theta_2)) = \theta_2$ (Definition 6), we obtain that $t_1(r_1) \leq \theta_2$ for every $r_1 \in [\theta_2, \bar{r}_1(0)]$, and therefore $\min\{r_1, t_1(r_1)\} \leq \theta_2$ for all $r_1 \in (0, \bar{r}_1(0)]$. Similarly, we get that $\max\{r_2, t_2(r_2)\} \geq \theta_1$ for all $r_2 \in [\underline{r}_2(0), 0)$. Therefore, we have that $p(\theta|r_1, r_2) = 0$ for every $\theta \notin [\max\{r_2, \bar{r}_1^{-1}(r_1), t_2(r_2)\}, \min\{r_1, \underline{r}_2^{-1}(r_2), t_1(r_1)\}]$, and by Proposition 3 we obtain that $p(\theta|r_1, r_2) \propto f(\theta) \cdot \phi_1(r_1, \theta) \cdot \phi_2(r_2, \theta) > 0$ otherwise.

Consider now the case where $r_2 = s(r_1)$ (or, by Lemma 4, $r_1 = s(r_2)$). By definition, in state $\theta' = t_1(r_1)$ we have that $s(\underline{r}_2(\theta')) = r_1$. Therefore, we get that $s(r_1) = \underline{r}_2(\theta') = r_2$ and $\underline{r}_2^{-1}(r_2) = \theta' = t_1(r_1)$. Similarly, we obtain that $\bar{r}_1^{-1}(r_1) = t_2(r_2)$. Therefore, for every pair of reports $(r_1, s(r_1))$ we have that $p(\theta|r_1, s(r_1)) > 0$ if and only if $\theta \in \left[\max\left\{r_2, \bar{r}_1^{-1}(r_1)\right\}, \min\left\{r_1, \underline{r}_2^{-1}(r_2)\right\}\right]$. $\square$

Proposition 5. *In a direct equilibrium, the swing report function $s(r_i)$ is implicitly defined for $i, j \in \{1, 2\}$, $i \neq j$, and $r_i \in \hat{R}$, as*

$$s(r_i) = \left\{ r_j \in R_j \;\middle|\; \int_{\max\left\{r_2, \bar{r}_1^{-1}(r_1)\right\}}^{\min\left\{r_1, \underline{r}_2^{-1}(r_2)\right\}} f(\theta) \frac{u_{dm}(\theta)}{u_1(\theta)u_2(\theta)} \frac{dC_j(r_j, \theta)}{dr_j} \frac{dC_i(r_i, \theta)}{dr_i} d\theta = 0 \right\}. \tag{4}$$

Proof. Given the equilibrium reporting strategies $\phi_j(r_j|\theta) = \delta(r_j - \theta)\alpha_j(\theta) + \psi_j(r_j|\theta)$, $j \in \{1, 2\}$ (Propositions 2, 3, and 4), the mixed probability distribution $p(r_1, r_2|\theta) = \phi_1(r_1, \theta)\phi_2(r_2, \theta)$ is

$$\begin{aligned} p(r_1, r_2|\theta) = {} & \delta(r_1 - \theta)\delta(r_2 - \theta)\alpha_1(\theta)\alpha_2(\theta) + \delta(r_1 - \theta)\alpha_1(\theta)\psi_2(r_2, \theta) \\ & + \delta(r_2 - \theta)\psi_1(r_1, \theta)\alpha_2(\theta) + \psi_1(r_1, \theta)\psi_2(r_2, \theta). \end{aligned}$$

Consider a pair of reports (r_1, r_2) such that $\bar{r}_1(0) \geq r_1 > 0 > r_2 \geq \underline{r}_2(0)$ and $r_2 = s(r_1)$ (as by Lemma 4 we have that if $r > 0$, then $s(r) < 0$). Since $\left.\frac{dC_j(r_j, \theta)}{dr_j}\right|_{r_j = \theta} = 0$ for every $\theta \in \Theta$, we obtain that $\psi_j(s(\theta), \theta) = 0$ for $i, j \in \{1, 2\}$, $i \neq j$, and therefore $p(r_1, s(r_1)|\theta) = \psi_1(r_1, \theta)\psi_2(s(r_1), \theta)$.

The swing report $s(r_1)$ is defined in Definition 5 to be the report $r_2 \in R_2$ such that $U_{dm}(r_1, r_2) = \int_\Theta u_{dm}(\theta)p(\theta|r_1, r_2)d\theta = 0$, and by Lemma 4 we know that $s(r_1) \in [\underline{r}_2(0), 0)$. By Lemma A.9 we have that $p(\theta|r_1, s(r_1)) > 0$ if and only if $\theta \in [\max\{r_2, \bar{r}_1^{-1}(r_1)\}, \min\{r_1, \underline{r}_2^{-1}(r_2)\}]$, and therefore by using Bayes' rule we can rewrite the condition $U_{dm}(r_1, s(r_1)) = 0$ as $G_s(r_1, s(r_1)) = 0$, where

$$G_s(r_1, r_2) = \frac{1}{p(r_1, r_2)} \int_{\max\left\{r_2, \bar{r}_1^{-1}(r_1)\right\}}^{\min\left\{r_1, \underline{r}_2^{-1}(r_2)\right\}} u_{dm}(\theta)f(\theta)\psi_1(r_1, \theta)\psi_2(r_2, \theta)d\theta.$$

By substituting for the equilibrium strategies $\psi_j(r_j, \theta)$ as described in Proposition 2, we obtain the implicit definition of the swing report given in equation (4). $\square$

Corollary 1. *Direct equilibria are essentially unique.*

Proof. The solution of equation (4) is unique and depends only on the model's primitives $u_{dm}(\theta)$, $f(\theta)$, $u_i(\theta)$, τ_i, k_i, $C_i(r_i, \theta)$, for $i \in \{1, 2\}$. Therefore, for every $r \in [\underline{r}_2(0), \bar{r}_1(0)]$, the swing report $s(r)$ is the same in every DE. It follows that the truthful cutoffs θ_1 and θ_2, and the senders' reporting strategies $\phi_j(\theta)$ and supports $S_j(\theta)$, $j \in \{1, 2\}$, are also the same in all DE. Thus, all DE are strategy- and outcome-equivalent. $\square$

Lemma A.10. *There exist direct equilibria with unprejudiced beliefs.*

Proof. Consider a DE and an off path pair of reports (r_1, r_2). By Propositions 2, 3, and 4, and by Lemma 5, we obtain that the only pair of reports such that $\phi_j(r_j, \theta) = 0$ for all $\theta \in \Theta$ and $j \in \{1, 2\}$ is $(0, 0)$. For every other off path pair of reports, there is always a sender i such that $\phi_i(r_i, \theta) > 0$ for some $\theta \in \Theta$. There are three types of off path pairs of reports that need to be considered: those that violate Lemma 1, such as when $r_1 > r_2$; those that violate Proposition 3, such as when $r_1 > s(\underline{r}_2(r_2))$; those that violate Lemma 5, such as when $r_1 \neq r_2$ for some $(r_1, r_2) \notin (\theta_1, \theta_2)^2$.

For beliefs to be unprejudiced, Definition 7 requires that for every such off path pair of reports we have that $p(\theta''|r_1, r_2) > 0$ if and only if there is a sender $i \in \{1, 2\}$ such that $\phi_i(r_i, \theta'') > 0$. Since $p(\theta''|r_1, r_2)$ can be arbitrarily small, I can focus on the decision maker's beliefs that only one sender is deviating. Specifically, I will consider some posterior beliefs p' which, given an off path pair of reports (r_1, r_2) and provided that $\phi_j(r_j, \theta) > 0$ for some $\theta \in \Theta$ and $j \in \{1, 2\}$, rationalize deviations as originating with certainty from one specific sender i. If there is a DE with such posterior beliefs p', then there exists a DE with posterior beliefs p'' (e.g., a small perturbation of p') that satisfy Definition 7 (and hence also Definition 2).

First, if $0 \leq r_1 < r_2$ (resp. $r_1 < r_2 \leq 0$), then set p' such that the decision maker believes that sender 1 (resp. 2) is the deviator. Given the equilibrium strategies, it must be that sender 2 (1) is reporting truthfully, and therefore p' leads to $\beta(r_1, r_2) = \oplus$ ($\ominus$). Consider now the case where $r_1 < 0 < r_2$, and set p' such that the decision maker believes that only sender 1 (or 2) is deviating. Therefore, it must be that sender 2 (1) is reporting truthfully, and therefore $\beta(r_1, r_2) = \oplus$ ($\ominus$). Second, consider an off path pair of reports such that, for $x \geq 0$, we have that $r_2 > x$ and $r_1 \geq s(\underline{r}_2(x))$ (resp. $r_1 < y \leq 0$ and $r_2 \leq s(\bar{r}_1(y))$). If the decision maker believes that sender 1 (2) is the deviator, then it must be that $\theta = r_2$ ($\theta = r_1$) and therefore $\beta(r_1, r_2) = \oplus$ ($\ominus$). Finally, consider an off path pair $(r_1, r_2) \notin (\theta_1, \theta_2)^2$ with $r_1 \neq r_2$. If both $r_1, r_2 \geq \theta_2$ (resp. $r_1, r_2 \leq \theta_1$), then, by inferring that only one sender is deviating, the decision maker believes that $\theta \geq \theta_2 > 0$ ($\theta \leq \theta_1 < 0$) and selects $\beta(r_1, r_2) = \oplus$ ($\ominus$).

Since p' is consistent with conditions (C) and (D), and since given p' no sender is better off deviating from the prescribed equilibrium strategies, it follows that there are DE with unprejudiced beliefs as defined in Definitions 2 and 7. $\square$

Corollary 2. *There are direct equilibria with unprejudiced beliefs that are also ε-robust.*

Proof. Consider an ε-perturbed game with sequence ε^n and full support distributions $\hat{G} = (\hat{G}_1, \hat{G}_2)$ such that $\hat{g}_1(r_1) \approx 0$ for all $r_1 < 0$ and $\hat{g}_2(r_2) \approx 0$ for all $r_2 > 0$. This means that it is relatively unlikely that the decision maker will misinterpret the report of sender 1 (resp. 2) to be negative (resp. positive). By equation (5), the limit beliefs $\hat{p}_{0^+}$ induced by the strategies of a DE after the decision maker observes a pair of reports (r_1, r_2) such that $0 \leq r_1 < r_2$ are

$$\hat{p}_{0^+}(\theta|r_1 \geq 0, r_2 > 0) \approx f(\theta)\frac{\delta(r_2 - \theta)\alpha_2(\theta)}{f(r_2)\alpha_2(r_2)}.$$

Therefore, the CDF $\hat{P}_{0^+} = \int p_{0^+}(\theta|r_1, r_2)d\theta$ is such that

$$\hat{P}_{0^+}(\theta|r_1 \geq 0, r_2 > 0) \approx \begin{cases} 0 & \text{if } \theta < r_2 \\ 1 & \text{if } \theta \geq r_2. \end{cases}$$

As $\varepsilon^n \to 0^+$ and for every off path pair of reports that are both positive, the decision maker is almost sure that the realized state coincides with the report of sender 2. Similarly, we obtain that $\hat{P}_{0^+}(r_1 < 0, r_2 \leq 0) \approx 0$ for all $\theta < r_1$ and ≈ 1 otherwise, and by Lemma 3 we have that $\hat{p}_{0^+}(\theta|r_1 < 0, r_2 > 0) > 0$ only for $\theta \in \{r_1, r_2\}$. Therefore, the limit beliefs $\hat{p}_{0^+}$ are arbitrarily close to the posterior beliefs p' in the proof of Lemma A.10, and therefore can support a DE. Since a DE is also a PBE, by Lemma 3 we obtain that some direct equilibria with unprejudiced beliefs are ε-robust. □

Corollary 3. *A direct equilibrium always exists.*

Proof. Given strategies $\phi_j(r_j, \theta) = \delta(r_j - \theta)\alpha_j(\theta) + \psi_j(r_j, \theta)$ as in Propositions 2 and 4, with support $S_j(\theta)$ as in Proposition 3, posterior beliefs $p(\theta|r_1, r_2)$ are such that the swing report function $s(r)$ is as in Proposition 5. Given $s(r)$, strategies $\phi_j(r_j, \theta)$ are optimal by construction, and therefore no sender $j \in \{1, 2\}$ is better off deviating from $\phi_j(r_j, \theta)$. Therefore, for every primitive of the model that satisfies the conditions outlined in Section 3, there must exist a direct equilibrium as defined by Definition 4. □

A.3 Example: Symmetric Environments

Corollary 4. *In a direct equilibrium of a symmetric environment, $s(r) = -r$ for every $r \in \hat{R}$.*

Proof. The proof follows directly from Proposition 5: consider a symmetric environment and suppose that $s(r) = -r$. Given a report $r \in (0, \bar{r}_1(0))$, the interval of integration in (4) has $\max\{-r, \bar{r}_1^{-1}(r)\} = -\min\{r, \underline{r}_2^{-1}(-r)\}$. Since the integrand in (4) is symmetric around zero, we obtain that $G_s(r, -r) = 0$, confirming that indeed $s(r) = -r$. □

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