Auctions with Frictions*

Stephan Lauermann[†] Asher Wolinsky[‡]

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Abstract

Much of the work on auction theory focuses on design aspects in situations motivated by formally organized auctions, like those held by government agencies. However, auction models are also convenient abstractions of less formal price-formation processes that arise in markets for assets or services. Existing models have to be enriched to capture certain frictions that are more salient in such informal situations. In particular, bidder participation may be the outcome of costly recruitment efforts, participation may be costly for the bidders as well, the seller's commitment abilities may be limited, and private information of the seller may be more consequential. This paper develops a model of auctions with such frictions and derives some novel predictions. In particular, outcomes are often inefficient, and the market sometimes unravels.

Much of the work on auction theory focuses on design aspects in situations in which the auctioneer has substantial commitment power and the potential bidders are known and ready to participate (bidding or first acquiring information). This is motivated by formally organized auctions, such as those held by government agencies. However, auction models are also convenient abstractions of less formal price

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[†]University of Bonn, Department of Economics, s.lauermann@uni-bonn.de.

[‡]Northwestern University, Department of Economics, a-wolinsky@northwestern.edu.

formation processes that arise in markets for assets or services.[1] In such situations, the commitment ability of the auctioneer may be limited, the recruitment of bidders might be a central issue, and the interaction might be affected by information that the auctioneer has or is trying to learn.

These "design" and "markets" agendas differ not only in some of their assumptions but also in the bigger questions that they are addressing. The "markets" agenda is interested in traditional economic questions concerning the efficiency of markets and how competition handles information asymmetries. It is not very interested in design questions (such as identifying a mechanism that performs well by some criteria) and, in fact, might prefer to think of situations in which the fine details of the design are not very important.

This paper belongs to the "markets" agenda. It explores the role of four aspects of such less formal auction scenarios: costly recruitment, costly bidder entry/information acquisition, seller's inability to commit to the level of recruitment effort, and bidders' inability to observe participation.

The model features a first-price auction with a random number of prospective bidders, which is the realization of a Poisson distribution, whose parameter is determined by the seller's costly recruitment effort. A prospective bidder thus contacted decides whether to participate. Participation may involve a cost, which can be motivated by information acquisition and other preparations. The bidders do not observe the seller's recruitment effort. We first look at the independent private-values version and within it examine two scenarios. In the *PO scenario* ("participation-observable"), bidders observe the number of participants before they bid. The unique equilibrium yields two related insights. First, it may involve a substantial inefficiency in the form of costly excessive recruitment effort, even when the cost per unit effort is small. Second, it may result in no trade, even when the recruitment and participation costs are low enough to facilitate trade if seller commitment or greater transparency were possible. In the alternative *PU scenario* ("participation–unobservable"), the bidders do not observe the extent of participation in the auction. The unobservability generates a new consideration—the seller's incentive to secretly reduce recruitment. This may give rise to multiple equilibria sustained by different levels of fulfilled expectations. One of them is an equilibrium with no trade. It

[1] The view of auction models as abstractions of free-form price formation motivated some of the earlier literature (Milgrom, 1979, and Wilson, 1977).

always exists and sometimes it is unique, even when the circumstances seem favorable for trade (bidders bear no cost of entry and the seller's recruitment cost is not prohibitive). Thus, the auction collapses under circumstances that could sustain beneficial trade in the presence of commitment or observable participation.

Some of the more interesting insights arise from the comparison of the two scenarios. First, if the bidders' cost of entry/information acquisition is sufficiently small to not constrain the equilibria, then equilibrium participation and profit are higher in the PO scenario. In particular, the PO equilibrium may involve active trade, even when no trade is the unique PU equilibrium outcome, as explained above. In contrast, when the bidders' costs are sufficiently high to constrain the equilibrium, then the total recruitment cost is larger and consequently profit and total surplus are lower in the PO than in the PU scenario. In particular, when bidders' cost is large enough, the unique PO involves no trade, whereas, for small enough recruitment costs, the PU scenario has an active trade equilibrium.

In terms of payoffs and costs, the PO equilibrium is equivalent to the dominant strategy equilibrium of the second-price auction format in either scenario (as observability does not matter). Therefore, these insights also apply verbatim to a comparison between the first-price and second-price auction formats. In the absence of the frictions (lack of commitment, costly recruitment, and costly participation), these two scenarios would be equivalent in terms of profit and surplus. In the presence of costs and absence of commitment power, they are not equivalent and their ranking depends on the recruitment and entry-cost conditions. Given our "informal auctions" perspective, we do not think of the choice of the auction format from the design angle, but rather as a proxy for some mix of bidding and bargaining that arises naturally in these situations. However, the results concerning the comparison between these two formats bring out elements that will be present to various degrees in hybrid formats and they may also shed light on circumstances under which one format or the other is more likely to emerge.

We then look at the PO scenario with the added feature of bidders' uncertainty over the seller's recruitment cost. It is a specific example for the broader issue of private seller's information that gets incorporated into behavior and affects incentives and equilibrium outcomes. This might be of greater relevance for the informal environments that we have in mind than, say, for a government sponsored auction. One insight that arises in this environment is the possibility of nearly complete unraveling of the market: Almost all seller types stay out of the market despite the fact that, if their type were commonly known, each of them would be active in equilibrium. Finally, we consider a number of extensions that illustrate the robustness of the qualitative findings. In particular, we consider the case in which bidders know their valuation before entering, as well as the seller's optimal entry fee/subsidy and reserve price.

Although the related literature is vast, some of the issues addressed by this paper may not have been extensively explored. In particular, we are not aware of references that contain the specific insights described above. Here are some of the most immediately related references. In Levin and Smith (1994), bidders' costly entry is also an important element, but that paper takes the auction design approach and focuses on its traditional questions that are orthogonal to our work. Bulow and Klemperer (1996) compares auctions to negotiations when entry is costly but they do not share our focus on the lack of commitment and recruitment efforts. Szech (2011) considers the optimal costly recruitment of bidders by a seller who can commit. Lauermann and Wolinsky (2017, 2021) also present auction models in which the seller incurs recruitment cost. However, they consider a common values environment and explore the extent of information aggregation by price when there is a privately informed seller.

1 The PO auction: Poisson recruitment, Observable participation

1.1 The model

One seller owns an indivisible object that has value 0 to her. The seller makes recruitment effort $\gamma \geq 0$, resulting in a random number of prospective bidders that is distributed Poisson with mean γ, i.e., the probability of contacting t bidders is $\frac{\gamma^t}{t!}e^{-\gamma t}$. The cost of effort γ is γs, for some $s > 0$.

The prospective bidders are ex-ante symmetric. A prospective bidder i who decides to participate incurs a cost $c \geq 0$. Afterward, the bidder observes his own value v_i for the good and the total number n of bidders who chose to enter the auction (including i). The v_i's are private values, independently and identically distributed with a cumulative distribution function (c.d.f.) G, with support $[0, 1]$, a continuous density g, and increasing "virtual values," $v - \frac{1-G(v)}{g(v)}$. The bidders do not observe γ.

Finally, the participating bidders submit bids. The highest bidder wins and pays his own bid.

When an auction ends with winning bid p, the payoff for the seller is $p - \gamma s$, for the winning bidder i it is $v_i - p - c$, for a participating bidder who lost it is $-c$, and for a contacted bidder who declines entry it is 0.

1.2 Interaction: Strategies and Equilibrium

The seller's strategy is the recruitment effort $\gamma \geq 0$. Bidder i's strategy is (q_i, β_i), where $q_i \in [0,1]$ is the entry probability and $\beta_i : [0,1] \times \{1,2,...\} \to [0,1]$ describes i's bid as a function of his information (v_i, n)— i's private value and the number of participating bidders. Bidder i's belief concerning the seller's effort, conditional on being contacted (but before observing (v_i, n)) is a probability distribution μ_i with finite support[2] in $[0, \infty)$. Thus, $\mu_i(\gamma)$ is the probability that bidder i assigns to the possibility that the seller chose effort γ.

We study symmetric behavior in which all bidders employ the same strategy (q, β) and hold the same belief μ.

Definition: An **equilibrium** is γ^*, q^*, and β^* such that:

1. γ^* maximizes the seller's expected payoff given q^* and β^*;

2. There exists a belief μ such that:

    - q^* and β^* maximize each bidder's payoff, given μ and the other bidders' strategy (q^*, β^*);
    - if $\gamma^* > 0$, then $\mu(\gamma^*) = 1$, i.e., the belief is confirmed on the path;
    - if $\gamma^* = 0$, then every $\hat{\gamma}$ in the support of μ maximizes the seller's payoff given q^* and β^*.

Thus, by definition, the equilibrium is symmetric and allows only pure recruitment and bidding strategies; mixing is allowed only in the bidders' entry decisions, $q \in [0,1]$.

[2] Finiteness will turn out to involve no loss of generality in this model.

Off-path beliefs arise only when $\gamma^* = 0$ but their role is not negligible since this is an important case of extreme market failure. The last bullet point in the equilibrium definition imposes a refinement on the off-path beliefs. This refinement allows us to rule out no-trade equilibria that rely on unfounded beliefs. It will be discussed in Section 6.5, where we present alternative ways to get the needed refinement.[3]

Notice that the refinement does not rule out $\mu(0) > 0$. This is the case in which a bidder is contacted off-path despite the fact that the seller's profit maximizing effort is 0. Essentially, the bidder believes that the seller only "trembled" slightly and most likely this bidder is the only one to have been contacted.[4]

The random number of actual participants in the auction is Poisson distributed with mean

$$\lambda := q\gamma$$

Given the Poisson distribution, λ is not just the expected number of participants from an outsider's perspective, it is also the expected number of competitors of a participating bidder (Myerson, 1998).

For convenience we will mostly use λ (instead of γ). Thus, bidders' belief μ will be over λ and the equilibrium will be expressed in terms of $\lambda^* := q^*\gamma^*$.

2 Equilibrium analysis for the PO Scenario

2.1 Solving backward

The interaction unfolds in three stages—recruitment, bidders' entry, bidding—and the equilibrium can be solved backward.

Stage 3: Bidding. Once the number of participants n is realized, it is a standard symmetric first-price auction (FPA) with independent private values drawn from the c.d.f. G. Such an auction has a unique symmetric equilibrium (see, e.g., Krishna 2010),

$$\beta_{FPA}(v, n) = v - \int_0^v \left[\frac{G(y)}{G(v)}\right]^{n-1} dy, \tag{1}$$

and so $\beta^* = \beta_{FPA}$ is the bidding strategy in every equilibrium.

[3] We chose this approach for the main text since it is easy to state and does not require any special notation or modification of the model.

[4] This point is also discussed in Section 6.5.

Stage 2: Bidders' Entry. Let $U_o(\lambda)$ be the bidders' ex-ante expected payoff (gross of the cost of entry), given a Poisson distributed number of participating bidders with mean λ who use β_{FPA}. The subscript "o" here and later stands for "observable" participation.

Claim 1 *U_o is strictly decreasing and continuous,*

$$U_o(\lambda) = \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]dv. \tag{2}$$

Observe that $U_0(0) = E[v]$ and $\lim_{\lambda\to\infty} U_0(\lambda) = 0$.

Given bidders' belief μ concerning λ—a probability distribution with finite support in $[0, \infty)$.[5]—their optimal entry decision q satisfies

$$\begin{aligned} E_\mu[U_o(\hat{\lambda})] > c \quad &\Rightarrow \quad q = 1, \\ E_\mu[U_o(\hat{\lambda})] < c \quad &\Rightarrow \quad q = 0. \end{aligned} \tag{3}$$

The case of $c \geq U_0(0)$ is uninteresting, since it means that the bidder stays out. We therefore assume from now on that

$$0 \leq c < U_0(0).$$

Since U_0 is continuous and strictly decreasing to 0, the equation $U_o(\lambda) = c$ has a unique solution if $c > 0$. We denote this solution by $\overline{\lambda}^c$, that is, for $c > 0$,

$$U_o(\overline{\lambda}^c) = c. \tag{4}$$

This is the bidders' break-even participation level: given λ, a bidder's expected payoff from entering is nonnegative if and only if $\lambda \leq \overline{\lambda}^c$. For $c = 0$, we set $\overline{\lambda}^c = \infty$. The upper bar in $\overline{\lambda}^c$ will serve as a reminder that this the maximal scale acceptable to bidders.

It follows that, in any equilibrium,

$$\lambda^* \leq \overline{\lambda}^c, \tag{5}$$

and, if $\lambda^* \in (0, \overline{\lambda}^c)$, then $q^* = 1$.

[5] We used μ before to denote belief over γ. From here on, it denotes beliefs over λ.

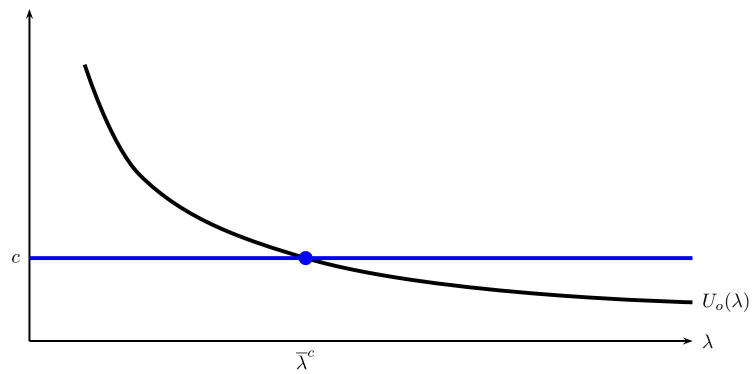

Figure 1: The function $U_o(\lambda)$.

Stage 1: Recruitment. Given q and β, the seller's problem is to choose recruitment effort γ to maximize profit. The choice of effort γ at cost s is equivalent to the choice of $\lambda = q\gamma$ at cost s/q. Letting $R_o(\lambda)$ be the seller's expected revenue given the participation level λ and β_{FPA}, the profit as a function of λ and $q > 0$ is

$$\Pi_o(\lambda, q) = R_o(\lambda) - \lambda\frac{s}{q};$$

with $\Pi_o(0,0) = 0$ and $\Pi_o(\lambda, 0) = -\infty$ for $\lambda > 0$.

In any equilibrium, $\lambda^* \in \arg\max \Pi_o(\lambda, q^*)$. The following discussion describes the solution to this maximization problem.

In Figure 2, $\Pi_o(\lambda, q)$ is captured by the vertical difference between the curves. (Although we provide analytical arguments, the reader might find it easier to just follow the graphical arguments that essentially capture everything.)

The revenue $R_o(\lambda)$ is an increasing function since a larger λ induces more aggressive bidding and higher maximal values. Owing to the former effect, R_o is not concave.

Figure 3 depicts properties of R_o that are relevant for solving the seller's problem. These properties are also established analytically by Claim 3 below

The figure depicts some notation that is used repeatedly:

$$\begin{aligned} &\bar{s}_o := \max_{\lambda} \frac{R_o(\lambda)}{\lambda}, \\ &\lambda_o(z) := \text{largest } \lambda \text{ st. } R_o'(\lambda) = z \text{ for } z \leq \bar{s}_o, \\ &\underline{\lambda}_o := \lambda_o(\bar{s}_0). \end{aligned} \tag{6}$$

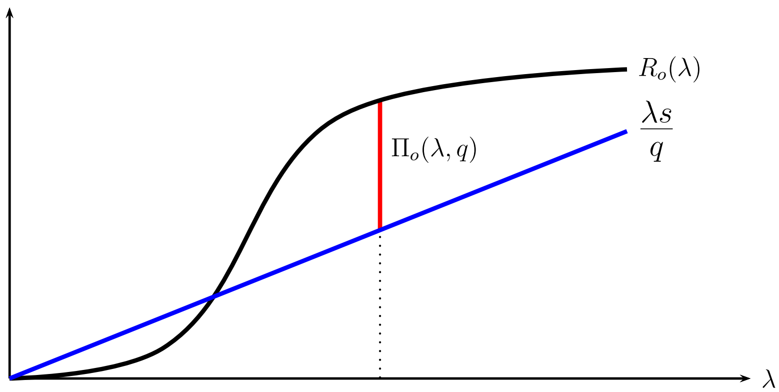

Figure 2: Revenue, cost, and profit.

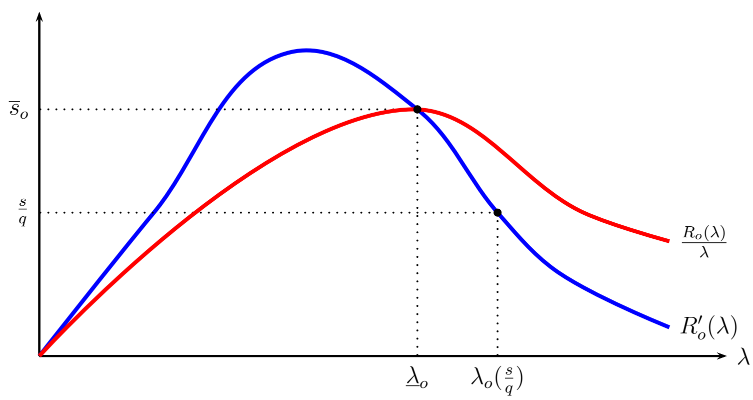

Figure 3: Marginal revenue, average revenue, and marginal cost.

Observe that

$$R_0'(\underline{\lambda}_o) = \frac{R_o\left(\underline{\lambda}_o\right)}{\underline{\lambda}_o} = \bar{s}_0.$$

These properties of $R_o(\lambda)$ imply the form of the solution to the maximization of $\Pi_o(\lambda, q)$ summarized by the following claim. In particular, the claim shows that $\underline{\lambda}_o$ is the minimal positive profit-maximizing scale. (The lower bar in $\underline{\lambda}_o$ serves as a reminder of that.)

Claim 2 $\Pi_o(\lambda, q)$ *is maximized either at* $\lambda = 0$ *or at some* $\lambda \geq \underline{\lambda}_o$*:*

$$\begin{aligned} \frac{s}{q} &> \bar{s}_o \Rightarrow \arg\max \Pi_o(\cdot, q) = 0, \\ \frac{s}{q} &< \bar{s}_o \Rightarrow \arg\max \Pi_o(\cdot, q) = \lambda_o(\frac{s}{q}) > \underline{\lambda}_o, \\ \frac{s}{q} &= \bar{s}_o \Rightarrow \arg\max \Pi_o(\cdot, q) = \{0, \underline{\lambda}_o\}. \end{aligned} \tag{7}$$

This claim follows immediately from the following claim that summarizes the observations depicted in Figure 3.

Claim 3 *Revenue and optimality:*

1. $R_o(\lambda)$ *is strictly increasing,* $R_o(0) = 0$*, and* $\lim_{\lambda \to \infty} R_o(\lambda) = 1$*.*

2. $R_o(\lambda)$ *is continuously differentiable,* $R_o'(0) = 0$*,* $R_o'(\lambda) \to 0$ *as* $\lambda \to \infty$*, and* R_o' *is single peaked.*

3. $\frac{R_o(\lambda)}{\lambda}$ *is single peaked; at its peak,* $\frac{R_o(\lambda)}{\lambda} = R_o'(\lambda)$*.*

Thus, in any equilibrium $\lambda^* = 0$ or $\lambda^* = \lambda_o(\frac{s}{q^*})$ depending on whether $\frac{s}{q^*} \gtreqless \bar{s}_o$.

2.2 Equilibrium

Solving backwards through the three stages above results in the following "reduced form" equilibrium definition.

Equilibrium is λ^* and q^* that satisfy one of the following:

1. $\lambda^* = \lambda_o(s) \in [\underline{\lambda}_o, \overline{\lambda}^c)$ and $q^* = 1$.

2. $\lambda^* = \overline{\lambda}^c$ and $q^* \in (0, 1]$, with $\lambda_o(\frac{s}{q^*}) = \overline{\lambda}^c$.

3. $\lambda^* = 0$ and q^* is a best response[6] to μ, where $\text{supp}(\mu) \subset \arg\max_\lambda \Pi_o(\cdot, q^*)$.

Thus, if an equilibrium λ^* is positive, it must satisfy $\underline{\lambda}_o \leq \lambda^* \leq \overline{\lambda}^c$. That is, λ^* is between the minimal profit-maximizing scale and the maximal acceptable scale for the bidders.

The essential information about the equilibrium is depicted by Figures 4 and 5 below and also stated by the following propositions.

[6] So, it satisfies (3).

Proposition 1 *An equilibrium exists and it is unique for almost all* (s, c)*.*

Proposition 1 follows from the characterization results of Propositions 2 and 3 below.

Proposition 2 *If* $s < \overline{s}_o$ *and* $\overline{\lambda}^c > \underline{\lambda}_o$*, then the unique equilibrium outcome has* $\lambda^* = \min\{\overline{\lambda}^c, \lambda_o(s)\}$*.*

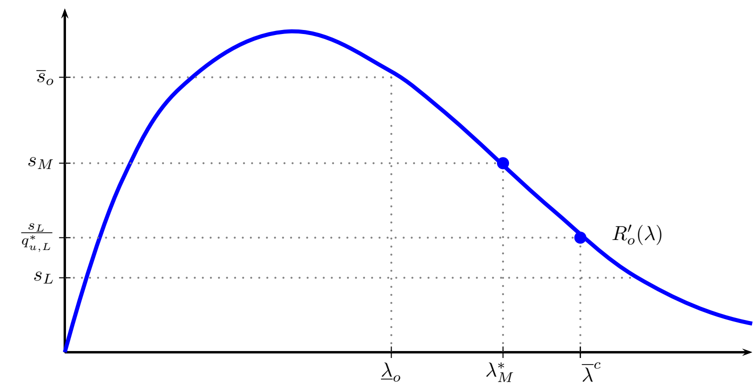

Figure 4: PO Scenario with Trade.

The figure illustrates R'_o and the two cutoffs, $\underline{\lambda}_o$ and $\overline{\lambda}^c$. It shows the two types of equilibria with trade, each obtaining for a different level of s:

- at s_M, the unique equilibrium outcome is with trade, $\lambda^*_M = \lambda_o(s_M) \in [\underline{\lambda}_o, \overline{\lambda}^c]$;
- at s_L, the unique equilibrium outcome is with trade, $\lambda^*_L = \overline{\lambda}^c$ and q^*_L satisfies $\lambda_o(\frac{s_L}{q^*_L}) = \overline{\lambda}^c$. In this case, $\lambda_o(s_L) > \overline{\lambda}^c$.

Proposition 3 *If* $s > \overline{s}_o$ *or* $\overline{\lambda}^c < \underline{\lambda}_o$*, then* $\lambda^* = 0$ *(no trade) is the unique equilibrium outcome.*

The first case is already illustrated by the previous Figure 4. At s_H, the unique equilibrium outcome is $\lambda^*_H = 0$ since $R'_o(\lambda) < s_H$ for all $\lambda \geq \underline{\lambda}_o$. The second case is illustrated by Figure 5.

If $\overline{\lambda}^c < \underline{\lambda}$, as depicted, no trade takes place even if s is low, as shown by s_L in the figure.

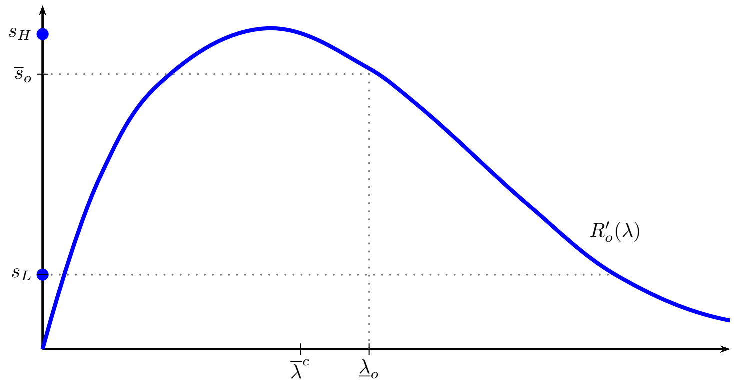

Figure 5: PO Scenario—no trade when $\overline{\lambda}^c < \underline{\lambda}_o$.

Entry is beneficial for bidders at $\lambda < \overline{\lambda}^c$, and, when s is low, such λ would be profitable for the seller as well. The problem is that the seller cannot commit to such low λ. She also cannot be incentivized through reduced bidders' entry ($q < 1$) that would raise the marginal recruitment cost s/q. If $s/q > \overline{s}_o$, the profit is maximized at $\lambda = 0$; if $s/q \leq \overline{s}_o$, the profit is maximized at $\lambda \geq \underline{\lambda}_o > \overline{\lambda}^c$ (and at both $\underline{\lambda}_o$ and 0 if $s/q = \overline{s}_o$). In this case, the market will be closed even in the face of substantial potential gains from trade.

Finally, if either one of the strict inequalities in Propositions 2 are replaced with an equality, i.e., when $s = \overline{s}_o$ or $\overline{\lambda}^c = \underline{\lambda}_o$, then both $\lambda^* = 0$ and $\lambda^* = \underline{\lambda}_o$ are equilibrium outcomes. This also holds if both of the inequalities in Proposition 3 are replaced, i.e., when $s = \overline{s}_o$ and $\overline{\lambda}^c = \underline{\lambda}_o$.

2.3 A qualitative insight

For the following corollary, we include s as an argument in the seller's payoff and write $\Pi_o(\lambda, q, s)$.

Corollary 1 *Consider a sequence $(s_k)_{k=1}^{\infty}$ with $s_k \to 0$, and let (λ_k^*, q_k^*) be the corresponding equilibrium outcomes.*

1. *If $c = 0$, then $q_k^* = 1$ for all s_k, $\lambda_k^* \to \infty$, and $s_k \lambda_k^* \to 0$.*

2. *If $c > 0$ and $\overline{\lambda}^c \geq \underline{\lambda}_o$, then, for all $s_k < R'_o(\overline{\lambda}^c)$,*

$$\lambda_k^* = \overline{\lambda}^c, \quad \frac{s_k}{q_k^*}\lambda_k^* = \text{constant} \quad \textit{and} \quad \Pi_o(\lambda_k^*, q_k^*, s_k) = \text{constant.}^7$$

Corollary 1 implies that, for $c > 0$ (but not prohibitively large) and all small enough s, total recruitment costs are constant and bounded away from zero.

To be willing to bear the cost of entry, bidders must believe that the seller is not recruiting too aggressively. This is achieved in equilibrium through bidders' sufficient reluctance to enter (sufficiently small q^*) that raises sufficiently the marginal recruitment cost to a level that induces the seller to stop at $\overline{\lambda}^c$.

In contrast, total recruitment cost becomes negligible when s is small, either when $c = 0$ or when $c > 0$ and the seller can commit to some recruitment effort γ. No matter how large such commitment γ is, q would adjust to achieve $\overline{\lambda}^c$ but $s\gamma \to 0$ as $s \to 0$. *Thus, inefficient costly recruitment effort is the consequence of a lack of commitment and costly bidder participation.*

2.4 Other Auction Formats and Bargaining

Consider the second-price auction (SPA) format in the same environment.

Claim 4 *The expected payoffs and the equilibrium magnitudes of λ^* and q^* are the same as they would be in the dominant strategy equilibrium of the SPA format.*

This result follows immediately from revenue equivalence. Therefore, the same insights and conclusions hold.[8]

More generally, the characterization of equilibrium relies only on the properties of the reduced form payoffs $U_o(\cdot)$ and $R_o(\cdot)$ derived in the Claims 1 and 3. Thus, the characterization extends from the FPA and SPA to any auction or "bargaining" scenario that implies these properties for the expected payoffs and revenue.

3 The PU auction: Poisson recruitment, Unobservable participation

This is the independent, private-values model considered so far, except that here bidders **cannot** observe the number of other participants at any stage (before or

[7] These constants are $\overline{\lambda}^c R_o'\left(\overline{\lambda}^c\right)$ and $R_o\left(\overline{\lambda}^c\right) - \overline{\lambda}^c R_o'\left(\overline{\lambda}^c\right)$, respectively.

[8] The distribution of the winning bid itself would be different but this does not affect the results we look at.

after entry). Besides being interesting in its own right, this scenario will help us distinguish the role of commitment from the role of observability. We call it PU and index its magnitudes with the subscript "u" (for "unobservable").

The equilibrium definition presented in Subsection 1.2 above remains the same, and we continue to work in terms of $\lambda = \gamma q$.

3.1 Solving backward (PU Scenario)

As before, the interaction unfolds in three stages—recruitment, bidders' entry, bidding—and the equilibrium is solved backward.

Stage 3: Bidding. Since bidders do not observe the actual participation, this is a first-price auction (FPA) with a random number of bidders distributed Poisson ($\hat{\lambda}$), where $\hat{\lambda}$ is bidders' point-belief[9] concerning the expected participation. This auction has a unique symmetric bidding equilibrium, denoted $\beta_{\hat{\lambda}} : [0,1] \to [0,1]$.[10]

Claim 5 *Given belief $\hat{\lambda}$, the unique symmetric equilibrium bidding strategy is*

$$\beta_{\hat{\lambda}}(v) = v - \int_0^v e^{-\hat{\lambda}(G(v)-G(x))} dx. \tag{8}$$

For $\hat{\lambda} > 0$, the bidding strategy $\beta_{\hat{\lambda}}$ is strictly increasing in v and differentiable; for $\hat{\lambda} = 0$, we have $\beta_{\hat{\lambda}}(v) = 0$.

In an equilibrium with participation λ^*, the bidders' equilibrium strategy is $\beta^* = \beta_{\lambda^*}$.

Stage 2: Bidders' entry. Let $U_u(\lambda)$ be a contacted bidder's ex-ante expected payoff, given λ and β_λ. Given bidders' point belief $\hat{\lambda}$, bidders' entry decision q is optimal if

$$\begin{aligned} U_u(\hat{\lambda}) > c \quad &\Rightarrow \quad q = 1, \\ U_u(\hat{\lambda}) < c \quad &\Rightarrow \quad q = 0. \end{aligned} \tag{9}$$

Claim 6 *$U_u(\lambda)$ is the same as $U_o(\lambda)$ of the previous PO scenario.*

[9] As in the PO part, we could describe beliefs as a distribution μ. But we need for this part only point beliefs, in the sense that we would not get any more equilibrium outcomes by allowing for non-degenrate beliefs off the equilibrium path. Therefore, we focus on them right away.

[10] This result is proven as a strightforward implication of payoff equivalence, just as in the analysis of other auction scenarios with an uncertain numbers of bidders; see, e.g., Krishna (2009, Section 3.2.2).

Since β_λ is monotonic, this claim is a consequence of payoff-equivalence and does not require a proof. Therefore, U_u is decreasing to 0 and continuous, and the break-even participation level $\overline{\lambda}^c$ (with $U_u(\overline{\lambda}^c) = c$) is also the same as in the PO scenario. Thus, as above, in equilibrium, $\lambda^* \leq \overline{\lambda}^c$, and, if the inequality is strict, then $q^* = 1$.

Stage 1: Recruitment. Let $R_u(\lambda, \beta)$ be the seller's expected revenue given participation level λ and bidding strategy β. The seller's expected payoff $\Pi_u(\lambda, \beta, q)$ is

$$\Pi_u(\lambda, \beta, q) = R_u(\lambda, \beta) - \lambda \frac{s}{q},$$

for $\lambda, q > 0$. It is 0 for $\lambda = q = 0$ and it is $-\infty$ for $q = 0$ and $\lambda > 0$.

In any equilibrium, $\lambda^* \in \arg\max_\lambda \Pi_u(\lambda, \beta_{\lambda^*}, q^*)$. It is shown below (Claim 7) that $\Pi_u(\lambda, \beta_{\hat{\lambda}}, q)$ is concave and differentiable in λ (for fixed $\hat{\lambda}$). Therefore, any λ that satisfies the first-order condition (with respect to λ) is a maximizer of $\Pi_u(\lambda, \beta_{\hat{\lambda}}, q)$.

3.2 Equilibrium (PU scenario)

Let

$$\xi(\lambda) := \frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda}.$$

This is the marginal revenue with respect to λ at a point where λ coincides with the given expectation $\hat{\lambda}$.

Proposition 4 *The strategies λ, β_λ, and q constitute an equilibrium if and only if q satisfies (9) and*

$$\frac{s}{q} \geq \xi(\lambda), \tag{10}$$

with equality holding for $\lambda > 0$.

Proof. The proof uses the following claim, which is proved in the appendix.

Claim 7 *(i) $R_u(\lambda, \beta_{\hat{\lambda}})$ is twice differentiable (in λ and $\hat{\lambda}$) and, for $\hat{\lambda} > 0$, it is strictly concave in λ;*

(ii) The function $\xi(\lambda)$ is continuous, $\xi(0) = 0$, and $\lim_{\lambda \to \infty} \xi(\lambda) = 0$.

Since for $\hat{\lambda} > 0$, $R_u(\lambda, \beta_{\hat{\lambda}})$ is strictly concave and differentiable in λ, so is $\Pi_u(\lambda, \beta_{\hat{\lambda}}, q)$. Therefore, the first-order condition $\frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}}) \leq \frac{s}{q}$ (with equality at $\lambda > 0$) is both, necessary and sufficient. If $\hat{\lambda} = 0$, then $R_u(\lambda, \beta_{\hat{\lambda}}) = 0$ for all λ, and so $\lambda = 0$ is the unique best response, with $\frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}}) = 0 < \frac{s}{q}$. ■

The essential information about the equilibrium is depicted by the following two figures and then stated formally by Corollary 2. Let

$$\overline{s}_u := \max_{\lambda} \xi(\lambda),$$

and, for $0 < z \leq \bar{s}_u$, let $\overline{\lambda}_u(z)$ and $\underline{\lambda}_u(z)$ be the maximal and minimal λ's satisfying $\xi(\lambda) = z$.

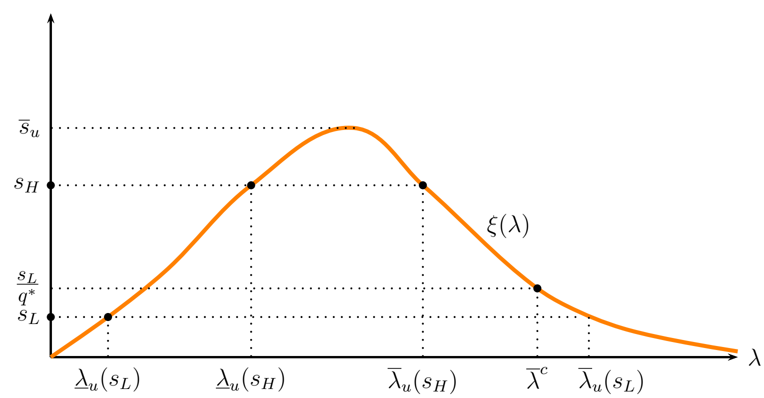

Figure 6: PU Scenario with small c.

Figure 6 depicts the equilibria (marked with a little circles) for two s values, $s_H > s_L$, and small c (which translates to a relatively large $\overline{\lambda}^c$). For each of the s levels there are three equilibria:

- For s_H, $\lambda^* = 0$, $\lambda^* = \underline{\lambda}_u(s_H)$, and $\lambda^* = \overline{\lambda}_u(s_H)$, respectively.
- For s_L, $\lambda^* = 0$, $\lambda^* = \underline{\lambda}_u(s_L)$, and $\lambda^* = \overline{\lambda}^c$, respectively.

Thus, c does not constrain the equilibria for s_H. It constrains only the largest equilibrium for s_L, in which q^* adjusts to achieve $\overline{\lambda}_u(s_L/q^*) = \overline{\lambda}^c$. In the other equilibria, $q^* = 1$.

The case of a larger c—implying a smaller $\overline{\lambda}^c$—is depicted by Figure 7.

In this case, for s_H there is a unique equilibrium with $\lambda^* = 0$, while for s_L there are still three equilibria as in the case of small c. The key difference is that $\overline{\lambda}^c < \underline{\lambda}_u(s_H)$, precluding trade in equilibrium with s_H.

More generally, if c and s are such that the market is not closed, then either c is relatively small to not constrain the equilibria, and so in all equilibria, $\lambda^* \leq \overline{\lambda}^c$

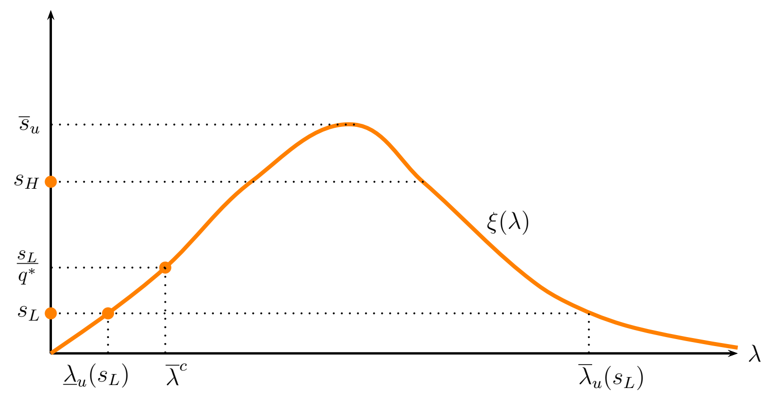

Figure 7: PU Scenario with large c.

and $q^* = 1$; or c is sufficiently large so the largest equilibrium λ^* is $\overline{\lambda}^c$ and the corresponding $q^* < 1$. In particular, given c, for small enough s there is always such an equilibrium.

Corollary 2 (to Proposition 4)[11] summarizes the above observations.

Corollary 2 *(i) For all s and c, there is a no-trade equilibrium with $\lambda^* = 0$. If $s > \overline{s}_u$ or $\overline{\lambda}^c < \underline{\lambda}_u(s)$, this is the unique equilibrium outcome.*

(ii) If $s < \overline{s}_u$ and $\overline{\lambda}^c > \underline{\lambda}_u(s)$, there are also (possibly multiple) equilibria with trade: $\lambda^* > 0$ is an equilibrium outcome if and only if either

$$\xi(\lambda^*) = s \qquad \text{and} \qquad \lambda^* < \overline{\lambda}^c,$$

or there is some $q^* \in (0, 1]$ such that

$$\lambda^* = \overline{\lambda}^c \qquad \text{and} \qquad \xi\left(\overline{\lambda}^c\right) = \frac{s}{q^*}.$$

Note that we have not established that ξ is single peaked.[12] Therefore, the possibility of more equilibria than the three shown in the figure is not ruled out.

Two conflicting observations on the equilibrium set are direct consequences of the bidders' inability to observe participation. First, the no-trade equilibrium al-

[11] Strictly speaking, this is just a restatement of Proposition 4, where condition (9) on the optimality of q^* is expressed in terms of $\overline{\lambda}^c$.

[12] We established analytically that ξ is continuous and that $\xi(0) = 0$, $\xi(\lambda) > 0$ for $\lambda > 0$, and $\lim_{\lambda\to\infty} \xi(\lambda) = 0$.

ways exists, even when s and c are low.[13] This equilibrium is sustained because bidders expect no competition and therefore intend to bid 0, making recruitment unprofitable. In the absence of the ability to commit to the level of recruitment, the seller cannot break out of this equilibrium. This logic also explains the low-trade equilibria (e.g., at $\underline{\lambda}_u(s_L)$). When $s > \overline{s}_u$, this effect is strong enough to leave no trade as the unique equilibrium outcome, even if there are potential gains from trade.

Second, for any $c < U(0)$ (which is assumed throughout), if s is sufficiently small, there still exists an equilibrium with trade. If bidders could observe participation, a low λ that is needed to induce bidders with high c to enter could not be sustained in equilibrium since the seller would have an incentive to recruit more aggressively. However, here the bidders' expectations of low participation induce low bids and discourage aggressive recruiting.

Let λ_u^* and q_u^* denote the equilibrium values with the largest λ (for a given s and c). We are interested in this equilibrium mainly as a useful reference in the comparison between the scenarios. However, when c is small so that $\overline{\lambda}^c$ is not binding, this equilibrium is also distinguished by being the seller's maximal profit equilibrium and by being pseudo-stable in the sense that the best response to a locally displaced λ points in the direction of the equilibrium.[14]

Proposition 5 *Suppose $s \leq \overline{s}_u$.*

1. *If $\overline{\lambda}^c < \underline{\lambda}_u(s)$, then $\lambda_u^* = 0$.*
2. *If $\overline{\lambda}^c \in [\underline{\lambda}_u(s), \overline{\lambda}_u(s)]$, then $\lambda_u^* = \overline{\lambda}^c$ and $\frac{s}{q_u^*} = \xi\left(\overline{\lambda}^c\right)$.*
3. *If $\overline{\lambda}^c > \overline{\lambda}_u(s)$, then $\lambda_u^* = \overline{\lambda}_u(s)$ and $q_u^* = 1$, and this is the seller's most profitable equilibrium.*
4. *If $c = 0$ (i.e., $\overline{\lambda}^c = \infty$), then $s \to 0$ implies $\lambda_u^* \to \infty$ and $s\lambda_u^* \to 0$.*

Parts 1-3 follow immediately from Corollary 2 and hence do not require a proof. Part 4 is also immediate. First, $\lambda_u^* \to \infty$ as $s \to 0$ since $\lambda_u^* = \overline{\lambda}_u(s)$ and, by Claim 7, $\overline{\lambda}_u(s) \to \infty$. Second, an analogous argument to that of Corollary 1 implies that,

[13] As opposed to only when s or c are too high, as it is the case in the PO scenario.

[14] We do not place much weight on this observation since, when $\overline{\lambda}^c$ is binding, the naive pseudo-stability argument is less clear.

when s is small enough, the seller can extract the whole surplus with a possibly suboptimal $\lambda = 1/\sqrt{s}$, and hence the equilibrium $s\lambda_u^*$ may not be bounded away from 0.

A qualitative insight. Corollary 1 of the PO scenario is valid for the present scenario as well, and so is the insight that, for $c > 0$ (but not prohibitively large) and all small enough s, the total recruitment cost is constant; that is,

$$\frac{s}{q_u^*}\lambda_u^* = \xi\left(\overline{\lambda}^c\right)\overline{\lambda}^c = \text{constant},$$

which follows from $\frac{s}{q_u^*} = \xi(\overline{\lambda}^c)$ for all small enough s.

4 Comparison of the PO and PU Scenarios

4.1 Ranking Reversals

With observable participation (PO scenario), the incentive to recruit is driven by two considerations: increasing the likelihood that high-value bidders might appear and inducing more aggressive bidding. With unobservable participation (PU scenario), only the former consideration is present. This difference is reflected by the stronger marginal incentive to recruit with observable participation. It is translated to ranking reversals: With "small" c and not "too small" s, the PO scenario generates higher participation and profit than the PU scenario; these relations are reversed with "large" c or "small" s.

Figure 8 combines Figures 4 and 6. Recall that $\bar{s}_0$ and $\bar{s}_u$ are the maximal s's for which an equilibrium with positive λ exists for the PO and PU scenarios, respectively. The essential features depicted in the figure are stated by the following Claim.

Claim 8 *(i)* $R_o'(\lambda) > \xi(\lambda)$*, for all* $\lambda > 0$*; (ii)* $\bar{s}_0 > \bar{s}_u$*.*

Figure 8 depicts the case of small c and medium/large s. It shows λ_o^* and λ_u^* (the unique maximal equilibrium λ's for the PO[15] and PU scenarios, respectively) for two recruitment cost levels, $s_H > s_M$. At s_H, there is trade only in the PO scenario: $\lambda_{o,H}^* > \lambda_{u,H}^* = 0$. At s_M, there is trade in both scenarios, $\lambda_{o,M}^* > \lambda_{u,M}^* > 0$, and the

[15] Note that, for this comparison, we add a subscript "o" to the unique λ^* of the PO scenario.

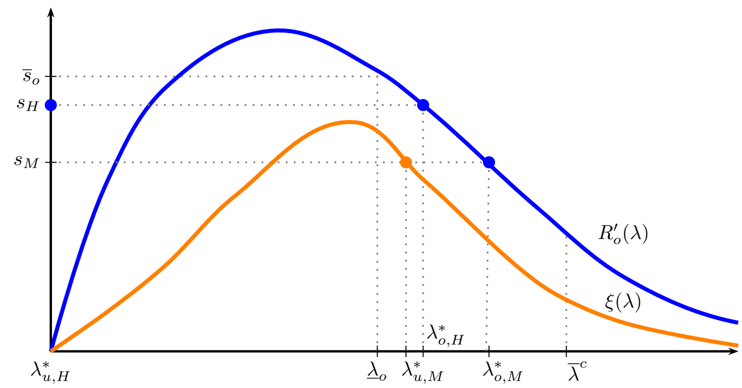

Figure 8: Comparison with small c and medium/large s.

participation is unconstrained by c. In both cases, the profit in the PO scenario is higher. Graphically, the profit in both scenarios is the area between the $R'_o(\lambda)$ curve and the appropriate s line. Thus, the conclusion about the profit is evident from inspection of the figure.

Figure 9 depicts the case of small c and small s. At s_L, the participation level is constrained by c in both scenarios, yielding the same equilibrium participation $\overline{\lambda}^c$ but with higher effective recruitment costs in the PO scenario, $\frac{s_L}{q^*_{o,L}} > \frac{s_L}{q^*_{u,L}}$. Obviously, the profit in the PO scenario is lower in this case.

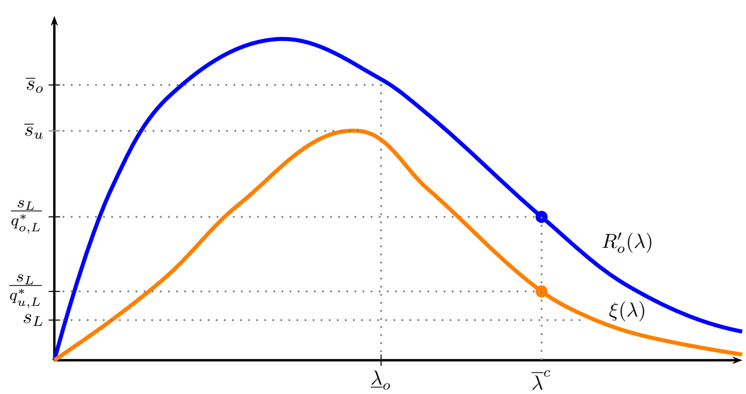

Figure 9: Comparison with small c and small s.

Figure 10 depicts the remaining case of large c. In this case, trade takes place only in the PU scenario: $\lambda^*_u = \overline{\lambda}^c > 0 = \lambda^*_{o}$, and the PU profit is of course higher.

Observe that the same levels of participation and profits will prevail for any lower s as well.

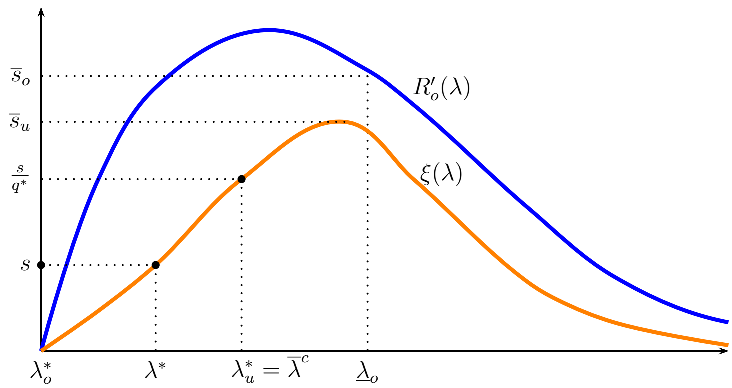

Figure 10: Comparison with large c.

The observations we made with the aid of the three diagrams are stated formally by the two claims further below.

The main take away is that there is **ranking reversal of profit and participation**.

- If s is non-prohibitive for the PO scenario ($s < \overline{s}_o$), then for small enough c, both participation and profit are higher in the PO scenario.

- If s is non-prohibitive for the PU scenario ($s < \overline{s}_u$), for c large enough, both participation and profit are higher in the PU scenario.

This reversal does not conflict with the revenue equivalence logic. If λ is the same in both scenarios and $\hat{\lambda} = \lambda$ (in the PU scenario), then by revenue equivalence, so is the revenue and, if q is also the same, so is the profit. This is seen in the diagram where the profit associated with λ and q (given that $\hat{\lambda} = \lambda$ in the PU case) in both scenarios corresponds to the area between the $R'_o(\lambda)$ curve and the s/q line over the interval $[0, \lambda]$.

The reversal owes to the significant role of the observability of participation. This is something that is often assumed automatically in auction models, although it is not so obvious when less formal situations are considered. When combined with costly recruitment and participation, the observability has important consequences.

In the presence of recruiting costs, the unobservability of participation has a retarding effect on the profitability of recruiting and, in the extreme, may result in no trade. Consider the case of $c = 0$, so that bidders' entry considerations are absent (i.e., $q^* = 1$ in all equilibria). In this case, the profit of both scenarios is maximized at λ_o^* (where $R_o'(\lambda_o^*) = s$). As just mentioned above, if the seller in the PU scenario could commit to λ_o^*, it would get the same profit. However, in the absence of commitment, this is not sustainable in the PU scenario. The seller would prefer to secretly reduce λ. Bidders anticipating this would plan to bid less aggressively than they would if they expected λ_o^*, thus augmenting the seller's incentive to secretly reduce λ even further. When $\overline{s}_o > s > \overline{s}_u$, these self-reinforcing considerations drive the maximal PU equilibrium participation λ_u^* to 0—complete "unraveling" of the market, even though $s < \overline{s}_o$ implies a positive λ_o^*. When $s < \overline{s}_u$, then λ_u^* settles at a positive level. albeit lower than λ_o^*. In either case, this implies lower profit in the PU scenario.[16]

When $c > 0$, the retarding effect of unobservable participation may help sustain trade by insuring bidders against excessive recruitment that will make their entry unprofitable.

The mix of these two effects, which depend on the relative sizes of s and c, explains the "reversal."

The following two claims just summarize the above observations formally (and hence do not require a proof). Recall the seller's profit functions $\Pi_o(\lambda, q) = R_o(\lambda) - \lambda\frac{s}{q}$ and $\Pi_u(\lambda, \beta_{\hat{\lambda}}, q) = R_u(\lambda, \beta_{\hat{\lambda}}) - \lambda\frac{s}{q}$.

Higher participation and profit in the PO equilibrium.

Claim 9 *Suppose* $\overline{\lambda}^c > \underline{\lambda}_o$.

1. *If* $\bar{s}_0 > s > \bar{s}_u$, *then* $\lambda_o^* > \lambda_u^* = 0$ *and* $\Pi_o(\lambda_o^*, q_o^*) > \Pi_u(\lambda_u^*, \beta_{\lambda_u^*}, q_u^*) = 0$.

2. *If* $\bar{s}_u > s > R_o'(\overline{\lambda}^c)$, *then* $\lambda_o^* > \lambda_u^* > 0$ *and* $\Pi_o(\lambda_o^*, q_o^*) > \Pi_u(\lambda_u^*, \beta_{\lambda_u^*}, q_u^*)$.

Higher participation and profit in the PU equilibrium.

[16] A slightly different explanation appeals to the seller's revealed preference. By choosing λ_u^* in the PO scenario, the seller could secure the PU equilibrium profit. Since $\Pi_o(\lambda, q)$ is increasing in λ at $\lambda = \lambda_u^*$ (as evident in the diagram from $R_o'(\lambda_u^*) > s$), the conclusion follows.

Claim 10

1. *If* $\overline{\lambda}^c < \underline{\lambda}_o$*, then* $\lambda_u^* \geq \lambda_o^* = 0$ *and* $\Pi_u(\lambda_u^*, \beta_{\lambda_u^*}, q_u^*) \geq \Pi_o(\lambda_o^*, q_o^*) = 0$*, with strict inequalities for* $s < \overline{s}_u$*.*

2. *If* $\overline{\lambda}^c \geq \underline{\lambda}_o$ *and* $s < \xi(\overline{\lambda}^c)$*,, then* $\lambda_o^* = \lambda_u^* = \overline{\lambda}^c$ *and* $\Pi_u(\lambda_u^*, \beta_{\lambda_u^*}, q_u^*) > \Pi_o(\lambda_o^*, q_o^*)$*.*

Remark. Claims 9 and 10 do not speak about the intermediate range of c where $\overline{\lambda}^c > \underline{\lambda}_o$ and $s \in (\xi(\overline{\lambda}^c), R_o'(\overline{\lambda}^c))$. The purpose of this omission was to avoid dealing with details that might complicate the presentation, though they do not affect the general message. Over this range, $\overline{\lambda}^c$ constrains participation only in the PO scenario. For s near the bottom of this range, the outcomes are close to those of Part 2 of Claim 10 ($\lambda_o^* = \overline{\lambda}^c$and λ_u^* just below it), and the PU equilibrium is more profitable. For s near the top, the outcomes are close to those of Part 2 of Claim 9 and the PO equilibrium is more profitable. The ranking switches somewhere in the interior of this range.

4.2 Comparison of FPA to SPA

Recall from Claim 4 that, in terms of payoffs and costs, the PO equilibrium is equivalent to the dominant strategy equilibrium of a second-price auction (SPA) format, where the observability does not matter:

Claim 11 *All the above insights concerning the comparison of the PO and PU hold for a comparison between SPA and FPA, respectively, with unobservable participation.*

Thus, while FPA and SPA formats are equivalent in terms of equilibrium profit and welfare when participation is observable, these formats are not equivalent with unobservable participation in this environment, and their ranking is affected by the magnitudes of the entry and recruitment costs.

Given our "informal auctions" perspective, we do not think of the choice of the auction format from a design angle but rather as a proxy for some mix of bidding and bargaining that arises naturally in these situations. In line with this, the results concerning the comparison between these two formats bring out elements that will be present to various degrees in hybrid formats, and they may also shed light on circumstances under which one format or the other is more likely to emerge.

Remark. We explained above the stronger incentive to recruit in the PO scenario by its effect on inducing "more aggressive bidding." We also noted that the PO equilibrium is equivalent (in terms of profit) to the dominant strategy equilibrium of the SPA scenario in which the bidding (one's own value) is independent of observability, and hence larger participation in the SPA scenario does not induce "more aggressive bidding." These two observations are not inconsistent with each other. What matters for the incentive to recruit is the effect on the expected price. Whereas, in the SPA the presence of more bidders does not induce more aggressive bidding, each given winning bid is translated into a higher price in expectation.

4.3 Disclosure

Suppose that the seller could credibly commit in advance to always disclose or always not disclose the number of participants prior to the bidding. This is equivalent to the seller choosing between the PO and PU scenarios. Thus, the above comparison applies also to the disclosure question (if such commitment is possible). In particular, it follows from the above discussion that the seller may prefer to commit in advance to disclosure or no disclosure depending on s and c.

5 Uncertainty about Seller's Type

It is natural to suppose that bidders are uncertain about the seller's recruitment effort (even in equilibrium). This is modeled here by assuming bidders' uncertainty about s.

5.1 Binary Setup

The above PO model is modified minimally to capture this uncertainty. Privately known seller's type ω has marginal recruitment cost s_ω and occurs with prior probability ρ_ω, $\omega = L, H$. Type L is more efficient, $s_H > s_L > 0$. Seller type ω selects recruitment effort γ_ω.

Contacted bidders decide on entry, then observe their own value and the number of participants, and finally submit bids in a FPA. Bidders' symmetric entry and bidding strategy (q,β) and the state-dependent participation rates $\boldsymbol{\lambda}:=(\lambda_L, \lambda_H)$, where $\lambda_\omega = q\gamma_\omega$, are just as in the PO scenario.

In any symmetric equilibrium, β must be the unique FPA symmetric equilibrium strategy $\beta_{FPA}(v,n)$ (see (1)). Therefore, for any given participation rate λ, the seller's revenue and the bidders' ex-ante expected payoff are the same as in the PO scenario. Hence, the profit of seller type ω is

$$\Pi_\omega(\lambda_\omega, q) = R_o(\lambda_\omega) - \lambda_\omega \frac{s_\omega}{q},$$

and, given bidders' belief μ (the distribution over λ conditional on being contacted), their expected payoff is $E_\mu(U_o(\lambda))$ and their optimal entry decision q satisfies (3).

For $\boldsymbol{\lambda} \neq 0$, let

$$\phi_\omega(\boldsymbol{\lambda}) = \frac{\rho_\omega \lambda_\omega}{\Sigma \rho_\omega \lambda_\omega}.$$

Since $\boldsymbol{\lambda} \neq 0$ implies $\boldsymbol{\gamma}{:=}(\gamma_L, \gamma_H) \neq 0$, it follows that $\phi_\omega(\boldsymbol{\lambda})$ is the probability of ω, conditional on a bidder being contacted by the seller.

Equilibrium is $\boldsymbol{\lambda}^* = (\lambda_L^*, \lambda_H^*)$, and q^* such that

(E1) $\lambda = \lambda_\omega^*$ maximizes $\Pi_\omega(\lambda_\omega, q^*)$.

(E2) There exists belief μ such that

(i) q^* is optimal given μ, i.e., it satisfies (3);

(ii) If $\boldsymbol{\lambda}^* \neq (0,0)$, then $\mu(\lambda_\omega^*) = \phi_\omega(\boldsymbol{\lambda}^*)$ (confirmation on path);

(iii) if $\boldsymbol{\lambda}^* = (0,0)$, then every λ in the support of μ maximizes $\Pi_\omega(\lambda, q^*)$ for some ω.

Claim 12 *There exists an equilibrium.*

The equilibrium analysis just imports what we know from the PO scenario to the current setting. The following discussion and the diagram prove Claim 12 above and the subsequent Claim 13. Recall from the PO scenario that $\overline{s}_o$ is the maximal s that sustains equilibrium with trade; that $\lambda_o(z)$ is the profit maximizing λ for a given $z \leq \overline{s}_o$ (i.e., the maximal solution of $R_o'(\lambda) = z$); that $\underline{\lambda}_o$ is the minimum profitable scale for the seller ($\underline{\lambda}_o = \lambda_o(\overline{s}_o)$), and that $\overline{\lambda}^c$ is the maximal λ with which bidder entry is beneficial ($U_o(\overline{\lambda}^c) = c$).

Let

$$\widehat{\lambda}_\omega(q) = \begin{cases} \lambda_o(\frac{s_\omega}{q}) & \text{if } \frac{s_\omega}{q} < \overline{s}_o, \\ 0 & \text{if } \frac{s_\omega}{q} > \overline{s}_o, \end{cases}$$

and $\widehat{\boldsymbol{\lambda}}(q) = \left(\widehat{\lambda}_L(q), \widehat{\lambda}_H(q)\right)$. It follows immediately from the PO analysis that $\lambda^*_\omega = \widehat{\lambda}_\omega(q^*)$. Therefore, an equilibrium with $\boldsymbol{\lambda}^* = (0,0)$ exists if and only if $\frac{s_L}{q^*} \geq \overline{s}_o$, which can occur if and only if $s_L \geq \overline{s}_o$ or $\overline{\lambda}^c \leq \underline{\lambda}_o$, and it is unique if one of these inequalities is strict.

To consider equilibrium with trade, $\boldsymbol{\lambda}^* \neq (0,0)$, let $V(\boldsymbol{\lambda})$ denote bidders' expected payoff at $\boldsymbol{\lambda} = (\lambda_L, \lambda_H)$,

$$V(\boldsymbol{\lambda}) = \Sigma\phi_\omega(\boldsymbol{\lambda})U_o(\lambda_w). \tag{11}$$

In an equilibrium with trade, q^* has to satisfy

$$\begin{array}{lcl} q^* \in (0,1) & \Rightarrow & V(\widehat{\boldsymbol{\lambda}}(q^*)) = c, \\ q^* = 1 & \Rightarrow & V(\boldsymbol{\lambda}^*) \geq c. \end{array} \tag{12}$$

Obviously, $s_H > \overline{s}_o$ implies $\lambda^*_H = 0$ in any equilibrium, and we are back in the PO scenario with commonly known $s = s_L$, for which existence and characterization are already established. Therefore, the only interesting case to consider is $\overline{s}_o > s_H > s_L > 0$.

The following diagram depicts $V(\widehat{\boldsymbol{\lambda}}(q))$ as a function of q. The intersection points between it and c correspond to (12), and therefore capture all the possible equilibria with trade. The maximal c that is compatible with equilibrium with trade is $\overline{c}$ st. $\overline{\lambda}^{\overline{c}} = \underline{\lambda}_o$, just as in the PO scenario. The minimal q that still facilitates a profitable positive scale for seller type ω is $\overline{q}_\omega$ s.t. $\overline{s}_o = \frac{s_\omega}{\overline{q}_\omega}$. At $\overline{q}_L$, type L becomes active with the minimal positive scale $\underline{\lambda}_o$; at $\overline{q}_H$, type H also joins with the minimal scale $\underline{\lambda}_o$, and this explains the discontinuity of $V(\widehat{\boldsymbol{\lambda}}(q))$ at $\overline{q}_H$. In other words, $\widehat{\boldsymbol{\lambda}}(q) = (0,0)$ for $q < \overline{q}_L$; it jumps to $(\underline{\lambda}_o, 0)$ at $\overline{q}_L$ and increases continuously with $q \in [\overline{q}_L, \overline{q}_H)$ according to $(\lambda_o(\frac{s_L}{q}), 0)$; it jumps again at $\overline{q}_H$ to $(\lambda_o(\frac{s_L}{\overline{q}_H}), \underline{\lambda}_o)$, and thereafter continues according to $(\lambda_o(\frac{s_L}{q}), \lambda_o(\frac{s_H}{q}))$. For V, note that it is decreasing until $\overline{q}_H$ given that $(\lambda_o(\frac{s_L}{q}), 0)$ is increasing and, for this range, $V(\lambda_o(\frac{s_L}{q}), 0) = U_0\left(\lambda_o(\frac{s_L}{q})\right)$. As note, at $\bar{q}_H$, V jumps up. Moreover, at this point, $\lambda_o(\frac{s_L}{\overline{q}_H}) > \underline{\lambda}_o$ implies that $V(\underline{\lambda}_o, 0) > V(\lambda_o(\frac{s_L}{\overline{q}_H}), \underline{\lambda}_o)$, which is seen in the diagram by V being higher at $\overline{q}_L$ than at $\overline{q}_H$.

Each of the panels shows all the equilibria for a given level of c. The left panel depicts a case in which there is only one equilibrium, and in it only type L is active. The right panel depicts a case in which there are are two equilibria: in one, only

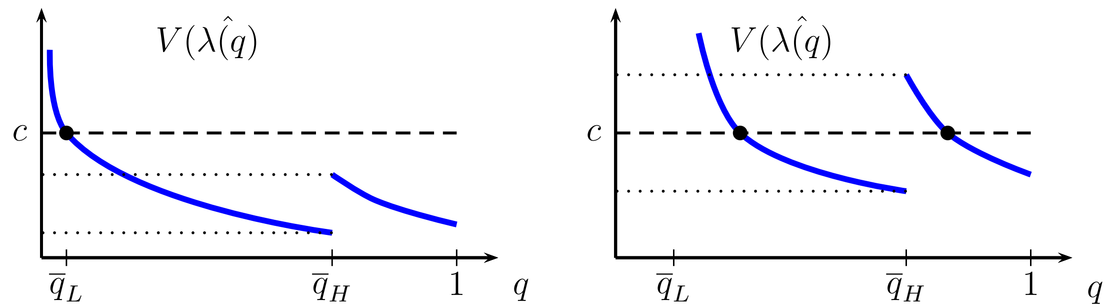

Figure 11: Equilibrium with two seller types.

type L is active; in the other, both types are active.

The location of the $V(\widehat{\boldsymbol{\lambda}}(q))$ curve depends on s_L and s_H. A lower s_L induces a downward shift of both parts of the curve.[17] Thus, the case depicted by the left panel might correspond to a lower s_L than the case depicted in the right panel.

The following claim summarizes what the above discussion and the diagram have established.

Claim 13 *For $\overline{s}_o > s_H > s_L > 0$, the equilibrium set is characterized by three cutoffs $\overline{c} > c_1 > c_2$:*

(i) For $c > \overline{c}$, the unique equilibrium has $\lambda_L^* = \lambda_H^* = 0$.

(ii) For $c \in (c_1, \overline{c})$, the unique equilibrium with trade has $\lambda_L^* > 0 = \lambda_H^*$.

(iii) For $c < c_2$, the unique equilibrium with trade has $\lambda_L^* > \lambda_H^* > 0$;

(iv) For $c \in (c_2, c_1)$, there are two equilibria with trade, one with $\lambda_L^* > 0 = \lambda_H^*$ and one with $\lambda_L^* > \lambda_H^* > 0$.

Remark. We restrict attention to pure strategies for the seller. However, if we admit randomized strategies for the seller, then for $c \in (c_2, c_1)$ there is also a third equilibrium in which $\lambda_L^* > 0$ and λ_H^* is randomized between a positive level and 0.

5.2 Unraveling

As noted above, when s_L is sufficiently small relatively to s_H, only type L is active in equilibrium (i.e., $\lambda_H^* = 0$). This is so even when s_H itself is small enough so that, if it were commonly known, the equilibrium would involve active recruiting.

[17] A lower s_H shifts downward only the right branch of the curve.

Claim 14 *Suppose $c > 0$. For any $s_H > 0$ and $\rho_H > 0$, there exists a threshold $S(s_H, \rho_H)$ such that $s_L < S(s_H, \rho_H)$ implies $\lambda_L^* > 0$ and $\lambda_H^* = 0$.*

When s_L is small, q^* must be small as well, for otherwise λ_L^* would be very large and bidders entry unprofitable. However, a given s_H combined with small q^* means high marginal recruiting cost s_H / q^* for type H, making participation unprofitable for this type. More formally, given s_H and ρ_H, for sufficiently small values of s_L, $V(\lambda_o(s_L/\overline{q}_H), \underline{\lambda}_o) < c$. Hence, for any $q \geq \overline{q}_H$ (that accommodates H's participation), $V(\lambda_o(s_L/q), \lambda_o(s_H/q)) \leq V(\lambda_o(s_L/\overline{q}_H), \underline{\lambda}_o) < c$. Thus, it must be that $q^* < \overline{q}_H$, and the unique equilibrium is with $\lambda_L^* = \overline{\lambda}^c$ and $\lambda_H^* = 0$.

In other words, if we start with the case depicted by the right panel of Figure 11 and lower s_L sufficiently, we will reach the case depicted by the left panel.

This outcome is inefficient: type s_H might fail to trade even when s_H is quite low and would result in active trade if it were known.

This insight does not hinge on the two types assumption. Nothing of importance in the analysis would change if there were $m > 2$ types: if the lowest s is low enough, still all types with higher s would be shut out of the market. However, this insight depends, of course, on the discreteness, since the argument relies on making the ratio of the lowest to the next lowest cost small enough while keeping their probabilities constant. The question is whether and under what circumstances a similar unraveling occurs in an environment where exceedingly lower costs are associated with exceedingly lower probabilities.

5.3 Continuum of Seller Types

We address the last question by a version of the model with a continuum of possible seller types. The marginal recruiting cost s is distributed uniformly on $[\underline{s}, \overline{s}_o]$, where $\underline{s} > 0$ and $\overline{s}_o$ is as defined above (the maximal s compatible with active recruitment in the commonly known type case).

The model extends immediately to this environment. Identifying ω with s itself, we write λ_s and $\boldsymbol{\lambda} = (\lambda_s)_{s \in [\underline{s}, \overline{s}_o]}$.

The equilibrium definition also extends almost directly. For each s, λ_s^* and q^* satisfy equilibrium conditions (E1) and (E2) above, with s and λ_s replacing s_ω and λ_ω, respectively. The equilibrium belief density μ also satisfies the analogous conditions. In particular, let

$$\phi_s(\boldsymbol{\lambda}) := \frac{\lambda_s}{\int_{\underline{s}}^{\overline{s}_o} \lambda_s ds}.$$

If $\lambda_s^* \neq 0$ for some s, then $\mu(\lambda_s^*) = \phi_s(\boldsymbol{\lambda}^*)$ for all s. Let

$$V(\boldsymbol{\lambda}) := \int_{\underline{s}}^{\overline{s}_o} \phi_s(\boldsymbol{\lambda}) U_o(\lambda_s)) ds = \frac{\int_{\underline{s}}^{\overline{s}_o} \lambda_s U_o(\lambda_s)) ds}{\int_{\underline{s}}^{\overline{s}_o} \lambda_s ds}.$$

In an equilibrium with trade, q^* satisfies (12).

We already know from the discrete type case about the possibility of partial unraveling, in the sense that trade might be shut down for some type, despite it being sustainable if that type was commonly known. The question is whether it is possible to have complete or nearly complete unraveling in equilibrium, even when c is low enough to allow trade when s is commonly known.

Obviously, if some seller type in $[\underline{s}, \overline{s}_o]$ is active in equilibrium, so is every lower type. Hence, the equilibrium has a cutoff structure, and, moreover, the cutoff must be $q^*\overline{s}_o$. It follows from the previous discussion that, for $s < q^*\overline{s}_o$, $\lambda_s^* = \lambda_o(s/q^*) > 0$, and, for $s > q^*\overline{s}_o$, $\lambda_s^* = 0$, where $\lambda_o(z)$ is the profit maximizing λ in the PO scenario when the marginal recruitment cost is z. Let $\boldsymbol{\lambda_o}$=$(\lambda_o(s))_{s\in[\underline{s},\overline{s}_o]}$, and recall that $\overline{c}$ is the maximal cost level compatible with trade in the PO scenario (i.e., $\overline{\lambda}^{\overline{c}} = \lambda_o(\overline{s}_o) = \underline{\lambda}_o$).

Claim 15 *The unique equilibrium outcome is:*

(i) $c \geq \overline{c}$*: no trade,* $\boldsymbol{\lambda}^* = 0$*;* $q^* = \underline{s}/\overline{s}_o$*;*

(ii) $c \leq V(\boldsymbol{\lambda_o})$*: all types are active,* $\boldsymbol{\lambda}^* = \boldsymbol{\lambda_o}$*;* $q^* = 1$*;*

(iii) $V(\boldsymbol{\lambda_o}) < c < \overline{c}$*: only* $s \in [\underline{s}, \overline{s}_o q^*]$ *are active, with*

$$\lambda_s^* = \begin{cases} \lambda_o(s/q^*) > 0 & \textit{for} \quad s \in [\underline{s}, \overline{s}_o q^*], \\ 0 & \textit{for} \quad s > \overline{s}_o q^*, \end{cases} \tag{13}$$

and $q^* \in (0,1)$ *is such that* $V(\boldsymbol{\lambda}^*) = c$*.*

Proof: The equilibrium has a cutoff structure with cutoff $q^*\overline{s}_o$ and λ_s^*, as in (13). It is also immediate that the configurations described in (i)-(iii) are equilibria.

Part (i). For $c > \overline{c}$, the equilibrium is just the same as the $\lambda = 0$ equilibrium of the PO scenario with $s = \underline{s}$. That is, the support of the off-path beliefs is $\{0, \lambda_o(\overline{s}_o)\}$, which are optimal for type $\underline{s}$ given $q^* = \underline{s}/\overline{s}_o$. The probabilities μ satisfy $\mu(0)U_o(0) + \mu(\lambda_o(\overline{s}_o))U_o(\lambda_o(\overline{s}_o)) = c$. The uniqueness is also just as it is in the corresponding PO scenario. For $c = \overline{c}$, besides the above equilibrium, there is also an equilibrium in which only type $\underline{s}$ might be active. Since type $\underline{s}$ is of zero measure, we think of this as a no-trade outcome as well.

Parts (ii) and (iii). The λ_s^* and q^* are optimal, and there are no off-path moves. To see that the equilibria in (ii) and (iii) are unique among those with $\lambda_s > 0$ for some s, suppose that, in either scenario, there are two equilibria with $q_1^* < 1$ and $q_2^* > q_1^*$. The corresponding equilibrium λ's, $\lambda_s^*(q_1^*)$ and $\lambda_s^*(q_2^*)$, are given by (13). Hence, $V(\boldsymbol{\lambda}^*(q_2^*)) < V(\boldsymbol{\lambda}^*(q_1^*)) = c$, in contradiction to $q_2^* > 0$. Therefore, to establish uniqueness, we only have to rule out the no-trade equilibrium. Such an equilibrium may be supported only by the beliefs μ described in the proof of Part (i). But $c < \overline{c} = U_o(\lambda_o(\overline{s}_o))$ implies that $\mu(0)U_o(0) + \mu(\lambda_o(\overline{s}_o))U_o(\lambda_o(\overline{s}_o)) = c$ cannot hold. ■

Since by definition $\overline{c} = U_o(\underline{\lambda}_o)$, for any $c < \overline{c}$ and commonly known $s < \overline{s}_o$, the equilibrium in the PO scenario involves trade. In contrast, Part (iii) of Claim 15 identifies a range of $c < \overline{c}$ and $s < \overline{s}_o$ for which there is no trade.

The extent of such unraveling depends on c and $\underline{s}$. The following claim identifies a threshold $\underline{c} < \overline{c}$ such that, if bidder entry cost c exceeds $\underline{c}$, then the unraveling is nearly complete when $\underline{s}$ is small; if $c < \underline{c}$, trade always takes place regardless of how small is $\underline{s}$.

The probability of equilibrium with no trade (given c and $\underline{s}$) is

$$\Pr(\text{no-trade}|c, \underline{s}) = \Pr(\{s : \lambda_s^* = 0\}|c, \underline{s}).$$

Proposition 6 *There exists $\underline{c} < \overline{c}$ s.t.,*

(i) for any $c \in (\underline{c}, \overline{c})$, $\lim_{\underline{s}\to 0} \Pr(\textit{no-trade}|c, \underline{s}) = 1$;

(ii) for any $c < \underline{c}$ and any $\underline{s} < \overline{s}_o$, $\Pr(\textit{trade}|c, \underline{s}) = 1$.

Proof: For the proof, we include $\underline{s}$ as an argument in $V(\boldsymbol{\lambda_o}, \underline{s})$. Let

$$\underline{c} := \lim_{\underline{s}\to 0} V(\boldsymbol{\lambda_o}, \underline{s}) = V(\boldsymbol{\lambda_o}, 0).$$

Since $V(\boldsymbol{\lambda_o})$ is monotone in $\underline{s}$ (if $\underline{s}$ is decreasing, bidders are facing higher λ; see the proof of Claim 15), it holds that $U_o(\lambda_o(\overline{s}_o)) > V(\boldsymbol{\lambda_o}, \underline{s}) > \lim_{\underline{s}\to 0} V(\boldsymbol{\lambda_o}, \underline{s})$. So, $\overline{c} = U_o(\lambda_o(\overline{s}_o)$ implies that

$$\underline{c} < \overline{c}.$$

If $c > \underline{c}$, then for small enough $\underline{s}$, $V(\boldsymbol{\lambda_o}, \underline{s}) < c$, and the equilibrium is given by Part (iii) of Claim 15. Therefore,

$$\Pr(\text{no-trade}|c, \underline{s}) = \frac{\overline{s}_o - q^* \overline{s}_o}{\overline{s}_o - \underline{s}}.$$

A change-of-variables shows that, for all $q > 0$,

$$V(\boldsymbol{\lambda_o}, 0) = \frac{\int_0^{\overline{s}_o} \lambda_0(s)\, U_o(\lambda_0(s)))ds}{\int_0^{\overline{s}_o} \lambda_0(s)\, ds} = \frac{\int_0^{\overline{s}_o q} \lambda_0\left(\frac{s}{q}\right) U_o(\lambda_0\left(\frac{s}{q}\right)))\frac{1}{q}ds}{\int_0^{\overline{s}_o q} \lambda_0\left(\frac{s}{q}\right)\frac{1}{q}ds}, \tag{14}$$

where the right-hand side equals $V(\boldsymbol{\lambda}(\cdot, q), 0)$ for $\boldsymbol{\lambda}(\cdot, q) = \lambda_0\left(\frac{s}{q}\right)$. Thus, $V(\boldsymbol{\lambda}(\cdot, q'), 0) = V(\boldsymbol{\lambda}(\cdot, q''), 0)$ for all $q', q'' > 0$. (The bidders' payoffs are independent of the cutoff $\overline{s}_o q$. This stationarity property utilizes the uniform distribution.)

Now, denote by $\boldsymbol{\lambda}_k^*$ and q_k^* the equilibrium magnitudes for $\underline{s}_k$. From (14), if $q_k^* \to q > 0$, then $\lim_{k\to\infty} V(\boldsymbol{\lambda}_k^*, \underline{s}_k) = V(\boldsymbol{\lambda_o}, 0)$. Since $c > \underline{c} = V(\boldsymbol{\lambda_o}, 0)$, this implies a contradiction to buyer optimality. Hence, it must be that $q_k^* \to 0$, which implies the claim.

When $c < \underline{c}$, then for any $\underline{s}$, $V(\boldsymbol{\lambda_o}, \underline{s}) > c$, and the equilibrium is given by Part (ii) of Claim 15. Therefore, $\Pr(\text{trade}|c, \underline{s}) = 1$. ■

Note again that the nearly complete unraveling occurs for a range of $c < \overline{c}$ for which trade would take place at any commonly known $s \in (0, \overline{s}_o]$.

6 Discussion and Extensions

6.1 Welfare

Welfare $W(\lambda, q)$ is identified with the total surplus,

$$W(\lambda, q) := T(\lambda) - \lambda \frac{s}{q} - \lambda c,$$

where $T(\lambda) = \int_0^1 v\lambda e^{-\lambda(1-G(v))} g(v)dv = \int_0^1 [1 - e^{-\lambda(1-G(v))}]dv$ is the expected value of the first-order statistic given Poisson(λ)-distributed participation. Let λ^w and q^w denote the welfare maximizing magnitudes.

Proposition 7 *(i)* $q^w = 1$. *(ii) If* $U_o(0) > s + c$, *then* λ^w *is the unique level satisfying*

$$U_o(\lambda) = c + s. \tag{15}$$

If $U_o(0) \leq s + c$, *then* $\lambda^w = 0$.

Proof: (i) Obvious. (ii) Note that

$$T'(\lambda) = \int_0^1 (1 - G(v)) [1 - e^{-\lambda(1-G(v))}]dv = U_o(\lambda),$$

using (2) for the second equality. Since U_o is strictly decreasing, T is strictly concave. It follows that, (15) is the first-order condition for welfare maximization and the condition is sufficient, proving the claim. ■

The critical equality is

$$T'(\lambda) = U_o(\lambda). \tag{16}$$

For intuition, recall the equivalence of the expected payoffs to those of the second-price auction where each bidder's payoff is equal to his marginal contribution to the total surplus.

There are two types of inefficiency in equilibrium. First, as we already know, we can have $q^* < 1$ in equilibrium, which immediately means wasted recruiting effort. Second, as shown below, for almost all (s, c) combinations in the PO scenario, $\lambda^* \neq \lambda^w$ and both excessive participation, $\lambda^* > \lambda^w$, and deficient participation, $\lambda^* < \lambda^w$, may arise in equilibrium.

For the equilibrium of the PO scenario to coincide with the welfare maximum, we must have $R'_o(\lambda^*) = s$ and $U_o(\lambda^*) = s + c$. Since both U_o and R'_o are independent of s and c, these equalities in general cannot be expected to hold simultaneously. Thus, in general, the equilibrium does not maximize welfare.

Figure 12 depicts a possible relation between $U_o(\lambda)$ and $R'_o(\lambda)$. The relevant features of this relation are consistent with the case of uniform distribution of values.

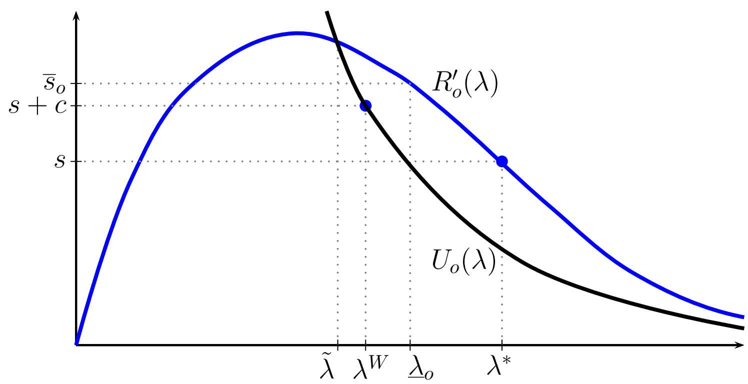

Figure 12: Welfare.

In this case, since for any $\lambda \geq \underline{\lambda}_o$, $U_o(\lambda) < R'_o(\lambda)$, it follows that $\lambda^* > \lambda^w$ in any equilibrium with trade. If $\lambda^* < \overline{\lambda}^c$, then $s + c > s = R'_o(\lambda^*) > U_o(\lambda^*)$; if $\lambda^* = \overline{\lambda}^c$, then $s + c > c = U_o(\lambda^*)$. In the case of $\lambda^* = \overline{\lambda}^c$, there is also the inefficiency of $q^* < 1$ (except when s is exactly equal to $R'_o(\overline{\lambda}^c)$). On the other hand, there is a range of (s, c) combinations such that $s + c < U_o(0)$ requires trade, $\lambda^w > 0$, but either $s > \overline{s}_o$ or $\overline{\lambda}^c < \underline{\lambda}_o$ precludes trade in equilibrium, meaning, $\lambda^w > \lambda^* = 0$.

We did not examine in detail the relationship between equilibrium and welfare maximum in the PU scenario. However, the observation that the equilibrium is generally inefficient should hold for that scenario as well. Since the maximal equilibrium in the PU scenario involves lower participation than in the equilibrium of the PO scenario, the extent of inefficient excessive recruiting would be more moderate.

For a general G (satisfying our assumptions), it was established before that the general shapes of $U_o(\lambda)$ is decreasing and R'_o is single peaked as shown in Figure 12. The fact that U_o intersect R'_o for the first time at some point $\tilde{\lambda}$ to the right of the maximum of R'_o holds for general G (see Claim 20 in the appendix). Some other details in the figure have not been established analytically for a general G,[18] but these details do not affect the general understanding of the sub-optimality of the equilibrium.

6.2 Fees/subsidies to affect participation

The question of optimal entry subsidies or fees is of secondary importance for this paper. First, it belongs more to the "design" paradigm that assumes significant seller commitment power, which we try to de-emphasize in this paper. Second, entry fees and subsidies might be abused by non-serious bidders and sellers, and their credible implementation may require commitment and enforcement capabilities.

Here, we put aside those issues and consider briefly the possibility of a flat subsidy/fee that is offered to, or collected from, all bidders who enter the auction in the PO scenario. Let D denote this fee ($D < 0$ means it is a subsidy). The subsequent interaction is formally equivalent to the PO scenario with bidders' cost given by $c+D$ and seller's marginal cost given by $\frac{s}{q}-D$. Let $\lambda^*(D)$ and $q^*(D)$ be the unique equilibrium magnitudes given D, and $\overline{\lambda}^{c+D}$ the solution for $U_o(\overline{\lambda}^{c+D}) = c+D$.

Claim 16 *(i) If the seller can commit to λ, then profit is maximized at λ^w with $D > 0$.*
(ii) Suppose that the seller cannot commit to λ. If there exists a D that facilitates trade (i.e., $s - D \leq \overline{s}_o$ and $\overline{\lambda}^{c+D} \geq \underline{\lambda}_o$), then profit is maximized with D^* that satisfies $s - D^* = R'_o(\overline{\lambda}^{c+D^*})$: $\lambda^*(D^*) = \overline{\lambda}^{c+D^*}$ and $q^*(D^*) = 1$.

Part (ii) implies that the profit maximizing fee is related to the equilibrium configuration that prevails when fees cannot be imposed (i.e., the case of $D = 0$). If $\lambda^*(0) < \overline{\lambda}^c$ (i.e., recruiting is unconstrained when fees are not allowed), then $D^* > 0$—a fee. If $\lambda^*(0) = \overline{\lambda}^c$, than $D^* < 0$—a subsidy.

Proof of Claim 16: (i) By committing to λ^w and imposing an entry fee D that satisfies $U_o(\lambda^w) = c + D$, the seller creates the maximal possible surplus and fully appropriates it since bidders' payoff is 0. Since $U_o(\lambda)$ is decreasing and $\overline{\lambda}^c > \lambda^w$, it follows that $D > 0$.

[18] For a uniform distribution, we have shown that $U_o(\lambda)$ and $R'_o(\lambda)$ intersect only once and that $\widetilde{\lambda}$ is below $\underline{\lambda}_o$. For general G, we have not established these properties. However, loosely, speaking, we expect $U_o(\lambda)$ to be mostly below $R'_o(\lambda)$ since $R_o(\lambda)$ is below $T(\lambda)$ and converging to it. We have verified that, for a uninform distribution, these properties hold (there is a unique intersection

(ii) Given D, we noted that this is the PO scenario with seller cost $\frac{s}{q} - D$ and bidders' cost $c + D$. Thus, in equilibrium given D, either $\lambda^*(D) \leq \overline{\lambda}^{c+D}$ and $R_o'(\lambda^*(D)) = s - D$, or $\lambda^*(D) = \overline{\lambda}^{c+D}$ and $R_o'(\lambda^*(D)) = \frac{s}{q^*} - D$.

If $\lambda^*(D) < \overline{\lambda}^{c+D}$, then $D' > D$ such the inequality still holds yields $\lambda^*(D') > \lambda^*(D)$ and higher profit.

If $\lambda^*(D) = \overline{\lambda}^{c+D}$ and $R_o'(\lambda^*(D)) > s - D$, then $q^*(D) < 1$. In this case, a fee $D' < D$ defined by

$$s - D' = \frac{s}{q^*(D)} - D,$$

results in $q^*(D') = 1$, $\overline{\lambda}^{c+D'} > \overline{\lambda}^{c+D}$ and $\lambda^*(D) = \lambda^*(D')$. This and the equality of the marginal recruitment costs imply that the profits for D and D' are equal as well. But then the argument of the previous paragraph implies that a slightly higher fee than D' would be even more profitable.

Thus, by elimination, D^* satisfies $\lambda^*(D^*) = \overline{\lambda}^{c+D^*}$ and $R_o'(\overline{\lambda}^{c+D^*}) = s - D^*$. ■

Part (i) is established by Levin and Smith (1994) in the context of their model.[19] In contrast, in the absence of commitment to λ, the availability of fees and subsidies does not necessarily improve welfare. For example, if $\lambda^w < \lambda^* < \overline{\lambda}^c$ with no fees or subsidies, then the profit maximizing entry fee is strictly positive and will drive the equilibrium λ further away from λ^w.

6.3 Reserve price

This subsection discusses the effects of a reserve price r—a minimum bid below which the good is not sold. Before we turn to the details, it should be mentioned that the imposition of a reserve price requires commitment power that might not be available in the less formal settings that we have in mind. However, it is still interesting to understand the role of such instruments even if their use is limited or imperfect.

Assume that the auctions in both scenarios are subject to a reserve price $r > 0$ (not necessarily the optimal one). The equilibrium would differ in some details from the case of $r = 0$ analyzed above but not in the main qualitative insights. Graphically, the marginal revenue curves in the diagrams would change somewhat:

[19] The critical argument is that (16), that is, bidder entry is surplus maximizing, and the seller can extract the full surplus by an appropriate fee.

for small values of λ they lie above the $r = 0$ curve; in particular, the intercept at $\lambda = 0$ is $r(1 - G(r))$ rather than 0, and, for large values of λ, they lie below the $r = 0$ curve. However, their general properties (like single peakedness of $dR_o/d\lambda$ and the relationship between the PO and PU curves) would remain the same, and the general relationship between the curves and the nature of the equilibria would also not change. One immediate implication of the intercept at $\lambda = 0$ being $r(1 - G(r))$ is that, in the PU scenario, the no-trade equilibrium $\lambda = 0$ will continue to exist only for $s \geq r(1 - G(r))$. For lower level of s, the equilibrium necessarily involves trade. Bidders entry decision is also affected since the reserve price lowers the benefit of entry for any level of anticipated participation.

Recall from the literature that, under the maintained assumptions on G, the revenue maximizing $r_{\max}$ for a standard auction satisfies $r = \frac{1-G(r)}{g(r)}$. It follows immediately that this is also true for the first-price auction with stochastic participation of the PU scenario. Therefore, if the seller commits to r only after bidders enter, then the profit maximizing r is $r_{\max}$. Of course, since $r_{\max}$ maximizes the revenue at any realized auction, it also maximizes the expected revenue in both scenarios, given any fixed participation rate λ.

Let us add r as an argument and write $U_o(\lambda; r)$, $R_o(\lambda; r)$, $\Pi_o(\lambda, q; r)$, etc.

Claim 17 *(i) For a given λ, $R_o(\lambda; r)$ (and hence*[20] *$R_u(\lambda, \beta_\lambda(r))$) is maximized at $r_{\max}$.*
(ii) If the seller commits to r only after bidders enter, the reserve price is $r_{\max}$ in any equilibrium.

If the seller can commit to a reserve price in advance, then it affects entry and, therefore, the profit maximizing r may be different from $r_{\max}$. Suppose that the seller commits to a reserve price r, and then the interaction proceeds according to the PO scenario. Essentially the same arguments presented in the $r = 0$ case establish that, in the subgame following the selection of r, there is a unique equilibrium. Let $\lambda^*(r)$, $q^*(r)$ and $\overline{\lambda}^c(r)$ denote the equilibrium magnitudes in the subgame following r, and let r^* denote the seller's profit maximizing r, i.e., $r^* = \arg\max_r \Pi_o(\lambda^*(r), q^*(r); r)$.

Claim 18 *In the PO scenario:*
(i) If $\lambda^*(r^*) > 0$ and bidders' entry does not constrain the equilibrium, i.e., $\lambda^*(r^*) <$

[20] By revenue equivalence, $R_u(\lambda, \beta_\lambda(r)) = R_o(\lambda, r)$.

$\overline{\lambda}^c(r^*)$, *then* $r^* = r_{\max}$.

(ii) If bidders' entry constrains the equilibrium, i.e., $\lambda^*(r^*) = \overline{\lambda}^c(r^*)$, *then* $r^* \neq r_{\max}$.

Both parts of this claim are almost immediate. In Part (i), bidders' entry does not constrain the seller, and so there is no reason to deviate from $r_{\max}$. In Part (ii), bidders' entry considerations do constrain the equilibrium, so the first-order effect of a change of r at $r = r_{\max}$ is its effect on entry, which does not vanish.

The introduction of $r > 0$ affects the seller's profit and the bidders' expected benefit at each participation level. First, it makes the auction more profitable. This increases the range of s for which an equilibrium with trade can be sustained, i.e., $\overline{s}_o(r) > \overline{s}_o(0)$. Second, it lowers bidders' benefit from entry for any expected level of participation resulting in a lower maximal level of participation for which entry is profitable, i.e., $\overline{\lambda}^c(r) < \overline{\lambda}^c(0)$.

Intuitively, it seems that it should be that $r^* < r_{\max}$ because lowering r slightly when it is above $r_{\max}$ increases the profitability of the auction and relaxes the bidders' entry constraint. However, this intuition is incomplete, since q^* would change at the same time and the total recruitment cost would increase. For this reason, although $r^* < r_{\max}$ might be true in general, we have been able to establish it only under additional conditions that guarantee that the $\overline{\lambda}^c(r)$'s corresponding to the r's in the relevant range are not too small. This would the case if c is not too large.[21]

Analogous results most likely hold for the equilibria with trade in the PU scenario, but we did not prove it. However, it is immediate that, if $s \leq r[1 - G(r)]$ and c is not prohibitive, the no-trade outcome is not an equilibrium in the PU scenario. Since $r[1 - G(r)]$ is maximized at $r_{\max}$, it follows that if $s < r_{\max}[1 - G(r_{\max})]$, the seller can avoid the no-trade outcome by selecting an appropriate reserve price.

6.4 Bidders learn their value before entering

In the models discussed so far, bidders learned their private values only after incurring the cost c. This is a scenario of costly information acquisition. If, however, the values are readily known and the main effort is bid preparation or other costs associated with the bidding, then a more suitable model would have the bidders' costly entry decision taking place with knowledge of their private values. This subsection

[21] The precise condition is $\overline{\lambda}^c(r)\,[2 - G(r)] > 1$.

outlines how the analysis can be expanded to cover this case. A full analysis of this case would take too much space, but the discussion suggests that this is doable and is not likely to change the main qualitative insights of the models discussed above. In particular, we show below for the case of small marginal recruitment cost s that the recruitment cost is higher under the PO scenario than under the PU scenario.

Consider the PO scenario in this case. If entry is profitable for a bidder with value v, then it is profitable for bidders with higher values. Therefore, bidders would enter if and only if their value v exceeds a certain cutoff $\underline{v} \in (0, 1)$ at which a prospective bidder is just indifferent about entry.

As before, let γ denote the Poisson rate of contacts made by the seller. The probability that a contacted bidder enters (the counterpart of q above) is $1 - G(\underline{v})$ and the effective Poisson rate of entry into the auction is $\lambda = \gamma(1 - G(\underline{v}))$. For a given $\underline{v}$, the seller's problem of choosing γ at marginal cost s is equivalent to choosing λ at marginal cost $s/(1 - G(\underline{v}))$ and, as before, it would be convenient to express the relevant magnitudes in terms of λ rather than in terms of γ.

The bidding game among entrants is a first-price auction with observable participation and private values independently drawn from $[\underline{v}, 1]$. In equilibrium, if there is only one participant, the winning bid is 0; if there are two or more participants, the bids lie in $(\underline{v}, 1]$ and are monotone in values. Therefore, the seller's revenue is 0 if fewer than two bidders enter and it is the appropriate equilibrium winning bid which lies in $(\underline{v}, 1]$ otherwise. Given λ and $\underline{v} < 1$, the seller's payoff $\Pi_0(\lambda, \underline{v})$ is

$$\Pi_0(\lambda, \underline{v}) = R_o(\lambda, \underline{v}) - \lambda s/(1 - G(\underline{v})). \tag{17}$$

Since the equilibrium bids in the bidding subgame with two or more bidders are monotone in values, the marginal entering bidder $\underline{v}$ will win the good only if he is the sole entrant, in which case he will pay 0. The probability that the bidder with value $\underline{v}$ is the sole entrant is $e^{-\lambda}$. Therefore, this bidder's payoff from entering is $\underline{v}e^{-\lambda}$ and the indifference of bidder $\underline{v}$ with respect to entry implies

$$\underline{v}e^{-\lambda} = c. \tag{18}$$

An equilibrium with trade is characterized by some $\lambda > 0$ and $\underline{v} < 1$ such that λ maximizes $\Pi_0(\lambda, \underline{v})$ and $\underline{v}$ satisfies (18).

Consider next the PU scenario. Here, too, bidders enter if their value v exceeds a threshold $\underline{v}$. Given the Poisson rate γ of contacts made by the seller, the effective Poisson rate of entry into the auction is $\lambda = \gamma(1 - G(\underline{v}))$. As before, it would be convenient to express the relevant magnitudes in terms of λ rather than in terms of γ. The bidding game among entrants is a first-price auction with unobservable participation and independent private values drawn from $[\underline{v}, 1]$. Given that bidders expect an effective Poisson rate $\widehat{\lambda}$ of entry, the equilibrium bidding strategy of the entering bidders, $\beta(v; \underline{v}, \widehat{\lambda})$, is strictly increasing in $v \in [\underline{v}, 1]$.

With probability $e^{-\lambda}$ no bidders enter and the seller's revenue is 0; otherwise it is the winning bid. Let $R_u(\lambda, \underline{v}, \widehat{\lambda})$ denote the expected winning bid given λ, $\widehat{\lambda}$ and $\underline{v} < 1$. The seller's payoff $\Pi_u(\lambda, \underline{v}, \widehat{\lambda})$ is

$$\Pi_u(\lambda, \underline{v}, \widehat{\lambda}) = R_u(\lambda, \underline{v}, \widehat{\lambda}) - \lambda s/(1 - G(\underline{v})). \tag{19}$$

Since $\beta(v; \underline{v}, \lambda)$ is strictly increasing in v, the marginal entering bidder $\underline{v}$ will win the good only if he is the sole entrant. Therefore, $\beta(\underline{v}; \underline{v}, \widehat{\lambda}) = 0$, and $\underline{v}$ satisfies the same entry condition as above,

$$\underline{v} e^{-\lambda} = c. \tag{20}$$

An equilibrium with trade is characterized by $\lambda > 0$ and $\underline{v} < 1$ such that λ maximizes $\Pi_u(\lambda, \underline{v}, \widehat{\lambda})$ with $\widehat{\lambda} = \lambda$ and $\underline{v}$ satisfies (20).

Existence of equilibrium is somewhat more complicated than in the previous scenarios of subsections 2.2 and 3.2 since, here, $\underline{v}$ varies with λ. We do not undertake the full equilibrium analysis for this case. Instead, we conjecture that, for sufficiently small s and c, there exists an equilibrium with trade in both scenarios and, under this assumption, compare the equilibrium outcomes in the limit as $s \to 0$.

Let $\lambda_i(s)$ and $\underline{v}_i(s)$ denote the equilibrium magnitudes in the equilibrium with maximal λ in the PO ($i = o$) and PU ($i = u$) scenarios, respectively.[22]

Claim 19 *(i)* $\lim_{s \to 0} \lambda_i(s) = -\ln c$ *for* $i = u$ *and* $i = o$.

(ii) In the limit, total recruitment cost is higher in the PO scenario

$$\lim_{s \to 0} \lambda_o(s) \frac{s}{1 - G(\underline{v}_o(s))} = (\ln c)^2 c > \lim_{s \to 0} \lambda_u(s) \frac{s}{1 - G(\underline{v}_u(s))}.$$

[22] In the PO scenario this is probably the unique equilibrium. However, we did not prove it because it would be essentially a repetition of the analysis in subsection 2.2.

Thus, in the limit as $s \to 0$, both scenarios give rise to the same level of effective participation but total recruitment cost is higher under the PO scenario. This ranking of the costs is the same as in the information acquisition case in which bidders learn their values only after incurring c.

6.5 Uniqueness of equilibrium in the PO scenario

The equilibrium outcome of the PO scenario is unique for almost all values of s and c (except when $s = \bar{s}_0$ or $\overline{\lambda}^c = \underline{\lambda}_o$) given the refinement imposed by Equilibrium Condition 2(iii). Without the refinement, the no-trade outcome is always an equilibrium. More precisely, without the condition,

- if $s > \bar{s}_0$ or $\overline{\lambda}^c < \underline{\lambda}_o$, then no-trade is the unique equilibrium outcome;
- if $s \leq \bar{s}_0$ and $\overline{\lambda}^c \geq \underline{\lambda}_o$, there two equilibrium outcomes: one with $\lambda^* > 0$ and one with $\lambda^* = 0$.

In the case with $s < \bar{s}_0$ and $\overline{\lambda}^c > \underline{\lambda}_o$, the additional no-trade equilibrium $\lambda^* = 0$ is supported by the off-path belief $\mu(\overline{\lambda}^c) = 1$ and $q^* \in (0, \frac{s}{\bar{s}_0})$. That is, bidders who are contacted off-path conjecture that $\lambda = \overline{\lambda}^c$, which makes them just indifferent among all choices of q, including q^* that makes it unprofitable for the seller to recruit. Such an equilibrium violates Condition 2 (iii) since the seller's best response to $q^* < \frac{s}{\bar{s}_0}$ is $\lambda = 0$, rather than the conjectured $\overline{\lambda}^c$.[23]

Observe that such a no-trade equilibrium is "not convincing" on other grounds as well.

First, when $s < \bar{s}_0$ and $\overline{\lambda}^c > \underline{\lambda}_o$, the no-trade equilibrium is Pareto dominated by the equilibrium with trade. Second, it is not robust to perturbations. Consider a perturbation in which the seller is required to choose at least an effort $\gamma \geq \varepsilon > 0$, for some small $\varepsilon > 0$. As $\varepsilon \to 0$, this perturbed game has a unique limit outcome that corresponds to the equilibrium with trade. This is because, for any $q \in (0, 1)$ that is small enough so that $\frac{s}{q} \geq \bar{s}_0$, the seller's best response is either $\lambda = \varepsilon$ or $\underline{\lambda}_o$ (or mixing between them). However, in all these cases, $\overline{\lambda}^c > \underline{\lambda}_o$ implies that the bidders would have a strict incentive to enter, implying $q = 1$.

Formally, since this game is not finite (there is both, a continuum of actions and an unbounded number of players), we cannot apply directly the concept of stability

[23] When $q^* = \frac{s}{\bar{s}_o}$, then $\lambda = \underline{\lambda}_o$ is also a best response but still $\underline{\lambda}_o \neq \overline{\lambda}^c$.

in the sense of Kohlberg-Mertens (1986). However, if we look at a discretized version in which the seller chooses λ from a finite grid (that contains 0, $\overline{\lambda}^c$, and $\underline{\lambda}_o$), we can define a refinement in the spirit of stability, requiring that the equilibrium be immune to all vanishing fully mixed perturbations. It is fairly immediate that the no-trade equilibrium will fail such refinement, while the unique equilibrium with trade will survive it.[24]

We can also confirm the instability of the no-trade equilibrium indirectly by observing that it fails the Invariance Property of Stable equilibrium. To see this, consider the equivalent extensive form in which the seller first chooses between $\lambda = 0$, which terminates the game, and another action, "$\lambda > 0$", which stands for all positive recruitment efforts. The action "$\lambda > 0$" is followed by the seller's choice of the specific λ and the subsequent bidders' decisions. The unique SPE in this extensive form is the equilibrium with trade by the same argument presented above for the variation that embodies the constraint $\gamma \geq \varepsilon$.

7 Appendix

7.1 Proof for the PO Scenario

7.1.1 Bidders ex-ante expected payoff

Claim 1: U_o is strictly decreasing and continuous,

$$U_o(\lambda) = \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]dv.$$

Proof: The expression for U_o can be derived directly. But we use instead an indirect argument noting that the total surplus (gross of the recruitment costs) is the expectation of the first-order statistic of Poisson(λ),

$$\text{Total Surplus}(\lambda) = \int_0^1 \left[1 - e^{-(1-G(v))\lambda}\right] dv$$

and is equal to the sum of the revenue, $R_o(\lambda)$, and total bidders' expected payoff,

[24] Note, however, that the no-trade equilibrium will survive a refinement in the spirit of perfect equilibrium that is defined in an anlogous way since we can focus on a sequence of perturbations for which the expectation conditional on $\lambda > 0$ is $\overline{\lambda}^c$.

$\lambda U_o(\lambda)$. Therefore,

$$U_o(\lambda) = \left(\int_0^1 \left[1 - e^{-(1-G(v))\lambda} \right] dv - R_o(\lambda) \right) / \lambda$$

Substituting $R_o(\lambda)$ from its expression presented in (22) below, we get the result. ■

7.1.2 PROOFS OF PROPOSITIONS ABOUT PO

Claim 3 Revenue and optimality:

1. $R_o(\lambda)$ is strictly increasing, $R_o(0) = 0$ and $\lim_{\lambda \to \infty} R_o(\lambda) = 1$.

2. $R_o(\lambda)$ is continuously differentiable, $R'_o(0) = 0$, $R'_o(\lambda) \to 0$ as $\lambda \to \infty$ and R'_o is single peaked.

3. $\frac{R_o(\lambda)}{\lambda}$ is single peaked; at its peak $\frac{R_o(\lambda)}{\lambda} = R'_o(\lambda)$.

Proof of Claim 3: Let F^{SPA} denote the price distribution arising in the dominant strategy equilibrium of the second-price auction (SPA) format, given the same participation process[25]

$$F^{SPA}(b|\lambda) = e^{-(1-G(b))\lambda} + e^{-(1-G(b))\lambda}\left((1 - G(b))\lambda\right) \tag{21}$$

By revenue equivalence,

$$R_o(\lambda) = \int_0^1 \left(1 - F^{SPA}(b|\lambda)\right) db = \int_0^1 \left[1 - e^{-(1-G(b))\lambda} - e^{-(1-G(b))\lambda}\left((1 - G(b))\lambda\right)\right] db \tag{22}$$

Therefore,

$$\frac{d}{d\lambda} R_o(\lambda) = \int_0^1 \frac{d}{d\lambda}\left(1 - F^{SPA}(b)\right) db = \int_0^1 \lambda \left(1 - G(b)\right)^2 e^{-(1-G(b))\lambda} db \tag{23}$$

Parts 1 and 2:

[25] F^{SPA} is not the same as winning bid distribution of the FPA format we consider, and this is not claimed.

- Positivity, continuity, and values at $\lambda = 0$ and $\lambda \to \infty$ are obvious from (23).
- To establish that R_o' is single-peaked, consider the second derivative

$$\frac{d^2}{d\lambda^2} R_o(\lambda) = \int_0^1 (1 - G(b))^2 e^{-(1-G(b))\lambda} db - \int_0^1 \lambda (1 - G(b))^3 e^{-(1-G(b))\lambda} db \tag{24}$$

$$= e^{-\lambda} \left(\frac{1}{g(0)} - \int_0^1 (1 - G(b))^2 e^{G(b)\lambda} \left[b - \frac{1 - G(b)}{g(b)} \right]_b' db \right)$$

  where the transition from 1st to 2nd line above uses integration by parts.

- Recall that by assumption, $\left[b - \frac{1-G(b)}{g(b)}\right]_b' > 0$. Thus, the integral on the last line of (24) is positive and increasing in λ, while the first term is positive and independent of λ. Therefore, $\frac{d^2}{d\lambda^2} R_o(\lambda) < 0$ for large λ and, once it turns negative, it stays negative. Inspection of the first line of (24) reveals that $\frac{d^2}{d\lambda^2} R_o(\lambda) > 0$ for $\lambda \in [0, \varepsilon]$ for some $\varepsilon > 0$. These two observations together imply that $\frac{d}{d\lambda} R_o(\lambda)$ is single peaked.

Part 3: This follows immediately from Parts 1 and 2 and from $d(R_o(\lambda)/\lambda)d\lambda = \left[R_o'(\lambda) - \frac{R_o(\lambda)}{\lambda}\right]/\lambda$. ■

7.2 Proofs of Propositions 3, 2 and Corollary 1

The following Lemma summarizes the implications of the backward induction of subsection 2.1 for the form of the equilibrium and will be used in the proofs of the propositions.

Lemma 1 *if (λ^*, q^*) is an equilibrium, then (i) Either $\lambda^* = 0$ or $\underline{\lambda}_o \leq \lambda^* \leq \overline{\lambda}^c$ and $R_o'(\lambda^*) = \frac{s}{q^*}$; (ii) if $q^* \in (0,1)$, then either $\lambda^* = \overline{\lambda}^c$ or $\lambda^* = 0$ and $E_\mu[U_o(\lambda)] = c$; (iii) if $\lambda^* = 0$, the support of μ is contained in $\{0, \underline{\lambda}_o\}$.*

Proof: Part (i). As noted in (5), $\lambda^* \leq \overline{\lambda}^c$. The rest follows immediately from Claim 2.

Part (ii) follows immediately from (3).

Part (iii). $\lambda^* = 0$ implies that $\max \Pi_o(\cdot, q^*) = 0$. Therefore, by the last equilibrium condition $\Pi_o(\lambda, q^*) = 0$, for any λ in the support of μ. The result then follows from Claim 2 and the fact that the only $\lambda \geq \underline{\lambda}_o$ s.t. $R_o'(\lambda) = \frac{s}{q}$ and $\Pi_o(\lambda, q^*) = 0$ is $\underline{\lambda}_o$. ■

Proposition 3 If $s > \bar{s}_o$ or $\overline{\lambda}^c < \underline{\lambda}_o$, then $\lambda^* = 0$ (no trade) is the unique equilibrium outcome.

Proof of Proposition 3: It follows from Lemma (1) that the only possible equilibrium outcome in these cases is $\lambda^* = 0$. It remains to establish that equilibria with $\lambda^* = 0$ indeed exist.

If $s > \bar{s}_o$, then $\lambda = 0$ is the uniquely optimal choice of the seller for any q^*. Therefore, $\lambda^* = 0$ with $q^* = 1$ and $\mu(0) = 1$ is an equilibrium.

If $s \leq \bar{s}_o$ and $\overline{\lambda}^c < \underline{\lambda}_o$, then the following is an equilibrium: $\lambda^* = 0$, $q^* = s/\bar{s}_o$ and μ with support on $\{0, \underline{\lambda}_o\}$ such that $\mu(0)U(0) + \mu(\underline{\lambda}_o)U(\underline{\lambda}_o) = U(\overline{\lambda}^c)$. The choice of q^* guarantees that $\max \Pi_o(\lambda, q^*) = 0$ and it is achieved at $\lambda = 0$ and $\lambda = \underline{\lambda}_o$. The choice of μ guarantees that $E_\mu(U(\lambda)) = U(\overline{\lambda}^c) = c$ so q^* is optimal for the bidders. ■

Proposition 2 If $s < \bar{s}_o$ and $\overline{\lambda}^c > \underline{\lambda}_o$, then the unique equilibrium has $\lambda^* = \min\{\overline{\lambda}^c, \lambda_o(s)\}$.

Proof of Proposition 2: If $\overline{\lambda}^c \geq \lambda_o(s)$, then $\lambda^* = \lambda_o(s)$ and $q^* = 1$ is an equilibrium. By definition $\lambda_o(s) = \arg\max \Pi_o(\cdot, 1)$. The optimality of $q^* = 1$ for bidders follows from $\overline{\lambda}^c > \lambda^*$ and (3).

If $\overline{\lambda}^c < \lambda_o(s)$, then $\lambda^* = \overline{\lambda}^c$ and q^* satisfying $\lambda_o(\frac{s}{q^*}) = \overline{\lambda}^c$ constitute an equilibrium. By the choice of q^*, $\overline{\lambda}^c = \arg\max \Pi_o(\cdot, q^*)$. Since by definition, $U\left(\overline{\lambda}^c\right) = c$, the optimality of q^* for bidders follows.

It follows from Lemma (1) that, if $\lambda^* > 0$, then $\lambda^* \geq \underline{\lambda}_o$ and $R_o'(\lambda^*) = \frac{s}{q^*}$ and that q^* may differ from 1 only if $\lambda^* = \overline{\lambda}^c$. Therefore, the only possibilities are $\lambda^* = \overline{\lambda}^c$ or $\lambda^* = \lambda_o(s)$. If $\overline{\lambda}^c > \lambda_o(s)$, then for any q, $R_o'\left(\overline{\lambda}^c\right) < \frac{s}{q}$ so $\overline{\lambda}^c$ cannot be an equilibrium outcome. If $\overline{\lambda}^c < \lambda_o(s)$, then $U(\lambda_o(s)) < c$ so $\lambda_o(s)$ cannot be an equilibrium outcome. Thus, if $\lambda^* > 0$, it must be that $\lambda^* = \min\{\overline{\lambda}^c, \lambda_o(s)\}$.

It remains to show that there is no equilibrium with $\lambda^* = 0$. It follows from Lemma (1) that the support of the belief μ in such an equilibrium is contained in $\{0, \underline{\lambda}_o\}$. Since $\overline{\lambda}^c > \underline{\lambda}_o$, q^* must be 1. But then $\Pi_o(\lambda_o(s), 1) > \Pi_o(\underline{\lambda}_o, 1) = \Pi_o(0, 1)$ contradicting the equilibrium condition on beliefs. ■

Corollary 1 Consider a sequence $(s_k)_{k=1}^{\infty}$ with $s_k \to 0$, and let (λ_k^*, q_k^*) be the corresponding equilibrium outcomes.

Corollary 3 *1. If $c = 0$, then $q_k^* = 1$ for all s_k, $\lambda_k^* \to \infty$, and $s_k\lambda_k^* \to 0$.*

2. If $c > 0$ and $\overline{\lambda}^c \geq \underline{\lambda}_o$, then, for all $s_k < R_o'(\overline{\lambda}^c)$,

$$\lambda_k^* = \overline{\lambda}^c, \quad \frac{s_k}{q_k^*}\lambda_k^* = \text{constant} \quad \textit{and} \quad \Pi_o(\lambda_k^*, q_k^*, s_k) = \text{constant}.$$ [26]

Proof of Corollary 1. Part 1. Since $c = 0$, $q_k^* = 1$ for all k. Let $\lambda_k = \frac{1}{\sqrt{s_k}}$. Since $\lim_{\lambda\to\infty} R_o(\lambda) = 1$, we have $\Pi_o(\lambda_k, q_k^*, s_k) = R_o(\lambda_k) - \sqrt{s_k} \to 1$. From optimality, $\Pi_o(\lambda_k^*, q_k^*, s_k) \geq \Pi_o(\lambda_k, q_k^*, s_k)$ for all k. Hence $\lim \Pi_o(\lambda_k^*, q_k^*, s_k) \geq 1$. This together with $\Pi_o(\lambda_k^*, q_k^*, s_k) \leq R_o(\lambda_k^*) \leq 1$ implies that $\lim_{k\to\infty} \lambda_k^* s_k = 0$.

Part 2. For all $s_k < R_o'\left(\overline{\lambda}^c\right)$, $\lambda_k^* = \overline{\lambda}^c$ and $\frac{s_k}{q^*} = R_o'\left(\overline{\lambda}^c\right)$. Therefore, $\frac{s_k}{q^*}\lambda_k^* = \overline{\lambda}^c R_o'\left(\overline{\lambda}^c\right) = \text{constant}$ and $\Pi_o(\lambda_k^*, q_k^*, s_k) = R_o\left(\overline{\lambda}^c\right) - \overline{\lambda}^c R_o'\left(\overline{\lambda}^c\right) = \text{constant}$. ■

7.3 Proofs for the PU Scenario

7.3.1 The Bidding Strategy

Claim 5 Given belief $\hat{\lambda}$, the unique symmetric equilibrium bidding strategy is

$$\beta_{\hat{\lambda}}(v) = v - \int_0^v e^{-\hat{\lambda}(G(v)-G(x))}dx. \tag{25}$$

Note that this expression is valid also for $\hat{\lambda} = 0$.

Proof: Let SPA stand for second-price auction when the its dominant strategy equilibrium is played. By revenue equivalence

$$\begin{aligned}\beta_{\hat{\lambda}}(v) &= E[\text{payment} \mid v; \text{win SPA}] \\ &= \sum_{i=0} \Pr(i \text{ other bidders} \mid v; \text{win SPA})\, E[\text{payment} \mid v; \text{win SPA; i other bidders}]\end{aligned}$$

Substituting

$$\Pr(i \text{ other bidders} \mid v; \text{win SPA}) = \frac{\frac{e^{-\lambda}\lambda^i}{i!}G^i(v)}{e^{-\lambda(1-G(v))}}$$

[26] These constants are $\overline{\lambda}^c R_o'\left(\overline{\lambda}^c\right)$ and $R_o\left(\overline{\lambda}^c\right) - \overline{\lambda}^c R_o'\left(\overline{\lambda}^c\right)$, respectively.

Let $v_1^{(i)}$ denote the first-order statistic of i values, where $v_1^{(0)} = 0$. The above yields

$$\begin{aligned}
\beta_{\hat{\lambda}}(v) &= \frac{1}{e^{-\lambda(1-G(v))}} \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} G^i(v) E[\text{payment} \mid v; \text{win SPA; i others}] \\
&= \frac{1}{e^{-\lambda(1-G(v))}} \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} G^i(v) E\left[v_1^{(i)} | v_1^{(i)} \leq v\right] \\
&= \frac{1}{e^{-\lambda(1-G(v))}} \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} G^i(v) \int_r^v x \frac{dG^i(x)}{G^i(v)} \\
&= \frac{1}{e^{-\lambda(1-G(v))}} \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} G^i(v) \left(v - \int_0^v \frac{G^i(x)}{G^i(v)} dx\right)
\end{aligned}$$

and rewriting further

$$\begin{aligned}
\beta_{\hat{\lambda}}(v) &= \frac{1}{e^{-\lambda(1-G(v))}} \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} G^i(v) \left(v - \int_0^v \frac{G^i(x)}{G^i(v)} dx\right) \\
&= \frac{e^{-\lambda}}{e^{-\lambda(1-G(v))}} \sum_{i=0}^{\infty} \frac{(\lambda G(v))^i}{i!} v - \frac{1}{e^{-\lambda(1-G(v))}} \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \int_0^v G^i(x)\, dx \\
&= \frac{e^{-\lambda}e^{\lambda G(v)}}{e^{-\lambda(1-G(v))}} v - \frac{1}{e^{-\lambda(1-G(v))}} \int_0^v \sum_{i=0}^{\infty} \frac{e^{-\lambda}(\lambda G(x))^i}{i!} dx \\
&= v - \frac{1}{e^{-\lambda(1-G(v))}} \int_0^v \sum_{i=0}^{\infty} \frac{e^{-\lambda}(\lambda G(x))^i}{i!} dx \\
&= v - \int_0^v e^{-\lambda(G(v)-G(x))} dx
\end{aligned}$$

■

7.3.2 PROOFS OF PROPOSITIONS ABOUT PU

Claim 7:(i) $R_u(\lambda, \beta_{\hat{\lambda}})$ is twice differentiable (in λ and $\hat{\lambda}$) and, for $\hat{\lambda} > 0$, it is strictly concave in λ.;

(ii) The function $\xi(\lambda) := \frac{\partial}{\partial\lambda} R_u(\lambda, \beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda}$ is continuous, $\xi(0) = 0$, and $\lim_{\lambda\to\infty} \xi(\lambda) = 0$.

Proof of Claim 7: Let $F_u(\cdot|\lambda, \beta_{\hat{\lambda}})$ be the distribution of the price received by the seller, given that actual participation is distributed Poisson (λ) and all bidders bid according to $\beta_{\hat{\lambda}}$, where the no-trade event is identified with price 0. Let $\widetilde{\beta}_{\lambda}^{-1}$ denote the "generalized inverse" of β_{λ} defined as follows: $\widetilde{\beta}_{\lambda}^{-1} = \beta_{\lambda}^{-1}$ over $[0, \beta_{\lambda}(1))$ and $\widetilde{\beta}_{\lambda}^{-1} \equiv 1$ over $[\beta_{\lambda}(1), 1]$. Note that this implies that $\widetilde{\beta}_{0}^{-1} \equiv 1$. Therefore,

$$F_u\left(b|\lambda, \beta_{\hat{\lambda}}\right) = e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)\right)\right)} \tag{26}$$

Observe that F_u is indeed a c.d.f. and is well defined for $\hat{\lambda} = 0$ as well: Since $\beta_{\hat{\lambda}}$ is non-decreasing for any $\hat{\lambda} \geq 0$, $\widetilde{\beta}_{\hat{\lambda}}^{-1}$ is non-decreasing and so is F; since $\widetilde{\beta}_{\hat{\lambda}}^{-1}(1) = 1$, $F_u\left(1|\lambda, \beta_{\hat{\lambda}}\right) = 1$, and $F_u\left(0|\lambda, \beta_{\hat{\lambda}}\right) = e^{-\lambda} < 1$. Then,

$$R_u(\lambda, \beta_{\hat{\lambda}}) = \int_0^1 [1 - F_u\left(b|\lambda, \beta_{\hat{\lambda}}\right)]db = \int_0^1 [1 - e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)\right)\right)}]db \tag{27}$$

where the last equality is obtained by substitution from (26). This and the functional form of $\beta_{\hat{\lambda}}$ presented in (**8**) imply that R_u is twice continuously differentiable in λ and $\hat{\lambda}$.

$$\frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}}) = \int_0^1 \left(1 - G\left(\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)\right)\right) e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)\right)\right)} db \tag{28}$$

If $\hat{\lambda} > 0$, then $\widetilde{\beta}_{\hat{\lambda}}^{-1}(b) < 1$ for all $b < 1$. Therefore,

$$\frac{\partial^2}{\partial \lambda^2} R_u(\lambda, \beta_{\hat{\lambda}}) < 0 \tag{29}$$

implying that $R_u\left(\lambda, \beta_{\hat{\lambda}}\right)$ is strictly concave in λ and so is $\Pi_u(\lambda, \beta_{\hat{\lambda}}, q)$.

By definition

$$\xi\left(\lambda\right) = \int_0^1 \left(1 - G\left(\widetilde{\beta}_{\lambda}^{-1}(b)\right)\right) e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\lambda}^{-1}(b)\right)\right)} db$$

The continuity of $\xi\left(\lambda\right)$ and its other properties follow directly from this functional form and the properties of $\widetilde{\beta}_{\lambda}^{-1}$. ■

7.4 Comparison of PO/PU Scenarios

Claim 8: (i) $R_o'(\lambda) > \xi\left(\lambda\right)$, for all $\lambda > 0$; (ii) $\bar{s}_0 > \bar{s}_u$.

Proof of Claim 8: (1) By revenue equivalence $R_o(\lambda) = R_u(\lambda, \beta_\lambda)$ for every λ. Hence,

$$\frac{d}{d\lambda}R_o(\lambda) = \frac{d}{d\lambda}R_u(\lambda,\beta_\lambda) = \underbrace{\frac{\partial}{\partial\lambda}R_u(\lambda,\beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda}}_{=\xi(\lambda)} + \frac{\partial}{\partial\hat{\lambda}}R_u(\lambda,\beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda}.$$

Now, using (27)

$$\begin{aligned}\frac{\partial}{\partial\hat{\lambda}}R_u(\lambda,\beta_{\hat{\lambda}})|_{\hat{\lambda}=\lambda} &= \left(\frac{\partial}{\partial\hat{\lambda}}\int_0^1 [1-e^{-\lambda\left(1-G\left(\widetilde{\beta}_{\hat{\lambda}}^{-1}(b)\right)\right)}]db\right)|_{\hat{\lambda}=\lambda} \\ &= -\int_0^1 \lambda g\left(\widetilde{\beta}_\lambda^{-1}(b)\right)\frac{\partial}{\partial\lambda}\widetilde{\beta}_\lambda^{-1}(b)e^{-\lambda\left(1-G\left(\widetilde{\beta}_\lambda^{-1}(b)\right)\right)}db\end{aligned}$$

and from (8),

$$\frac{\partial}{\partial\lambda}\widetilde{\beta}_\lambda^{-1}(b) = -\frac{\frac{\partial}{\partial\lambda}\beta_\lambda(v)}{\frac{\partial}{\partial v}\beta_\lambda(v)} = -\frac{\int_0^v (G(v)-G(x))\,e^{-\lambda(G(v)-G(x))}dx}{\lambda g(v)\int_0^v e^{-\lambda(G(v)-G(x))}dx} < 0$$

where $v = \widetilde{\beta}_\lambda^{-1}(b)$. Therefore, we have $\frac{\partial}{\partial\hat{\lambda}}R_u(\lambda,\beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda} > 0$ for all λ, which implies Part (i) of the claim.

(ii)

$$R_u(\lambda,\beta_\lambda) = \int_0^\lambda \frac{\partial}{\partial t}R_u(t,\beta_\lambda)dt > \lambda\frac{\partial}{\partial\lambda}R_u(\lambda,\beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda}, \tag{30}$$

since (**29**) implies that $\frac{\partial}{\partial t}R_u(t,\beta_\lambda)$ is strictly decreasing in t. Since by revenue equivalence $R_u(\lambda,\beta_\lambda) = R_o(\lambda)$, for all λ, it follows from (30) that

$$\frac{R_o(\lambda)}{\lambda} > \frac{\partial}{\partial\lambda}R_u(\lambda,\beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda}.$$

The claim then follows from $\overline{s}_o = \max\frac{R_o(\lambda)}{\lambda}$ and $\overline{s}_u = \max\frac{\partial}{\partial\lambda}R_u(\lambda,\beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda}$. ■

7.5 Discussion and Extensions Section 6

7.5.1 Welfare

Claim 20 *(i) For any Λ, there is $\lambda > \Lambda$ such that $U_o(\lambda) < R_o'(\lambda)$. (ii) There is $\widetilde{\lambda} > \overline{\lambda}$ such that $U_o(\lambda) \geq R_o'(\lambda)$ for $\lambda \leq \widetilde{\lambda}$ and $U_o(\lambda) < R_o'(\lambda)$ at least over some* *interval just above* $\widetilde{\lambda}$.

Proof: Obviously, $R_o(\lambda)$ is also the residual surplus not received by the bidders

$$R_o(\lambda) = T(\lambda) - \lambda U_o(\lambda)$$

and $R_o(\lambda) \to T(\lambda)$ as $\lambda \to \infty$.

(i) If there was Λ s.t. $U_o(\lambda) > R_o'(\lambda)$ for all $\lambda \geq \Lambda$, then, by (16), for all such λ, $T(\lambda) - R_o(\lambda) > T(\Lambda) - R_o(\Lambda) > 0$ in contradiction to $R_o(\lambda) \to T(\lambda)$ as $\lambda \to \infty$.

(ii) By (16)

$$R_o'(\lambda) = -\lambda U_o'(\lambda) = \lambda \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]^2 dv \tag{31}$$

and

$$U_o(\lambda) - R_o'(\lambda) = U_o(\lambda) + \lambda U_o'(\lambda) = \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]\left[1 - (1 - G(v))\lambda\right] dv$$

$$\begin{aligned} R_o''(\lambda) &= -U_o'(\lambda) - \lambda U_o''(\lambda) = \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]^2 dv \\ &\quad -\lambda \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]^3 dv \\ &= \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]^2 \left[1 - (1 - G(v))\lambda\right] dv \end{aligned} \tag{32}$$

Recall that $R_o'(\lambda)$ is single peaked and let $\overline{\lambda}$ denote the argument of the peak. Thus, $R_o''(\overline{\lambda}) = 0$ and it follows from (32) that there must be x s.t. $(1 - G(x))\,\overline{\lambda} = 1$, the integrand on the RHS of (32) is positive for $v > x$ and is negative for $v < x$. Therefore,

$$\begin{aligned} 0 &= R_o''(\overline{\lambda}) < \int_0^x e^{-(1-G(v))\overline{\lambda}}[1 - G(x)][1 - G(v)]\left[1 - \left(1 - G(v)\right)\overline{\lambda}\right] dv \\ &\quad + \int_x^1 e^{-(1-G(v))\overline{\lambda}}[1 - G(x)][1 - G(v)]\left[1 - \left(1 - G(v)\right)\overline{\lambda}\right] dv \\ &= [1 - G(x)] \int_0^1 e^{-(1-G(v))\overline{\lambda}}[1 - G(v)]\left[1 - \left(1 - G(v)\right)\overline{\lambda}\right] dv = [1 - G(x)][U_o(\overline{\lambda}) - R_o'(\overline{\lambda})] \end{aligned}$$

where the first inequality follows from $1-G(x)<1-G(v)$ for the range $v<x$ of negative integrand and $1-G(x)>1-G(v)$ for the range $v>x$ of positive integrand. Therefore, $U_o(\overline{\lambda})>R_o'(\overline{\lambda})$. Since U_o is decreasing and R_o' is increasing for $\lambda<\overline{\lambda}$, it follows that $U_o(\lambda)>R_o'(\lambda)$ form all $\lambda\leq\overline{\lambda}$. This and Part (i) imply that U_o and R_o' first intersect at some $\widetilde{\lambda}>\overline{\lambda}$. ■

7.5.2 Reserve price

Claim 18 In the PO scenario:
(i) If $\lambda_o^*(r^*)>0$ and bidders' entry does not constrain the equilibrium, i.e., $\lambda_o^*(r^*)<\overline{\lambda}^c(r^*)$, then $r^*=r_{\max}$.
(ii) If bidders' entry constrains the equilibrium, i.e., $\lambda_o^*(r^*)=\overline{\lambda}^c(r^*)$, then $r^*\neq r_{\max}$.

Proof: Obviously, r^* satisfies $\frac{d\Pi_o(\lambda_o^*(r),q^*(r);r)}{dr}|_{r=r^*}=0$. Observe that

$$\begin{aligned}\frac{d\Pi_o(\lambda_o^*(r),q^*(r);r)}{dr} &= \frac{d}{dr}\left[R_o(\lambda_o^*(r);r)-\frac{s}{q^*(r)}\lambda_o^*(r)\right]\\ &= \left(\frac{\partial R_o(\lambda_o^*(r);r)}{\partial\lambda}-\frac{s}{q^*(r)}\right)\frac{\lambda_o^*(r)}{dr}+\frac{\lambda_o^*(r)s}{(q^*(r))^2}\frac{dq^*(r)}{dr}+\frac{\partial R_o(\lambda_o^*(r);r)}{\partial r}\\ &= \frac{\lambda_o^*(r)s}{(q^*(r))^2}\frac{dq^*(r)}{dr}+\frac{\partial R_o(\lambda_o^*(r);r)}{\partial r}\end{aligned}$$

where the first term on the second line vanishes because it is the first-order condition with respect to λ. Also observe that, using integration by parts,

$$R_o(\lambda;r)=1-e^{-\lambda(1-G(r))}[r-\frac{1-G(r)}{g(r)}]-\int_r^1 e^{-(1-G(b))\lambda}\left[b-\frac{1-G(b)}{g(b)}\right]_b' db$$

and therefore

$$\frac{\partial}{\partial r}R_o(\lambda;r)=-g(r)\lambda e^{-(1-G(r))\lambda}\left[r-\frac{1-G(r)}{g(r)}\right]$$

Hence, $\frac{\partial}{\partial r}R_o(\lambda;r)=0$ iff $r=r_{\max}$.

Now if $\lambda_o^*(r^*)<\overline{\lambda}^c(r^*)$, then $q^*(r)=1$ in a neighborhood of r^*. Hence $\frac{dq^*(r)}{dr}|_{r=r^*}=0$ and

$$\frac{d\Pi_o(\lambda_o^*(r),q^*(r);r)}{dr}=\frac{\partial R_o(\lambda_o^*(r);r)}{\partial r}$$

Therefore, $\frac{d\Pi_o(\lambda_o^*(r),q^*(r);r)}{dr}=0$ iff $r=r_{\max}$ implying $r^*=r_{\max}$.

If $\lambda_o^*(r) = \overline{\lambda}^c(r)$, then $\frac{dq^*(r)}{dr}$ is obtained from total differentiation of the FOC w.r.t. λ, $\frac{\partial R_o(\lambda_o^*(r);r)}{\partial\lambda} - \frac{s}{q^*(r)} = 0$. Thus,

$$\frac{dq^*(r)}{dr} = -\frac{\frac{\partial^2 R_o(\lambda_o^*(r);r)}{\partial\lambda^2}\frac{d\lambda_o^*(r)}{dr} + \frac{\partial^2 R_o(\lambda_o^*(r);r)}{\partial\lambda\partial r}}{\frac{s}{(q^*(r))^2}}$$

Now, $\frac{d\lambda_o^*(r)}{dr} = \frac{d\overline{\lambda}^c(r)}{dr} = -\frac{\frac{\partial U_o(\lambda_o^*(r);r)}{\partial r}}{\frac{\partial U_o(\lambda_o^*(r);r)}{\partial\lambda}} < 0$ and $\frac{\partial^2 R_o(\lambda_o^*(r);r)}{\partial\lambda^2} < 0$ from the second-order condition of profit maximization with respect to λ. Furthermore, at $r = r_{\max}$ both $\frac{\partial^2}{\partial\lambda\partial r}R_o(\lambda;r) = 0$ and $\frac{\partial R_o(\lambda_o^*(r);r)}{\partial r} = 0$. Therefore, at $r = r_{\max}$,

$$\frac{d\Pi_o(\lambda_o^*(r), q^*(r); r)}{dr} = \lambda_o^*(r)\frac{\partial^2 R_o(\lambda_o^*(r); r)}{\partial\lambda^2}\frac{\frac{\partial U_o(\lambda_o^*(r);r)}{\partial r}}{\frac{\partial U_o(\lambda_o^*(r);r)}{\partial\lambda}} < 0$$

implying that $r^* \neq r_{\max}$. ■

7.5.3 Bidders learn their value before entering

Claim 19: (i) $\lim_{s\to 0}\lambda_i(s) = -\ln c$. (ii) In the limit total recruitment cost is higher in the PO scenario

$$\lim_{s\to 0}\lambda_o(s)\frac{s}{1-G(\underline{v}_o(s))} = (\ln c)^2 c > \lim_{s\to 0}\lambda_u(s)\frac{s}{1-G(\underline{v}_u(s))}$$

Proof of Claim 19: (i) In both scenarios, $\underline{v}_i(s) \to 1$ as $s \to 0$. Therefore, the entry condition $\underline{v}e^{-\lambda} = c$ for both scenarios implies $\lim_{s\to 0}\lambda_i(s) = -\ln c$.

(ii) For a given s, the respective equilibria (with trade) of the two scenarios satisfy the FOC $\partial\Pi_0(\lambda_o(s), \underline{v}_o(s))/\partial\lambda = 0$ and $\partial\Pi_u(\lambda_u(s), \underline{v}_u(s), \widehat{\lambda})/\partial\lambda|_{\widehat{\lambda}=\lambda_u(s)} = 0$, where

$$\partial R_0(\lambda_o(s), \underline{v}_o(s))/\partial\lambda = \frac{s}{1-G(\underline{v}_o(s))} \tag{33}$$

and

$$\partial R_u(\lambda_o(s), \underline{v}_o(s), \widehat{\lambda})/\partial\lambda|_{\widehat{\lambda}=\lambda_o(s)} = \frac{s}{1-G(\underline{v}_u(s))} \tag{34}$$

Thus, in each of the scenarios, the total recruiting cost is

$$\lambda_i(s)\frac{s}{1-G(\underline{v}_i(s))} = \lambda_i(s)\partial R_i/\partial\lambda \tag{35}$$

By revenue equivalence, $R_o(\lambda, \underline{v})$ and hence $\partial R_0(\lambda, \underline{v}_o)/\partial\lambda$ is the same as it would be with SPA. Therefore,

$$\partial R_0(\lambda, \underline{v})/\partial\lambda = \lambda \underline{v} e^{-\lambda} + \int_{\underline{v}}^{1} \left(\left(\frac{(1 - G(b))}{1 - G(\underline{v})} \right)^2 \lambda e^{-\frac{(1-G(b))}{1-G(\underline{v})}\lambda} \right) db$$

Since as $s \to 0$, $\underline{v}_o(s) \to 1$, $\lim_{s\to 0} \partial R_0(\lambda_o(s), \underline{v}_o(s))/\partial\lambda = \lim_{s\to 0} \lambda_o(s) e^{-\lambda_o(s)} = -c \ln c$. Therefore, $\lim_{s\to 0} \lambda_o(s) \frac{s}{1-G(\underline{v}_o(s))} = (\ln c)^2 c$.

The inequality in Part (ii) of the claim will follow from $\lim_{s\to 0} \lambda_i(s) = -\ln c$ and (35) after establishing

$$\lim_{s\to 0} \partial R_u(\lambda, \underline{v}, \widehat{\lambda})/\partial\lambda|_{\widehat{\lambda}=\lambda} < \lim_{s\to 0} \partial R_o(\lambda, \underline{v})/\partial\lambda \tag{36}$$

This follows from observing that, by revenue equivalence, $R_o(\lambda, \underline{v}) = R_u(\lambda, \underline{v}, \lambda)$ and hence

$$\partial R_o(\lambda, \underline{v})/\partial\lambda = dR_u(\lambda, \underline{v}, \lambda)/d\lambda = \partial R_u(\lambda, \underline{v}, \widehat{\lambda})/\partial\lambda|_{\widehat{\lambda}=\lambda} + \partial R_u(\lambda, \underline{v}, \widehat{\lambda})/\partial\widehat{\lambda}|_{\widehat{\lambda}=\lambda}$$

Then by adapting somewhat the arguments used in subsection 3.2, it can be shown that

$$\lim_{\underline{v}\to 1} \partial R_u(\lambda, \underline{v}, \widehat{\lambda})/\partial\widehat{\lambda}|_{\widehat{\lambda}=\lambda} = \lim_{\underline{v}\to 1} \int_0^1 \left[e^{-\lambda[1-G(\beta^{-1}(b;\underline{v},\lambda))]/[1-G(\underline{v})]} \right] \frac{\left(G(\beta^{-1}(b; \underline{v}, \lambda)) - G(\underline{v})\right)}{(1 - G(\underline{v}))} db > 0$$

which implies (36) and hence Part (ii) of the claim. ■

8 References

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