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Dynamic Tax Evasion and Capital Misallocation in General Equilibrium*

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Abstract

We study tax evasion in a tractable macroeconomic model with productive public expenditure financed by a fixed-rate income tax. Taxpayers are heterogeneous in their productivity and subject to borrowing constraints. They can lower their fiscal burden by evading taxes at the risk of being audited (and fined) by the government. We solve the model for its competitive equilibrium and characterize entrepreneurs' optimal policies contingent on their individual productivity and the endogenous price levels. The model predicts that enforcing tax compliance stimulates the productivity of public expenditure, thus making less productive enterprises viable. At the same time, however, fewer evasion opportunities alleviate borrowing constraints by offsetting the advantage of low-productivity (and highly-evasive) entrepreneurs, thereby re-allocating capital to more productive users. On the demand side, decreasing tax evasion reduces consumption levels by curbing private capital accumulation. However, it fosters consumption rates by mitigating entrepreneurs' precautionary motif against auditing risk. (JEL: E25, E26, H23, H26)

Keywords: Dynamic Tax Evasion; Financial Frictions; General Equilibrium; Misallocation

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1 Introduction

Tax evasion is an ever-present phenomenon across the world. While it is particularly pronounced in developing countries (Beck et al., 2014), it affects more mature economies too. In the US, for instance, the IRS estimates the gap between total taxes owed and taxes paid to be above 15% (Slemrod, 2007). In the EU, this figure ranges from 8 to 30%, with large disproportions even between the most advanced countries, with Germany scoring about 10% and Italy 24% (see Murphy, 2019).

A compelling stylized fact of tax evasion is the negative co-variation between the size of a country's shadow economy and its aggregate productivity, an observation that is consistent across both developed and developing economies. This pattern is evident in Figure 1 (Panel (a)), which displays a scatter plot of the shadow economy's size, expressed as a % of GDP, and the Total Factor Productivity (TFP) of 23 developing and 34 developed countries.[1] A second persistent phenomenon is the positive correlation between the size of a country's informal sector and its financial outreach. Figure 1 (Panel (b)) highlights this relationship by pairing bank credit spreads, a common measure of financial frictions, and the size of the shadow economy in 50 developing countries at the end of 2007.[2]

In light of these empirical regularities, this paper develops a tractable macroeconomic model of tax evasion featuring heterogeneous entrepreneurs, a public sector, and financial frictions. The model allows us to explore how individual-level tax evasion and government expenditure jointly affect aggregate productivity.[3] As a main result, we characterize the equilibrium mechanisms through which the shadow economy interacts with the distribution of capital across entrepreneurs and the supply of public goods to exacerbate (or alleviate) capital misallocations due to financial frictions. In this respect, we contribute to the literature by extending the framework of Chen (2003), who introduces public goods in an endogenous

[1]The productivity and credit spreads data includes observations from the World bank and the Penn World datasets. The shadow economy data come from Schneider et al. (2010).

[2]Evidence on the association between financial frictions and tax evasion also appears in La Porta and Shleifer (2014) and Beck et al. (2014), who point out that the lack of access to finance is one of the main determinants of informality. Similarly, Dabla-Norris et al. (2008) provide evidence of the connection between financing constraints and informality, especially among smaller firms.

[3]The crucial role of government spending in affecting the private sector has been highlighted by several works since the seminal contribution of Aschauer (1989), for instance Agénor (2010) and Agénor and Neanidis (2015).

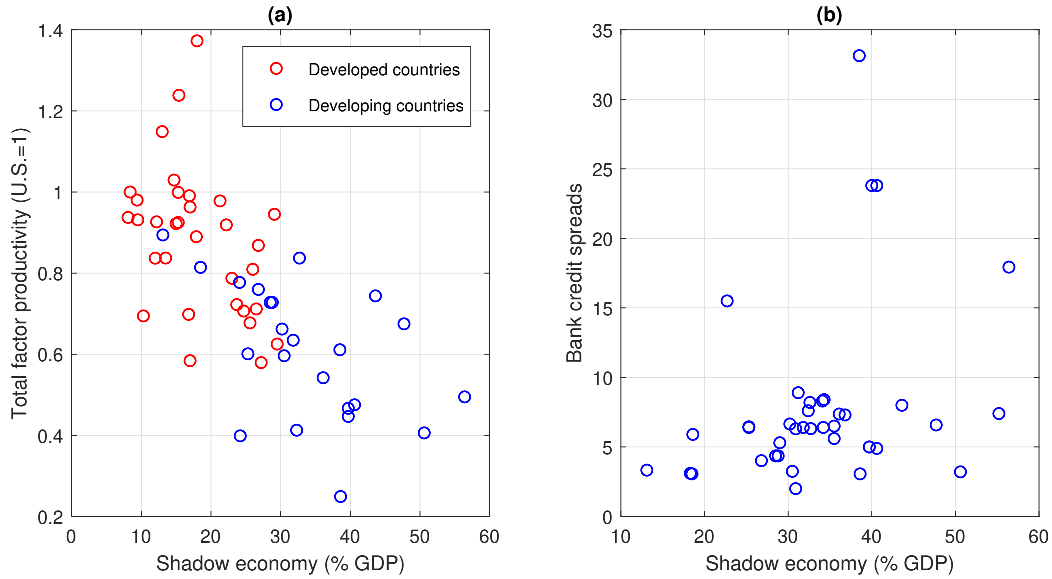

Figure 1: Stylized facts of the relationship between TFP and the size of the shadow economy (Panel (a)) and between financial frictions and the shadow economy (Panel (b)). The TFP level of each country is normalized by using that of the US as a benchmark. Sources: the World Bank (credit spreads); the Penn World dataset (TFP); Schneider et al. (2010) (shadow economy).

growth model with tax evasion but does not consider heterogeneous producers. Moreover, we calibrate the model and show that it is able to replicate the stylized facts summarized in Figure 1.

Following Moll (2014), we build our model in continuous time. This allows us to improve on analytical tractability relative to other closely related studies working in a discrete-time environment (e.g., Ordonez, 2014; López, 2017; Di Nola et al., 2021). The model features a continuum of small-sized entrepreneurs and a public sector, which finances its expenditure by imposing a flat-rate income tax. In the same spirit of Barro (1990), public expenditure (i.e., total tax revenues) acts as an input to production. Risk-averse entrepreneurs, the taxpayers, are heterogeneous in their productivity levels and face frictions in the form of credit constraints and incomplete financial markets. At each point in time, they take three decisions. First, they choose their consumption to maximize expected inter-temporal utility. Second, they choose to either (actively) finance their own enterprise or – if their firm's productivity is too low – to (passively) rent capital to their peers. Third, they can relieve their tax burden by hiding a share of income but carry the risk of being audited and, in turn, fined by the government. Auditing revenues contribute to finance the aggregate public expenditure. Our focus on small-sized enterprises and the assumption that they evade by concealing capital income are motivated by the empirical evidence that roughly half of aggregate evasion consists of business profits under-reporting (Slemrod, 2019), and that self-employed activities are misreported by about one third (Hurst et al., 2014).

Similarly to the classical (Allingham and Sandmo, 1972; Andreoni, 1992; Lin and Yang, 2001) and more recent (e.g., Levaggi and Menoncin, 2013, 2016) tax evasion literature, our first result characterizes entrepreneurs' optimal policies at the individual level. Our framework differentiates from previous studies because tax evasion is state-contingent and time-varying, depending on both entrepreneurs' individual productivity and their cross-sectional distribution. The model predicts that, everything else constant, a higher individual productivity associates to lower tax evasion, which is in line with the empirical findings in Gradstein et al. (2019).[4]

Next, we solve the model for its competitive equilibrium and, thanks to its analytical tractability, we derive the level of macroeconomic aggregates and factor prices explicitly. Moreover, we characterize the transition dynamics of the equilibrium towards its steady-state. Equipped with these results, we take a partial equilibrium perspective and explain the relation between factor prices and the economy's TFP by means of two different but strictly related mechanisms. The former one, labelled as "threshold channel", relates aggregate TFP changes to variations in the productivity threshold above which entrepreneurs become active. According to this channel, increments in (private) factor prices, capital and labour, foster the economy's TFP because higher production costs crowd inefficient enterprises out of the economy. On the contrary, a higher public expenditure diminishes TFP because it allows low-productivity entrepreneurs to become active. Reminiscent of the "selection channel" identified by Di Nola et al. (2021), these forces affect the size of the shadow economy due to the interaction between entrepreneurs' production and tax evasion decisions. The latter mechanism is the "net worth channel", which links the dynamics of TFP growth to changes in the average saving rate across active entrepreneurs and, thus, to their net worth distribution. Due to this channel, the individual evasion strategies of different entrepreneurs affect the

[4] A very different mechanism that generates a similar result can be found in Gillman and Kejak (2014), who develop an endogenous growth model in which human capital enhances productivity, thereby reducing tax evasion incentives.

cross-sectional dynamics of capital and, in turn, the aggregate amount of resources that are available for production. In particular, since low-productivity entrepreneurs are those who evade the most, the net worth channel "redistributes" capital from more to less productive agents, by boosting the saving capacity of the latter at the expenses of the former.[5]

Finally, the paper explores numerically the joint effect of tax evasion and financial frictions on the macroeconomic dynamics in general equilibrium. In particular, we calibrate the model and compute the steady-state equilibrium of a benchmark economy with tax evasion. Then, we study the transition dynamics following a permanent unanticipated ("MIT") shock that makes taxes perfectly enforceable. This exercise allows us to visualize and investigate the dynamic interaction between the threshold and net worth channels and to uncover several additional insights. First, improving tax enforcement depresses the overall size of the economy; that is, aggregate output and capital stock. This happens because, when they can no longer enjoy the benefit of tax evasion, the least productive entrepreneurs are left out of the market. Moreover, more taxes stem into a higher public expenditure to the detriment of private capital accumulation. However, in line with the stylized facts of Figure 1, reducing tax evasion improves efficiency because those entrepreneurs who remain active are less financially constrained and more productive on average. Second, removing evasion opportunities increases the average consumption rate of the economy, because it relaxes entrepreneurs' precautionary motive against (uninsurable) auditing risk. Higher consumption rates impair the capital accumulation process, thereby enforcing the adverse effect described in the first insight. The third effect is that, even though the supply of the public good decreases due to the reduction in aggregate capital, its positive effect on aggregate TFP increases. This result has important policy implications because it highlights a negative connection between tax evasion and the productivity of government expenditure.

Last, we carry out a sensitivity analysis of the benchmark equilibrium to variations in some key model parameters. The analysis highlights that heterogeneous tax evasion policies interact with financial constraints in a non-trivial way. For instance, after relaxing entrepreneurs' leverage constraints, those who are either inactive or very productive evade

[5] This result is akin to the "subsidy" channel described in Di Nola et al. (2021), who build a model in which tax evasion opportunities have a non-linear effect across (self-employed) entrepreneurs with different business abilities.

less, whereas the majority of those who are active evade more. As a consequence, average tax evasion increases.

The paper unfolds as follows. Section 2 presents the model and characterizes its steady-state equilibrium and transitional dynamics. Section 3 exploits analytical and numerical methods to investigate different aspects of the relationship between tax evasion and capital misallocation. Section 4 concludes.

2 Model

2.1 Production entities and financial frictions

Time is continuous and indexed $t \in [0, \infty)$. There is a continuum of firms that are heterogeneous in their idiosyncratic TFP $z_t \in \mathbb{Z} \equiv [0, z^{\max}]$. Similarly to Barro (1990), they are endowed with a constant-return-to-scale technology that inputs three factors of production: private capital k_t (henceforth just "capital"), labour l_t, and public good g_t. Firms' output y_t is given by

$$y_t = z_t^{\alpha+\beta} k_t^{\alpha} g_t^{\beta} l^{1-\alpha-\beta}, \tag{1}$$

in which α and β denote the (constant) output elasticity of capital and public good, respectively. As in Moll (2014), the TFP of individual firms is stochastic and its log-returns evolve according to the following Ornstein-Uhlenbeck process:

$$d \ln z_t = -\nu \ln z_t + \sqrt{\nu} dW_t, \tag{2}$$

in which W_t is a standard Brownian motion and $\nu \in [0, \infty)$ is a parameter capturing the degree of auto-correlation.[6,7] The economy does not feature financial markets to negotiate risky

[6] An Ornstein-Uhlenbeck process can be considered as the continuous-time analogue of a discrete-time $AR(1)$ process. In particular, it is possible to show (see Stokey, 2008) that, over a unit time interval ($dt = 1$), $\nu = -\ln \mathbb{C}orr\,(z_{t+1}, z_t)$.

[7] By applying Itô's lemma to $e^y = z$, we can infer that adopting the process in Eq. (2) is equivalent to assuming that the productivity level z follows a geometric mean-reverting process

$$dz = z \left[\nu \,(0.5 - \ln z)\, dt + \sqrt{\nu} dW \right].$$

claims written on their individual productivity shocks; accordingly, firms can be interpreted as small- or medium-sized enterprises.

The stock of capital k is supplied by entrepreneurs, each of whom is endowed with a net worth n and owns property rights in one firm with productivity z. Firms' productivity can be freely assessed by their owners, meaning that their relationship entails no moral hazard problems. At each instant in time, entrepreneurs decide whether to directly engage in the business activity or not, depending on the level of z. In the first case, they supply their entire net worth as capital to their firm and are entitled to the firms' proceeds. Else, they allocate it in risk-free bonds issued by other entrepreneurs and receive an endogenous rental rate. From now on, we label the former group of entrepreneurs as "active" and the latter as "inactive". Active entrepreneurs can issue bonds and thus leverage their balance sheets. In doing so, they face a borrowing constraint in the following form:

$$k_t \leq \phi n_t, \tag{3}$$

in which $\phi \geq 1$ defines their maximal leverage capacity.[8] Jointly with the assumption of incomplete financial markets, the credit constraint in Eq. (3) constitutes the only financial friction in the model. Since capital markets are perfectly competitive and there is no aggregate risk, direct firm financing and bond holdings are remunerated at the same (risk-free) rate r_t, determined in equilibrium.

For the sake of simplicity, the labour side of the economy is parsimoniously modelled. In particular, human capital l is inelastic and supplied by a unit mass of "hand-to-mouth" workers, who have no capital endowment and own no claims on firms' profits. While they are allowed to save, they limit themselves to consume their wage w, endogenously determined, and play no additional role in the economy.[9]

[8] Eq. (3) can be obtained as a result of limited-enforcement frictions in capital markets as in Gertler and Kiyotaki (2010). These frictions materialize when levered entrepreneurs are allowed to instantaneously divert a fraction $1/\phi \in [0, 1]$ of their capital assets, thereby defaulting on their liabilities. The constraint is "incentive-compatible" in that it dispels diversion incentives by ensuring that the value of entrepreneurs' diversion revenues $V(k/\phi)$ is always smaller than or equal to their continuation $V(n)$.

[9] As we show in Appendix A.1, even if they are allowed to collect savings, households exhibit a "hand-to-mouth" behavior under the assumption that their inter-temporal discount rate is higher than the rental rate of capital. We verify numerically in Section 3 that these conditions always hold in equilibrium.

The public good supply g is financed by a flat-rate tax on entrepreneurs' capital income (firms' profits plus bond returns). We postulate and verify later that g enters the production function proportionally to the capital level of each firm; that is

$$g_t = \omega_t k_t. \tag{4}$$

in which ω_t is an endogenous quantity determined in equilibrium. From an economic perspective, Eq. (4) describes a pure public good (i.e., non-rivalrous and non-excludable), such as broadband and mobility infrastructures, which benefits firms' output proportionally to the volume of their activity.

By substituting Eq. (4) into Eq. (1), individual firms' output simplifies as

$$y_t = \omega_t^{\beta} \left(z_t k_t\right)^{\epsilon} l_t^{1-\epsilon}, \tag{5}$$

in which $\epsilon = \alpha + \beta$. Given Eq. (5), individual firms choose capital and labour to maximize their profits

$$\Pi_t \equiv \max_{k_t \leq \phi n_t, l_t} \left\{ y_t - r_t k_t - l_t w_t \right\}. \tag{6}$$

Firms' optimal decisions are derived in the following proposition.

Proposition 1. ***Production strategies and profit flows***

1. *The optimal capital and labour demand policies that solve the problem in Eq. (6) are*

$$k(z_t, n_t) = \mathbb{I}_{z_t \geq \bar{z}} \phi n_t \quad \text{and} \quad l(z_t, n_t) = z_t k(z_t, n_t) \frac{\pi_t}{w_t} \left(\frac{1-\epsilon}{\epsilon} \right), \tag{7}$$

   in which $\pi = \epsilon \omega^{\beta/\epsilon} \left(\left(1-\epsilon\right)/w \right)^{(1-\epsilon)/\epsilon}$ *and the threshold level* $\bar{z} = r/\pi$.

2. *By implementing the strategy in Eq. (7), an entrepreneur with productivity* z_t *receives the instantaneous profit flow*

$$\Pi(z_t, n_t) = n_t \varphi(z_t), \tag{8}$$

   in which $\varphi(z_t) = \phi \mathbb{I}_{z_t \geq \bar{z}} \left(z_t \pi_t - r_t \right)$.

Proof. See Appendix A.2. □

The first implication of Proposition 1 is that the choice of whether to produce or not depends exclusively on the firms' productivity level and not on the net worth of the entrepreneurs. The second one is that entrepreneurs are willing to actively engage in their firm's business as long as their z is higher than or equal to a threshold level $\bar{z}$. In that case, they choose to commit their net worth n to production and leverage their balance sheet up to maximal limit ϕn. Conversely, when their firms' productivity is too low, entrepreneurs find more convenient to lend capital to their more productive peers at the endogenous rate r.[10] Finally, it is important to highlight that the threshold $\bar{z}$, i.e. the productivity level that makes entrepreneurs indifferent between running their own business and lending capital on the market, is jointly determined with the other equilibrium quantities (i.e, the risk-free rate r, the labour wage w, and the public good multiplier ω).

2.2 Entrepreneurs, taxation, and tax evasion

The economy features a continuum of entrepreneurs, who are endowed with a different net worth n and map one-to-one to the firms with heterogeneous TFP levels z. They share the same relative risk aversion γ and inter-temporal discount rate ρ.

Conditional on their firms' production strategy, entrepreneurs face the dynamic problem of maximizing the inter-temporal utility of their consumption. At each instant, they either earn capital rental revenues and profits from firms or interests from bonds, depending on whether they are active or not. Then, they consume a fraction of their disposable net worth. Any source of income is taxed by the government at the flat rate $\tau \in [0, 1]$. Accordingly, entrepreneurs' net worth after taxes evolves with dynamics

$$dn_t = (1 - \tau) \left[(b_t + k_t) r_t + \Pi_t \right] dt - c_t dt, \tag{9}$$

in which b labels bond holdings and c instantaneous consumption. By using $b + k = n$ and

[10] Note that, as in Moll (2014), we do not model financial intermediation directly. Instead, we let inactive entrepreneurs finance other firms in a perfectly competitive rental (or bond) market. In the market, each bond is remunerated "as if" its underlying was invested in a diversified portfolio across the universe of active firms.

considering that firms' profits satisfy Eq. (8), Eq. (9) can be conveniently re-written as

$$dn_t = \left[n_t \left(1-\tau\right)\left(r_t + \varphi_t\right) - c_t\right] dt. \tag{10}$$

While taking their consumption decision, entrepreneurs can also relieve their tax burden by concealing a fraction e_t of their gross income from the tax authority. By doing so, they expose themselves to the possibility of being audited by the government and paying a fine. Following Levaggi and Menoncin (2013), tax audits follow a Poisson process with a constant rate λ. Audits are pair-wise independent across entrepreneurs. For the sake of tractability, evasion fines are equal to a fixed share η of the evaded capital revenues.[11]

By taking into account the possibility of tax evasion, the dynamics of entrepreneurs' net worth in Eq. (10) modifies as

$$dn_t = \left[n_t \left(1-\tau+\tau e_t\right)\left(r_t + \varphi_t\right) - c_t\right] dt - n_t e_t \eta \left(r_t + \varphi_t\right) dJ_t, \tag{11}$$

in which J_t is a standard Poisson process with parameter λ. Importantly, the possibility of tax evasion introduces a trade-off between higher disposable income and more predictable net worth dynamics, because, if evading, the uncertain fine may materialize. Entrepreneurs solve the following maximization problem:

$$\max_{\{c_t, e_t\}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt\right] \tag{12}$$

subject to the dynamics balance sheet constraint in Eq. (11); their optimal strategies are summarized in the following proposition.

Proposition 2. ***Optimal consumption and tax evasion***

The optimal consumption and tax evasion strategies that solve the problem in Eq. (12) are

$$\frac{c_t}{n_t} \equiv \zeta_t = \mathbb{E}_t \left[\left(\int_t^\infty e^{-\int_t^s \hat{\rho}_u du} ds\right)^\gamma\right]^{-\frac{1}{\gamma}} \quad \textit{and} \quad e_t = \frac{1-\Xi}{\left(\varphi_t + r_t\right)\eta}, \tag{13}$$

[11] The model preserves its analytical tractability as long as the evasion fine is linear in entrepreneurs' individual net worth n, while it can be non-linear in any other variable in the model.

in which $\Xi = (\lambda\eta/\tau)^{\frac{1}{\gamma}}$ *and* $\gamma\hat{\rho} = \rho + \lambda\left(1 - \Xi^{1-\gamma}\right) - (1-\gamma)\left[(1-\tau)(\varphi + r) + \frac{\tau}{\eta}(1-\Xi)\right]$.

Proof. See Appendix A.3. □

The optimal consumption strategy is similar to that obtained in any infinite-horizon Merton problem with a diffusion-type state variable other than agent's net worth (see for example Steffensen, 2011). The optimal evasion strategy is similar to that derived in Levaggi and Menoncin (2012) for a representative agent. However, it substantially differs in two aspects. First, it varies over the cross-section of entrepreneurs, depending on the TFP level of their firm. Second, it is contingent on the endogenous (and time-varying) quantities r, w, ω, and $\bar{z}$. In this respect, our model predicts that, everything else constant, the evasion share decreases in productivity and in the rental rate on capital.

With these results in mind, note that, from a macroeconomic perspective, tax evasion allows some low z enterprises to be viable, thereby diverting capital from more profitable allocations. As we will see next, this pattern has additional and non-trivial implications on the aggregate economy due to its repercussion on equilibrium prices and public good supply.

2.3 Competitive equilibrium and aggregation

We now solve the model for its competitive equilibrium and characterize the transition dynamics towards its balanced-growth path (or "steady state").

Let us define as $F_t(z, n)$ the joint distribution of the entrepreneurs' net worth and their firms' productivity at time t, and let $f_t(n, z)$ be its density function. From now on, for the sake of clear notation we omit the time subscripts of micro-level quantities (such as n, k, and z) but maintain them to denote aggregate-level objects (such as r_t and w_t). Moreover, without loss of generality we normalize aggregate labor supply to one unit ($L_t = 1$). Formally, we define the economy's competitive equilibrium as follows.

Definition 1. ***Competitive equilibrium***
A competitive equilibrium is a set of aggregate output (Y_t), capital (K_t), and public good supply (G_t or, equivalently, its multiplier ω_t), factor prices (w_t and r_t), as well as consumption and evasion strategies such that:

1. *Firms maximize profits and entrepreneurs maximize their value in Eqs. (6) and (12).*

2. *The public good supply equals the sum of taxes and auditing revenues:*

$$G_t = \int_0^\infty \int_{\mathbb{Z}} \left[\tau \left(nr_t + \Pi\right) + n\left(1 - \Xi\right)\lambda\right] f_t(z,n)dndz. \tag{14}$$

3. *All markets clear:*

   (a) *Capital*

$$K_t = \int_0^\infty \int_{\mathbb{Z}} k(n,z) f_t(z,n)dndz = \int_0^\infty \int_{\mathbb{Z}} n f_t(z,n)dndz \tag{15}$$

   (b) *Labour*

$$1 = \int_0^\infty \int_{\mathbb{Z}} l(n,z) f_t(z,n)dndz. \tag{16}$$

Equipped with Definition 1, we can use Eqs. (7) and (15) and rewrite Eq. (14) as

$$G_t = \int_0^\infty \int_{\mathbb{Z}} \left[z\pi - (1-\tau)\, r_t + (1-\Xi)\,\lambda\right] k(z,n) f_t(z,n)dndz, \tag{17}$$

which verifies our initial guess in Eq. (4); that is, the public good supply is proportional to that of private capital.

Following Moll (2014), we characterize the equilibrium by defining the auxiliary function

$$\theta_t(z) \equiv \frac{\int_0^\infty n f_t(z,n)dn}{K_t} \geq 0, \tag{18}$$

which denotes the aggregate capital share held by entrepreneurs whose firm's productivity level is z. Similarly to $f(\bullet)$, Eq. (18) is a density function but it describes the net worth distribution across entrepreneurs in a way that is convenient in the aggregation process.[12] Moreover, it is useful to (implicitly) identify the endogenous productivity threshold $\bar{z}$, conditional on how capital is distributed across entrepreneurs. In fact, by substituting firms'

[12] The fact that $\theta_t(z)$ is a density function can be verified by substituting Eq. (18) in the market clearing condition for capital in Eq. (15), which gives: $\int_{\mathbb{Z}} \theta_t(z)dz = 1$; and by considering that entrepreneurs' net worth and aggregate capital are always greater than zero. The analytical tractability provided by the use of $\theta_t(z)$ derives from the fact that, given entrepreneurs' optimal strategies in Eq. (13), the drift and jump terms of the state variable's dynamics in Eq. (11) are linear in its level n.

capital policy in Eq. (7) in the market clearing condition in Eq. (15) and using the definition in Eq. (18), we obtain that

$$\phi \left(1 - \int_0^{\bar{z}} \theta_t(z) dz\right) = 1. \tag{19}$$

Eq. (19) clarifies the link between the productivity threshold $\bar{z}$ and the credit constraint ϕ (i.e., the "financial depth" of the economy). In particular, it tells us that the higher the leverage capacity of each entrepreneur, the higher the productivity threshold that sorts those who decide to be active from those who lend them capital. In the limit, when $\phi \to \infty$, the whole stock of capital is always handled by the most productive entrepreneurs (whose $z = z^{\max}$), and heterogeneity plays no role. Conversely, when no leverage is allowed ($\phi = 1$) all the entrepreneurs are always active and the distribution of their net worth maps 1-to-1 to that of their productivity.

For a given couple $\{K, \theta(\bullet)\}$, we are able to analytically characterize the level of all macroeconomic aggregates (output, factor prices, and public good supply) at any point in time as follows.

Proposition 3. ***Macroeconomic aggregates***
Given an aggregate stock of capital K_t with distribution $\theta_t(z)$:

1. *The total output at time t equals*

$$Y_t = \omega_t^{\beta} \left(\phi Z_t K_t\right)^{\epsilon}, \tag{20}$$

   in which $Z_t = \mathbb{E}_t^{\theta}\left[z \mid z > \bar{z}\right]$.

2. *Wages w_t, rental rates r_t, and the public good multiplier ω_t satisfy the following system of equations:*

$$\begin{cases} w_t = (1-\epsilon) Y_t, \\ r_t = \epsilon \bar{z} \left(\phi Z_t K_t\right)^{\epsilon - 1} \omega_t^{\beta}, \\ \omega_t = \epsilon \tau \left(\phi Z_t\right)^{\epsilon} K_t^{\epsilon - 1} \omega_t^{\beta} + \lambda \left(1 - \Xi\right). \end{cases} \tag{21}$$

Proof. See Appendix A.4. □

Proposition 3 shows that the aggregate TFP ($\equiv \omega^{\beta} (\phi Z)^{\epsilon}$) is a non-linear function of the public good multiplier, the credit constraint, and the average productivity across active firms. Moreover, it highlights that wages are linear in output, which is a result of the assumption of an inelastic labour supply. Finally, it reveals that the rental rate on capital crucially depends on both the productivity threshold and the public good multiplier. As we will see in Section 3, this detail has an important role in channelling capital misallocation as a result of tax evasion.

To complete the equilibrium's characterization, we are able to derive the dynamics of its determinants, K and $\theta(\bullet)$. Their laws of motion towards the balanced-growth path (steady state) are described in the following proposition.

Proposition 4. ***Transition dynamics***

1. *The transition dynamics of K towards its steady-state, denoted as K_∞, is given by*

$$\frac{d \ln K_t}{dt} = \frac{Y_t}{K_t} (1-\tau)\,\epsilon + \lambda \left(\varXi^{-\gamma} - 1 \right) (1 - \varXi) - \bar{\zeta}_t, \tag{22}$$

in which $\bar{\zeta}_t = \mathbb{E}_t^{\theta} [\zeta(z)]$ labels the economy's average consumption rate.

2. *At each point in time, the auxiliary density function $\theta(\bullet)$ satisfies the following PDE:*

$$\frac{\partial \theta_t(z)}{\partial t} = - \underbrace{\frac{d \ln K_t}{dt} \theta_t(z)}_{\textit{Aggregate capital}} + \underbrace{\left[(1-\tau)(\varphi_t + r_t) + \lambda \varXi^{-\gamma} (1 - \varXi) - \zeta(z) \right] \theta_t(z)}_{\textit{Retained earnings}} + \\ - \underbrace{(1 - \Xi)\lambda\theta(z)}_{\textit{Auditing flows}} + \underbrace{\frac{\partial}{\partial z} \left[\theta_t(z) z \nu \left(\ln z - \frac{1}{2} \right) \right]}_{\textit{Autocorrelation}} + \underbrace{\frac{\nu}{2} \frac{\partial^2}{\partial z^2} \left[\theta_t(z) z^2 \right]}_{\textit{Randomness}} \tag{23}$$

with boundary and mass preservation conditions $\theta_t(0) = 0$ and $\int_{\mathbb{Z}} \theta_t(z) dz = 1$, $\forall t$.

Proof. See Appendix A.5. □

The transition dynamics (i.e., the growth rate) of aggregate capital stock in Eq. (22) can be decomposed into three terms. The first one is the marginal after-tax productivity of capital. The second is the aggregate gain from evading taxes net of the associated auditing fines. The third term reduces the growth rate by the average consumption rate.

Similarly, we can decompose Eq. (23) into five components. The first one implies that, everything else constant, a higher growth rate in aggregate capital stock results in a lower concentration of net worth across entrepreneurs. The second component highlights that, conditional on firms' productivity, evasion opportunities increase the capital share of their entrepreneurs by fostering their retained earnings. The "auditing flow" component indicates that net worth concentration decreases in both the auditing intensity parameter λ and the tax evasion rate $1-\Xi$. The last two terms of Eq. (23) capture the effects of firms' individual productivity (auto-correlation and randomness), according to Eq. (2).

The last implication of Proposition 4 is that, in the long run, the aggregate capital stock, its distribution, and the associated macroeconomic aggregates drift towards their balance-growth path (or steady-state level); that is,

$$\lim_{t\to\infty}\left\{K_t,\theta_t(z),\nu_t(z),r_t,w_t,\omega_t\right\}=\left\{K_\infty,\theta_\infty(z),\nu_\infty(z),r,w,\omega\right\}.$$

The following Corollary characterizes the steady-state output-to-capital ratio.

Corollary 1. ***Balance-growth path (steady-state equilibrium)***
In the steady-state, the output-to-capital ratio is given by

$$\frac{Y_\infty}{K_\infty}=\frac{\bar{\zeta}_\infty-\lambda\left(\varXi^{-\gamma}-1\right)\left(1-\varXi\right)}{\left(1-\tau\right)\epsilon}, \tag{24}$$

whose associated density $\theta_\infty(z)$ is the stationary solution of Eq. (23). All remaining macroeconomic aggregates are defined as in Proposition 3.

3 Tax evasion and capital misallocation

In this section, we first investigate the mechanisms by which factor prices (r and w) and public good supply (through ω) affect the economy's allocative efficiency in partial equilibrium. Then, we solve the model numerically and explore how tax evasion affects the level and dynamics of these quantities and, in turn, of all macroeconomic aggregates, in general equilibrium. To perform the second step, we calibrate a benchmark economy featuring tax evasion and compute its steady-state equilibrium as described by Corollary 1.

Then, we study the transition dynamics of the economy following an unanticipated ("MIT") shock that permanently shifts the government auditing parameters to obtain perfect tax enforcement.

In the final part of the section, we conduct a comparative statics analysis. In particular, we investigate the sensitivity of the results to positive and negative variations in the auditing fine, output elasticity of public good, and financial friction level.

3.1 Analytical results in partial equilibrium

As we emphasize in Proposition 3, the economy's aggregate TFP is an endogenous object that depends on individual entrepreneurs' strategies and on their net-worth distribution. Importantly, these quantities are affected by the government's taxation and auditing policies through the government's supply of public good that, in turn, contributes to determine the prices of capital and labour through the system in Eq. (21). The presence of this "feedback loop" effect, which is the central contribution of our general equilibrium framework, may however make it difficult to disentangle the various forces at play from each other.

To tackle this problem, this section investigates how variations in factor prices and public good multiplier affect the economy, from a partial equilibrium perspective. This approach allows us to clearly identify two different channels through which variations in these quantities affect entrepreneurs' saving rates and, relatedly, the aggregate TFP. The "threshold" channel is due to changes in the threshold level $\bar{z}$ that sorts passive from active entrepreneurs. The "net worth" channel captures the effects of changes in the shape of the net worth distribution across entrepreneurs with different productivity.

The first channel can be formalized by substituting Eq. (19) in the aggregate TFP in Eq. (20), and taking the partial derivatives with respect to $x_t \equiv \{r, w, \omega\}$ while keeping everything else constant. The results are summarized in the following proposition.

Proposition 5. ***Capital misallocation and the threshold channel.***
In partial equilibrium, increasing factor prices always improves the economy's allocative ef-

ficiency by increasing the productivity threshold $\bar{z}$. Formally:

$$\frac{\partial TFP_t}{\partial r_t}\frac{1}{TFP_t} = \underbrace{\epsilon\theta_t(\bar{z})\left(\frac{\phi Z_t - \bar{z}}{Z_t}\right)}_{\frac{\partial TFP_t}{\partial \bar{z}}\frac{1}{TFP_t}} \times \underbrace{\pi_t^{-1}}_{\frac{\partial \bar{z}}{\partial r_t}} > 0, \tag{25}$$

$$\frac{\partial TFP_t}{\partial w_t}\frac{w_t}{TFP_t} = \underbrace{\epsilon\theta_t(\bar{z})\left(\frac{\phi Z_t - \bar{z}}{Z_t}\right)}_{\frac{\partial TFP_t}{\partial \bar{z}}\frac{1}{TFP_t}} \times \underbrace{\frac{r_t}{\pi_t}\frac{1-\epsilon}{\epsilon}}_{\frac{\partial \bar{z}}{\partial w_t}w_t} > 0. \tag{26}$$

Conversely, increasing the public good multiplier has an ambiguous effect on the economy's allocative efficiency: on the one hand, it increases individual TFP across all entrepreneurs; on the other hand, it decreases the productivity threshold $\bar{z}$. Formally:

$$\frac{\partial TFP_t}{\partial \omega_t}\frac{\omega_t}{TFP_t} = \beta - \underbrace{\epsilon\theta_t(\bar{z})\left(\frac{\phi Z_t - \bar{z}}{Z_t}\right)}_{\frac{\partial (\phi Z_t)^\epsilon}{\partial \bar{z}}TFP_t^{-1}} \times \underbrace{\frac{\beta}{\epsilon}\frac{r_t}{\pi_t}}_{-\frac{\partial \bar{z}}{\partial \omega_t}\omega_t} \gtreqless 0. \tag{27}$$

Proof. See Appendix A.6. □

The first implication of Proposition 5 is that, everything else constant, increasing the remuneration of either capital or labour increases total factor productivity. This happens because higher production costs "crowd out" less productive entrepreneurs, who find more convenient to lend capital than actively engaging in their firms' business. As a consequence, more competitive enterprises enjoy the benefits of this capital re-allocation, which improves the average productivity among the active firms. In other words, increasing factor prices enhances the productivity (or entry level) threshold $\bar{z}$, which mitigates the effect of financial frictions in the economy by re-allocating capital in the hands of more productive users.

The second implication of Proposition 5 is that the effect of an increase in the public good multiplier is ambiguous. On the one hand, a higher supply of public good fosters productivity across *all* entrepreneurs, independently on their productivity levels, through the endogenous term ω^β. At the same time, however, such productivity gains make profitable for less productive entrepreneurs to become active, as the productivity threshold $\bar{z}$ reduces. This lower level of $\bar{z}$ exacerbates capital misallocation due to financial frictions and scales down the average productivity Z. The overall effect depends on which of these two opposing forces dominates.

To describe the net worth channel we follow González et al. (2022) and derive the sensitivity of TFP growth to changes in the endogenous quantities x_t (r_t, w_t, ω_t), keeping the threshold $\bar{z}$ constant. The result is summarized in the following proposition.

Proposition 6. ***Capital misallocation and the net worth channel***
In partial equilibrium, the impact of changes in factor prices and public good multiplier on the growth rate of aggregate TFP depends on how these changes affect the drift of the average saving rate across active entrepreneurs. Formally:

$$sign\left(\frac{\partial}{\partial x_t}\frac{d\ln TFP_t}{dt}\right) = sign\left(\mathbb{E}^{\bar{\theta}}\left[\frac{\partial \mu_n(z)}{\partial x_t}\,|z_t > \bar{z}\right]\right), \tag{28}$$

in which $\bar{\theta}_t(z) \equiv \frac{z\theta_t(z)\mathbb{I}_{z\geq\bar{z}}}{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}$ *is a probability density function defined over* $[\bar{z}, z^{max}]$.

Proof. See Appendix A.7. □

Proposition 6 draws a clear connection between changes in the average saving rate (capital revenues plus firms' profits minus consumption rate) of active entrepreneurs and the economy's allocative efficiency. In particular, it predicts that the sensitivity of aggregate TFP growth to changes in x goes hand-in-hand with that of the average net worth across active entrepreneurs. Indeed, changes in factor prices and the public good multiplier relate to how capital is distributed across active entrepreneurs, which is a result of their individual evasion strategies.

3.2 Numerical results in general equilibrium

Even though our partial equilibrium results are useful in disentangling the model's main mechanisms, they fail in capturing the complexity resulting from their interplay. Being unable to analytically characterize the interaction between tax evasion and the macro economy in general equilibrium, we now calibrate our model and solve it numerically to analyze its transition dynamics from a tax-evasion (benchmark) economy towards the steady-state equilibrium in which taxes are perfectly enforced.

Parameter	Meaning	Value	Source
β	Public good parameter	0.1	Glomm and Ravikumar (1997)
ϕ	Borrowing constraint	1.43	González et al. (2022)
λ	Evasion - auditing intensity	0.15	Bernasconi et al. (2020)
η	Evasion - fine rate	1.5	-
τ	Tax rate	0.24	-
ν	Productivity auto-correlation	0.40	Gilchrist et al. (2014)
ρ	Discount rate	0.1	Standard
γ	Risk aversion	2.5	-
α	Capital share	0.33	-

Table 1: Parameters in the benchmark model.

3.2.1 Parametric restrictions and calibration

The model requires some restrictions on the parameter domain. In particular, we set the tax rate (τ) and auditing parameters (η and λ) so that the optimal evasion rates e remain in the interval $[0, 1]$.

The restriction guaranteeing that tax evasion remains positive is obtained by looking at Eq. (13), which implies $\eta\lambda \leq \tau$. Indeed, tax evasion is a viable option as long as the marginal gain from tax evasion (τ) is greater than its expected marginal cost ($\lambda\eta$). The restriction guaranteeing that $e \leq 1$ is $r\eta \geq 1 - \Xi$, meaning that tax evasion is bounded at or below 100% as long as the rental rate of capital r is sufficiently high. Notice that r in the last inequality is an endogenous object determined in equilibrium. Accordingly, we are not able to verify analytically that the latter parametric restriction holds ex ante, but we check that it is not violated at any step in our numerical simulations.

With these restrictions in mind, we select our benchmark parameter values as reported in Table 1. Empirical estimates of the output elasticity to public good range between 0.06 (Ratner, 1983) and 0.2 (Binswanger and Rosenzweig, 1993).[13] Accordingly, we set $\beta = 0.1$. The financial friction parameter ϕ matches the corporate leverage (debt-to-net worth) of 43% reported by FRED (see also González et al., 2022); the tax rate τ and auditing parameters η and λ are broadly in line with the calibration exercise in Bernasconi et al. (2020) and the

[13] A comprehensive review of the early literature concerned with the estimate of the output elasticity to public good appears in Glomm and Ravikumar (1997).

literature referred therein. Finally, the calibration of ν is based on Gilchrist et al. (2014), who estimate the yearly auto-correlation of firms' idiosyncratic productivity based on the AR process $x_{t+1} = \beta x_t + \epsilon_t$, and obtain $\beta \approx 0.8$ at the quarterly level, which implies $\beta = 0.41$ yearly. Accordingly, we set $\nu = -\log(0.41/(1-0.41))$ to be around 0.4. The discount rate $\rho = 0.1$, the relative risk aversion $\gamma = 2.5$, and the capital share $\alpha = 0.33$ are standard values in the literature. The time interval $dt \approx \Delta t = 1/12$; that is, we simulate the model on a monthly basis.

3.2.2 Numerical solution and analysis

This section runs a numerical experiment to showcase the model's dynamics in general equilibrium. In particular, it approximates the solution of the PDEs that describe the model's equilibrium by using an up-wind scheme; all details concerning the numerical methods and solution algorithm appear in Appendix B.

As a starting point for our exercise, we initialize the economy at its steady-state equilibrium, given the benchmark parameters collected in Table 1. Then, we study the transition dynamics which follows an unanticipated ("MIT") shock that permanently shifts the government's auditing parameters to obtain perfect tax enforcement (i.e., $\eta\lambda = \tau$).

Macroeconomic aggregates Figures 2 and 3 depict the transition of the macroeconomic aggregates, factor prices, and the public good multiplier from the benchmark to the perfect-tax-enforcement steady-state.

The first result of our analysis is that eliminating tax evasion reduces both aggregate output and capital stock, but increases the output-to-capital ratio (Figure 2, Panels (a)-(c)). These outcomes are associated with the following two developments on the supply side of the economy. First, without the opportunity brought by tax evasion, less productive entrepreneurs remain out of the market. Hence, the corresponding share of their firms' factor demand vanishes, thereby reducing the aggregate stock of capital by about 10%. Second, tax enforcement counterbalances credit market constraints and reallocates resources towards more productive entrepreneurs. Accordingly, the output-to-capital ratio increases (i.e., aggregate output reduces less than capital) because the firms that remain active have a higher average z. All in all, removing tax evasion improves the economy's TFP by almost

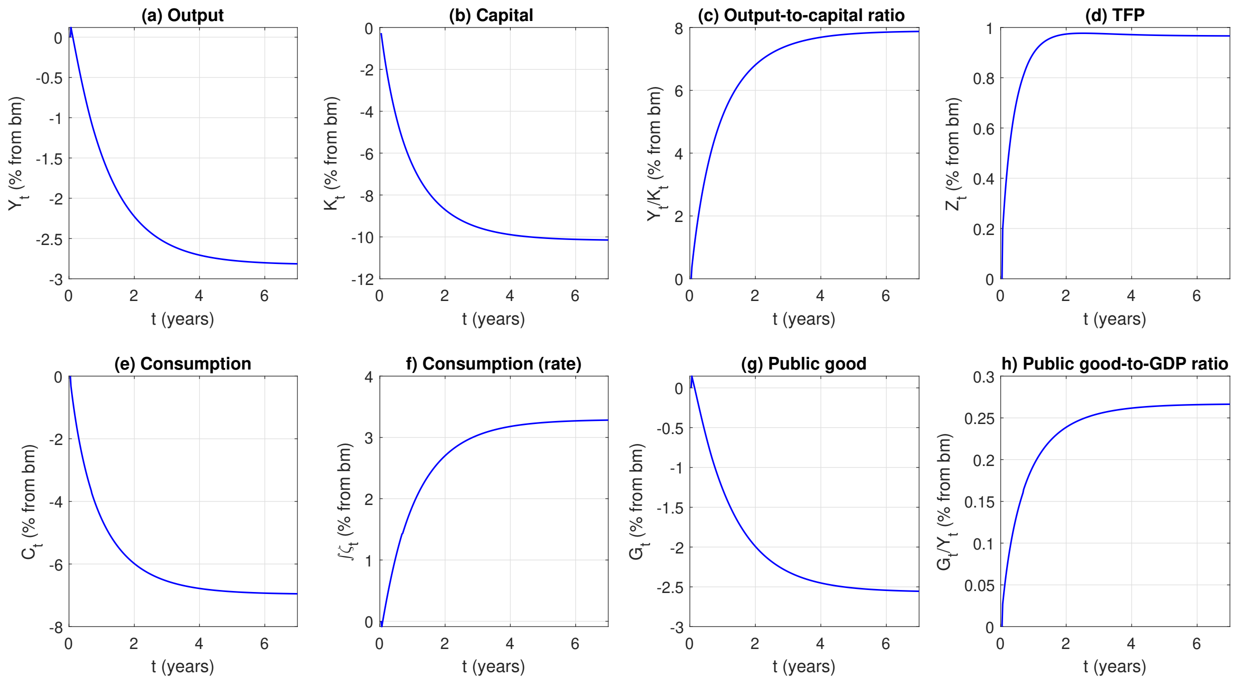

Figure 2: Transition dynamics of macroeconomic aggregates from the benchmark to the perfect-tax-enforcement economy (% variation from the steady state of the benchmark economy).

1% (Figure 2, Panel (d)).

By looking at the demand side of the economy, our second result is that perfect tax enforcement reduces the aggregate consumption level but increases its rate (Figure 2, Panels (e) and (f)). While the former effect is driven by the overall decrease in the aggregate stock of capital, the latter is the combination of two different forces that we label as "productivity" and "precautionary" effect, respectively. The productivity effect materializes as a consequence of the increase in aggregate TFP, which generates a higher willingness to consume. The precautionary effect is a consequence of tax evasion and relates to the net worth uncertainty due to government audits and fines. Implementing a perfect tax enforcement cancels out such uncertainty, thereby eliminating the need to retain capital buffers against (un-insurable) auditing events.

The third result of our numerical exercise concerns the government's public good supply. Similarly to what happens to other macroeconomic aggregates, perfect tax enforcement reduces public good supply in absolute terms (see Figure 2, Panel (f)). What is relevant to

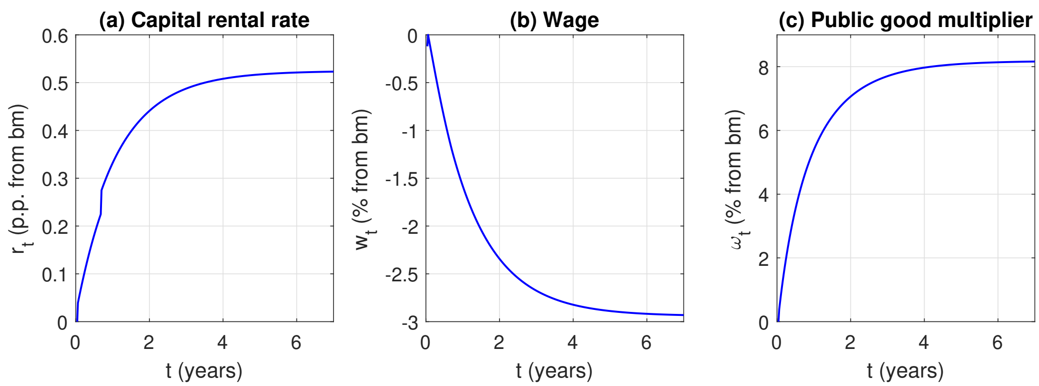

Figure 3: Transition dynamics of factor prices and the public good multiplier from the benchmark to the perfect-tax-enforcement economy (percentage points or % variation from the steady state of the benchmark economy).

stress is that, however, ex ante a higher level of enforcement does not trivially result into lower tax revenues (and thus public expenditure). On the contrary, it may either increase or reduce the production of public good, depending on the entrepreneurs' individual decisions. On the one hand, perfect tax compliance improves G because no income is concealed from the tax authority; on the other hand, it decreases it because both K and Y drop due to the threshold, productivity, and precautionary effects.[14] What we observe in our simulations is the case in which the latter effects prevail. In particular, we obtain that, despite the public good supply reduces by about 2.5%, its production per unit of output increases by more than 0.25%, which translates into a substantial increment (8%) in the public good multiplier ω (Figure 3, Panel (c)).

Before we investigate how perfect tax enforcement affects different entrepreneurs over the cross-section, we briefly comment on its effects on factor prices. While the response of r after removing tax evasion is straightforward because a lower level of K generates higher rental rates (see Figure 3, Panel (a)), the effect on w is less intuitive. In particular, our simulations suggest that removing tax evasion generates competitive wages that are almost 3 percent lower than in the benchmark economy (Figure 3, Panel (c)). This outcome is not trivial. It comes as a result of three different forces affecting wages in different directions; the increase in both aggregate Z and ω, which affect them positively, and the decrease in K,

[14]Keep in mind that the public good supply G is always proportional to that of aggregate capital stock K by a factor of ω (see Eq. (17)).

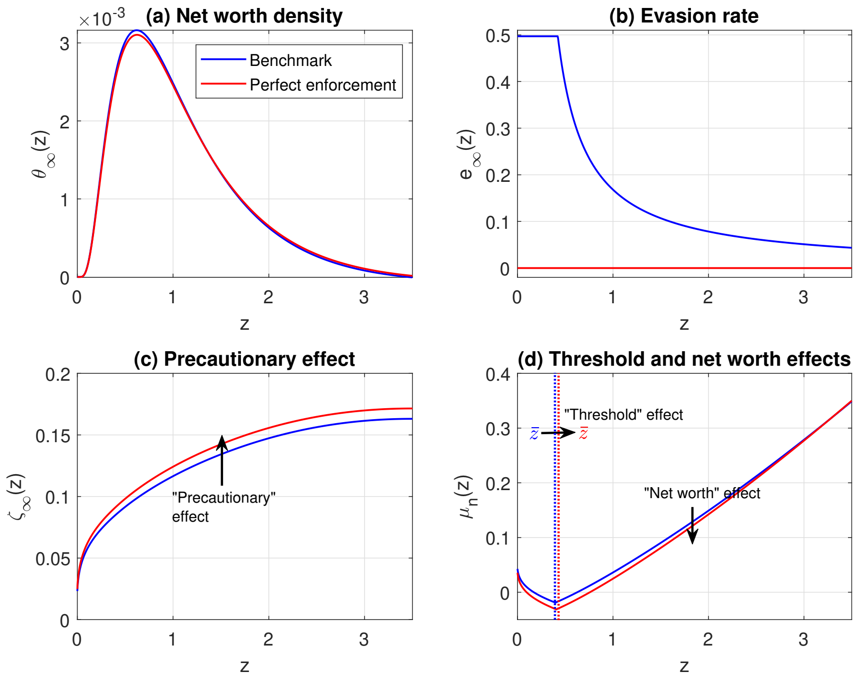

Figure 4: Stationary density, cross-sectional tax evasion and consumption rates, and net worth drift across entrepreneurs as functions of their productivity level z. The blue and red lines depict the benchmark and the perfect-tax-enforcement economy, respectively.

which affects them negatively (see Eq. (21)). In our transition, the third effect dominates over the first two.[15]

Cross-sectional distribution and capital misallocation As a next step of the analysis, Figure 4 shows the steady-state densities, net worth drift, and the tax evasion and consumption rates of the entrepreneurs before and after the tax-enforcement shock.

These cross-sectional outcomes allow us to clearly visualize the net worth channel, by showing that, when taxes are perfectly enforced, the drift of all entrepreneurs' net worth decreases for any productivity level (Figure 4, Panel (d)). This reduction is more pronounced for low productivity enterprises, since their tax evasion strategies are decreasing in the idiosyncratic productivity level z (Figure 4, Panel (b)). This implies that evasion opportunities exacerbate capital misallocation by fostering the growth rate of less productive entrepreneurs relative to their more productive (and thus less elusive) peers. As a consequence, eliminating tax evasion undoes the relative advantage of low-productivity entrepreneurs, thereby

[15]Notice also that the reduction in wages is particularly pronounced because the aggregate supply of labour is inelastic to variations in w.

Parameters	η			β			ϕ		
	1.4	1.5	1.6	0.08	0.1	0.12	1.2	1.43	1.65
Stocks									
Y	2.1%	-	(1.4%)	11.4%	1.5	(10.5%)	(3.7%)	-	2.3%
K	4.2%	-	(2.5%)	19.2%	4.1	(15.2%)	1.6%	-	(2.2%)
Productivity									
$\bar{z}$	(1.1%)	-	0%	1.1%	0.7	(2.1%)	(26.6%)	-	17.0%
Z	(0.6%)	-	0.4%	3.3%	0.9	(4.0%)	(8.9%)	-	6.5%
ω^{β}	(0.6%)	-	0.5%	5.8%	0.7	(5.1%)	(0.4%)	-	0.5%
TFP	(0.3%)	-	0.3%	6.7%	0.8	(6.3%)	(5%)	-	3%
Factor prices									
r	(2.5%)	-	0.7%	(12.8%)	0.1	12.6%	(23.6%)	-	14.9%
w	2.1%	-	(1.4%)	15.3%	0.9	(13.7%)	(3.7%)	-	2.3%
Rates									
$\bar{\zeta}$	(1.5%)	-	1%	(10.8%)	0.1	10.5%	(5.1%)	-	4.7%
ω	(5.9%)	-	4.9%	(9.4%)	0.04	10.5%	(4.05%)	-	5.1%
$\bar{e}$	132.3%	-	(89.6%)	32.3%	0.2	(3.1%)	17.8%	-	7.4%

Table 2: Comparative static analysis with respect to the tax fine (η), the output elasticity of public expenditure (β), and the credit constraint (ϕ) parameters.

re-allocating wealth from the left-hand to the right-hand side tail of the stationary net worth density (Figure 4, Panel a)) and, ultimately, increasing the economy's aggregate productivity.

The other three phenomena captured by the model that we can identify by looking at Figure 4 are the productivity, precautionary, and threshold effects. In particular, we observe that, in line with the aggregate-level results discussed above, perfect tax enforcement increases entrepreneurs' consumption rates over the whole cross-section (Panels (c)). Moreover, removing tax evasion shifts the productivity threshold $\bar{z}$ to the right (Panel (d)), suggesting that the negative effect of ω ($\frac{\partial \bar{z}}{\partial \omega} < 0$) is overtaken by that of r and w ($\frac{\partial \bar{z}}{\partial r}$ and $\frac{\partial \bar{z}}{\partial w} > 0$).

3.3 Comparative statics

To conclude our study, we perform a comparative statics analysis. In particular, we investigate how variations in the auditing fine, the output elasticity of public good, and the financial friction parameters affect our numerical results.

Table 2 reports the equilibrium's aggregates and factor prices in the steady state for

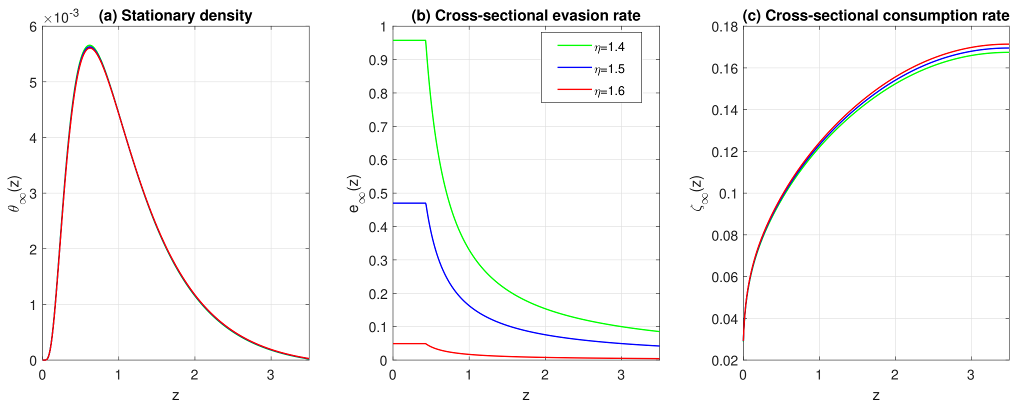

Figure 5: Comparative statics of the steady-state density, cross-sectional tax evasion and consumption with respect to variations in the auditing fine parameter η.

different values of η (left), β (mid), and ϕ (right); the numbers with (without) brackets indicates a % reduction (increment) with respect to the benchmark economy collected in the central column. In a complementary manner, Figures 5-7 display how these changes affect the steady-state density, consumption, and evasion rates across entrepreneurs.

When looking at the variations of η, what stands out is that, in line with the previous section's analysis, higher (lower) evasion fines affect negatively (positively) both capital and output levels. At the same time, however, costlier (cheaper) evasion opportunities improve (reduce) the economy's aggregate productivity Z, while they increase (decrease) the capital rental rate and reduce (increase) wages. As intuition suggests, increasing (decreasing) η also fosters (shrinks) aggregate consumption and public expenditure, both in the aggregate and over the whole cross-section of entrepreneurs, while it hinders (fosters) tax evasion (see Figure 5, Panels (b) and (c)).

We now look at the effect of changing the output elasticity of public goods. From a broad perspective, increasing (decreasing) β generates lower (higher) stock of aggregate capital and output. The latter, however, decreases (increases) less than proportionally because $Y \propto K^{\alpha+\beta}$ and, thus, increments in β increase (decrease) the marginal productivity of each unit of K (see r in Row 6). Also, positive (negative) variations of β decrease (increase) both the average productivity Z and the threshold level $\bar{z}$. This happens because, by generating a higher (lower) level of the public good multiplier ω, more productive entrepreneurs are crowded out (in) in favor of their less competitive peers by means of the threshold chan-

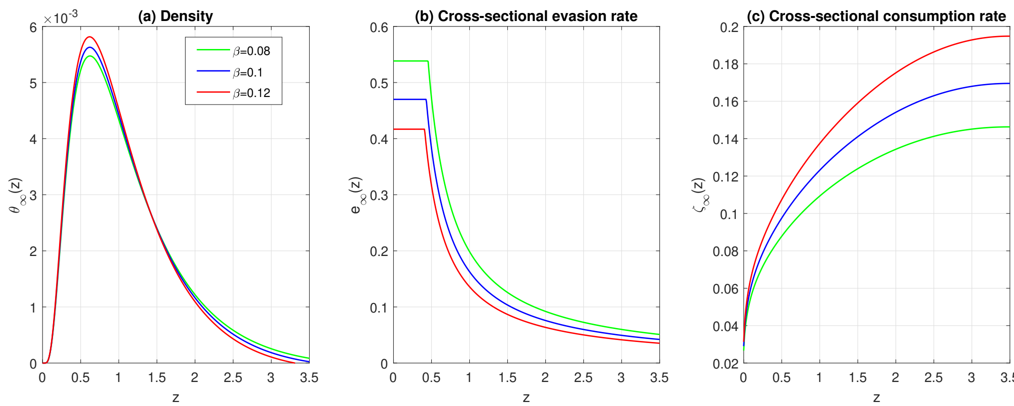

Figure 6: Comparative statics of the steady-state density, cross-sectional tax evasion and consumption with respect to variations in the output-to-public-good elasticity parameter β.

nel (Figure 6, Panel (a)). The third outcome of changing β is that positive and negative variations generate higher and lower consumption rates over the whole cross-section of entrepreneurs and, thus, also at the aggregate level (see Figure 6, Panels (b) and (c) and Table 2, Rows 9 and 11). The negative relationship between these two quantities is consistent with the prevalence of the precautionary channel.

A few other results emerge when looking at the variations of the financial friction parameter ϕ. As intuition suggests, releasing (exacerbating) financial frictions fosters (hinders) the economy's productivity in all its dimensions, from output to the level of both factor prices and the public good multiplier. The effect on tax evasion is ambiguous and substantially depends on the level of z (Figure 7, Panel (b)). In fact, while weaker (stronger) frictions associate to lower evasion rates at the left- and right-hand side of the productivity distribution, they do increase it in its middle, where most of the firms are located. Overall, these non-linear effects result into an increase in the average level of tax evasion for both low and high values of ϕ, even though in the latter case the phenomenon is milder.

From a distributional standpoint, higher productivity levels and lower evasion rates among low productivity entrepreneurs generate a transfer of mass from the left- to the right-hand side of the distribution and vice versa (Figure 7, Panel (a)), because of the net worth channel. Moreover, higher (lower) productivity levels boost (diminish) consumption rates across all entrepreneurs as an effect of the productivity channel. (Figure 7, Panel (c)).

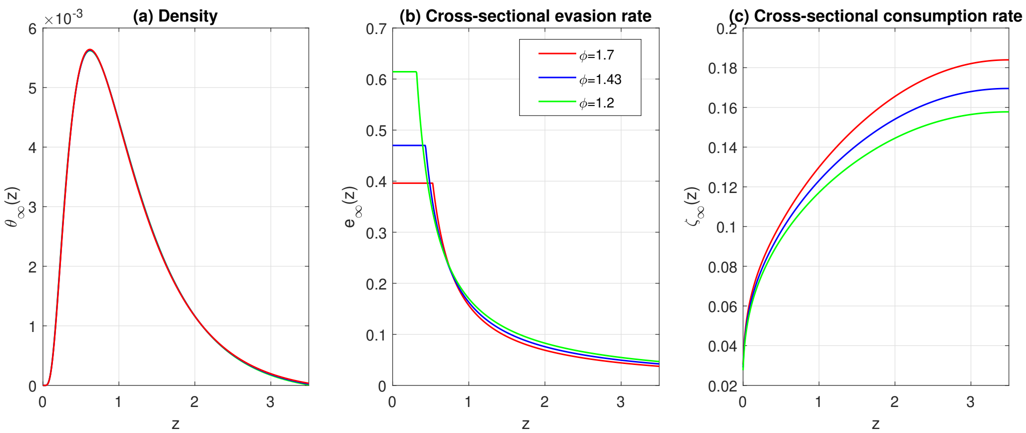

Figure 7: Comparative statics of the steady-state density, cross-sectional tax evasion and consumption with respect to variations in the financial friction parameter parameter ϕ.

4 Conclusions

We develop a macroeconomic model to study the relation between tax evasion, financial frictions, and aggregate productivity. In the model, the government finances productive public expenditure by imposing an income tax on individual enterprises. Entrepreneurs are heterogeneous in their productivity and face frictions in the form of borrowing constraint and incomplete financial markets; they can evade taxes at the risk of being audited (and fined) by the government.

Our analysis complements the macroeconomics and public finance literature by showing that, by means of their joint effects on the distribution of private capital and the public good supply, tax evasion opportunities play an important role in exacerbating capital misallocation due to financial frictions. In particular, we characterize the channels through which the interplay between tax evasion and financial frictions affect the economy's aggregate productivity. The model – calibrated to real-world data – is able to reproduce the observed negative relation between tax evasion and aggregate TFP, and the positive one between tax evasion and financial frictions.

While the model is analytically tractable, the macroeconomic dynamics it describes is rich. However, tractability requires a few simplifications. First, we assume that workers' labour supply is fixed and perfectly elastic. This prevents us from drawing more general welfare implications. Second, we assume that the government operates with no budget deficit, thus abstracting from several important channels that relate tax evasion with the government tax capacity and, in turn, the level of public debt. In this paper, we also neglect aggregate risk and, as a consequence, the issue of optimal taxation in the presence of tax evasion over the business cycle. We leave the study of these important issues to future research.

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A Appendix: proofs and derivations

A.1 Workers' problem

In the same spirit of Moll (2014), this appendix provides a micro-foundation of the "hand-to-mouth" behaviour of the labour suppliers (henceforth, simply "workers") in the model. Each worker is endowed with one unit of labour and choose its labour supply l and consumption c to solve the following problem:

$$\max_{\{c,l\in[0,1]\}} \int_0^\infty e^{-\rho t} \frac{c^{1-\gamma}}{1-\gamma} dt, \tag{29}$$

subject to $\dot{n} = nr + wl - c$, in which w denotes the economy's competitive wage. Since they do not gain utility from leisure, and wages are strictly non-negative, the optimal labour supply of the workers is always equal to one. Concerning their consumption choices, the Hamiltonian of the problem in Eq. (29) is

$$H = \frac{c^{1-\gamma}}{1-\gamma} + \vartheta \left(nr + w - c\right),$$

in which ϑ denotes the co-state variable. Taking the first-order conditions yields

$$\frac{\partial H}{\partial c} = 0 \to c = \vartheta^{-\frac{1}{\gamma}} \text{ and } \dot{\vartheta} = \rho\vartheta - \frac{\partial H}{\partial n} \to \dot{\vartheta} = \left(\rho - r\right)\vartheta, \tag{30}$$

which can be rearranged to obtain the Euler equation $\dot{c} = ((r-\rho)\gamma)c$ or, equivalently,

$$c_t = c_0 e^{\int_0^t \frac{r_s - \rho}{\gamma} ds}. \tag{31}$$

By re-writing workers' balance sheet constraint as

$$\int_0^\infty c_t e^{-\int_0^t r_s ds} dt = n_0 + \underbrace{\int_0^\infty w_t e^{-\int_0^t r_s ds} dt}_{\text{Human capital}}$$

and substituting Eq. (31) while imposing that $n_0 = 0$ (no intial net worth), one obtains that

$$c_0 = \frac{\int_0^\infty w_t e^{-\int_0^t r_s ds} dt}{\int_0^\infty e^{\int_0^t \frac{r_s(1-\gamma)-\rho}{\gamma} ds} dt}, \tag{32}$$

which, in the steady state, simplifies as $c = w(\rho + r(\gamma - 1))/r\gamma$. The hand-to-mouth behaviour of the workers emerges under the condition that $c \geq w$, implying the parametric restriction that $\rho \geq r$, which ensures that they are always willing to consume (at least) their whole wage.

A.2 Proof of Proposition 1

Firms' labour demand is unconstrained; thus, the associated first-order condition is

$$w = (1-\epsilon)\,\omega^\beta\,(zk)^\epsilon\, l^{-\epsilon}. \tag{33}$$

By first substituting Eq. (33) into firms' problem in Eq. (6) and then factoring our k, one obtains that their profits are linear in k

$$\max_{k \leq \phi n} k \left[\omega^\beta z \left[\frac{(1-\epsilon)\,\omega^\beta}{w} \right]^{\frac{1-\epsilon}{\epsilon}} - r - z\omega^\beta \left[\frac{(1-\epsilon)\,\omega^\beta}{w} \right]^{\frac{1}{\epsilon}} \right],$$

which implies a corner solution so that either $k_t = 0$ or $k_t = \phi n_t$. Eq. (7) follows suit.

A.3 Proof of Proposition 2

By standard stochastic control arguments (see for instance Pham, 2009, , Chapter 2), the value function V of an entrepreneur facing the problem in Eq. (12) satisfy the Hamilton-Jacobi-Bellman Equation (by dropping all unnecessary time subscripts and functional dependence on z)

$$\rho V = \max_{c}\left\{\frac{c^{1-\gamma}}{1-\gamma} - \frac{\partial V}{\partial n}c\right\} + \frac{\partial V}{\partial z}\mu_z + \frac{\partial^2 V}{\partial z^2}\frac{\sigma_z^2}{2} + \frac{\partial V}{\partial t} + \\ + \max_{e}\left\{\begin{array}{c} \frac{\partial V}{\partial n}n\left[\left(1-\tau-\tau e\right)\left(r+\varphi\right)\right] + \\ +\lambda\left[V\left(n\left(1-e\eta\left(r+\varphi\right)\right)\right) - V\right] \end{array}\right\}. \tag{34}$$

According to the first-order conditions with respect to c and e, the optimal controls of the problem satisfy

$$c^{-\gamma} = \frac{\partial V}{\partial n} \text{ and } \frac{\partial V}{\partial n}n\tau = \lambda\frac{\partial V\left(n\left(1-e\eta\left(r+\varphi\right)\right)\right)}{\partial\left(n\left(1-e\eta\left(r+\varphi\right)\right)\right)}\eta. \tag{35}$$

To further characterize the problem's solution, we look for a candidate solution to the HJBE in the form

$$V_t(z,n) \equiv \nu_t(z)\frac{n^{1-\gamma}}{1-\gamma} \tag{36}$$

for some unknown function ν. By substituting Eq. (36) into the HJBE and in its first-order conditions, we obtain

$$c^{-\gamma} = n^{-\gamma}\nu \text{ and } \tau = \eta\lambda\left[1-e\eta\left(r+\varphi\right)\right]^{-\gamma}, \tag{37}$$

that can be rearranged to obtain the policy functions in Eq. (13). Given these policies, the unknown function ν satisfies the following quasi-linear PDE:

$$0 = \underbrace{\frac{\partial \nu}{\partial t} - \hat{\rho}\nu}_{\equiv h} + \gamma\nu^{1-\frac{1}{\gamma}} + \frac{\partial \nu}{\partial z}\underbrace{z\nu\left(0.5-\ln z\right)}_{\equiv \mu_z} + \frac{1}{2}\frac{\partial^2 \nu}{\partial^2 z}\underbrace{\nu z^2}_{\equiv \sigma_z^2}, \tag{38}$$

with boundary conditions[16]

$$\lim_{T\to\infty}\mathbb{E}_t\left[e^{-\int_t^T \hat{\rho}_s ds}\nu_T\right] = \left.\frac{\partial \nu_t}{\partial z}\right|_{z=z^{\max}} = 0, \tag{39}$$

[16]As we discuss in the numerical Appendix B.2.1, we are able to solve Eq. (38) without specifying the left-hand side boundary condition because at $z=0$ both drift and diffusion terms of dz equal zero and thus the solution algorithm "takes care" of it by itself (see Moll, 2014). The right-hand side boundary condition, instead, can be interpreted as a reflecting barrier in z^{max} (see Dixit, 2013).

and

$$\hat{\rho} = \rho + \lambda\left(1 - \Xi^{1-\gamma}\right) - (1-\gamma)\left[(1-\tau)(\varphi + r) + \tau(1-\Xi)\eta\right].$$

Even if the solution to the partial differential in Eq. (38) cannot be found in closed form, there is a Feynman-Kac representation of its (unique) solution (a formal derivation of this result appears in Yong and Zhou, 1999, Theorem 4.5) that is expressed as

$$\nu_0(z) = \mathbb{E}_0\left[X_0(z)\right]$$

in which X satisfies $\dot{X} = -h(z, X)$ with boundary condition $\lim_{t\to\infty} e^{-\int_0^t \hat{\rho}_u du} X_t = 0$. In our specific case,

$$X_0 = \left[\int_t^\infty e^{-\int_0^s \frac{\hat{\rho}_u}{\gamma} du} dt\right]^\gamma$$

and thus

$$\nu_0 = \mathbb{E}_0\left[\left(\int_0^\infty e^{-\int_0^s \hat{\rho}_u du} ds\right)^\gamma\right]^{-\frac{1}{\gamma}}.$$

A.4 Proof of Proposition 3

By substituting Eq. (7) into Eq. (5) and rearranging, individual firms' output can be written as

$$y = \omega^\beta z k \left(\frac{\pi}{\epsilon}\left(\frac{1-\epsilon}{w}\right)\right)^{1-\epsilon}.$$

Multiplying both sides of this equation by $f(\bullet)$, integrating over the state space, and using the auxiliary density defined in Eq. (18) and the expression in Eq. (8), one obtains that

$$Y = \omega^\beta \left(\frac{\pi}{\epsilon}\left(\frac{1-\epsilon}{w}\right)\right)^{1-\epsilon} \phi K Z. \tag{40}$$

Similarly, aggregating the individual labour supply using Eq. (7) yields

$$w\epsilon = \pi(1-\epsilon)\phi K Z. \tag{41}$$

eq:consumption Eq. (20) is obtained by raising both terms of the equation for π in Proposition 1 to the power of $1-\epsilon$, rearranging, and substituting the result into Eq. (40).

The system in Eq. (21) is derived as follows. The first equation comes from matching Eqs. (40) and (41) and rearranging. The second equation is obtained by substituting Eqs. (41) and $w = (1 - \epsilon)Y$ into $\bar{z}\pi = r$. The third equation is obtained by rewriting Eq. (14) as

$$\omega = \tau r + \tau\phi\pi Z - \tau r\phi \int_{\mathbb{Z}} \mathbb{I}_{z \geq \bar{z}} z\theta(z)dz + \lambda\left(1 - \Xi\right)$$

that, by using Eq. (19) jointly with Eq. (41) and $w = (1 - \epsilon)\,Y$, simplifies as

$$\omega = \epsilon\tau\left(\phi Z\right)^{\epsilon} K^{\epsilon-1}\omega^{\beta} + \lambda\left(1 - \Xi\right).$$

A.5 Proof of Proposition 4

To obtain Eq. (22), we can first aggregate the dynamics of individual entrepreneurs' net worth in Eq. (11) given their optimal policies in Eq. (13), which yields

$$dK = \int_0^{\infty} \int_{\mathbb{Z}} n\left[(1 - \tau)\left(r + \varphi\right) + \lambda\Xi^{-\gamma}\left(1 - \Xi\right) - \zeta - \left(1 - \Xi\right)\lambda\right] f(z, n)dndz. \tag{42}$$

By using the definition in Eq. (18), the mass preservation condition in Eq. (**??**), and the market clearing in Eq. (19), we can then rewrite Eq. (42) as

$$\frac{dK}{dt}\frac{1}{K} = \left(1 - \tau\right)\phi\pi Z\lambda\left(\Xi^{-\gamma} - 1\right)\left(1 - \Xi\right) - \bar{\zeta} \tag{43}$$

As a final step, by using Eq. (41) and considering that $w = (1 - \epsilon)Y$, Eq. (43) can be expressed as

$$d\ln K = \left(1 - \tau\right)\epsilon Y/K + \lambda\left(\Xi^{-\gamma} - 1\right)\left(1 - \Xi\right) - \bar{\zeta}.$$

Similarly to Moll (2014), we obtain the PDE for the dynamics of θ by using that, given the dynamics in Eq. (11) and the optimal policies in Eq. (13), the density function $f(\bullet)$

satisfies the so-called Fokker-Plank equation (for a formal derivation, see Stokey, 2008)

$$\frac{\partial f(z,n)}{\partial t} = -\frac{\partial}{\partial n}\left[nf(z,n)\mu_n(z)\right] - \frac{\partial}{\partial z}\left[f(z,n)\mu_z(z)\right] + \\ + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left[f_t(z,n)\sigma_z^2(z)\right] - \lambda f(z,n) + \lambda f(z,n\Xi). \quad (44)$$

Given the auxiliary density function in Eq. (18), we obtain that (omitting functional dependence on z)

$$\frac{\partial \theta}{\partial t} = \frac{\int_0^\infty n\frac{\partial f}{\partial t}dnK - \dot{K}\int_0^\infty nfdn}{K^2}. \quad (45)$$

Then, by substituting Eq. (45) in Eq. (44) we have that

$$\frac{\partial \theta}{\partial t} = \frac{\int_0^\infty -n\left[\frac{\partial}{\partial n}\left(nf\mu_n\right) + \frac{\partial}{\partial z}\left(f\mu_z\right)\right]dn}{K} + \\ + \frac{\int_0^\infty n\left[\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(f\sigma_z^2\right) - \lambda f\right]dn}{K} + \\ - \frac{dK}{K}\frac{1}{dt}\left[\frac{\int_0^\infty nfdn}{K} + \int_0^\infty n\lambda f\left(n\Xi\right)dn\right], \quad (46)$$

in which μ_n denotes the drift of the diffusion dynamics in Eq. (11).

Integrating by parts the first three addends to the right hand-side of Eq. (46), and using the definition of θ, one can see that

$$\int_0^\infty n\frac{\partial}{\partial n}\left(nf\mu_n\right)dn = -\mu_n\int_0^\infty nfdn, \quad (47)$$

$$\frac{\partial}{\partial n}\left(\int_0^\infty nfdn\mu_z\right) = K\partial_z\left(\theta\mu_z\right), \quad (48)$$

$$\int_0^\infty n\frac{\partial^2}{\partial z^2}\left(f\sigma_z^2\right)dn = K\frac{\partial^2}{\partial z^2}\left(\theta\sigma_z^2\right), \quad (49)$$

and that

$$\lambda\int_0^\infty nfdn = K\theta\lambda. \quad (50)$$

Since entrepreneurs' evasion decisions are independent of their individual net worth n, we can also rewrite the fourth and last term of Eq. (46) as

$$\lambda \int_0^\infty nf\left(n\Xi\right)dn = \Xi\lambda \int_0^\infty nf dn. \tag{51}$$

By substituting Eqs. (47)-(51) in Eq. (46) and rearranging, we obtain that the unknown function θ is a solution to the following PDE:

$$\frac{\partial \theta}{\partial t} = \left[\mu_n - \frac{d\ln K}{dt} - (1-\Xi)\lambda\right]\theta_t - \frac{\partial}{\partial z}\left(\theta\mu_z\right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\theta\sigma_z^2\right), \tag{52}$$

whose steady-state value, denoted as θ_∞, solves the following ODE:

$$0 = \left[\mu_n - (1-\Xi)\lambda\right]\theta_\infty - \frac{\partial}{\partial z}\left(\theta_\infty\mu_z\right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\theta_\infty\sigma_z^2\right).$$

A.6 Proof of Proposition 5

By matching Eqs. (19) and (20), we can express the economy's TFP as

$$TFP_t \equiv \omega_t^\beta \left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}\right)^\epsilon. \tag{53}$$

Then, by using that $\bar{z} = r\omega^{-\frac{\beta}{\epsilon}}\epsilon^{-1}\left(\left(1-\epsilon\right)/w\right)^{\frac{\epsilon-1}{\epsilon}}$, the sensitivity of Eq. (53) variations in factor prices and public good multiplier $x_t \equiv \{r_t, w_t, \omega_t\}$ is given by

$$\begin{aligned}\frac{\partial}{\partial x_t}\left[\omega_t^\beta \left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}\right)^\epsilon\right] &= \frac{\partial \omega_t^\beta}{\partial x_t} \times \left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}\right)^\epsilon + \\ &+ \epsilon\omega_t^\beta \left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}\right)^{\epsilon-1} \frac{\partial}{\partial \bar{z}}\left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}\right)\frac{\partial}{\partial x_t}\left(\frac{r_t}{\omega_t^{\frac{\beta}{\epsilon}}\epsilon\left(\frac{1-\epsilon}{w_t}\right)^{\frac{1-\epsilon}{\epsilon}}}\right).\end{aligned} \tag{54}$$

By using that

$$\frac{\partial\left(\frac{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz}{1-\int_0^{\bar{z}}\theta_t(z)dz}\right)}{\partial \bar{z}} = \frac{\theta_t(\bar{z})\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz - \bar{z}\theta_t(\bar{z})\left[1-\int_0^{\bar{z}}\theta_t(z)dz\right]}{\left(1-\int_0^{\bar{z}}\theta_t(z)dz\right)^2} = \phi\theta_t(\bar{z})\left(\phi Z_t - \bar{z}\right),$$

we can rearrange Eq. (54) as

$$\frac{\partial TFP_t}{\partial x_t} = TFP_t \left[\frac{\partial \omega_t^{\beta}}{\partial x_t} \frac{1}{\omega_t^{\beta}} + \epsilon \theta_t(\bar{z}) \frac{(\phi Z_t - \bar{z})}{Z_t} \frac{\partial}{\partial x_t} \left(\frac{r_t}{\omega_t^{\frac{\beta}{\epsilon}} \epsilon \left(\frac{1-\epsilon}{w_t} \right)^{\frac{1-\epsilon}{\epsilon}}} \right) \right].$$

A.7 Proof of Proposition 6

Similarly to González et al. (2022), the growth rate of the economy's TFP can be expressed as

$$\frac{dTFP_t}{dt} \frac{1}{TFP_t} = \frac{d \ln TFP_t}{dt} = \frac{d}{dt} \left[\beta \ln \omega_t + \epsilon \ln \left(\int_{\bar{z}}^{z^{\max}} z \theta_t(z) dz \right) - \epsilon \ln \left(\int_{\bar{z}}^{z^{\max}} \theta_t(z) dz \right) \right],$$

which can be conveniently rearranged as

$$\frac{d \ln TFP_t}{dt} = \beta \frac{\dot{\omega}_t}{\omega_t} + \epsilon \left[\frac{\int_{\bar{z}}^{z^{\max}} z \dot{\theta}_t(z) dz}{\int_{\bar{z}}^{z^{\max}} z \theta_t(z) dz} - \frac{\int_{\bar{z}}^{z^{\max}} \dot{\theta}_t(z) dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z) dz} \right]. \tag{55}$$

By taking the partial derivative of Eq. (55) with respect to the price vector x_t and keeping $\dot{\omega}$ constant, we obtain

$$\frac{\partial}{\partial x} \frac{d \ln TFP_t}{dt} = \epsilon \left[\frac{\int_{\bar{z}}^{z^{\max}} z \frac{\partial \dot{\theta}_t(z)}{\partial x_t} dz}{\int_{\bar{z}}^{z^{\max}} z \theta_t(z) dz} - \frac{\int_{\bar{z}}^{z^{\max}} \frac{\partial \dot{\theta}_t(z)}{\partial x_t} dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z) dz} \right],$$

which by using Eq. (23) and rearranging yields

$$\begin{aligned} \frac{\partial}{\partial x_t} \frac{d \ln TFP_t}{dt} = \epsilon & \left[\frac{\int_{\bar{z}}^{z^{\max}} z \frac{\partial \mu_n}{\partial x_t} \theta_t(z) dz}{\int_{\bar{z}}^{z^{\max}} z \theta_t(z) dz} - \frac{\int_{\bar{z}}^{z^{\max}} \frac{\partial \mu_n}{\partial x_t} \theta_t(z) dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z) dz} \right] + \\ & - \frac{\partial \left(\frac{d \ln K_t}{dt} - (1 - \Xi) \lambda \right)}{\partial x_t} \underbrace{\left(\frac{\int_{\bar{z}}^{z^{\max}} z \theta_t(z) dz}{\int_{\bar{z}}^{z^{\max}} z \theta_t(z) dz} - \frac{\int_{\bar{z}}^{z^{\max}} \theta_t(z) dz}{\int_{\bar{z}}^{z^{\max}} \theta_t(z) dz} \right)}_{=0} \end{aligned}$$

or, by using that $\frac{z\theta_t(z)\mathbb{I}_{z\geq\bar{z}}}{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz} \equiv \bar{\theta}_t(z)$ and $\frac{\theta_t(z)\mathbb{I}_{z\geq\bar{z}}}{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz} \equiv \tilde{\theta}_t(z)$ can be interpreted as probability density functions over $[\bar{z}, z^{max}]$,

$$\frac{\partial}{\partial x}\frac{d\ln TFP_t}{dt} = \epsilon\left\{\mathbb{E}^{\bar{\theta}}\left[\frac{\partial\mu_n(z)}{\partial x_t}\,|z_t \geq \bar{z}\right] - \mathbb{E}^{\tilde{\theta}}\left[\frac{\partial\mu_n(z)}{\partial x_t}\,|z_t \geq \bar{z}\right]\right\}. \tag{56}$$

Lemma 1. ***Sign of the aggregate TFP growth***

The sign of the partial derivative in Eq. (56) is uniquely determined by the sign of $\mathbb{E}^{\bar{\theta}}\left[\frac{\partial\mu_n(z)}{\partial x_t}\,|z_t \geq \bar{z}\right]$. *This happens because* $\tilde{\theta}$ *is stochastically dominated by* $\bar{\theta}$.

Proof. The result can be proved by noticing that the monotone likelihood ratio

$$\frac{\bar{\theta}_t(z)}{\tilde{\theta}_t(z)} = z\frac{\int_{\bar{z}}^{z^{\max}} \theta_t(z)dz}{\int_{\bar{z}}^{z^{\max}} z\theta_t(z)dz},$$

is non-decreasing in z, meaning that for each couple $z_1 \geq z_0$ it holds that

$$\frac{\bar{\theta}_t(z_1)}{\tilde{\theta}_t(z_1)} \geq \frac{\bar{\theta}_t(z_0)}{\tilde{\theta}_t(z_0)}. \tag{57}$$

By rearranging Eq. (57) and integrating with respect to z_0 over $[\bar{z},\, z^{max}]$, one gets that

$$\int_{\bar{z}}^{z_1} \bar{\theta}_t(z_1)\tilde{\theta}_t(z_0)dz_0 \geq \int_{\bar{z}}^{z_1} \bar{\theta}_t(z_0)\tilde{\theta}_t(z_1)dz_0;$$

that is,

$$\frac{\bar{\theta}_t(z_1)}{\tilde{\theta}_t(z_1)} \geq \frac{\bar{\Theta}_t(z_1)}{\tilde{\Theta}_t(z_1)}. \tag{58}$$

By integrating the same equation with respect to z_1 over the interval $[z_0, z^{max}]$, we obtain instead that

$$\int_{z_0}^{z^{\max}} \bar{\theta}_t(z_1)\tilde{\theta}_t(z_0)dz_1 \geq \int_{z_0}^{z^{\max}} \bar{\theta}_t(z_0)\tilde{\theta}_t(z_1)dz_1;$$

that is,

$$\frac{\bar{\theta}_t(z_0)}{\tilde{\theta}_t(z_0)} \leq \frac{1-\bar{\Theta}_t(z_0)}{1-\tilde{\Theta}_t(z_0)}. \tag{59}$$

By matching Eqs. (58) and (59) for $z_0 = z_1 = z$, we obtain the following first-order stochastic dominance relationship:

$$\bar{\Theta}_t(z) \leq \tilde{\Theta}_t(z)$$

$\square$

B Solution algorithm and numerical approximation

B.1 Overview of the solution algorithm

Following Moll (2014), we numerically compute the model's steady-state equilibrium and its transition dynamics by recursively implementing the following steps.

1. For a given stock of capital K_t (a scalar) and its distribution across entrepreneurs $\theta_t(z)$ (a vector of size $1 \times I$), approximate the productivity threshold $\bar{z}$ and the average productivity Z_t by solving Eq. (19) over a grid $\mathbb{Z}^I = \begin{bmatrix} z_1 = 0 & \cdots & z_i & \cdots & z_I = z_{\max} \end{bmatrix}$ of size I.

2. Given the outcome of Point 1, compute market prices and public good supply $\{r_t, w_t, \omega_t\}$ by solving numerically the non-linear system in Eq. (21) over $\mathbb{Z}^I$.

3. Given the outcome of Point 2, compute entrepreneurs' consumption rate $\zeta(z)$ by approximating the solution of Eq. (38) over $\mathbb{Z}^I$ (see Appendix B.2.1 for details).

4. Given the outcome of Point 3 and some discrete time interval $dt \approx \Delta t$, obtain the density $\theta_{t+dt}(z)$ by approximating over $\mathbb{Z}^I$ the solution of Eq. (52), in which $d \ln K_t/dt \approx (\ln K_{t+\Delta t} - \ln K_t)/\Delta t$ is given by Eq. (22) (see Appendix B.2.2 for details).

5. Given the outcome of Point 4, verify if $\|K_{t+dt} - K_t\| \wedge \|\theta_{t+dt}(z) - \theta_t(z)\| \leq 10^{-4}$. If yes, stop and define the steady-state equilibrium as $\{K_t, \theta_t(z), r_t, w_t, \omega_t\} \equiv \{K_\infty, \theta_\infty(z), r, w, \omega\}$. If not, update $K_t \to K_{t+dt}$ and $\theta_t(z) \to \theta_{t+dt}(z)$ and repeat from Point 1.

B.2 Numerical methods

B.2.1 Approximating entrepreneurs' consumption rate

Entrepreneurs' consumption rate is given by $\zeta_t(z) = \nu_t(z)^{-\frac{1}{\gamma}}$, whose value solves Eq. (38). We approximate function $\nu_{t_j}(z_i) \approx \nu_{i,j}$ over a uniformly-spaced grid of $I \times J$ points. Grid elements span with distance $\Delta z = z^{\max}/(I-1)$ and Δt in space and time, respectively. To approximate the derivatives of ν, we follow Candler (1999) and implement the following upwind scheme

$$\begin{aligned}\frac{\nu_{i,j+1} - \nu_{i,j}}{\Delta t} + \hat{\rho}\nu_{i,j} = \gamma {\nu_{i,j}}^{1-\frac{1}{\gamma}} + \frac{\nu_{i,j} - \nu_{i-1,j}}{\Delta z} \max\{\mu_z, 0\} + \\ + \frac{\nu_{i+1,j} - \nu_{i,j}}{\Delta z} \min\{\mu_z, 0\} + \frac{1}{2}\frac{\nu_{i+1,j} - 2\nu_{i,j} + \nu_{i-1,j}}{\Delta z^2}\sigma_z^2, \quad (60)\end{aligned}$$

in which we adopt backward and forward finite-difference approximation in space depending on whether the drift of the state variable z is positive or negative. Since Eq. (60) moves backward in time, we use a forward finite-difference approximation for its time dimension. Equipped with this scheme, we can rearrange Eq. (60) as

$$\nu_{i,j+1} = \nu_{i-1,j}a(z_i) + \nu_{i,j}b(z_i) + \nu_{i+1,j}c(z_i), \quad (61)$$

where

$$a(z) = -\frac{\max\{\mu_z, 0\}\Delta t}{\Delta z} + \frac{1}{2}\frac{\sigma_z^2 \Delta t}{\Delta z^2},$$

$$b(z_i, \nu_{i,j}) = \gamma {\nu_{i,j}}^{-\frac{1}{\gamma}}\Delta t + 1 - \hat{\rho}\Delta t + \frac{\max\{\mu_z, 0\}\Delta t}{\Delta z} - \frac{\min\{\mu_z, 0\}\Delta t}{\Delta z} - \frac{\sigma_z^2 \Delta t}{\Delta z^2},$$

and

$$c(z_i) = \frac{\min\{\mu_z, 0\}\Delta t}{\Delta z} + \frac{1}{2}\frac{\sigma_z^2 \Delta t}{\Delta z^2}$$

for $i = 1, 2, ..., I$. The upper boundary condition in Eq. (39) is enforced by imposing $\nu_{I+1,j} = \nu_{I,j}$, which implies[17]

$$\frac{\nu_{I,j+1}}{\Delta t} = \nu_{I-1,j} a(z_I) + \left[b(z_I) + c(z_I)\right] \nu_{I,j}.$$

We compute $\nu_{i,j+1}$ recursively for $j = 1, 2, ..., J$ given an initial guess $\nu_{i,0} = \rho^{-\gamma}$; that is, the optimal steady-state consumption in the limit case in which $\gamma = 1$ (log utility). We implement the numerical solution as follows.

Given a collection of equilibrium quantities $\{\nu_{i,j}, \omega_{i,j}, r_{i,j}, w_{i,j}\}$ we compute $\nu_{i,j+1}$ for $i = 1, 2, ..., I-1$ as

$$\underbrace{\begin{bmatrix} \nu_{2,j+1} \\ \nu_{2,j+1} \\ \vdots \\ \nu_{I-1,j+1} \end{bmatrix}}_{\nu_{I-1}^{J+1}} = \underbrace{\begin{bmatrix} b(z_2, \nu_{2,j}) & c(z_2) & 0 & \cdots & 0 \\ a(z_3) & b(z_3, \nu_{3,j}) & c(z_3) & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & a(z_{I-1}) & b(z_{I-1}) + c(z_{I-1}) \end{bmatrix}}_{A_{I-1}^{J}} \underbrace{\begin{bmatrix} \nu_{2,j} \\ \nu_{3,j} \\ \vdots \\ \nu_{I-1,j} \end{bmatrix}}_{\nu_{I-1}^{J}} \Delta t,$$

which we iterate with a convergence rule $\left\| \nu_{I-1}^{J+1} - \nu_{I-1}^{J} \right\|_\infty \leq 10^{-4}$.

B.2.2 Approximating entrepreneurs' density function

Similarly to entrepreneurs' value function in Section B.2.1, we approximate the density $\theta_{t_j}(z_i) \approx \theta_{i,j}$ in Eq. (52) by means of the following upwind finite-difference scheme

$$\begin{aligned} \frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} &= l(z_i)\theta_{i,j+1} + \frac{\theta_{i+1,j+1} - \theta_{i,j+1}}{\Delta z} \min\left\{m(z_i), 0\right\} + \\ &\quad + \frac{\theta_{i,j+1} - \theta_{i-1,j+1}}{\Delta z} \max\left\{m(z_i), 0\right\} + n(z_i) \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{\Delta z^2}, \end{aligned} \quad (62)$$

where

$$l(z) = \left[\mu_n(z) - \frac{K_{t+\Delta t} - K_t}{K_t} \frac{1}{\Delta t} - (1 - \Xi)\lambda + \nu\left(\frac{3}{2} + \ln z\right)\right],$$

[17] As it is discussed in Moll (2014), we do not need to specify any lower boundary condition in $z_1 = 0$ because there both drift and diffusion terms of the the state process are equal to zero.

$$m(z) = \nu z \left(\ln z + \frac{3}{2} \right),$$

and

$$n(z) = \frac{z^2 \nu}{2},$$

We approximate the dynamics of aggregate capital stock in Eq. (22) as

$$\frac{K_{t+\Delta t} - K_t}{K_t} \frac{1}{\Delta t} = -\sum_{i=1}^{I} \zeta_t(z_i)\theta_t(z_i)\Delta z + \epsilon(1 - \tau)\frac{Y_t(Z_t)}{K_t} + \lambda \left(\varXi^{-\gamma} - 1 \right) (1 - \varXi),$$

and the economy's aggregate productivity as

$$Z_t = \sum_{i=1}^{I} \mathbb{I}_{z_i \geq \bar{z}} z_i \theta_t(i) \Delta z.$$

Eq. (62) can be rearranged in the following linear system

$$\begin{bmatrix} p(z_i) & q(z_i) & r(z_i) \end{bmatrix} \begin{bmatrix} \theta_{i-1,j+1} \\ \theta_{i,j+1} \\ \theta_{i+1,j+1} \end{bmatrix} = \theta_{i,j},$$

for $i = 2, ..., I$, where

$$p(z_i) = \Delta t \left[\frac{\max\{m(z_i), 0\}}{\Delta z} - \frac{n(z_i)}{\Delta z^2} \right],$$

$$q(z_i) = 1 - \Delta t \left(l(z_i) - \frac{\min\{m(z_i), 0\}}{\Delta z} + \frac{\max\{m(z_i), 0\}}{\Delta z} - \frac{2n(z_i)}{\Delta z^2} \right),$$

and

$$r(z_i) = -\Delta t \left[\frac{\min\{m(z_i), 0\}}{\Delta z} + \frac{n(z_i)}{\Delta z^2} \right],$$

which can be solved iterating over $j = 1, 2, ..., J$ given an initial distribution $\theta_{i,0}$. The boundary conditions are set so that $\theta_{1,j} = 0$ and the mass preservation condition $\sum_{i=2}^{I} \theta_{i,j} dz = 1$ holds for all $j = 1, 2, ..., J$. We implement the numerical algorithm as follows.

Given a collection of quantities $\{\nu_{i,j}, \theta_{i,j}, \omega_{i,j}, r_{i,j}, w_{i,j}\}$, we compute $\theta_{i,j+1}$ by recursively solving forward in time the linear system

$$\underbrace{\begin{bmatrix} \theta_{2,j+1} \\ \theta_{3,j+1} \\ \vdots \\ \theta_{I-1,j+1} \\ \theta_{I,j+1} \end{bmatrix}}_{\theta^{j+1}_{(I-1)\times 1}} = \underbrace{\begin{bmatrix} q(z_2) & r(z_2) & 0 & \cdots & 0 \\ p(z_3) & q(z_3) & r(z_3) & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ dz & dz & dz & dz & dz \end{bmatrix}^{-1}}_{B^{-1}_{(I-1)\times(I-1)}} \underbrace{\begin{bmatrix} \theta_{2,j} \\ \theta_{3,j} \\ \vdots \\ \theta_{I-1,j} \\ 1 \end{bmatrix}}_{\theta^{j}_{(I-1)\times 1}},$$

with a convergence rule $\left\|\theta^J - \theta^{J-1}\right\|_\infty < 10^{-4}$.