What is Missing in Asset-Pricing Factor Models?*

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Abstract

Our objective is to price the cross-section of asset returns. Despite considering hundreds of systematic risk factors ("factor zoo"), factor models still have sizable pricing errors. A limitation of these models is that returns compensate only for systematic risk. We allow *compensation also for unsystematic risk* while imposing no arbitrage. The resulting stochastic discount factor (SDF) dominates traditional factor models in pricing assets. Empirically, about 70% of this SDF's variation is explained by its unsystematic-risk component, which is correlated with strategies reflecting market frictions and behavioral biases. Our findings provide an avenue for resolving the factor zoo.
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1 Introduction

A major challenge in asset pricing is to explain the cross-section of asset returns. To address this challenge, the literature has examined a large number of systematic or common risk factors, leading to a factor zoo (Cochrane, 2011). However, virtually all models featuring factors from this zoo have sizable pricing errors (Bryzgalova, Huang, and Julliard, 2023), called alpha. A limitation of these models is that they allow expected excess returns to be related only to *systematic* sources of risk and preclude compensation for unsystematic risk comprised of asset-return shocks unexplained by systematic risk factors. In our work, we allow *compensation also for unsystematic risk* and show that this single departure from the traditional risk-return tradeoff provides an avenue for resolving the factor zoo.

To price a cross-section of assets, we use as a foundation for our analysis the Arbitrage Pricing Theory (APT) of Ross (1976, 1977), Chamberlain (1983), and Chamberlain and Rothschild (1983). The APT provides an ideal framework because it allows expected excess returns to contain asset-specific components with two properties. First, these asset-specific components are unrelated to the compensation for asset exposures to systematic risk factors. Second, these asset-specific components satisfy an asymptotic no-arbitrage restriction. Thus, the APT permits us to entertain, in a no-arbitrage setting, the possibility that these asset-specific components in expected excess returns represent compensation for unsystematic risk. In our work, we allow unsystematic risk to include both pure asset-specific risk and weak factors (Lettau and Pelger, 2020).

Our first contribution is to derive an *admissible* SDF, namely an SDF that prices a given cross-section of assets correctly, assuming asset returns are described by the APT. In particular, we show theoretically how systematic and unsystematic risk appear in this admissible SDF and demonstrate that the asset-specific components in expected excess returns represent compensation for unsystematic risk. Thus, we depart from the conventional wisdom that financial markets compensate investors only for exposure to systematic sources of risk and, therefore, investors should diversify away unsystematic risk. To provide microfoundations for our unconventional idea, we derive the SDF in the equilibrium model of Merton (1987) that has nonzero compensation for unsystematic risk when the number of assets is finite. We show that even when the number of assets is asymptotically large, like in the APT, compensation for unsystematic risk remains nonzero, and the functional form of the equilibrium SDF is the same as the one we derive based on the APT.

Our second contribution is providing empirical support for our insight that unsystematic risk is priced and quantifying its importance. To do this, we develop a projection-based SDF that is a version of the admissible SDF projected on a set of basis assets and a risk-free asset. This step is necessary because the admissible SDF, which depends on latent systematic factors and unsystematic shocks, is empirically infeasible. Furthermore, we show that the projection-based SDF converges in probability to the admissible SDF as the number of basis assets increases.

Then, using data for monthly returns on 202 portfolios of stocks, we identify and characterize the SDF and its components. One component reflects systematic risk ("systematic SDF component"), and the other unsystematic risk ("unsystematic SDF component"). Two central findings emerge from our empirical analysis. First, the compensation for unsystematic risk is nonzero. Second, the unsystematic SDF component explains 72.60% of the admissible SDF's variation. Thus, unsystematic risk plays a major role in pricing the cross-section of asset returns, despite the risk premia associated with unsystematic shocks being small on average. Furthermore, a back-of-the-envelope calculation suggests that the two types of unsystematic risk, pure asset-specific risk and weak factors, contribute about equally to the variation of the unsystematic SDF component.

The quantitative importance of the unsystematic SDF component implies that candidate factor models with proxies for only systematic risk factors *cannot* lead to an admissible SDF. We show that *by construction*, the unsystematic SDF component satisfies the definition of a weak factor in a cross-section of basis assets. Thus, statistical methods of factor analysis, such as Principal Component Analysis (PCA), cannot identify it in the data. Instead, to identify the unsystematic SDF component and subsequently obtain an admissible SDF, one must recognize that unsystematic risk is priced, and we show how to do this.[1] This insight paves the way to resolving the factor zoo.

We study 457 trading strategies to understand which ones reflect sizable compensation for unsystematic risk. We find that, of the strategies with high compensation for unsystematic risk, some can be interpreted as being behavioral—for example, the performance factor (Stambaugh and Yuan, 2017), the long-horizon financial factor (Daniel, Hirshleifer, and Sun, 2020a), the factor reflecting expectations about future earnings (La Porta, 1996),

[1] Our findings emphasize the arguments of MacKinlay (1995) and Daniel and Titman (1997) about the importance of assets' characteristics for understanding risk premia and the inability of a factor model to explain a cross-section of stock returns, but with two crucial differences. First, our model ensures asymptotic no-arbitrage. Second, we demonstrate that, in our framework, the asset-specific components in expected returns represent compensation for unsystematic risk.

and the momentum factor (Jegadeesh and Titman, 1993)—while others as reflecting market frictions—for example, the betting-against-beta factor (Frazzini and Pedersen, 2014) and distress risk (Campbell, Hilscher, and Szilagyi, 2008). The compensation offered by these strategies for bearing unsystematic risk is large; for instance, the premium for bearing the unsystematic risk for the 12-month momentum strategy of Jegadeesh and Titman (1993) is 8.27% per annum.

Turning next to the analysis of the *systematic* SDF component, we find that the market return explains 95% of its variation.[2] The substantial contribution of the market return in explaining the variation of the systematic SDF component is because it plays an essential role in determining the *level* of stock returns, as shown by Clarke (2022), among others. However, because the systematic SDF component accounts for only 27.40% of the variation in the admissible SDF, the contribution of the market return to the overall SDF's variation is only 26.03% (that is, 95% $\times$ 27.40%).

Our third contribution is to use our framework to shed light on the poor performance of popular candidate factor models used to price a cross-section of stock returns, and to show how to use our methodology to correct them. We consider: (i) a model with the market return, as suggested by the CAPM of Sharpe (1964), (ii) a model with the consumption-mimicking portfolio, as implied by the Consumption Capital Asset Pricing Model (C-CAPM) of Breeden (1979), and (iii) the three-factor model (FF3) of Fama and French (1993).[3] These candidate factor models may be misspecified because they omit systematic sources of risk, the search for which has been the focus of the existing literature. But, these candidate models may be misspecified also because they omit compensation for unsystematic risk, the focus of our work. Both theoretically and empirically, we identify and characterize the wedge between the admissible SDF and the SDFs implied by these candidate factor models.

The central insight from our empirical analysis of these three candidate models is that their implied SDFs represent less than 40% of the variation in the admissible SDF. The principal source of missing variation is *unsystematic* risk, which, in these models, similar to virtually all other factor models, is assumed to have zero compensation. These candidate factor models also omit sources of systematic risk, and we show how to correct for this type

[2] To explain 99% of variation in the systematic SDF component, we need the returns on four extra trading strategies, all highly correlated with the variable known as the size factor.

[3] We do not analyze other candidate factor models because the conclusions we draw from our empirical analysis of these three candidate factor models apply to virtually any candidate factor model that is based on the premise that only systematic risk is compensated in financial markets.

of misspecification as well. Moreover, once we use our approach to include what is missing in each of the three analyzed candidate models, we obtain admissible SDFs that are almost perfectly correlated.

When traditional asset-pricing models, such as the CAPM, fail to explain a cross-section of stock returns, the response has been to search for additional proxies for systematic factors. For instance, momentum (Jegadeesh and Titman, 1993), value (Fama and French, 2015), and investment (Hou, Xue, and Zhang, 2015) have attracted attention as successful proxies. We find that these variables correlate more highly with the unsystematic rather than the systematic SDF component. This finding has two important implications. First, these variables appear to be weak factors in the cross-section of basis assets, as opposed to systematic factors. Second, these variables contribute to explaining the cross-section of expected returns because they reflect largely the compensation for unsystematic risk. Moreover, we show empirically that including even a large number of such observable factors in a candidate factor model fails to fully capture the unsystematic SDF component.

Our work is related to several strands of the literature. First, we contribute to the literature that uses a large cross-section of asset returns to examine the risk-return tradeoff implied by factor models. To handle a large number of assets, Kozak, Nagel, and Santosh (2020), Lettau and Pelger (2020), Pelger (2020), Giglio and Xiu (2021), and Giglio, Xiu, and Zhang (2021b) develop methods based on PCA for estimating the SDF, identifying factors that price the cross-section of expected returns, and estimating prices of these risk factors even in the presence of model misspecification. Our approach, founded on the APT, also allows for a large number of assets to estimate an admissible SDF. In contrast to these papers, our approach explicitly allows compensation not only for factor risk but also for asset-specific shocks.

Second, because we correct the misspecified SDF implied by a given candidate factor model, we contribute to the literature that studies misspecification of the SDF and develops methods to characterize the wedge between the misspecified and admissible SDFs. Hansen and Jagannathan (1997) provide the smallest additive nonparametric adjustment (in a least-squares sense) required to make a given SDF admissible. Almeida and Garcia (2012) provide an additive correction term based on minimum-discrepancy projections. Sandulescu and Schneider (2021) build on Almeida and Schneider (2021) to construct an SDF that is a sum of a linear part, which is identical to that from Hansen and Jagannathan (1997), and a nonlinear part, which ensures the positivity of the SDF and leads to an admissible SDF. Korsaye, Quaini, and Trojani (2021) construct a minimum-dispersion SDF subject to a convex pricing constraint and study the tradeoff between the SDF's pricing accuracy and its comovement with standard proxies for systematic risk. Ghosh, Julliard, and Taylor (2017) provide a multiplicative SDF correction using a Kullback-Leibler entropy-minimization approach. Just like Ghosh et al. (2017), we ensure the positivity of the SDF by specifying it in an exponential form. Complementary to the previous approaches, our method allows for unsystematic risk, in addition to omitted sources of systematic risk, to explain the wedge between the misspecified SDF and admissible SDF.

Third, we contribute to the factor-zoo literature (see, e.g., Cochrane, 2011; Harvey, Liu, and Zhu, 2015; Kogan and Tian, 2015), which has proposed hundreds of variables that can potentially proxy for systematic risk priced in the cross-section of asset returns. Our contribution is in identifying that what asset-pricing factor models are missing is compensation for unsystematic risk rather than a yet-undiscovered proxy for systematic risk.

Furthermore, our paper complements methodological advances aimed at taming the factor zoo. Feng, Giglio, and Xiu (2020), Freyberger, Neuhierl, and Weber (2020), Giglio, Liao, and Xiu (2021a), and Bryzgalova et al. (2023) propose model-selection methods to discipline the proliferation of factors, and account for data snooping when performing multiple-hypothesis testing in linear asset-pricing models. Our focus is different: as a by-product of our analysis, we provide a method that establishes whether an arbitrary variable is a proxy for a systematic or weak factor and quantifies the price of this factor risk.

Clearly, our work is also related to the literature on the idiosyncratic-volatility puzzle, which studies the relation between the compensation for asset-specific risk and the *volatility* of asset-specific shocks; see, for example, Fama and MacBeth (1973) and Ang, Hodrick, Xing, and Zhang (2006), with a comprehensive review provided by Bali, Engle, and Murray (2016). Complementary to this empirical literature, we construct an SDF, where the compensation for unsystematic risk represents the negative covariance between this SDF and unsystematic shocks (rather than their *volatility*). Furthermore, we find that the returns of the idiosyncratic-volatility factors of Ali, Hwang, and Trombley (2003) and Ang et al. (2006) explain less than 10% of the variation in the unsystematic SDF component. Thus, what is missing in asset-pricing factor models is not just the idiosyncratic-volatility factor.

The rest of the paper is organized as follows. Section 2 presents our theoretical results for constructing an admissible SDF. Section 3 explains how to estimate an admissible SDF. Section 4 describes the data we use to illustrate our approach. Section 5 presents the empirical findings from applying our approach to this data. Section 6 provides an example of an equilibrium model in which unsystematic risk is priced. We conclude in Section 7. The Internet Appendix reports proofs, the estimation algorithm, and additional results.

2 Constructing an Admissible SDF

In this section, we first derive the SDF under the classical APT. Then, we explain how to ensure the positivity and empirical feasibility of this SDF. Next, we show how to correct the SDF implied by an arbitrary candidate linear factor model for misspecification caused by omitted (i) sources of systematic risk, (ii) compensation for unsystematic risk, and (iii) time variation in risk premia.

Throughout the manuscript, we use the following notation. Let an N-dimensional vector $R_{t+1} = (R_{1,t+1}, R_{2,t+1}, \ldots, R_{N,t+1})'$ denote the vector of gross returns of N risky assets between t and $t+1$. Let R_f be the gross return on a risk-free asset over the same period.[4] Let $\mathbb{E}(\cdot)$ denote the expectation operator and 1_N indicate an $N \times 1$ vector of ones, so that $\mathbb{E}(R_{t+1} - R_f 1_N)$ represents the vector of expected excess returns on the N assets. Let f_{t+1} be a $K \times 1$ vector of systematic risk factors, with $K < N$ and a $K \times K$ positive definite covariance matrix $V_f > 0$.[5] Let $\beta = (\beta_1, \beta_2, \ldots, \beta_N)'$ be an $N \times K$ full-rank matrix of loadings of asset returns on the systematic factors f_{t+1}. The notation 0_N indicates an $N \times 1$ vector of zeros. For deterministic sequences $\{a_N\}$ and $\{b_N\}$ the notation $a_N = O(b_N)$ means that, as $N \to \infty$, $|a_N|/b_N < \delta$, where $\delta > 0$ is some finite number, and $a_N = o(b_N)$ means that $|a_N|/b_N \to 0$.

2.1 The SDF under the Arbitrage Pricing Theory (APT)

The APT of Ross (1976, 1977), Chamberlain (1983), and Chamberlain and Rothschild (1983) is our working assumption about the true data-generating process for asset returns. There are several advantages to choosing the APT as the null model. First, the APT is a flexible model that does not take a stand on systematic risk factors. Second, it is a no-arbitrage model; the absence of arbitrage opportunities implies the existence of an SDF. Third, and more importantly for our purpose, the APT allows for asset-specific components in expected excess returns unrelated to systematic risk.

[4] If a risk-free asset does not exist, one can use the return on the minimum-variance portfolio or the return on a zero-beta portfolio instead.

[5] Note that, for an arbitrary matrix A, the expression $A > 0$ means that A is positive-definite.

The classical APT builds on the following two assumptions.

Assumption 2.1 (Linear Factor Model)**.** *The vector R_{t+1} of gross asset returns satisfies*

$$R_{t+1} = \mathbb{E}(R_{t+1}) + \beta\big(f_{t+1} - \mathbb{E}(f_{t+1})\big) + e_{t+1}, \tag{1}$$

where the vector e_{t+1} has $\mathbb{E}(e_{t+1}) = 0_N$ and an $N \times N$ covariance matrix $V_e > 0$, whose eigenvalues are uniformly bounded and bounded away from zero. The shocks e_{t+1} are uncorrelated with the K systematic factors f_{t+1}, implying the covariance matrix of returns is

$$V_R = \beta V_f \beta' + V_e. \tag{2}$$

Recall that in the APT of Chamberlain (1983) and Chamberlain and Rothschild (1983), asset returns satisfy an approximate factor structure; that is, the covariance matrix V_e is not restricted to be diagonal. The case of a non-diagonal matrix V_e with uniformly bounded eigenvalues can accommodate the presence of weak factors f_{t+1}^{weak} in the shocks e_{t+1}, or equivalently, in returns R_{t+1}. We define weak factors similarly to Lettau and Pelger (2020) as factors that affect only a subset of the underlying assets or all assets but marginally. Mathematically, if β^{weak} is a matrix of loadings of returns R_{t+1} on weak factors f_{t+1}^{weak}, then, as $N \to \infty$, this matrix satisfies $\beta^{\text{weak}\prime}\beta^{\text{weak}} \to E > 0$, where E is some symmetric matrix.

Assumption 2.2 (Asymptotic No Arbitrage)**.** *There is no sequence of portfolios containing N risky assets with weights $w = (w_1, w_2, \ldots, w_N)'$, for which, as $N \to \infty$:*

$$\operatorname{var}(R_{t+1}'w) \to 0 \qquad \textit{and} \qquad (\mathbb{E}(R_{t+1}) - R_f 1_N)'w \geq \delta > 0,$$

where δ denotes an arbitrary positive scalar.

Assumptions 2.1 and 2.2 imply that, under the APT, asset excess returns are

$$R_{t+1} - R_f 1_N = a + \beta\lambda + \beta(f_{t+1} - \mathbb{E}(f_{t+1})) + e_{t+1}, \tag{3}$$

with expected *excess* returns

$$\mathbb{E}(R_{t+1} - R_f 1_N) = a + \beta\lambda \tag{4}$$

containing two components: a and $\beta\lambda$. The $K \times 1$ vector of risk premia λ represents the compensations for a unit of assets' exposures to the factors f_{t+1}. Ingersoll (1984) derives the precise condition for λ to exist and shows that $\lambda = \lim_{N\to\infty} \left(\beta' V_e^{-1} \beta\right)^{-1} \beta' V_e^{-1}(\mathbb{E}(R_{t+1}) -$

$R_f 1_N$). Ross (1976), Huberman (1982), Chamberlain (1983), Chamberlain and Rothschild (1983), and Ingersoll (1984) show that the $N \times 1$ vector $a = (\mathbb{E}(R_{t+1}) - R_f 1_N) - \beta\lambda$, typically referred to as the vector of pricing errors, satisfies the following *no-arbitrage restriction*

$$a' V_e^{-1} a \leq \delta_{\text{apt}} < \infty, \tag{5}$$

where δ_{apt} is some arbitrary positive scalar.

We now provide the SDF under the classical APT.

Proposition 1 (An Admissible SDF). *The SDF M_{t+1} implied by the APT model of asset returns is*

$$M_{t+1} = M_{t+1}^{\beta} + M_{t+1}^{a}, \quad \textit{where} \tag{6}$$

$$M_{t+1}^{\beta} = \frac{1}{R_f} - \frac{\lambda' V_f^{-1}}{R_f}(f_{t+1} - \mathbb{E}(f_{t+1})) \quad \textit{and} \quad M_{t+1}^{a} = -\frac{a' V_e^{-1}}{R_f} e_{t+1},$$

with $\text{cov}(M_{t+1}^{\beta}, M_{t+1}^{a}) = 0$.

The term M_{t+1}^{β} is a linear function of the systematic risk factors f_{t+1}, and therefore we refer to M_{t+1}^{β} as the *systematic* SDF component. The term M_{t+1}^{a} is a linear function of the asset-return shocks e_{t+1} that are orthogonal to systematic risk, and therefore we refer to M_{t+1}^{a} as the *unsystematic* SDF component. The unsystematic SDF component captures the pricing implications of both asset-specific risk and weak factors if they are present in the asset-return shocks e_{t+1}.

The presence of the unsystematic component M_{t+1}^{a} in the admissible SDF M_{t+1} leads to the main insight underlying our approach, which is the *interpretation* of the vector a. When viewed through the lens of the systematic SDF component, M_{t+1}^{β}, the vector a is typically interpreted as a pricing error:

$$a = \mathbb{E}(M_{t+1}^{\beta}(R_{t+1} - R_f 1_N)) \times R_f.$$

However, when viewed through the lens of the admissible SDF, the expression

$$a = -\text{cov}(M_{t+1}, e_{t+1}) \times R_f = -\text{cov}(M_{t+1}^{a}, R_{t+1}) \times R_f,$$

indicates that the vector a should be interpreted as the risk premium for unsystematic shocks e_{t+1}, rather than pricing errors. This novel interpretation paves the way for a quantitative assessment of *priced* unsystematic risk in financial markets that we undertake in our empirical analysis.

2.2 Constructing an Admissible SDF in Practice

In practice, there are two challenges in constructing the admissible SDF (6). First, this linear SDF may not always be strictly positive, which could result in negative asset prices leading to arbitrage opportunities. Second, the admissible SDF (6) is not feasible empirically because it depends on the unobserved factors f_{t+1} and shocks e_{t+1}. We address both challenges. To construct a feasible positive SDF, we rely on an exponential function of the linear projections of M^a_{t+1} and M^β_{t+1} on the set of the risk-free and risky assets to obtain[6]

$$\hat{M}_{\exp,t+1} = \hat{M}^\beta_{\exp,t+1} \times \hat{M}^a_{\exp,t+1}, \quad \text{where} \tag{7}$$

$$\hat{M}^\beta_{\exp,t+1} = \frac{1}{R_f} \times \exp\Big(-(\beta\lambda)' V_R^{-1}(R_{t+1} - \mathbb{E}(R_{t+1})) - \frac{1}{2}(\beta\lambda)' V_R^{-1}\beta\lambda \Big) \quad \text{and} \tag{8}$$

$$\hat{M}^a_{\exp,t+1} = \exp\Big(-a' V_R^{-1}(R_{t+1} - \mathbb{E}(R_{t+1})) - \frac{1}{2} a' V_R^{-1} a \Big). \tag{9}$$

In expressions (8) and (9), the covariance matrix of asset returns V_R satisfies equation (2) and the expected excess returns $\mathbb{E}(R_{t+1}) - R_f 1_N$ satisfy equation (4).

The next proposition shows that, as $N \to \infty$, our feasible SDF in equation (7) recovers the admissible SDF formulated as an exponential function of payoffs:[7]

$$M_{\exp,t+1} = M^\beta_{\exp,t+1} \times M^a_{\exp,t+1}, \quad \text{where} \tag{10}$$

$$M^\beta_{\exp,t+1} = \frac{1}{R_f} \times \exp\Big(-\lambda' V_f^{-1}(f_{t+1} - \mathbb{E}(f_{t+1})) - \frac{1}{2}\lambda' V_f^{-1}\lambda \Big) \quad \text{and}$$

$$M^a_{\exp,t+1} = \exp\Big(-a' V_e^{-1} e_{t+1} - \frac{1}{2} a' V_e^{-1} a \Big).$$

Proposition 2 (Asymptotic Properties of the Feasible SDF)**.** *Under Assumptions 2.1 and 2.2 of the APT and the assumption that the factors f_{t+1} and unsystematic shocks e_{t+1} are jointly Gaussian, the SDF in equation (10) is admissible. Furthermore, if the following two conditions—(i) $N^{-1}\beta' V_e{}^{-1}\beta \longrightarrow E > 0$, as $N \to \infty$, where E is some arbitrary symmetric $K \times K$ matrix, and (ii) $\beta' V_e{}^{-1} a = o(N^{\frac{1}{2}})$—are satisfied, then as $N \to \infty$, the following results hold*

$$\hat{M}^a_{\exp,t+1} - M^a_{\exp,t+1} \xrightarrow{p} 0, \quad \hat{M}^\beta_{\exp,t+1} - M^\beta_{\exp,t+1} \xrightarrow{p} 0, \quad \operatorname{cov}(\hat{M}^\beta_{\exp,t+1}, \hat{M}^a_{\exp,t+1}) \to 0.$$

The proof of Proposition 2 does not rely on V_e being diagonal and, therefore, allows for the presence of weak factors in asset-return shocks e_{t+1}. Internet Appendix IA.5 discusses

[6] In what follows, the symbol $\hat{\cdot}$ denotes a projection.

[7] See, for example, Gourieroux and Monfort (2007) and Ghosh et al. (2017) for modeling an SDF as an exponential function of payoffs to guarantee the SDF's positivity.

explicitly the case of weak factors in shocks e_{t+1} and shows that we can construct the unsystematic SDF component even in the presence of weak factors.

2.3 What is Missing in Popular SDF Models

In this section, we discuss how to identify what is missing in an arbitrary *candidate* linear factor model and correct this model for misspecification. We use the superscript "can" for variables related to a candidate model.

A standard candidate factor model has K^{can} observable risk factors f_{t+1}^{can} implying that the assets' expected excess returns reflect compensation for exposures to these risk factors. A classic example of a candidate factor model is the CAPM with $K^{\text{can}} = 1$ systematic risk factor represented by the market excess return and compensation for unsystematic risk $a^{\text{can}} = 0_N$. Viewed through the lenses of the APT, candidate factor models suffer from possibly three sources of misspecification. First, these models may omit systematic risk factors. Second, these models may omit compensation for unsystematic risk. Of course, candidate models may also be misspecified because they do not account for time variation in prices of risk or risk exposures. We address the first two sources of misspecification in Section 2.3.1 and the third source of misspecification in Section 2.3.2.

2.3.1 Accounting for omitted systematic risk factors or omitted compensation for unsystematic risk

Let β^{can} denote an $N \times K^{\text{can}}$ matrix of loadings of asset returns on the candidate factors f_{t+1}^{can} and λ^{can} denote a $K^{\text{can}} \times 1$ vector of risk premia for unit exposures to these factors. The candidate factor model implies

$$R_{t+1} - R_f 1_N = \alpha + \beta^{\text{can}}\lambda^{\text{can}} + \beta^{\text{can}}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})) + \varepsilon_{t+1}, \tag{11}$$

where the vector $\alpha = (\mathbb{E}(R_{t+1}) - R_f 1_N) - \beta^{\text{can}}\lambda^{\text{can}}$ captures the cross-sectional variation in expected excess returns left unexplained by compensation for asset exposures to systematic risk factors f_{t+1}^{can}, and the vector ε_{t+1} with the covariance matrix V_ε captures the returns' variation that is not explained by the set of candidate factors f_{t+1}^{can}.

The candidate factor model implies a linear SDF

$$M_{t+1}^{\beta,\text{can}} = \frac{1}{R_f} - \frac{(\lambda^{\text{can}})' V_{f^{\text{can}}}^{-1}}{R_f}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})),$$

which values asset returns with pricing errors $\alpha = \mathbb{E}(M_{t+1}^{\beta,\text{can}}(R_{t+1} - R_f 1_N))$.

The proposition below shows that even if the candidate factor model omits systematic risk factors, that is, $K^{\text{mis}} > 0$, asymptotic no-arbitrage implies that the vector α satisfies a similar restriction to that in expression (5) on the vector a in the classical APT. This implication of no-arbitrage allows us to work with misspecified candidate factor models and correct their implied SDFs to obtain the admissible SDF.

Proposition 3 (No-arbitrage Restriction in the Presence of Model Misspecification)**.** *Suppose that the vector of asset returns R_{t+1} satisfies Assumptions 2.1 and 2.2 of the APT. Given a candidate factor model with K^{can} factors f_{t+1}^{can}, suppose the first K^{mis} eigenvalues of the covariance matrix V_ε are unbounded when $N \to \infty$, the remaining eigenvalues are uniformly bounded, and the smallest eigenvalue is strictly positive. Then, by no arbitrage, there exist an $N \times 1$ vector a, $N \times K^{\text{mis}}$ matrix β^{mis}, and $K^{\text{mis}} \times 1$ vector λ^{mis}, such that*

$$\alpha = \beta^{\text{mis}}\lambda^{\text{mis}} + a \qquad \textit{and} \qquad V_\varepsilon = \beta^{\text{mis}} V_{f^{\text{mis}}} \beta^{\text{mis}\prime} + V_e, \tag{12}$$

and the candidate model's vector of pricing errors α satisfies

$$\alpha' V_\varepsilon^{-1} \alpha \leq \tilde{\delta}_{\text{apt}}, \tag{13}$$

for some constant $\tilde{\delta}_{apt} = \delta_{apt} + \lambda^{\text{mis}\prime} V_{f^{\text{mis}}}^{-1} \lambda^{\text{mis}} + o(1)$, with δ_{apt} defined in equation (5). The no-arbitrage restriction specified in equation (13) is asymptotically equivalent to that in equation (5), in the sense that when $\tilde{\delta}_{apt}$ is finite, then δ_{apt} is finite, and vice versa.

Equations (12) delivers two messages. First, if K^{mis} eigenvalues of V_ε are unbounded, then the vector ε_{t+1} has a factor structure, that is, shocks ε_{t+1} include K^{mis} latent factors f_{t+1}^{mis} with β^{mis} denoting the matrix of exposures of asset returns to these factors f_{t+1}^{mis}, that is,

$$\varepsilon_{t+1} = \beta^{\text{mis}}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) + e_{t+1}. \tag{14}$$

Without loss of generality, we assume that these latent factors are orthogonal to the candidate factors.[8] Second, the vector α in equation (12) includes compensation for unsystematic risk, a, and compensation for the exposures of asset returns to the missing systematic risk factors f_{t+1}^{mis}, that is, $\beta^{\text{mis}}\lambda^{\text{mis}}$, where λ^{mis} is a vector of compensations for a unit of asset exposures to the factors f_{t+1}^{mis}.

[8]Internet Appendix IA.7 shows that the admissible SDF is invariant to our assumption about the correlation structure between candidate and missing factors.

Next, we show how to correct the linear SDF $M_{t+1}^{\beta,\text{can}}$ implied by the misspecified candidate factor model of asset returns to obtain the admissible SDF M_{t+1}, which is the SDF implied by the APT model of asset returns.

Proposition 4 (SDF: Correcting a Misspecified Linear SDF)**.** *Under the assumptions of Proposition 3, given the candidate SDF $M_{t+1}^{\beta,\text{can}}$, there exists an admissible SDF M_{t+1}, such that*

$$M_{t+1} = M_{t+1}^{\beta,\text{can}} + M_{t+1}^{\alpha} = M_{t+1}^{\beta,\text{can}} + \underbrace{(M_{t+1}^{\beta,\text{mis}} + M_{t+1}^{a})}_{=M_{t+1}^{\alpha}}, \quad \textit{where}$$

$$M_{t+1}^{\beta,\text{mis}} = -\frac{(\lambda^{\text{mis}})' V_{f^{\text{mis}}}^{-1}}{R_f}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) \qquad \textit{and} \qquad M_{t+1}^{a} = -\frac{a' V_e^{-1}}{R_f} e_{t+1},$$

with $\text{cov}(M_{t+1}^{\beta,\text{can}}, M_{t+1}^{a}) = 0$, $\text{cov}(M_{t+1}^{a}, M_{t+1}^{\beta,\text{mis}}) = 0$, *and* $\text{cov}(M_{t+1}^{\beta,\text{can}}, M_{t+1}^{\beta,\text{mis}}) = 0$.

The wedge between the admissible SDF and the candidate SDF is a correction term labeled M_{t+1}^{α} that includes two components: $M_{t+1}^{\beta,\text{mis}}$ and M_{t+1}^{a}. The first component $M_{t+1}^{\beta,\text{mis}}$ captures pricing of the *systematic* risk factors f_{t+1}^{mis} omitted in the candidate factor model. The second component M_{t+1}^{a} captures pricing of *unsystematic* sources of risk e_{t+1}.

Given the growing interest in the role of weak factors in asset pricing (e.g., Lettau and Pelger, 2020; Giglio, Xiu, and Zhang, 2021b), we highlight the following result.

Proposition 5 (Unsystematic SDF Component is a Weak Factor for Basis Assets)**.** *If the no-arbitrage bound δ_{apt} is bounded away from zero, then the unsystematic SDF component M_{t+1}^{a} satisfies the definition of a weak factor (Lettau and Pelger, 2020) in the cross-section of basis assets, regardless of whether or not V_e is diagonal.*

Observe that the exposures of asset returns to the unsystematic SDF component M_{t+1}^{a} are equal to

$$\beta^a = \frac{\text{cov}(M_{t+1}^{a}, R_{t+1} - R_f 1_N)}{\text{var}(M_{t+1}^{a})} = \frac{\text{cov}\left(-\frac{a' V_e^{-1}}{R_f} e_{t+1}, R_{t+1} - R_f 1_N\right)}{\text{var}(M_{t+1}^{a})} = -\frac{a' R_f}{a' V_e^{-1} a}.$$

Thus, $\beta^{a\prime}\beta^a = R_f^2 (a'a)/(a' V_e^{-1} a)^2$, which together with the no-arbitrage restriction (5), the boundedness of δ_{apt} away from zero, and the boundedness of the eigenvalues of the covariance matrix V_e, implies that $\beta^{a\prime}\beta^a = O(1)$, that is, $\beta^{a\prime}\beta^a$ is bounded. As a result, M_{t+1}^{a} satisfies the definition of a weak factor in the cross-section of returns on basis assets.

Note that $M^a_{t+1} = -a'V_e^{-1}e_{t+1}/R_f$ satisfies the definition of a weak factor even if V_e is diagonal because M^a_{t+1} loads on a *combination* of asset-specific shocks e_{t+1}. By construction, e_{t+1} are orthogonal to *all* systematic factors. Therefore, the factor M^a_{t+1} consisting of e_{t+1} can only be a weak factor. Thus, conventional methods of factor analysis, e.g., PCA, cannot identify M^a_{t+1} in the data. The only way to identify the unsystematic SDF component, as can be seen from the definition of M^a_{t+1}, is by measuring the compensation a for unsystematic risk e_{t+1}.

The presence of the unsystematic SDF component M^a_{t+1} in the correction term M^α_{t+1} changes the direction of the quest for an asset-pricing model that explains the cross-section of expected excess returns. Candidate factor models may be misspecified because of missing systematic risk, the search for which has been the focus of the existing literature. But these models may be misspecified also because they omit compensation for unsystematic risk. What matters most—omitted systematic risk or nonzero compensation for unsystematic risk—is an empirical question we answer in this paper.

In practice, for the same reasons as explained in Section 2.2, recovering the positive admissible feasible SDF after correcting a misspecified candidate SDF requires (i) using the exponential function of the linear projection of M^a_{t+1} and $M^{\beta,\text{mis}}_{t+1}$ on the set of the risk-free and risky assets and (ii) specifying $M^{\beta,\text{can}}_{\exp,t+1}$ in exponential form. Thus, the feasible SDF takes the following form:

$$\hat{M}_{\exp,t+1} = M^{\beta,\text{can}}_{\exp,t+1} \times \hat{M}^{\beta,\text{mis}}_{\exp,t+1} \times \hat{M}^a_{\exp,t+1} \qquad \text{with} \tag{15}$$

$$M^{\beta,\text{can}}_{\exp,t+1} = \frac{1}{R_f} \times \exp\Big(- \lambda^{\text{can}\prime} V^{-1}_{f^{\text{can}}}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})) - \frac{1}{2}\lambda^{\text{can}\prime} V^{-1}_{f^{\text{can}}}\lambda^{\text{can}}\Big), \tag{16}$$

$$\hat{M}^{\beta,\text{mis}}_{\exp,t+1} = \exp\Big(- (\beta^{\text{mis}}\lambda^{\text{mis}})' V_R^{-1}(R_{t+1} - \mathbb{E}(R_{t+1})) - \frac{1}{2}(\beta^{\text{mis}}\lambda^{\text{mis}})' V_R^{-1}\beta^{\text{mis}}\lambda^{\text{mis}}\Big), \tag{17}$$

$$\hat{M}^a_{\exp,t+1} = \exp\Big(- a'V_R^{-1}(R_{t+1} - \mathbb{E}(R_{t+1})) - \frac{1}{2}a'V_R^{-1}a\Big), \tag{18}$$

and where, from Proposition 3,

$$\mathbb{E}(R_{t+1} - R_f 1_N) = a + \beta^{\text{mis}}\lambda^{\text{mis}} + \beta^{\text{can}}\lambda^{\text{can}},$$

and
$$V_R = \beta^{\text{can}} V_{f^{\text{can}}} \beta^{\text{can}\prime} + \beta^{\text{mis}} V_{f^{\text{mis}}} \beta^{\text{mis}\prime} + V_e.$$

In Internet Appendix IA.4, we extend Proposition 2 to the case in which we start with a misspecified candidate factor model that we correct in order to build an admissible SDF implied by the APT. We show that as $N \to \infty$, the projection version of the corrected SDF specified in equation (15) recovers the admissible SDF specified in exponential form, similar to that in equation (10). This result implies that when constructing an admissible SDF from a misspecified SDF, we do not need to pre-estimate the shocks e_{t+1} or the missing systematic factors f_{t+1}^{mis} that may be omitted in the candidate factor model.

2.3.2 Accounting for Time-Variation in Risk Premia

In Section 2.1, to identify an admissible SDF, we use the classical APT in which the prices of risk and the asset exposures to systematic risk factors are constant, and the vector a represents the compensation for unsystematic risk. Similarly, so far we considered candidate factor models with constant prices of risk and risk exposures. However, in practice one may wonder whether the vector a is a consequence of time variation in asset factor exposures or prices of risk. Below, we demonstrate that an arbitrary model of asset returns that has time-varying prices of risk or risk exposures is nested in the classical APT and that the interpretation of a as compensation for unsystematic risk is preserved. To distinguish models with constant parameters from those with time-varying parameters, we use a tilde $\tilde{\cdot}$ to denote the elements of the models with time variation. To facilitate our discussion, we consider a model with only time-varying risk exposures $\tilde{\beta}_t$; the analysis of a model with time-varying prices of risk is similar and is omitted for brevity.

Without loss of generality, assume that the true model for asset returns is a conditional model with a single factor $\tilde{f}_{t+1}$ and zero compensation for unsystematic risk $\tilde{e}_{t+1}$

$$R_{t+1} - \mathbb{E}_t(R_{t+1}) = \tilde{\beta}_t \tilde{f}_{t+1} + \tilde{e}_{t+1}, \tag{19}$$

where $\tilde{f}_{t+1}$ is a factor with unconditional risk premium $\tilde{\lambda}$, $\mathbb{E}_t(\tilde{f}_{t+1}) = 0$, $\tilde{\beta}_t$ is an $N \times 1$ vector of risk exposures of asset returns R_{t+1} to the factor $\tilde{f}_{t+1}$, and $\tilde{e}_{t+1}$ is an $N \times 1$ vector of unsystematic shocks with a covariance matrix $V_{\tilde{e}}$. We consider two cases.

Case 1: Common source of variation in risk exposures. Assume that

$$\tilde{\beta}_t = \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{g}_t,$$

where $\tilde{g}_t$ is a common source of time-variation in assets' exposures $\tilde{\beta}_t$ to the risk factor $\tilde{f}_{t+1}$. Without loss of generality, assume that $\mathbb{E}(\tilde{g}_t) = 0$. Given these assumptions, the true data-generating process for asset returns is

$$R_{t+1} - R_f 1_N = \tilde{\beta}_0 \tilde{\lambda} + \tilde{\beta}_1 \tilde{\lambda} \tilde{g}_t + \tilde{\beta}_0 \tilde{f}_{t+1} + \tilde{\beta}_1 \tilde{g}_t \tilde{f}_{t+1} + \tilde{e}_{t+1},$$

or equivalently

$$R_{t+1} - \mathbb{E}(R_{t+1}) = \tilde{\beta}_0 \tilde{f}_{t+1} + \tilde{\beta}_1 \tilde{\lambda} \tilde{g}_t + \tilde{\beta}_1 (\tilde{g}_t \tilde{f}_{t+1} - \mathbb{E}(\tilde{g}_t \tilde{f}_{t+1})) + \tilde{e}_{t+1}.$$

Thus, the true factor model (19) with the single factor $\tilde{f}_{t+1}$, time variation in risk premia driven by one common variable $\tilde{g}_t$, and zero compensation for unsystematic risk $\tilde{e}_{t+1}$, is observationally equivalent to the APT model of asset returns with $a = 0_N$ and three systematic factors, $f_{t+1} = (\tilde{f}_{t+1},\ \tilde{g}_t,\ \tilde{g}_t\tilde{f}_{t+1})'$.[9] Therefore, if one assumes a candidate model with the single factor $f^{\text{can}}_{t+1} = \tilde{f}_{t+1}$ and constant risk exposures $\beta^{\text{can}} = \tilde{\beta}_0$ and uses our approach to correct this candidate model, one obtains the admissible SDF, in which the component $M^{\beta,\text{mis}}_{t+1}$ is a function of the omitted factors $f^{\text{mis}}_{t+1} = (\tilde{g}_t,\ \tilde{g}_t\tilde{f}_{t+1})'$. Furthermore, $M^{\beta,\text{mis}}_{t+1}$ captures completely the wedge between the admissible SDF and the SDF implied by the candidate factor model, so that, $M^a_{t+1} = 0$.

Case 2: Asset-specific source of time-variation in risk exposures. Now, assume that

$$\tilde{\beta}_t = \tilde{\beta}_0 + \tilde{\beta}_1 \odot \tilde{G}_t,$$

where $\tilde{G}_t = (\tilde{g}_{1t},\ \tilde{g}_{2t},\ \cdots,\ \tilde{g}_{Nt})'$ is a vector of asset-specific sources of time-variation in risk exposures $\tilde{\beta}_t$ to the risk factor $\tilde{f}_{t+1}$, and the symbol $\odot$ denotes the Hadamard element-wise product. Without loss of generality, assume that $\mathbb{E}(\tilde{g}_{it}) = 0$ for each $1 \leq i \leq N$. Given these assumptions, the true data-generating process for asset returns is

$$R_{t+1} - R_f 1_N = (\tilde{\beta}_0 + \tilde{\beta}_1 \odot \tilde{G}_t)\tilde{\lambda} + (\tilde{\beta}_0 + \tilde{\beta}_1 \odot \tilde{G}_t)\tilde{f}_{t+1} + \tilde{e}_{t+1}$$

or equivalently

$$R_{t+1} - \mathbb{E}(R_{t+1}) = \tilde{\beta}_0\tilde{f}_{t+1} + \underbrace{\tilde{\lambda}\tilde{\beta}_1 \odot \tilde{G}_t + \tilde{\beta}_1 \odot (\tilde{G}_t\tilde{f}_{t+1} - \mathbb{E}(\tilde{G}_t\tilde{f}_{t+1})) + \tilde{e}_{t+1}}_{\tilde{\eta}_{t+1}}.$$

Thus, the true factor model (19) with the single factor $\tilde{f}_{t+1}$, time variation in risk premia driven by asset-specific variables $\tilde{G}_t$, and zero compensation for unsystematic risk $\tilde{e}_{t+1}$, is observationally equivalent to the APT model of asset returns with the single systematic factor $f_{t+1} = \tilde{f}_{t+1}$ and unsystematic shocks $\tilde{\eta}_{t+1}$. In the APT, unsystematic shocks $\tilde{e}_{t+1}$ have zero compensation, exactly as in the true factor model, while unsystematic shocks $e_{t+1} = \tilde{\eta}_{t+1} - \tilde{e}_{t+1}$ are compensated if $\mathbb{E}(\tilde{G}_t\tilde{f}_{t+1}) \neq 0_N$.[10] Therefore, if one assumes a candidate model with the single factor $f^{\text{can}}_{t+1} = \tilde{f}_{t+1}$ and constant risk exposures $\beta^{\text{can}} = \tilde{\beta}_0$ and uses our approach to correct this candidate model, one obtains the admissible SDF with the component M^a_{t+1}, which is a function of unsystematic shocks e_{t+1} instead of the shocks $\tilde{e}_{t+1}$. Furthermore, M^a_{t+1} captures completely the wedge between the admissible

[9] Note that $\tilde{g}_t\tilde{f}_{t+1}$ can be interpreted as a scale factor of Gagliardini, Ossola, and Scaillet (2016).

[10] This example illustrates that priced unsystematic shocks may represent the product of a systematic risk factor with asset-specific drivers of risk premia.

SDF implied by the APT and the SDF implied by the candidate factor model, so that $M_{t+1}^{\beta,\mathrm{mis}} = 0$.

Thus, the misspecification of a candidate factor model due to omitted variation in risk premia is observationally equivalent, in the context of unconditional pricing, to misspecification arising from omitted systematic risk factors or compensation for unsystematic risk.

3 Estimation Details

In this section, we describe our approach for estimating the model of asset returns, which is a prerequisite for constructing an admissible SDF, and the role played by the no-arbitrage restriction. We explain how to identify the number of latent factors K in the APT model of asset returns and how to identify the corresponding no-arbitrage bound δ_{apt} in equation (5). Similarly, for the case in which we start with a candidate model with K^{can} observable factors, we explain how to identify the number of missing factors K^{mis} and choose the corresponding no-arbitrage bound δ_{apt}. We also explain how to estimate V_e, the covariance matrix for unsystematic shocks.

3.1 Our Estimation Approach

We recover the admissible SDF implied by the APT in two steps. In the first step, we use a Gaussian maximum-likelihood estimator to estimate the APT model of asset returns given in (3), subject to the no-arbitrage restriction (5). At the estimation stage, we impose the necessary and sufficient conditions to identify the latent factors f_{t+1}. We adopt a standard identification scheme described in Internet Appendix IA.6. In the second step, we recover the positive feasible admissible SDF using formulas (7), (8), and (9), where the covariance matrix of unsystematic risk satisfies equation (2) and the expected excess returns satisfy equation (4).

Alternatively, if we are correcting an arbitrary candidate factor model with K^{can} observable factors, we use a Gaussian maximum-likelihood estimator to estimate the model of asset returns given in expression (11), where α and V_ε are defined in (12) and impose the no-arbitrage restriction specified in expression (5).[11] Without loss of generality, we consider candidate models with tradable factors that represent either factor returns (for example,

[11] We use asymptotic equivalence of the no-arbitrage restrictions in equations (5) and (13).

the market factor and returns on long-minus-short strategies) or excess returns on factor-mimicking portfolios.[12] Similar to the case of the APT model, we impose the necessary and sufficient conditions to identify the latent factors f_{t+1}^{mis} (see Internet Appendix IA.6). In the second step, we use the extended version of Proposition 2, which is formally presented and proved in Internet Appendix IA.4, and formulas (15), (16), (17), and (18) to recover the positive feasible admissible SDF.

3.2 The No-Arbitrage Restriction

The no-arbitrage restriction in (5) on the vector a serves several purposes. First, economically, it rules out asymptotic arbitrage. Also, the no-arbitrage restriction constrains the Sharpe ratio of the so-called alpha portfolio of Raponi, Uppal, and Zaffaroni (2022). In our setting, $a'V_e^{-1}a$ is approximately equal to the square of the Sharpe ratio associated with investing in a portfolio that represents the unsystematic SDF component $M_{\text{exp},t+1}^{a}$, that is, $\delta_{\text{apt}} \approx \text{var}(\log(M_{\text{exp},t+1}^{a}))$.[13]

Statistically, when estimating a candidate factor model, the no-arbitrage restriction (when binding) leads to the identification of the vectors λ^{mis} and a.[14] Specifically, at the estimation stage, the no-arbitrage restriction provides the N conditions that allow us to identify separately the estimates of $\beta^{\text{mis}}\lambda^{\text{mis}}$ and a. Identification of $\beta^{\text{mis}}\lambda^{\text{mis}}$ and a is necessary for constructing the missing systematic and unsystematic components of the admissible SDF, $\hat{M}_{\text{exp},t+1}^{\beta,\text{mis}}$ and $\hat{M}_{\text{exp},t+1}^{a}$, respectively. When estimating the APT, the no-arbitrage condition similarly leads to the identification of $\beta\lambda$ and a, and therefore the systematic and unsystematic SDF components $\hat{M}_{\text{exp},t+1}^{\beta}$ and $\hat{M}_{\text{exp},t+1}^{a}$.

Finally, under the no-arbitrage restriction, the estimator of a has the form of a ridge estimator, as shown in Proposition IA.6.5 of Internet Appendix IA.6. The ridge estimator has the appealing property of mitigating estimation noise. In our case, this property is especially valuable given that the vector a represents a component of expected returns, which, as is well known (Merton, 1980), are difficult to estimate.

[12]It is straightforward to extend the estimation algorithm to the case of candidate factor models with nontradable systematic risk factors.

[13]In the same spirit, Kozak, Nagel, and Santosh (2018, 2020) rule out near-arbitrage opportunities by restricting the maximum squared Sharpe ratio implied by the overall SDF.

[14]Even in population, the no-arbitrage restriction can be influenced by the presence of financial frictions (Korsaye et al., 2021, sec. 2).

3.3 Identifying the Number of Systematic Factors and No-arbitrage Bound

To estimate the APT model of asset returns, we need to determine the number of systematic risk factors K and the bound δ_{apt} on the no-arbitrage restriction specified in equation (5). The APT theory, however, is silent about the value of δ_{apt}; Ross (1977) suggests using a bound that is a multiple of the Sharpe ratio for the market portfolio, which is about 0.12 per month. Similarly, when correcting a candidate factor model of asset returns, we need to determine the number K^{mis} of missing systematic risk factors f_{t+1}^{mis} and the bound δ_{apt} on the no-arbitrage restriction. In both cases, we estimate the number of latent risk factors and the no-arbitrage bound using cross-validation with the Hansen and Jagannathan (1997) (HJ) distance as a selection metric.

Our cross-validation procedure uses ten folds.[15] We split the sample into ten folds and estimate the model on all but one fold, which we use for validation. We repeat this procedure ten times and compute the HJ distance on the validation folds. We fix a grid for δ_{apt} from 0 to 0.25 that corresponds to Sharpe ratios ranging from 0 to $\sqrt{0.25} = 0.5$ per month for the portfolio associated with unsystematic risk. We vary the number of systematic factors in the APT model from 1 to 10 and the number of missing factors, when evaluating a particular candidate model, from 0 to 5. We then choose K or K^{mis} and the value of δ_{apt} that deliver the smallest HJ distance in the validation folds. Finally, we re-estimate the model on the entire sample using the optimal number of systematic risk factors and the optimal no-arbitrage bound.

To guide our model selection, we choose the HJ distance, a widely recognized economically meaningful metric of pricing performance. Being a function of the SDF, the HJ distance summarizes how competing asset-pricing models fit the first and second moments of the return distribution. This is in contrast to other metrics, for example, the cross-sectional R^2, that assess how competing models fit only average excess returns.[16] Thus, as a model-diagnostic measure, the HJ distance sets a higher hurdle for competing models. Furthermore, a suitable diagnostic metric for models featuring compensation for unsystematic risk must embed the correct interpretation of the vector a, which is only possible if the diagnostic metric depends on the SDF, as the HJ distance does.

[15] Our results are similar if we use twenty folds.

[16] Given that for tractability we assume that asset returns are Gaussian, the SDF and HJ distance depend only on the mean and variance of excess returns. Notice that formulas (8)–(9) and (17)–(18) contain the model-implied quantities for $\mathbb{E}(R_{t+1} - R_f 1_N)$ and V_R.

The prior literature has used other methods for selecting the number of systematic risk factors in SDF models. For example, Giglio and Xiu (2021) use a statistical information criterion similar to Bai and Ng (2002). Lettau and Pelger (2020) and Kozak et al. (2020) use economic restrictions relating expected returns to the covariance of returns with factors, in addition to time-series information on the variation in asset returns.[17] Because none of these approaches applies directly to a model with nonzero compensation for unsystematic risk, a, we face a choice: either to use a two-stage estimation approach that pins down K or K^{mis} in the first step and δ_{apt} in the second step or to design our own method. We choose the latter and optimize an objective function that explicitly incorporates the no-arbitrage restriction while selecting simultaneously the number of systematic risk factors and the no-arbitrage bound to minimize the HJ distance.

3.4 Estimating the Covariance Matrix of Unsystematic Shocks

As pointed out earlier, the covariance matrix V_e of unsystematic shocks e_{t+1} in equations (1) or (14) does not have to be diagonal but must have bounded eigenvalues. That is, the shocks e_{t+1} do not have to be uncorrelated across basis assets but may include weak latent factors. Because it is not possible to obtain consistent estimates of weak factors (Lettau and Pelger, 2020, prop. 2), estimating V_e in the presence of weak factors is challenging.

Motivated by the shrinkage approach of Ledoit and Wolf (2004a,b), we develop the following two-step estimator for the matrix V_e. In the first step, we assume that V_e is a diagonal matrix. Given this assumption, for each K (if estimating the APT model of asset returns) or K^{mis} (if correcting a candidate factor model) and each value of δ_{apt}, we optimize the log-likelihood of asset returns subject to the no-arbitrage restriction. As a result, we obtain the first-step estimate $V_e^{(1)}$ of V_e. Next, we check whether the covariance matrix $V_{e^{\text{fit}}}$ of the fitted residuals e^{fit} of the asset-return model, is diagonal. If it is not, we proceed to the second step, in which we estimate $V_e = V_e^{(2)}$ as a linear combination of $V_e^{(1)}$ and $V_{e^{\text{fit}}}$,

$$V_e^{(2)} = \theta V_e^{(1)} + (1-\theta) V_{e^{\text{fit}}},$$

[17] Even though our objective function is similar to that of Lettau and Pelger (2020), our approach has two crucial differences. First, as explained above, from the perspective of the corrected model, α is not a pricing error; rather, it represents a crucial component of the SDF. Thus, α does not need to be the null vector, and therefore, our aim is not to compress it as much as possible but only to ensure that the no-arbitrage restriction holds. Second, our objective function does not explicitly include a pricing metric measuring goodness of fit. If we included such a pricing metric in the objective function, we would have had to augment our log-likelihood function with an additional penalty term represented by the HJ distance. Instead, we use the HJ distance only in the cross-validation procedure to determine the number of systematic risk factors in the cross-section of asset returns and the bound of the no-arbitrage restriction.

where we choose θ so that $\delta_{\text{apt}} \approx \text{var}(\log(\hat{M}^a_{\exp,t+1}))$. Effectively, we shrink the empirical covariance matrix of shocks e_{t+1} towards a diagonal matrix $V_e^{(1)}$, and we choose the degree of shrinkage, $1-\theta$, to preserve the economic interpretation of δ_{apt} as the squared Sharpe ratio of the portfolio that loads only on unsystematic risk.

4 Data

This section describes the data we use for our empirical analysis. First, we describe the basis assets we use to estimate the SDF. Then, we describe variables that could potentially span the estimated SDF and its components.

4.1 Basis Assets

We construct a projection of the SDF on a large set of standard characteristics-based portfolios of U.S. stocks. As in Giglio and Xiu (2017), we use monthly returns data for 202 portfolios from Kenneth French's website, which we label Dataset 1. The data includes returns on 25 portfolios sorted by size and book-to-market ratio (ME & BM), 17 industry portfolios (Ind), 25 portfolios sorted by operating profitability and investment (OP & INV), 25 portfolios sorted by size and variance (ME & VAR), 35 portfolios sorted by size and net issuance (ME & NetISS), 25 portfolios sorted by size and accruals (ME & ACCR), 25 portfolios sorted by size and beta (ME & BETA), and 25 portfolios sorted by size and momentum (ME & MOM). We use portfolios rather than individual assets because portfolios exhibit a more stable factor structure (Lettau and Pelger, 2020; Giglio and Xiu, 2021).

We also consider two other sets of assets, labeled Dataset 2 and Dataset 3, in the cross-sectional out-of-sample analysis we undertake in Section 5.3 below. Dataset 2, used in Korsaye et al. (2021), has a total of 199 assets consisting of 100 portfolios sorted by size and book-to-market, 25 portfolios sorted by size and long-term reversal, 25 portfolios sorted by size and short-term reversal, and 49 industry portfolios.[18] Dataset 3 has 349 assets, consisting of 100 portfolios sorted by size and book-to-market, 100 by size and operating profitability, 100 by size and investment, and 49 industry portfolios. For all three datasets, our sample runs from July 1963 to August 2019.

[18] The dataset of Korsaye et al. (2021) also includes twenty-five momentum portfolios that we exclude because they are present in our Dataset 1.

4.2 Variables Potentially Spanning the SDF

To examine which economic variables may explain the admissible SDF's variation, we collect a comprehensive set of variables available at a monthly frequency. Our dataset includes both macroeconomic and financial indicators and returns on trading strategies. In the factor-zoo literature, the returns on these trading strategies are also known as factors or anomalies. We briefly describe these variables below, with details regarding the data sources and construction of these variables provided in Internet Appendix IA.8.

We consider returns on 457 trading strategies studied in Novy-Marx (2013), Kozak et al. (2020), Chen and Zimmermann (2022), Jensen, Kelly, and Pedersen (2022), Hou, Mo, Xue, and Zhang (2021), and Bryzgalova et al. (2023).

Furthermore, we consider 103 macroeconomic and financial indicators. We include those analyzed in Bryzgalova et al. (2023), augment them with the first three principal components (PCs) of 279 macro variables from Jurado, Ludvigson, and Ng (2015), and the first eight PCs of 128 macro variables from the FRED-MD dataset of McCracken and Ng (2015). In addition, we include consumption growth and inflation constructed from real per capita consumption data on nondurables and services and the corresponding price index from the Bureau of Economic Analysis. We also include the market-dislocation index (Pasquariello, 2014), the disagreement index (Huang, Li, and Wang, 2021), the Chicago Board Options Exchange volatility index (VIX), the U.S. economic-policy-uncertainty (EPU) index (Baker, Bloom, and Davis, 2016), the equity-market-volatility (EMV) tracker (Baker, Bloom, Davis, and Kost, 2019), the credit-spread index (Gilchrist and Zakrajšek, 2012), the Chicago Fed National Financial Condition Index from FRED, the consumer-sentiment index, the U.S. business-confidence index, the U.S. consumer-confidence index, the U.S. composite-leading indicator, the coincident-economic-activity index, the NBER recession index, the TED spread, the effective federal-funds rate, and the real federal-funds rate. For persistent variables, we include their levels and first-order differences and, where appropriate, the AR(1) or VAR(1) innovations.

5 Empirical Analysis

In this section, first, we analyze the estimated admissible SDF implied by the APT model of asset returns and characterize its components, thereby establishing the relative importance of systematic versus unsystematic risk. Then, we examine three commonly used candidate factor models of asset returns: the market model, the model with consumption growth as the sole factor, and the FF3 model. For each candidate model, we characterize the *missing* systematic and unsystematic components of the corresponding SDFs. Next, we provide results of time-series and cross-sectional out-of-sample analyses. Finally, we explain that our conclusion regarding what is missing in the three analyzed candidate factor models applies to virtually any other asset-pricing model with only systematic risk factors.

5.1 The SDF under the APT Model of Asset Returns

To analyze the SDF implied by the APT, we first estimate the APT model of asset returns specified in equations (3) and (5). As mentioned earlier, we use a cross-validation procedure to determine the number of latent systematic factors K and the no-arbitrage bound, δ_{apt}.

5.1.1 Number of Latent Systematic Risk Factors and the No-Arbitrage Bound

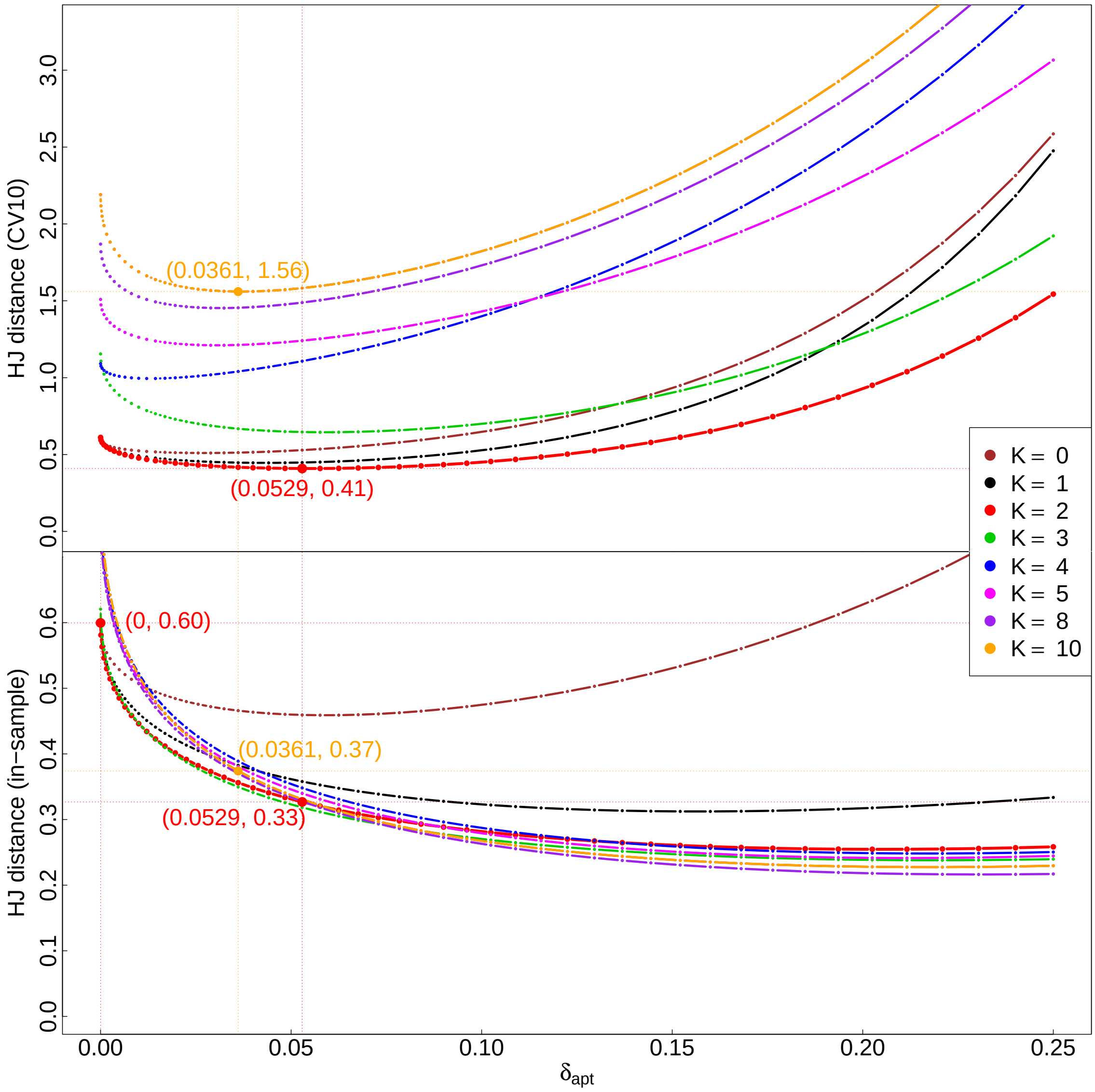
The top panel of Figure 1 illustrates how the HJ distance changes in the cross-validation procedure as we vary K and δ_{apt}. We see that the combination of $K = 2$ latent factors and $\delta_{\text{apt}} = 0.0529$ achieves the smallest HJ distance of 0.41, consistent with the evidence on low-dimensional latent factor models in Kozak et al. (2018) and Lettau and Pelger (2020). The nonzero value of the optimal δ_{apt} indicates that unsystematic risk is priced in the stock market, that is, $a \neq 0_N$. Figure 2 illustrates the estimated elements of the vector a for the 202 basis assets. Our finding that unsystematic risk is priced challenges the conventional view that expected asset returns compensate only for exposures to systematic risk factors.

The bottom panel of Figure 1 shows that if we were to naively choose K and δ_{apt} based on an *in-sample* analysis, we would have selected much larger values for these parameters. This is because the larger number of factors K fits the in-sample covariance matrix of returns better, while the larger δ_{apt} fits the in-sample cross-sectional variation in expected excess returns better. However, choosing K and δ_{apt} based on in-sample fit leads to overfitting and, consequently, an inferior fit of asset returns out-of-sample. The top panel shows the result of this overfitting—as we pick larger K or δ_{apt}, the cross-validation HJ distance *increases.*

To assess further the importance of nonzero compensation for unsystematic risk, one may wonder whether increasing the number of systematic factors would reduce the optimal δ_{apt} to zero. The top panel in Figure 1 shows that even if we were to assume that the APT

Figure 1: Model selection using the HJ distance

This figure illustrates how the HJ distance changes with K and δ_{apt}. The top panel shows the estimation results based on cross-validation with ten folds. The bottom panel shows the in-sample results. The numbers reported in the two plots are (δ_{apt}, HJ distance).



model had a much larger number of factors than the optimal $K = 2$, the compensation for unsystematic risk would remain sizable. For example, if we set the number of systematic factors to be $K = 10$ and then choose only δ_{apt} in a cross-validation exercise, the HJ distance is minimized at $\delta_{\text{apt}} = 0.0361$ rather than 0.

To understand the economic importance of accounting for compensation for unsystematic risk, we explore how the HJ distance changes in a model with $K = 2$ latent systematic factors if we set $a = 0_N$. From the bottom panel of Figure 1, we see that the HJ distance

Figure 2: Estimated compensation for unsystematic risk

This figure illustrates the estimated elements of the vector a (annualized and in percent) for the 202 basis assets, which we split into eight groups based on characteristics by which stocks are sorted into portfolios.

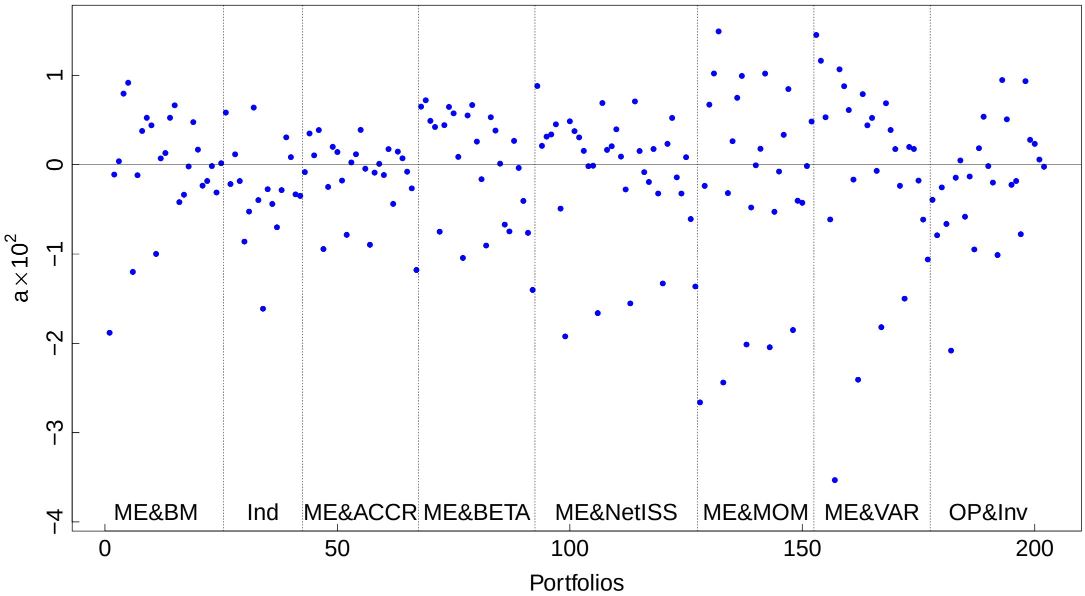


Figure 3: Pricing errors in the APT model with two latent factors with and without compensation for unsystematic risk

We split the 202 basis assets into eight groups based on characteristics by which stocks are sorted into portfolios. Red dots indicate the pricing errors for the 202 basis assets in a model with two latent factors and no compensation for unsystematic risk, $K = 2$ and $a = 0_N$. Blue dots indicate the pricing errors in a model with two latent factors and nonzero compensation for unsystematic risk, $K = 2$ and $a \neq 0_N$.

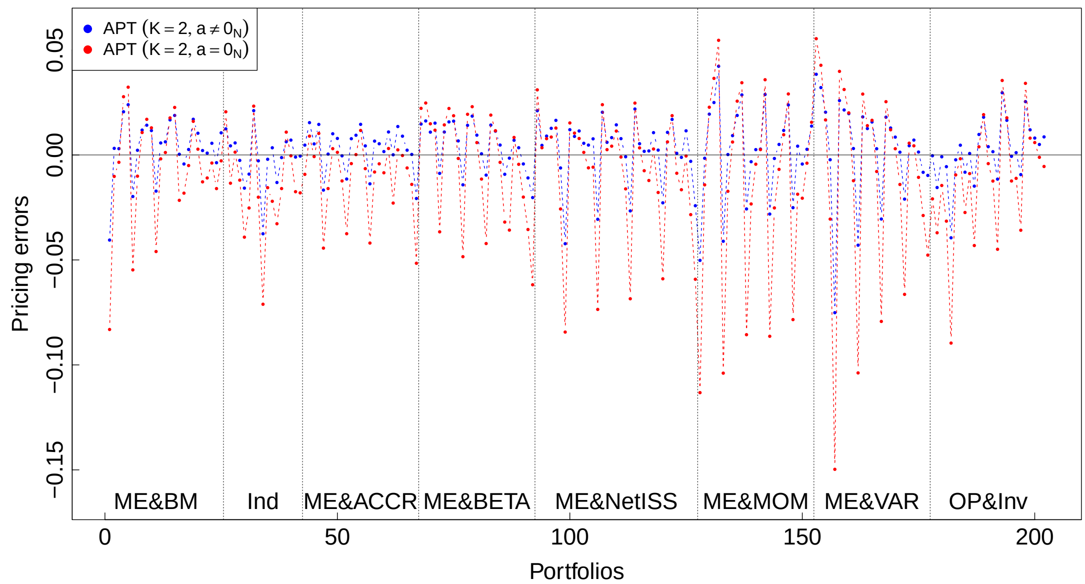
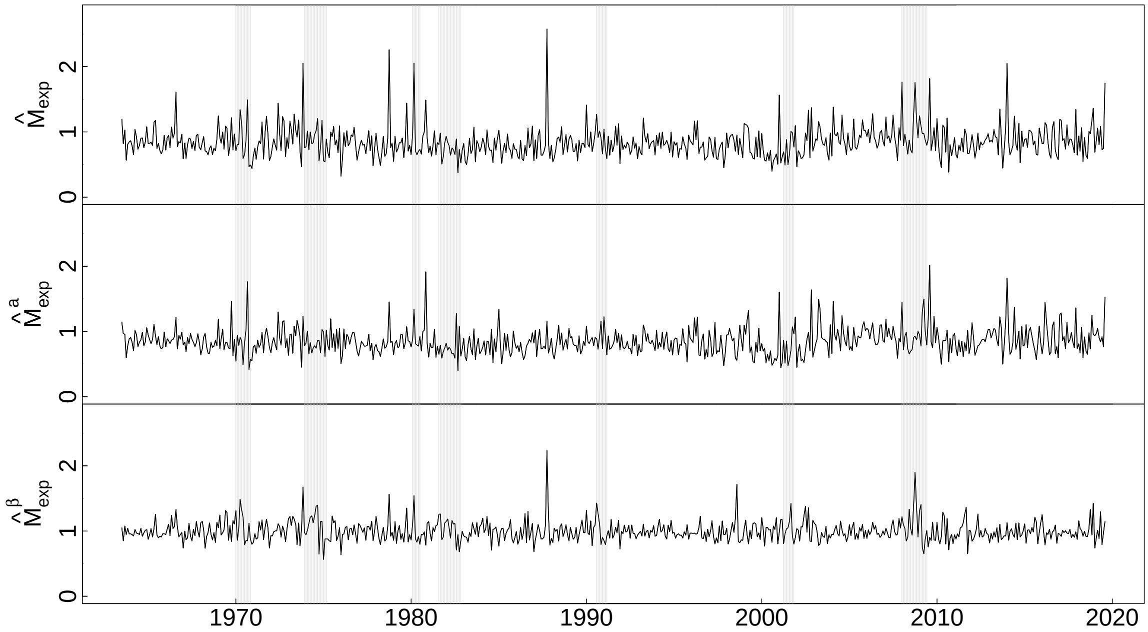


Figure 4: Time-series behavior of the SDF and its components

This figure has three panels. The top, middle, and bottom panels show the dynamics of the estimated SDF $\hat{M}_{\exp,t+1}$, its unsystematic component $\hat{M}^{a}_{\exp,t+1}$, and its systematic component $\hat{M}^{\beta}_{\exp,t+1}$, respectively. Gray bars indicate the NBER recession periods.



increases by a lot—about 81.81% = (0.60/0.33−1). In Figure 3, we report the pricing errors across the 202 basis assets. We find that the largest increase in pricing errors from setting $a = 0_N$ is for the portfolios sorted by size and variance (ME&VAR), size and momentum (ME&MOM), and size and net issuance (ME&NetISS).

5.1.2 Time-Series and Business-Cycle Properties of SDF and its Components

Having estimated the APT model of asset returns, we study the time-series properties of the implied SDF, $\hat{M}_{\exp,t+1}$ specified in (7), and its components, $\hat{M}^{\beta}_{\exp,t+1}$ and $\hat{M}^{a}_{\exp,t+1}$, specified in (8) and (9), respectively. Figure 4 shows that both $\hat{M}^{\beta}_{\exp,t+1}$ and $\hat{M}^{a}_{\exp,t+1}$, exhibit sizable volatility during recessions and also during normal times. Furthermore, we see that different components of the SDF dominate its variation in different periods. For example, the increase in $\hat{M}^{\beta}_{\exp,t+1}$ in October 1987 shows that systematic risk factors were responsible for the dramatic increase in the level and volatility of the SDF. On the other hand, in the early 2000s (following the dot-com bubble), the increase in the unsystematic component $\hat{M}^{a}_{\exp,t+1}$ generated the spike in the volatility of the SDF. Thus, both systematic and unsystematic risk contribute to explaining asset valuations.

Next, we explore the business-cycle properties of the estimated SDF and its components. To this end, we run a regression analysis of the log SDF and its components on macroeconomic and financial indicators; we do the log transformation because our SDF is in exponential form. We find that $\log(\hat{M}^{a}_{\exp,t+1})$ is largely acyclical: it does not significantly correlate with the NBER recession indicator.[19] The macroeconomic and financial indicators it correlates most with are intermediate constraints (He, Kelly, and Manela, 2017), the sentiment indices (Baker and Wurgler, 2006; Huang, Jiang, Tu, and Zhou, 2015), shocks in VIX, and shocks in credit spread (Gilchrist and Zakrajšek, 2012); individually, each of these variables explain less than 3.5% of the variation in $\log(\hat{M}^{a}_{\exp,t+1})$. Panel A of Table IA.1 in Internet Appendix IA.9 provides these results.

In contrast to $\log(\hat{M}^{a}_{\exp,t+1})$, the systematic component $\log(\hat{M}^{\beta}_{\exp,t+1})$ significantly correlates with the NBER recession indicator. In addition, the systematic SDF component correlates with the Chicago Fed National Financial Condition index, intermediary constraints (He et al., 2017), shocks in aggregate liquidity (Pástor and Stambaugh, 2003), shocks in credit spread (Gilchrist and Zakrajšek, 2012), shocks in dividend yield, shocks in financial uncertainty (Jurado et al., 2015), shocks in VIX, and shocks in the TED spread. Among these variables, shocks in intermediary constraints have the largest explanatory power for $\log(\hat{M}^{\beta}_{\exp,t+1})$: $R^2 = 55\%$; see Panel B of Table IA.1 in Internet Appendix IA.9.

We conclude the time-series analysis of the SDF by analyzing the relative importance of the two SDF components for the admissible SDF's variance. Table 1 reports the standard deviation of the SDF and its components for the APT model of asset returns. These standard deviations correspond to annual Sharpe ratios associated with exposure to the overall SDF and its unsystematic and systematic components, and are 0.89, 0.79, and 0.51, respectively.[20] Thus, strikingly, a unit exposure to unsystematic risk is compensated more prominently in financial markets than a unit exposure to systematic risk. Similarly, we find that of the total variation of the SDF, the unsystematic component contributes 72.60%, while the systematic component contributes only 27.40%. Thus, any model based on only systematic risk factors implies an SDF that is too smooth. The dominant role of unsystematic risk in the admissible SDF's variation that we document is consistent also with the puzzling evidence in Daniel and Titman (1997), Herskovic, Moreira, and Muir

[19]Throughout the manuscript, we use the 5% statistical significance level.

[20]In population, the sum of the squares of the standard deviations of the components of the log SDF must add up to the square of the standard deviation of the log SDF itself. But, in a finite sample, the components of the log SDF are not perfectly orthogonal to one another. Therefore, the sum of the squares of their standard deviations deviates slightly from the square of the standard deviation of the SDF.

Table 1: Analysis of APT model

This table reports two sets of quantities for the APT model: (1) The Sharpe ratio of the SDF along with its components, where the Sharpe ratios are approximated by the standard deviation of the SDF and its components in log; and (2) the variance decomposition of the log SDF.

	Std Dev or Sharpe ratio (p.a.)			Variance decomp. (%)	
	log of			log of	
Model	$\hat{M}_{\exp,t+1}$	$\hat{M}^{a}_{\exp,t+1}$	$\hat{M}^{\beta}_{\exp,t+1}$	$\hat{M}^{a}_{\exp,t+1}$	$\hat{M}^{\beta}_{\exp,t+1}$
APT	0.89	0.79	0.51	72.60	27.40

(2019), Chaieb, Langlois, and Scaillet (2021), and Lopez-Lira and Roussanov (2022) that a substantial portion of expected excess returns is left unexplained by factor risk premia. Our work shows that expected excess returns are explained largely by compensation for *unsystematic* risk.

5.1.3 The Unsystematic SDF Component

Given the dominance of unsystematic risk in explaining variation in the SDF, it is natural to ask whether weak factors or shocks specific to individual basis assets, which in our exercise are characteristic-sorted portfolios, play a major role in the unsystematic SDF component. Unfortunately, answering this question is not straightforward because weak factors cannot be estimated consistently (Lettau and Pelger, 2020). To circumvent this problem, we assume that the returns on the trading strategies described in Section 4.2 represent an exhaustive set of potential weak factors in the cross-section of our basis assets. Armed with this assumption, we split the fitted residuals e^{fit}_{t+1} from the estimated APT model specified in equations (3) and (5) into two parts: one representing weak factors and the other characteristic-sorted portfolio-specific shocks (CSP-specific shocks). To identify the CSP-specific shocks, we use the key property of these shocks that, by definition, they have a diagonal covariance matrix.

In practice, we regress the fitted residuals e^{fit}_{t+1} of the APT model on the returns of all trading strategies (orthogonalized with respect to the two latent factors of the APT) that substantially reduce the cross-sectional dependence in these residuals:

$$e^{\text{fit}}_{t+1} = \gamma_0 + \gamma' f^{\text{weak}}_{t+1} + \xi_{t+1}, \tag{20}$$

where f^{weak}_{t+1} are observable proxies for weak factors represented by the returns on the above-mentioned trading strategies after being orthogonalized with respect to the two latent factors of the APT. We find that out of 325 trading strategies available for the entire sample, 35 reduce the number of the significant off-diagonal terms in the covariance matrix of e^{fit}_{t+1} by 68%, leaving only 21% of the off-diagonal elements in the 202×202 covariance matrix of the fitted residuals ξ^{fit}_{t+1} statistically significantly different from zero.[21] We use this regression to split the fitted residuals e^{fit}_{t+1} in equation (20) into two parts: one explained by 35 trading strategies, or weak factors e^{weak}_{t+1}, and the other representing CSP-specific shocks e^{csp}_{t+1}, where the vectors e^{weak}_{t+1} and e^{csp}_{t+1} are the estimated values of $\gamma_0 + \gamma' f^{\text{weak}}_{t+1}$ and ξ_{t+1}, respectively.

Next, we decompose $M^a_{\exp,t+1}$ as

$$M^a_{\exp,t+1} = \exp\left(-a'V_e^{-1}e^{\text{weak}}_{t+1} - a'V_e^{-1}e^{\text{csp}}_{t+1} - \frac{1}{2}a'V_e^{-1}a\right)$$

and compute the standard deviations of $-a'V_e^{-1}e^{\text{weak}}_{t+1}$ and $-a'V_e^{-1}e^{\text{csp}}_{t+1}$, that approximately equal the Sharpe ratios of the investment strategies that include weak factors and CSP-specific shocks, respectively. We obtain the values 0.55 and 0.56 per annum, respectively. This result has the following implications. First, an investor earns sizable compensation for exposure to both types of unsystematic risk. Second, CSP-specific risk and weak factors contribute almost equally to the overall variation in the unsystematic SDF component.[22]

Having established the quantitative importance of the unsystematic SDF component, we look for the trading strategies whose returns reflect compensation for exposures to unsystematic risk. First, we run individual regressions of $\log(\hat{M}^a_{\exp,t+1})$ on the excess returns of the 457 strategies described in Section 4.2. We find that a large number—335 out of 457—correlate statistically significantly with the unsystematic SDF component. The returns on the five trading strategies that have the highest explanatory power for $\log(\hat{M}^a_{\exp,t+1})$ and are available for the entire sample are: one-year share issuance (Pontiff and Woodgate, 2008) with $R^2 = 17.82\%$, one-year momentum (Jegadeesh and Titman, 1993) with $R^2 = 14.08\%$, residual momentum (Blitz, Huij, and Martens, 2011) with $R^2 = 13.22$, betting-against-beta (Frazzini and Pedersen, 2014) with $R^2 = 13.19\%$, and net payout yield (Richardson, Sloan, Soliman, and Tuna, 2005) with $R^2 = 13.03\%$.

We also check the explanatory power of returns of the idiosyncratic-volatility factors of Ali et al. (2003) and Ang et al. (2006) for the unsystematic SDF component and find it

[21] Here, we are working with the sample covariance matrix of the fitted residuals. A sample estimator of the covariance matrix is known to be noisy, and therefore we consider a matrix with 21% significantly different from zero off-diagonal elements close to being diagonal.

[22] Weak factors contribute $0.55^2/(0.55^2 + 0.56^2) = 49.10\%$, while CSP-specific shocks contribute $0.56^2/(0.55^2 + 0.56^2) = 50.90\%$. Given that the unsystematic SDF component explains 72.60% of the variation in the estimated SDF, we obtain that of the total variation of the SDF, weak factors contribute $49.10\% \times 72.60\% = 35.65\%$, and CSP-specific shocks contribute $50.90\% \times 72.60\% = 36.95\%$.

Table 2: Strategies with high unsystematic risk premium RP^a

This table reports 25 selected trading strategies whose returns reflect large premia for unsystematic risk. The first column, using the classification scheme in Jensen et al. (2022), gives the name of the cluster to which the strategy belongs. If a strategy is not in the list of Jensen et al. (2022), we assign it to the cluster Unclassified. The second column gives the source. The third column shows the name of the variable, as in Chen and Zimmermann (2022), Jensen et al. (2022), or Bryzgalova et al. (2023). The last column reports the risk premium per annum in %. The clusters, and within each cluster, the sources, are listed in alphabetical order.

Cluster name	Source	Variable name	RP^a (%)
Low risk	Ang, Hodrick, Xing, and Zhang (2006)	IdioVol3F	6.07
	Bali, Cakici, and Whitelaw (2010)	MaxRet	6.45
	Bradshaw, Richardson, and Sloan (2006)	XFIN	5.95
	Bradshaw, Richardson, and Sloan (2006)	NetEquityFinance	4.98
	Frazzini and Pedersen (2014)	BAB	3.34
Momentum	Avramov, Chordia, Jostova, and Philipov (2007)	Mom6mJunk	5.43
	Jegadeesh and Titman (1993)	Mom12m	8.27
	Jegadeesh and Titman (1993)	Mom6m	7.21
	Moskowitz and Grinblatt (1999)	indmom	4.07
Profit Growth	Heston and Sadka (2008)	Mom12mOffSeason	7.24
	Novy-Marx (2013)	valmom	4.87
	Novy-Marx (2013)	valmomprof	4.62
Profitability	Chordia, Subrahmanyam, and Anshuman (2001)	std_turn	4.65
Quality	Haugen and Baker (1996)	VolMkt	4.32
	Stambaugh and Yuan (2017)	PERF	4.10
Value	Boudoukh, Michaely, Richardson, and Roberts (2007)	NetPayoutYield	4.24
Unclassified	Campbell, Hilscher, and Szilagyi (2008)	DISSTR	6.27
	Cooper, Gulen, and Schill (2008)	betaarb	4.82
	Daniel, Hirshleifer, Sun (2019)	BEH_FIN	4.38
	Dichev and Piotroski (2001)	CredRatDG	4.33
	Diether, Malloy, and Scherbina (2002)	ForecastDispersion	4.85
	Easley, Hvidkjaer, and O'Hara (2002)	ProbInformedTrading	5.91
	La Porta (1996)	fgr5yrLag	5.36
	Prakash and Sinha (2012)	DelDRC	4.95
	Ritter (1991)	AgeIPO	4.56

to be low: 6.44% and 9.57%, respectively. This quantitative result is important because if one relied on the insights of the idiosyncratic-volatility literature, one would expect an idiosyncratic-volatility factor to span the unsystematic SDF component completely. Thus, what is missing in asset-pricing factor models is not *just* the idiosyncratic-volatility factor.

Second, instead of looking at individual trading strategies, we ask whether returns on the universe of available trading strategies can span the variation in $\log(\hat{M}^a_{\exp,t+1})$.[23] To answer this question, we run 325 regressions, as many as trading strategies with returns available over the entire sample. In each regression, the dependent variable is $\log(\hat{M}^a_{\exp,t+1})$, whereas the number of independent variables grows from 1 to 325. The first regression includes

[23] See Internet Appendix IA.4 for a formal result justifying the spanning exercise for the unsystematic SDF component.

the return on a trading strategy that explains most of the unsystematic SDF component. Each subsequent regression includes an extra trading strategy whose return adds the most in explaining the dependent variable. Next, we select a linear model with the smallest Bayesian information criterion (BIC). We find that 39 trading strategies must be included to explain 66.45% of variation in the unsystematic SDF component. Any further increase in R^2 leads to overfitting because then BIC deteriorates.[24] We find that these 39 trading strategies explain 80.22% of the variation in the component of $\log(\hat{M}^a_{\exp,t+1})$ driven by the weak factors, $a'V_e^{-1}e^{\text{weak}}_{t+1}$, but only 28.28% of the variation in the component of $\log(\hat{M}^a_{\exp,t+1})$ driven by the CSP-specific shocks, $a'V_e^{-1}e^{\text{csp}}_{t+1}$. Because the CSP-specific shocks drive about half of the variation in the unsystematic SDF component, a large proportion of its variation is left unexplained by these trading strategies, implying that $\log(\hat{M}^a_{\exp,t+1})$ cannot be spanned even by a large number of trading strategies.

Finally, we compute the risk premia associated with compensation for the exposures of the trading strategies to the unsystematic SDF component as the negative covariance of the return on the strategy and $\hat{M}^a_{\exp,t+1}$:[25]

$$\text{RP}^a_{\text{strategy}} = -\operatorname{cov}(R_{\text{strategy},t+1}, \hat{M}^a_{\exp,t+1}) \times \mathbb{E}(M^{\beta,\text{can}}_{\exp,t+1}\hat{M}^{\beta,\text{mis}}_{\exp,t+1})/\mathbb{E}(\hat{M}_{\exp,t+1}).$$

Table 2 lists 25 selected strategies with high compensation for unsystematic risk. In the literature, some of these 25 strategies have been interpreted as being behavioral—for example, the performance factor (Stambaugh and Yuan, 2017), the long-horizon financial factor (Daniel et al., 2020a), the factor reflecting expectations about future earnings in growth (La Porta, 1996), and the momentum factor (Jegadeesh and Titman, 1993)—while others as reflecting market frictions—for example, the betting-against-beta factor (Frazzini and Pedersen, 2014) and distress risk (Campbell et al., 2008).

Summarizing our analysis of the unsystematic SDF component, we emphasize four novel findings about unsystematic risk, which is priced in the stock market. First, many trading

[24] Figure IA.1 in Internet Appendix IA.10 shows how the R^2 and BIC change as we increase the number of trading strategies in the regression for the unsystematic SDF component.

[25] We use the definition of risk premia and the result in Brillinger (2001, thm. 2.3.2) to obtain the risk premium decomposition on an asset i as compensation for candidate systematic risk, missing systematic risk (in a candidate factor model), and unsystematic risk, as follows

$$\begin{aligned}\mathbb{E}(R_{it+1} - R_f) &= -\operatorname{cov}(M_{\exp,t+1}, (R_{it+1} - R_f)) \\ &= -\mathbb{E}(M^a_{\exp,t+1}M^{\beta,\text{mis}}_{\exp,t+1}) \times \operatorname{cov}(M^{\beta,\text{can}}_{\exp,t+1}, R_{it+1} - R_f)/\mathbb{E}(M_{\exp,t+1}) \\ &\quad - \mathbb{E}(M^a_{\exp,t+1}M^{\beta,\text{can}}_{\exp,t+1}) \times \operatorname{cov}(M^{\beta,\text{mis}}_{\exp,t+1}, R_{it+1} - R_f)/\mathbb{E}(M_{\exp,t+1}) \\ &\quad - \mathbb{E}(M^{\beta,\text{can}}_{\exp,t+1}M^{\beta,\text{mis}}_{\exp,t+1}) \times \operatorname{cov}(M^a_{\exp,t+1}, R_{it+1} - R_f)/\mathbb{E}(\hat{M}_{\exp,t+1}).\end{aligned}$$

strategies featured in the existing literature correlate with the unsystematic SDF component. Second, the strategies correlated with the unsystematic SDF component that earn high-risk premia are related to market frictions and behavioral biases. Third, weak factors and purely CSP-specific shocks contribute almost equally to the unsystematic SDF component. And fourth, the strategies with the highest explanatory power for the unsystematic SDF component are primarily related to weak factors as opposed to CSP-specific shocks. Given these insights, it is clear why the prior literature that only investigated pricing of *systematic risk* could not explain the cross-section of expected returns.

5.1.4 The Systematic SDF Component

We now turn our attention to the systematic SDF component. We find that the strategy exhibiting the highest explanatory power for $\log(\hat{M}^{\beta}_{\exp,t+1})$ is the return on the market portfolio exhibits, with an $R^2 = 95.22\%$. It is remarkable that, despite all the criticism of the CAPM, when we consider only the systematic component of the SDF, the market return explains a large proportion of its time-series variation. We discover that four other trading strategies—sales-to-market (Barbee Jr, Mukherji, and Raines, 1996), dollar trading volume (Brennan, Chordia, and Subrahmanyam, 1998), bid-ask spread (Amihud and Mendelson, 1986), and days with zero trades (Liu, 2006)—explain an additional 4% of the variation in $\log(\hat{M}^{\beta}_{\exp,t+1})$, bringing the overall R^2 to 99.05%; see Table IA.2 in Internet Appendix IA.9 for further details. Any further increase in R^2 requires a large number of trading strategies to be used as factors.

We run a spanning exercise for the systematic SDF component similar to that when analyzing the explanatory power of returns on trading strategies for the unsystematic SDF component.[26] Figure IA.2 in Internet Appendix IA.10 shows that we need 54 strategies to explain 99.73% of the systematic SDF. These results are in line with the findings of Feng et al. (2020), Kozak et al. (2020), Lewellen (2022), and Bryzgalova et al. (2023) about the nonsparsity of the SDF in characteristics.

Next, we explore how sixteen variables often used as systematic tradable factors in popular asset-pricing models correlate with our SDF components. Table 3 reports these correlations. Four factors stand out: the market factor is almost perfectly negatively correlated with the systematic SDF component, the adjusted profitability factor (RMW*) of

[26] See Internet Appendix IA.4 for a formal result justifying the spanning exercise for the systematic SDF component.

Table 3: Correlations of tradable factors with SDF components

This table reports correlations of 16 selected tradable factors with the unsystematic and systematic SDF components.

Variable name	Correlation with $\log(\hat{M}^{a}_{\exp,t+1})$	Correlation with $\log(\hat{M}^{\beta}_{\exp,t+1})$	Source
Market	0.15	−0.98	Sharpe (1964), Lintner (1965)
Size	0.00	−0.36	Fama and French (1992)
Value	−0.23	0.14	Fama and French (1992)
Momentum	−0.36	0.18	Jegadeesh and Titman (1993)
Illiquidity	0.00	−0.27	Amihud (2002)
Operating profitability (RMW)	−0.25	0.17	Fama and French (2015)
Investment (CMA)	−0.32	0.31	Fama and French (2015)
Management (MGMT)	−0.37	0.48	Stambaugh and Yuan (2016)
Performance (PERF)	−0.37	0.27	Stambaugh and Yuan (2016)
Short-horizon underreaction (PEAD)	−0.23	0.14	Daniel, Hirshleifer, and Sun (2019)
Financing (FIN)	−0.41	0.42	Daniel, Hirshleifer, and Sun (2019)
Market*	−0.10	−0.56	Daniel, Mota, Rottke, and Santos (2020)
Size*	−0.13	−0.12	Daniel, Mota, Rottke, and Santos (2020)
Value*	−0.14	0.09	Daniel, Mota, Rottke, and Santos (2020)
RMW*	−0.13	0.02	Daniel, Mota, Rottke, and Santos (2020)
CMA*	−0.10	0.21	Daniel, Mota, Rottke, and Santos (2020)

Daniel, Mota, Rottke, and Santos (2020b) has nearly zero correlation with the systematic SDF component, while the size factor (Fama and French, 1992) and illiquidity factor (Amihud, 2002) have zero correlation with the unsystematic SDF component. The other tradable factors correlate sizably with both SDF components. These findings are consistent with the results of Holcblat, Lioui, and Weber (2022) that the market and size factors seem to represent risk in a frictionless economy, whereas most of the other tradable factors reflect frictions.

5.2 Candidate Factor Models

To illustrate what is missing in popular asset-pricing factor models, we consider three traditional candidate models—those implied by the CAPM of Sharpe (1964), the Consumption-CAPM (C-CAPM) of Breeden (1979), and the three-factor model of Fama and French (1993). For the SDFs $M^{\beta,\text{can}}_{\exp,t+1}$ implied by each of these candidate factor models, we estimate the correction terms $\hat{M}^{a}_{\exp,t+1}$ and $\hat{M}^{\beta,\text{mis}}_{\exp,t+1}$ that are required to obtain the admissible SDFs. We limit our analysis to only three candidate factor models because we find that the primary source of misspecification is omitted compensation for unsystematic risk, and, therefore, other candidate factor models with only systematic risk factors would be subject to the same misspecification.

Table 4: Analysis of models before and after correction for misspecification
The first column of the table lists the candidate factor models considered: CAPM, C-CAPM, and FF3. Then, the table reports three sets of quantities: (1) The HJ distances of alternative models, relative to the HJ distance of the APT model, $(\text{HJ}^{\text{model}}/\text{HJ}^{\text{APT}} - 1) \times 100\%$, before and after the model is corrected for misspecification; (2) the Sharpe ratio of the corrected SDF for each of the models along with its components, where the Sharpe ratios are approximated by the standard deviation of the SDF and its components in log; and (3) the variance decomposition of the log SDF.

	Relative HJ (%)		Std. Dev. or Sharpe ratio (p.a.)				Variance decomposition (%)		
	Before correction	After correction	log of				log of		
Model			$\hat{M}_{\text{exp},t+1}$	$\hat{M}^{a}_{\text{exp},t+1}$	$\hat{M}^{\beta,\text{can}}_{\text{exp},t+1}$	$\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1}$	$\hat{M}^{a}_{\text{exp},t+1}$	$\hat{M}^{\beta,\text{can}}_{\text{exp},t+1}$	$\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1}$
CAPM	82.96	4.09	0.89	0.80	0.42	0.27	74.14	18.48	7.38
C-CAPM	83.12	0.90	0.92	0.79	0.36	0.42	66.05	15.92	18.03
FF3	84.11	0.16	0.99	0.80	0.67	0.27	55.49	38.30	6.21

5.2.1 The CAPM

We consider a candidate model with the market return as its sole factor ($K^{\text{can}} = 1$) and the vector $a^{\text{can}} = 0_N$, which we refer to as the CAPM. When the candidate model is the CAPM, our estimation procedure selects $K^{\text{mis}} = 1$ and $\delta_{\text{apt}} = 0.0529$ (see Figure IA.3 in Internet Appendix IA.10). The obtained number of missing factors to correct the market model is consistent with our earlier finding that two latent factors summarize the common variation in asset returns, with one factor being a proxy for the market factor. The nonzero value of δ_{apt} indicates that the CAPM is misspecified not only because of missing systematic risk factors but also because it omits compensation for unsystematic risk. The value of $\delta_{\text{apt}} = 0.0529$, which is the same as for the APT model, implies an annual Sharpe ratio associated with the exposure to the unsystematic SDF component equal to 0.80.

The importance of allowing nonzero compensation for unsystematic risk and accounting for an additional source of systematic risk when correcting the CAPM is evident from Table 4. This table's second and third columns show that *after* we correct the CAPM for misspecification, the relative HJ distance drops by 78.87 percentage points (= 82.96% − 4.09%). The last three columns of this table show that the lion's share of the reduction in the HJ distance is attributable to nonzero compensation for unsystematic risk. Specifically, of the variation in $\log(\hat{M}_{\text{exp},t+1})$, 74.14% is due to the *unsystematic* component, while only 18.48% is due to market risk and 7.38% to missing systematic risk in the CAPM.[27]

[27]Figure IA.4 in Internet Appendix IA.10 shows the estimated time-series of the admissible SDF and its components obtained after correcting the candidate CAPM model. Figure IA.5 in Internet Appendix IA.10 shows the pricing errors before and after correcting the CAPM. We find that the correction brings the largest improvement in pricing for the portfolios consisting of small-cap stocks with low beta, small-cap stocks with

Table 5: The missing systematic SDF component in the candidate factor models and observable variables
This table reports the explanatory power of selected variables for the missing systematic SDF component, $\log(M^{\beta,\text{mis}}_{\text{exp},t+1})$, for the three candidate factor models: CAPM, C-CAPM, and FF3.

Variable	$R^2(\%)$	p-value
Panel A: CAPM		
NBER recession indicator	0.18	0.27
Illiquidity (Amihud, 2002)	88.29	0.00
Shocks in the credit spread (Gilchrist and Zakrajšek, 2012)	4.41	0.00
Size factor (Fama and French, 2015)	87.65	0.00
Panel B: C-CAPM		
NBER recession indicator	0.46	0.08
Market factor	92.36	0.00
Shocks in intermediary constraints (He et al., 2017)	55.08	0.00
Shocks in VIX	55.24	0.00
Panel C: FF3		
NBER recession indicator	0.34	0.13
Operating profitability (Fama and French, 2006)	31.20	0.00
Return on equity (Haugen and Baker, 1996)	30.24	0.00
Total accruals (Richardson et al., 2005)	28.22	0.00

Next, we analyze which variables can explain the variation in the missing *systematic* SDF component. Panel A of Table 5 shows that the variables known in the literature as the size factor (Fama and French, 1993) and illiquidity factor (Amihud, 2002) explain most of the variation in $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$: the R^2 of a linear regression of $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$ on the size factor or the illiquidity factor is 88%. Such a prominent role of the size factor in $\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1}$ explains the success of the models developed in Fama and French (1993, 2015) relative to the CAPM of Sharpe (1964).[28] Among business-cycle indicators, shocks in the credit spread (Gilchrist and Zakrajšek, 2012) have the largest, yet very small, explanatory power for the missing systematic SDF component, while the NBER recession indicator does not significantly correlate with it (because the CAPM already includes the market factor).

The properties of the unsystematic SDF component $\hat{M}^{a}_{\text{exp},t+1}$ are similar to those obtained when analyzing the SDF implied by the APT model of asset returns, as evident from

low net issuances, big-cap stocks with high beta, big-cap stocks with high net issuances, and portfolios formed by sorting stocks by size and momentum (ME&MOM) and size and variance (ME&VAR).

[28]Notice that the illiquidity factor of Amihud (2002) and size factor of Fama and French (1993) are highly correlated, at about 92.63%. According to Jensen et al. (2022), these two variables belong to the same cluster of trading strategies labeled "Risk."

Table 6: Correlation matrix of the corrected SDFs

This table reports the correlation matrix of admissible SDFs and their unsystematic components either obtained under the APT or after correcting different candidate models: CAPM, C-CAPM, and FF3.

		$\log(\hat{M}_{\text{exp},t+1})$				$\log(\hat{M}^{a}_{\text{exp},t+1})$			
			Corrected				Corrected		
		APT	CAPM	C-CAPM	FF3	APT	CAPM	C-CAPM	FF3
	APT	1.00	0.99	0.97	0.98	1.00	0.97	1.00	0.94
Corrected	CAPM	0.99	1.00	0.96	0.97	0.97	1.00	0.97	0.93
Corrected	C-CAPM	0.97	0.96	1.00	0.94	1.00	0.97	1.00	0.93
Corrected	FF3	0.98	0.97	0.94	1.00	0.94	0.93	0.93	1.00

the right panel of Table 6, which reports a correlation of 0.97 between the unsystematic SDF components of the SDFs implied by the APT and the corrected candidate factor models.

We conclude our analysis of the CAPM by highlighting that our approach successfully corrects this model's SDF to obtain an admissible SDF. We see from the left panel of Table 6 that the corrected SDF is almost perfectly correlated with the admissible SDF implied by the APT model.

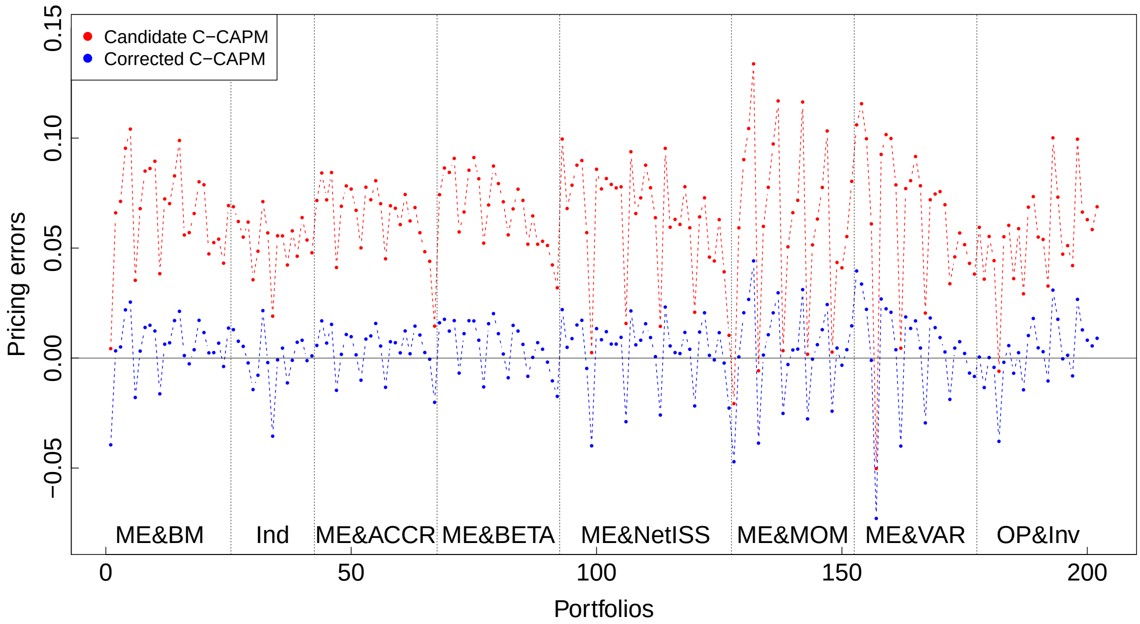
5.2.2 The C-CAPM

We now consider a candidate model with the return on a consumption-mimicking portfolio as its sole factor and the vector $a^{\text{can}} = 0_N$, which we refer to as the C-CAPM. We follow the standard approach of Breeden, Gibbons, and Litzenberger (1989) for constructing the consumption-mimicking portfolio.[29]

If the candidate factor model of asset returns is the C-CAPM, then the estimation procedure selects $K^{\text{mis}} = 2$ latent factors and $\delta_{\text{apt}} = 0.0529$ (see Figure IA.6 in Internet Appendix IA.10). The consumption-mimicking portfolio does not correlate highly with either of the latent factors of the APT model of asset returns (the correlations are 0.30 and -0.02), which explains why we still require two additional latent factors to capture the common variation in asset returns. The value of $\delta_{\text{apt}} = 0.0529$, which is the same as for the

[29] As outlined in Giglio and Xiu (2021), construction of factor-mimicking portfolios can be sensitive to the choice of basis assets. They propose a three-stage procedure, insensitive to the choice of basis assets. However, their procedure does not allow for compensation for unsystematic risk, which we document plays a major role in the risk-return tradeoff.

Figure 5: Pricing errors in the candidate and corrected C-CAPM
This plot displays the pricing errors in the candidate and corrected C-CAPM models. The red dots indicate the pricing errors for the 202 basis assets using the candidate C-CAPM model. The blue dots indicate the pricing errors using the corrected C-CAPM model.



APT and CAPM models, implies an annual Sharpe ratio for exposure to the unsystematic SDF component equal to 0.80.

The second and third columns of Table 4 show that augmenting the consumption-mimicking-portfolio factor with two latent factors and allowing for compensation for unsystematic risk lead to a large drop in the relative HJ distance by 82.22 percentage points (= 83.12% − 0.90%). The last three columns of the table show that, just like for the corrected CAPM, most of this drop is accounted for by the SDF's *unsystematic* component: missing systematic risk explains a much smaller proportion of the variation in the admissible SDF, compared to its unsystematic component: 18.03% versus 66.05%.[30] However, compared to the CAPM, the missing systematic risk in the C-CAPM is larger.

When we analyze the pricing errors, we observe from Figure 5 that the C-CAPM is missing a level factor: the pricing errors are centered around 6% in the candidate C-CAPM, whereas they are centered around zero in the corrected model. Next, we explore which observable variable explains most of the variation in $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$ and find, not surprisingly,

[30]Figure IA.7 in Internet Appendix IA.10 shows the estimated time-series of the admissible SDF and its components obtained after correcting the candidate C-CAPM.

it is the market factor, with $R^2 = 92.36\%$. Panel B of Table 5 shows that among financial and macroeconomic indicators, shocks to intermediary constraints (He et al., 2017) and VIX innovations explain most of the variation in the missing systematic SDF component, with $R^2 = 55.08\%$ and $R^2 = 55.24\%$, respectively. The missing systematic SDF component has only a modest correlation with the NBER recession indicator, which is unsurprising, given that the candidate factor model already includes consumption growth.

The left-hand-side panel of Table 6 shows that our approach for correcting misspecification in the C-CAPM model leads to an admissible SDF highly correlated with that implied by the APT and the corrected-CAPM models. The right-hand-side panel shows that the unsystematic SDF components obtained when correcting the C-CAPM for misspecification and when estimating the APT model of asset returns are perfectly correlated.

5.2.3 The Three-Factor Model of Fama and French (1993)

We consider a candidate model with the three factors of Fama and French (1993), market, size, and value, and the vector $a^{\text{can}} = 0_N$, and we refer to this model as FF3. When FF3 is the candidate model for asset returns, our estimation method selects $K^{\text{mis}} = 1$ systematic missing latent factor and an optimal $\delta_{\text{apt}} = 0.0529$ (see Figure IA.8 in Internet Appendix IA.10). This value of δ_{apt}, which is the same as for the previously discussed models, implies an annual Sharpe ratio associated with the exposure to the unsystematic SDF component equal to 0.80.

The third row of Table 4 shows that augmenting the FF3 model with one latent factor and nonzero compensation for unsystematic risk leads to a substantial improvement in pricing performance: the relative HJ distance drops by 83.95 percentage points $(= 84.11\% - 0.16\%)$. The main improvement is attributable to the inclusion of nonzero compensation for unsystematic risk, as suggested by the variance decomposition of the (log of the) corrected SDF in the last three columns of Table 4. Thus, similar to Stambaugh and Yuan (2017), Clarke (2022), and Bryzgalova et al. (2023), among others, we document sizable misspecification in the FF3 model. In contrast to these papers, however, we attribute the misspecification mainly to omitted compensation for unsystematic risk e_{t+1}, i.e., assuming $a = 0_N$, rather than missing systematic risk.

We analyze which observable variables can explain the variation in the missing systematic SDF component. Panel C of Table 5 shows that the operating profitability factor

(Fama and French, 2006), return on equity (Haugen and Baker, 1996), and total accruals (Richardson et al., 2005) explain most of the variation in $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$: the R^2 of a univariate linear regression of $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$ on one of these variables is about 30%. We find no relation of $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$ with the NBER recession indicator.[31]

The left-hand-side panel of Table 6 shows that our approach for correcting misspecification in the original FF3 model leads to an admissible SDF highly correlated with that implied by the APT model and also those obtained after correcting the CAPM and C-CAPM candidate factor models. The right-hand-side panel of the table confirms that the unsystematic SDF components obtained when correcting the FF3 for misspecification and when estimating the APT model of asset returns are almost perfectly correlated.

5.3 Out-of-Sample Analysis

To illustrate the robustness of our conclusions, we undertake two out-of-sample exercises in which we compare the performance of the SDF implied by the APT model of asset returns to models based on PCs and the original and fully corrected candidate factor models. We include models based on PCs in this analysis because they are agnostic about systematic risk factors, and therefore, nest linear candidate factor models that feature different proxies for systematic risk factors.

The first exercise is a *time-series* out-of-sample analysis. First, we split the sample into two equal parts: one part includes all the odd-numbered observations, and the other all the even-numbered observations. Next, we estimate each model of asset returns on one part of the sample and use the corresponding parameter estimates to form the SDF on the other part of the sample. We evaluate the performance of the SDF on the part of the sample not used in the estimation. Then, we swap the subsamples that we use for estimation and evaluation. Finally, we average the results from these two out-of-sample evaluations and report them in the second column of Table 7.

The second exercise we undertake is to run a *cross-sectional* out-of-sample analysis by evaluating how the set of models described above and estimated using the basis assets (Dataset 1) price two different cross-sections of portfolio returns (Datasets 2 and 3).

[31] Figure IA.9 in Internet Appendix IA.10 displays the time-series behavior of the admissible SDF obtained from correcting the original FF3 model. Figure IA.10 in Internet Appendix IA.10 displays the pricing errors before and after correcting the FF3 model. We find that the correction brings the largest improvement in pricing for the portfolios formed by sorting stocks by size and momentum (ME&MOM) and size and variance (ME&VAR).

Table 7: Time-Series and Cross-Sectional Out-Of-Sample Pricing Performance
This table reports the HJ distances of alternative models, relative to the HJ distance of the APT model, $(\mathrm{HJ}^{\mathrm{model}}/\mathrm{HJ}^{\mathrm{APT}} - 1) \times 100\%$. For the time-series out-of-sample analysis, we estimate the models on half of the available observations (odd or even) and then evaluate them on the other half. Then we swap the estimation and evaluation subsamples of the data and repeat our exercise. Finally, we compute the average performance, which is reported in the second column. For the cross-sectional out-of-sample analysis, we estimate all models on Dataset 1 consisting of 202 basis assets described in Section 4.1. The performance of these models is then evaluated using two datasets not used in the estimation ("Dataset 2" and "Dataset 3"). Any positive number indicates that the corresponding model performs worse than our benchmark APT model of asset returns.

Model	Relative HJ distance (%)		
	Time Series	Cross section	
	Odd/Even	Dataset 2	Dataset 3
Panel A: PC-based models			
PC1	11.32	43.23	39.82
PC2	14.47	43.49	40.05
PC3	16.32	49.96	45.33
PC4	62.21	74.11	65.69
PC5	92.06	73.91	65.57
Panel B: Candidate models			
CAPM	10.68	42.89	39.51
C-CAPM	10.72	43.40	39.06
FF3	13.52	45.36	42.21
Panel C: Candidate models after correction			
CAPM	0.31	2.11	2.15
C-CAPM	1.30	1.53	0.97
FF3	1.71	1.25	1.37

Datasets 2 and 3 include portfolios formed by sorting stocks on the same or similar characteristics as those used to form the set of basis assets, but these sorts have a different level of granularity or the sorts are undertaken in a different order. Thus, the test assets in Datasets 2 and 3 allow us to evaluate if the estimated SDF successfully captures the risks associated with the characteristics on which the basis portfolios are formed.[32] We report the results of pricing the assets from Datasets 2 and 3 in the third and fourth columns of Table 7, respectively.

Table 7 shows how much larger the HJ distance is for various models of asset returns relative to that for the APT model. We see that the HJ distance implied by the APT

[32]Note that the SDF we estimate is a *projection* of the marginal utility on a set of basis assets rather than the marginal utility per se. Thus, by construction, the estimated SDF would not price test assets whose returns are orthogonal to those of the basis assets (Cochrane, 2005), and hence, we use test assets that are not perfectly orthogonal to the basis assets in Dataset 1.

model is the smallest in both time-series and cross-sectional out-of-sample exercises. Only the candidate models fully corrected for missing sources of systematic risk and omitted compensation for unsystematic risk exhibit pricing behavior at par with that of the APT model of asset returns. This result showcases that our methodology is successful in correcting alternative candidate factor models.

There is another interesting observation from Table 7. We see that including extra principal components in a PC-based model leads to a deterioration in the fit out of sample. This suggests that candidate factor models with alternative observable factors that necessarily have to be correlated with higher-order PCs do not perform at par with the APT model. If anything, these models are subject to in-sample overfitting. Also, and perhaps more importantly, the out-of-sample performance of the PC-based models suggests that one cannot substitute for the unsystematic SDF component with higher-order PCs.

5.4 Discussion: Compensation for Unsystematic Risk

So far, we have shown the importance of including compensation for unsystematic risk in three popular asset-pricing factor models. Of course, we could repeat our empirical analysis for other candidate factor models. However, our main conclusion will not change—the unsystematic component in the SDF accounts for the lion's share of pricing of the cross-section of asset returns. It remains unspanned by virtually all known proxies for risk factors proposed in the literature so far. This insight can be confirmed by looking at principal-component-based models, which are agnostic about the choice of risk factors. Our main conclusion is also consistent with the empirical finding in Bryzgalova et al. (2023), who undertake a large-scale search for a *factor* model that prices a cross-section of asset returns but find none.

We observe that the standard deviations of $\log(\hat{M}^{a}_{\exp,t+1})$, 0.80 per annum, which also represents the Sharpe ratio of the strategy exposed to unsystematic risk, is stable across the APT (Table 1) and the three corrected candidate models (Table 4).[33] Given a candidate asset-pricing model, we have also shown that adding extra *systematic* risk factors to this

[33] Looking at the variance decomposition of the SDF, the quantitative importance of the unsystematic SDF component may appear to be different across admissible SDFs implied by alternative candidate models of asset returns corrected for misspecification. However, this is misleading because if a candidate factor model includes noisy proxies of the true systematic factors in a cross-section of asset returns, then the admissible SDF inherits the noise that inflates the SDF's variance and therefore biases down the proportional contribution of the unsystematic SDF component to the overall variation in the SDF, as we can see from the third-last column of Table 4.

model, without including compensation for unsystematic risk, cannot proxy for the SDF's unsystematic component. This is because the unsystematic component of the SDF behaves like a weak factor in the cross-section of asset returns.

6 Microfoundations for Priced Unsystematic Risk

In the previous section, we have presented strong empirical evidence that factor models need to include compensation for unsystematic risk and that it is the unsystematic component of the SDF that accounts for most of its variation. At this point, one may wonder in what kind of economic environment will unsystematic risk be compensated. Below we present an example of an equilibrium model that provides microfoundations for the notion that unsystematic risk is priced. Our example relies on the well-known static model of Merton (1987), which has a finite number of assets N. We show that, if N is asymptotically large, then the equilibrium asset returns and SDF in this model have the same functional forms as those we have in our APT model.

In Merton (1987), investors are aware of only a subset of the available securities in which they invest. This type of "incomplete information" implies that not only systematic risk factor but also shocks specific to each security are priced. The modeling framework of Merton (1987) can be viewed as a reduced-form representation of different microfoundations, such as market segmentation, institutional restrictions, transaction costs, illiquidity, or imperfect divisibility of securities, that lead investors to invest in only a subset of available securities. While the incomplete information of Merton (1987) may not be the only reason why the unsystematic SDF component plays a dominant role in the pricing assets, it is an appealing argument given the ample empirical evidence documenting that both retail (Polkovnichenko, 2005; Campbell, 2006; Goetzmann and Kumar, 2008) and institutional investors (Koijen and Yogo, 2019, table 2) invest in only a small number of available stocks.

In Merton (1987), and as shown in Internet Appendix IA.4, equilibrium asset returns satisfy

$$R_i - R_f = a_i - \beta_i a_m + \beta_i(\mathbb{E}(R_m) - R_f) + \frac{b_i}{b}(R_m - \mathbb{E}(R_m)) + e_i,$$

where R_m representing the market return is the only systematic risk factor, β_i denotes the beta of asset i with respect to the market return, $e_i = \sigma_i \epsilon_i$ are asset-specific shocks with the diagonal covariance matrix containing the elements σ_i^2 on its diagonal, b_i and σ_i are functions of the parameters of the firm's i production technology, $b = \sum_{i=1}^{N} x_i b_i$ with x_i

being the fraction of the market portfolio invested in asset i, $a_i = (1 - q_i)(\mathbb{E}(R_i) - R_f - b_i(\mathbb{E}(R_{N+1} - R_f))$ with q_i denoting the fraction of investors who know about the security i, R_{N+1} being the return on the $(N + 1)$th security which is in zero net supply and which combines the risk-free security and a forward contract with cash settlements on the only systematic risk factor R_m, and $a_m = \sum_{i=1}^{N} x_i a_i$.

We now derive the SDF in this economy when $N \to \infty$. We assume that x_i, the fraction of the market portfolio invested in asset i, is infinitesimally small.

Proposition 6. *When the number of assets $N \to \infty$, equilibrium asset returns are*

$$R_i - R_f = a_i + \beta_i(\mathbb{E}(R_m) - R_f) + \beta_i(R_m - \mathbb{E}(R_m)) + e_i, \tag{21}$$

$$= a_i + \beta_i(R_m - R_f) + e_i, \tag{22}$$

with the market return asymptotically orthogonal to asset-specific shocks e_i, and the equilibrium SDF is

$$M = \underbrace{\frac{-a'V_e^{-1}}{R_f} e}_{M^a} + \underbrace{\frac{1}{R_f} - \frac{\mathbb{E}(R_m) - R_f}{R_f \times \mathrm{var}(R_m)} (R_m - \mathbb{E}(R_m))}_{M^\beta}. \tag{23}$$

Note that the model of asset returns (22) coincides with the APT model of asset returns in equation (3) with $K = 1$ systematic factor, $f = R_m$. Similarly, the SDF in (23) coincides with the SDF in (6), given that the price of market risk is $\lambda = \mathbb{E}(R_m) - R_f$. Thus, in the model of Merton (1987) with an infinite number of assets, the SDF consists of two components: one representing unsystematic risk, M^a, and the other systematic risk, M^β, exactly as under the APT.

Note that a_i in (21) represents the compensation for unsystematic risk, because

$$a_i = -\operatorname{cov}\left(R_i - R_f, -\frac{a'V_e^{-1}}{R_f} e_i)\right) \times R_f,$$

which coincides with the elements of the vector a in the APT. Naturally, the other part of the risk premium in (21), $\beta_i(\mathbb{E}(R_m) - R_f)$, is compensation for exposure to systematic risk, represented by market risk because of the assumption of a single systematic factor:

$$\beta_i(\mathbb{E}(R_m) - R_f) = -\operatorname{cov}\left(R_i - R_f, -\frac{\mathbb{E}(R_m) - R_f}{R_f \times \mathrm{var}(R_m)} (R_m - \mathbb{E}(R_m))\right) \times R_f.$$

If all investors are fully informed about all N assets, that is, $q_i = 1$, then $a_i = 0$, and the results in (21) and (23) simplify to the expressions for security returns and the SDF under the CAPM, respectively. On the other hand, the no-arbitrage APT restriction in expression (5) is equivalent to stating that in the Merton (1987) model when $N \to \infty$ there are only a small number of assets that do not belong to the common information set of investors, that is, $q_i < 1$ for some of the assets but not all, or that there are only a small number of investors who are unaware of each asset, that is, for each i, q_i is approximately 1.

The above discussion shows that there are equilibrium models that support the notion that unsystematic risk is priced. Moreover, Proposition 6 shows that this result is not limited to an economy with a finite number of assets.

7 Conclusion

A fundamental challenge in finance is to price the cross-section of assets. The main difficulty when pricing assets is to determine the relevant sources of risk and quantify how to adjust assets' returns for these risks. The literature has proposed many proxies for *systematic risk* factors and developed factor models based on these proxies to explain the cross-sectional risk-return tradeoff. However, despite the proliferation of systematic risk factors, referred to as the factor zoo (Cochrane, 2011), there is still a sizable pricing error called alpha. This conundrum leads to the question posed in the title of this paper: "What is missing in asset-pricing factor models?"

We answer this question by challenging the conventional wisdom that only systematic sources of risk receive compensation in financial markets by showing that also *unsystematic risk* is compensated. That is, the pricing error alpha implied by factor models includes compensation not only for missing systematic risk factors but also for unsystematic risk. Theoretically, we demonstrate this key insight through the lens of the SDF under the assumptions of the APT and support it by demonstrating that an equilibrium model such as Merton (1987) is consistent with our insight. Empirically, we show that the component of the admissible SDF reflecting unsystematic risk, which is a linear combination of *unsystematic shocks*, accounts for more than 70% of the variation in the admissible SDF. Furthermore, the Sharpe ratio associated with the investment strategy exposed to only unsystematic risk is 0.8 per annum. Thus, what is missing in asset-pricing factor models is compensation for this unsystematic risk. This insight is crucial both for empiricists wanting to resolve the factor zoo and theorists wishing to develop microfounded asset-pricing models.

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Internet Appendix

In Section IA.1, we define the notation we will use in the Internet Appendix. Section IA.2 lists the assumptions used to prove the lemmas in Section IA.3 and propositions in Section IA.4. Section IA.5 presents the results for weak factors. Section IA.6 gives the details of how we estimate the APT model of asset returns. Section IA.7 discusses the case where the candidate factors are not assumed to be orthogonal to the missing sources of systematic risk. Section IA.8 provides the details of the data we use in our analysis. Section IA.9 collects additional tables and Section IA.10 additional figures that are related to the results reported in the main text of the manuscript.

IA.1 Notation

We adopt the following notation in the manuscript and appendix. $\mathbb{E}(\cdot)$ denotes the expectation operator. Capital letters denote matrices, while lowercase letters denote scalars or vectors. The notation 0_N and 1_N indicates an $N \times 1$ vector of zeros and ones, respectively. The notation I_K and O_K denotes the $K \times K$ identity matrix and matrix of zeros, respectively. For an arbitrary matrix A, the expression $A > 0$ means that A is a positive-definite matrix, $\|A\|$ denotes the Frobenius norm $\|A\| = (\text{tr}(A'A))^{\frac{1}{2}}$, where $\text{tr}(\cdot)$ is the trace operator, and $|A|$ is the determinant when A is a square matrix. For deterministic sequences $\{a_N\}$ and $\{b_N\}$, the notation $a_N = O(b_N)$ means that $|a_N|/b_N < \delta$, where $\delta > 0$ is a finite constant, and $a_N = o(b_N)$ means that $|a_N|/b_N \to 0$, as $N \to \infty$. The notation $a_N = O(b_N)$ and $a_N = o(b_N)$ is adopted for scalars and finite-dimensional vectors and matrices (whose number of rows and columns are not a function of N). Finally, the notation $a_N = O_p(b_N)$ and $a_N = o_p(b_N)$ means that the previous statements hold in probability. The notation $vec(A)$ for an arbitrary matrix A stands for an operator that transforms the matrix A into a column vector by vertically stacking the columns of the matrix. The notation $vech(A)$ for an arbitrary symmetric matrix A indicates an operator that transforms the symmetric matrix into a column vector that collects the elements in the lower triangular part of A. We use $\otimes$ to denote the Kronecker product.

IA.2 Assumptions

This section provides a set of assumptions we use in the lemmas and propositions of Sections IA.3 and IA.4, respectively.

Assumption IA.2.1 (Systematic candidate factors)**.** *We assume that a candidate model contains* K^{can} *systematic factors* f_t^{can}*, that is,* $\beta^{\text{can}\prime} V_e^{-1} \beta^{\text{can}}/N \to D$ *, where* $D > 0$ *is a* $K^{\text{can}} \times K^{\text{can}}$ *matrix.*

Assumption IA.2.2 (Asymptotic orthogonality of β^{can} and a)**.** *We assume that* $\beta^{\text{can}\prime} V_e^{-1} a = o(N^{\frac{1}{2}})$*.*

Assumption IA.2.3 (Systematic missing factors)**.** *We assume that a candidate factor model is omiting* K^{mis} *systematic factors* f_t^{mis}*, that is,* $\beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}}/N \to E$*, where* $E > 0$ *is some* $K^{\text{mis}} \times K^{\text{mis}}$ *matrix.*

Assumption IA.2.4 (Asymptotic orthogonality of β^{mis} and a)**.** *We assume that* $\beta^{\text{mis}\prime} V_e^{-1} a = o(N^{\frac{1}{2}})$*.*

Remark. Assumptions IA.2.2 and IA.2.4 represent asymptotic orthogonality conditions because they imply that as $N \to \infty$, $\beta^{\text{can}\prime} V_e^{-1} a/N \to 0$ and $\beta^{\text{mis}\prime} V_e^{-1} a/N \to 0$.[34]

IA.3 Lemmas

We now provide a set of lemmas that will be useful for proving our propositions.

Lemma IA.3.1. *For a normally-distributed vector* $z \sim N(\mu_z, \Sigma_z)$*, and any constant vector* d*:*

$$\mathbb{E}(ze^{d'z}) = \mu^* e^{\frac{1}{2}(\mu^{*\prime}\Sigma_z^{-1}\mu^* - \mu_z'\Sigma_z^{-1}\mu_z)}, \quad \textit{where} \quad \mu^* = (\mu_z + \Sigma_z d).$$

Proof: Denote by n_z the dimension of the vector z. Use the definition of the mathematical expectation to obtain

$$\mathbb{E}(ze^{d'z}) = \frac{1}{(\sqrt{2\pi})^{n_z}|\Sigma_z|^{\frac{1}{2}}} \int_{-\infty}^{\infty} ze^{d'z} e^{-\frac{1}{2}(z-\mu_z)'\Sigma_z^{-1}(z-\mu_z)} dz.$$

Note that

$$\begin{aligned}
e^{d'z} e^{-\frac{1}{2}(z-\mu_z)'\Sigma_z^{-1}(z-\mu_z)} &= e^{d'z - \frac{1}{2}z'\Sigma_z^{-1}z - \frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z + \mu_z'\Sigma_z^{-1}z} \\
&= e^{-\frac{1}{2}z'\Sigma_z^{-1}z - \frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z + (\Sigma_z d + \mu_z)'\Sigma_z^{-1}z} \\
&= e^{-\frac{1}{2}z'\Sigma_z^{-1}z - \frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z + \mu^{*\prime}\Sigma_z^{-1}z} \\
&= e^{-\frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z + \frac{1}{2}\mu^{*\prime}\Sigma_z^{-1}\mu^*} e^{-\frac{1}{2}z'\Sigma_z^{-1}z + \mu^{*\prime}\Sigma_z^{-1}z - \frac{1}{2}\mu^{*\prime}\Sigma_z^{-1}\mu^*} \\
&= e^{-\frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z + \frac{1}{2}\mu^{*\prime}\Sigma_z^{-1}\mu^*} e^{-\frac{1}{2}(z-\mu^*)'\Sigma_z^{-1}(z-\mu^*)},
\end{aligned}$$

[34] Assumptions IA.2.1 and IA.2.3, together with asymptotic no arbitrage, by the Cauchy-Schwarz inequality, imply that $\beta^{\text{can}\prime} V_e^{-1} a = O(N^{1/2})$ and $\beta^{\text{mis}\prime} V_e^{-1} a = O(N^{1/2})$ but we need a slightly slower rate.

implying that

$$\mathbb{E}(ze^{d'z}) = e^{-\frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z+\frac{1}{2}\mu^{*\prime}\Sigma_z^{-1}\mu^*} \times \left(\frac{1}{(\sqrt{2\pi})^{n_z}|\Sigma_z|^{\frac{1}{2}}}\int_{-\infty}^{\infty} ze^{-\frac{1}{2}(z-\mu^*)'\Sigma_z^{-1}(z-\mu^*)}dz\right)$$
$$= e^{-\frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z+\frac{1}{2}\mu^{*\prime}\Sigma_z^{-1}\mu^*}\mu^*. \qquad \square$$

Lemma IA.3.2. *Under Assumptions IA.2.1 and IA.2.3:*

$$\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{can}} = O(N).$$

Proof: We apply the Cauchy-Schwarz inequality for matrices and obtain

$$0 \leq ||\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{can}}|| \leq ||\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}||^{\frac{1}{2}} \times ||\beta^{\text{can}\prime}V_e^{-1}\beta^{\text{can}}||^{\frac{1}{2}} = O(N). \qquad \square$$

Lemma IA.3.3. *Under Assumptions IA.2.1 and IA.2.3:*

$$\beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}} = O(N).$$

Proof: Recall that $V_\varepsilon = \beta^{\text{mis}}V_{f^{\text{mis}}}\beta^{\text{mis}\prime} + V_e$ and apply the Sherman-Morrison-Woodbury formula to V_ε^{-1} to obtain

$$\beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}} = \beta^{\text{can}\prime}V_e^{-1}\beta^{\text{can}} - \beta^{\text{can}\prime}V_e^{-1}\beta^{\text{mis}}(V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{can}}$$
$$= O(N) + O(N) \times [O(1) + O(N)]^{-1} \times O(N) = O(N). \qquad \square$$

Lemma IA.3.4. *Under Assumption IA.2.3:*

$$\beta^{\text{mis}\prime}V_\varepsilon^{-1}\beta^{\text{mis}} \to V_{f^{\text{mis}}}^{-1} \quad as \quad N \to \infty,$$

implying $\beta^{\text{mis}\prime}V_\varepsilon^{-1}\beta^{\text{mis}} = O(1)$.

Proof: Recall that $V_\varepsilon = \beta^{\text{mis}}V_{f^{\text{mis}}}\beta^{\text{mis}\prime} + V_e$ and apply the Sherman-Morrison-Woodbury formula to V_ε^{-1} to obtain

$$\beta^{\text{mis}\prime}V_\varepsilon^{-1}\beta^{\text{mis}} = \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}} - \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}(V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}$$
$$= V_{f^{\text{mis}}}^{-1}(V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}$$
$$= V_{f^{\text{mis}}}^{-1} \times [O(1) + O(N)]^{-1} \times O(N) \to V_{f^{\text{mis}}}^{-1}. \qquad \square$$

Lemma IA.3.5. *Under Assumptions IA.2.1 and IA.2.3:*

$$\beta^{\text{mis}\prime}V_\varepsilon^{-1}\beta^{\text{can}} = O(1).$$

Proof: Recall that $V_\varepsilon = \beta^{\text{mis}} V_{f^{\text{mis}}} \beta^{\text{mis}\prime} + V_e$ and apply the Sherman-Morrison-Woodbury formula to $V_\varepsilon{}^{-1}$ to obtain

$$\begin{aligned}
\beta^{\text{mis}\prime} V_\varepsilon{}^{-1} \beta^{\text{can}} &= \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{can}} - \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{mis}} (V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{can}} \\
&= V_{f^{\text{mis}}}^{-1} (V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{can}} \\
&= O(1) \times [O(1) + O(N)]^{-1} \times O(N) = O(1).
\end{aligned}$$

□

Lemma IA.3.6. *Under Assumptions IA.2.3 and IA.2.4:*

$$a' V_\varepsilon{}^{-1} a - a' V_e{}^{-1} a \to 0 \quad \textit{as} \quad N \to \infty.$$

Proof: Recall that $V_\varepsilon = \beta^{\text{mis}} V_{f^{\text{mis}}} \beta^{\text{mis}\prime} + V_e$ and apply the Sherman-Morrison-Woodbury formula to $V_\varepsilon{}^{-1}$ to obtain

$$\begin{aligned}
a' V_\varepsilon{}^{-1} a &= a' V_e{}^{-1} a - a' V_e{}^{-1} \beta^{\text{mis}} (V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e{}^{-1} a \\
&= a' V_e{}^{-1} a + o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times o(N^{\frac{1}{2}}) \\
&= a' V_e{}^{-1} a + o(1),
\end{aligned}$$

where $a' V_e{}^{-1} a = O(1)$. □

Lemma IA.3.7. *Under Assumptions IA.2.1, IA.2.2, IA.2.3 and IA.2.4:*

$$\beta^{\text{can}\prime} V_\varepsilon{}^{-1} a = o(N^{\frac{1}{2}}).$$

Proof: Recall that $V_\varepsilon = \beta^{\text{mis}} V_{f^{\text{mis}}} \beta^{\text{mis}\prime} + V_e$ and apply the Sherman-Morrison-Woodbury formula to $V_\varepsilon{}^{-1}$ to obtain

$$\begin{aligned}
\beta^{\text{can}\prime} V_\varepsilon{}^{-1} a &= \beta^{\text{can}\prime} V_e{}^{-1} a - \beta^{\text{can}\prime} V_e{}^{-1} \beta^{\text{mis}} (V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e{}^{-1} a \\
&= o(N^{\frac{1}{2}}) + O(N) \times [O(1) + O(N)]^{-1} \times o(N^{\frac{1}{2}}) \\
&= o(N^{\frac{1}{2}}).
\end{aligned}$$

□

Lemma IA.3.8. *Under Assumptions IA.2.3 and IA.2.4:*

$$\beta^{\text{mis}\prime} V_\varepsilon{}^{-1} a = o(N^{-\frac{1}{2}}).$$

Proof: Recall that $V_\varepsilon = \beta^{\text{mis}} V_{f^{\text{mis}}} \beta^{\text{mis}\prime} + V_e$ and apply the Sherman-Morrison-Woodbury formula to $V_\varepsilon{}^{-1}$ to obtain

$$\begin{aligned}
\beta^{\text{mis}\prime} V_\varepsilon{}^{-1} a &= \beta^{\text{mis}\prime} V_e{}^{-1} a - \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{mis}} (V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e{}^{-1} a \\
&= V_{f^{\text{mis}}}^{-1} (V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime} V_e{}^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e{}^{-1} a \\
&= O(1) \times [O(1) + O(N)]^{-1} \times o(N^{\frac{1}{2}}) = o(N^{-\frac{1}{2}}).
\end{aligned}$$

□

Lemma IA.3.9. *Let e be an $N \times 1$ random vector with zero mean and covariance matrix V_e. Under Assumptions IA.2.1 and IA.2.3:*

$$\beta^{\text{can}\prime} V_{\varepsilon}^{-1} e = O_p(N^{\frac{1}{2}}).$$

Proof: For any random variable X with a finite second moment, we have that $X = O_p((\mathbb{E}(X^2))^{\frac{1}{2}})$. If $X = \beta^{\text{can}\prime} V_e^{-1} e$, then

$$\mathbb{E}(\beta^{\text{can}\prime} V_e^{-1} e e' V_e^{-1} \beta^{\text{can}}) = \beta^{\text{can}\prime} V_e^{-1} \beta^{\text{can}} = O(N),$$

and therefore, $\beta^{\text{can}\prime} V_e^{-1} e = O_p(N^{\frac{1}{2}})$. Similarly, we can show that $\beta^{\text{mis}\prime} V_e^{-1} e = O_p(N^{\frac{1}{2}})$. Apply the Sherman-Morrison-Woodbury formula to V_{ε}^{-1} and use Lemma IA.3.2 to obtain

$$\begin{aligned}
\beta^{\text{can}\prime} V_{\varepsilon}^{-1} e &= \beta^{\text{can}\prime} V_e^{-1} e - \beta^{\text{can}\prime} V_e^{-1} \beta^{\text{mis}} (V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e^{-1} e \\
&= O_p(N^{\frac{1}{2}}) + O(N) \times [O(1) + O(N)]^{-1} \times O_p(N^{\frac{1}{2}}) = O_p(N^{\frac{1}{2}}). \qquad \square
\end{aligned}$$

Lemma IA.3.10. *Under Assumption IA.2.3:*

$$\beta^{\text{mis}\prime} V_{\varepsilon}^{-1} e = O_p(N^{-\frac{1}{2}}).$$

Proof: From the proof of Lemma IA.3.9, $\beta^{\text{mis}\prime} V_e^{-1} e = O_p(N^{\frac{1}{2}})$. Apply the Sherman-Morrison-Woodbury formula to V_{ε}^{-1} and use Lemma IA.3.2 to obtain

$$\begin{aligned}
\beta^{\text{mis}\prime} V_{\varepsilon}^{-1} e &= \beta^{\text{mis}\prime} V_e^{-1} e - \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}} (V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e^{-1} e \\
&= V_{f^{\text{mis}}}^{-1} (V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e^{-1} e \\
&= O(1) \times [O(1) + O(N)]^{-1} \times O_p(N^{\frac{1}{2}}) = O_p(N^{-\frac{1}{2}}). \qquad \square
\end{aligned}$$

IA.4 Proofs of Propositions

In this section, we provide the proofs for the propositions in the manuscript.

Proof of Proposition 1

We use a guess-and-verify method to derive the SDF. Specifically, we guess that the SDF has the following functional form

$$M_{t+1} = \mathbb{E}(M_{t+1}) + b'(f_{t+1} - \mathbb{E}(f_{t+1})) + c' e_{t+1},$$

where b is a $K \times 1$ vector and c is an $N \times 1$ vector. We identify the unknown vector b and c by using the Law of One Price. Specifically, because we assume the existence of the risk-free asset, to determine the mean of the SDF we use the following condition:

$$\mathbb{E}(M_{t+1}) = \frac{1}{R_f}.$$

Next, because λ represents the vector of prices of risk of f_{t+1}, we have that

$$-\operatorname{cov}(M_{t+1}, f_{t+1}) \times R_f = \lambda.$$

These K conditions identify b:

$$b = -\frac{V_f^{-1}\lambda}{R_f}.$$

Finally, it must be the case that the SDF M_{t+1} prices the N assets:

$$\mathbb{E}\big(M_{t+1}(R_{t+1} - R_f 1_N)\big) = 0_N.$$

These N equations identify c:

$$c = -\frac{V_e^{-1}a}{R_f}.$$

Taken together

$$M_{t+1} = M_{t+1}^{\beta} + M_{t+1}^{a},$$

where

$$M_{t+1}^{\beta} = \frac{1}{R_f} - \frac{\lambda' V_f^{-1}}{R_f}(f_{t+1} - \mathbb{E}(f_{t+1})) \quad \text{and}$$

$$M_{t+1}^{a} = -\frac{a' V_e^{-1}}{R_f} e_{t+1}.$$

Pairwise uncorrelatedness of f_t and e_t implies that the covariance between M_{t+1}^{β} and M_{t+1}^{a} is zero. $\square$

Proof of Proposition 2

First, we prove that the exponential SDF specified in formula (10) is an admissible SDF. We use a guess-and-verify method. We guess that the SDF has the following functional form:

$$M_{\exp,t+1} = \exp\left[\mu_+ + b_+'(f_{t+1} - \mathbb{E}(f_{t+1})) + c_+' e_{t+1}\right],$$

with unknown vectors b_+ and c_+, as well as an unknown scalar μ_+.

To identify the unknowns and verify our guess we use the following $K+N+1$ equations, which are implications of the Law of One Price:

$$-\operatorname{cov}(M_{\exp,t+1}, f_{t+1}) \times R_f = \lambda,$$
$$\mathbb{E}(M_{\exp,t+1}(R_{t+1} - R_f 1_N)) = 0_N,$$
$$\mathbb{E}(M_{\exp,t+1}) = \frac{1}{R_f}.$$

The first K equations imply that

$$-\mathbb{E}(M_{\exp,t+1}(f_{t+1}-\mathbb{E}(f_{t+1})))=\mathbb{E}(M_{\exp,t+1})\times\lambda,$$

which, along with Lemma IA.3.1, gives

$$b_+=-V_f^{-1}\lambda.$$

The next N equations and Lemma IA.3.1 imply that

$$\begin{aligned}
0_N&=\mathbb{E}(M_{\exp,t+1}(R_{t+1}-R_f1_N))=\mathbb{E}(M_{\exp,t+1}(a+\beta\lambda+\beta(f_{t+1}-\mathbb{E}(f_{t+1}))+e_{t+1}))\\
&=(a+\beta\lambda)\mathbb{E}(M_{\exp,t+1})+\mathbb{E}(M_{\exp,t+1}e_{t+1})+\mathbb{E}(M_{\exp,t+1}\beta(f_{t+1}-\mathbb{E}(f_{t+1})))\\
&=(a+\beta\lambda)\mathbb{E}(M_{\exp,t+1})+V_ec_+\mathbb{E}(M_{\exp,t+1})-\beta\lambda\mathbb{E}(M_{\exp,t+1})=(a+V_ec_+)\mathbb{E}(M_{\exp,t+1}).
\end{aligned}$$

As a result,

$$c_+=-V_e^{-1}a.$$

Finally, the last identifying condition implies

$$\begin{aligned}
R_f^{-1}&=\mathbb{E}(M_{\exp,t+1})\\
&=\mathbb{E}(\exp[\mu_++b_+'(f_{t+1}-\mathbb{E}(f_{t+1}))+c_+'e_{t+1}])=\exp\left[\mu_++b_+'V_fb_+/2+c_+'V_ec_+/2\right].
\end{aligned}$$

Thus,

$$\exp\left(\mu_+\right)=R_f^{-1}\times\exp\big[-\lambda'V_f^{-1}\lambda/2-a'V_e^{-1}a/2\big].$$

Collecting all these results, we obtain

$$M_{\exp,t+1}=M^{\beta}_{\exp,t+1}\times M^{a}_{\exp,t+1},$$

where

$$\begin{aligned}
M^{\beta}_{\exp,t+1}&=R_f^{-1}\times\exp\left[-\lambda'V_f^{-1}(f_{t+1}-\mathbb{E}(f_{t+1}))-\lambda'V_f^{-1}\lambda/2)\right],\\
M^{a}_{\exp,t+1}&=\exp\left[-a'V_e^{-1}e_{t+1}-a'V_e^{-1}a/2\right].
\end{aligned}$$

Next, we prove that, as $N\to\infty$, the feasible SDF given in equation (7) recovers the exponential SDF (10). We start by analyzing the exponent of $\hat{M}^{a}_{\exp,t+1}$:

$$-a'V_R^{-1}(R_{t+1}-\mathbb{E}[R_{t+1}])-\frac{1}{2}a'V_R^{-1}a=-a'V_R^{-1}\beta(f_{t+1}-\mathbb{E}(f_{t+1}))-a'V_R^{-1}e_{t+1}-\frac{1}{2}a'V_R^{-1}a.$$

We apply the Sherman-Morrison-Woodbury formula to V_R^{-1}, use Assumptions IA.2.1 and IA.2.2, and Lemma IA.3.9 to obtain[35]

$$a'V_R^{-1}\beta=a'V_e^{-1}\beta-a'V_e^{-1}\beta(V_f^{-1}+\beta'V_e^{-1}\beta)^{-1}\beta'V_e^{-1}\beta$$

[35] The APT model contains K systematic factors f_{t+1}, so it satisfies Assumptions IA.2.1 and IA.2.2, where one replaces K^{can} by K, f^{can}_{t+1} by f_{t+1}, and β^{can} by β.

$$
\begin{aligned}
&= a'V_e^{-1}\beta(V_f^{-1} + \beta'V_e^{-1}\beta)^{-1}V_f^{-1} \\
&= o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times O(1) \\
&= o(N^{-1/2}), \\
a'V_R^{-1}e_{t+1} &= a'V_e^{-1}e_{t+1} - a'V_e^{-1}\beta(V_f^{-1} + \beta'V_e^{-1}\beta)^{-1}\beta'V_e^{-1}e_{t+1} \\
&= a'V_e^{-1}e_{t+1} + o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times O_p(N^{\frac{1}{2}}) \\
&= a'V_e^{-1}e_{t+1} + o_p(1), \\
a'V_R^{-1}a &= a'V_e^{-1}a - a'V_e^{-1}\beta(V_f^{-1} + \beta'V_e^{-1}\beta)^{-1}\beta'V_e^{-1}a \\
&= a'V_e{}^{-1}a + o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times o(N^{\frac{1}{2}}) \\
&= a'V_e^{-1}a + o(1).
\end{aligned}
$$

These three results imply that

$$
- a'V_R^{-1}(R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2}a'V_R^{-1}a = -a'V_e^{-1}e_{t+1} - \frac{1}{2}a'V_e^{-1}a + o_p(1),
$$

and therefore, by Slutzky's theorem,

$$
\hat{M}^a_{\exp,t+1} - M^a_{\exp,t+1} \xrightarrow{p} 0 \qquad \text{as} \quad N \to \infty.
$$

Next, we analyze the exponent of $\hat{M}^\beta_{\exp,t+1}$:

$$
\begin{aligned}
&- (\beta\lambda)'V_R^{-1}(R_{t+1} - \mathbb{E}(R_{t+1})) - \frac{1}{2}(\beta\lambda)'V_R^{-1}\beta\lambda \\
&\quad = -(\beta\lambda)'V_R^{-1}\beta(f_{t+1} - \mathbb{E}(f_{t+1})) - (\beta\lambda)'V_R^{-1}e_{t+1} - \frac{1}{2}(\beta\lambda)'V_R^{-1}\beta\lambda.
\end{aligned}
$$

We apply the Sherman-Morrison-Woodbury formula to V_R^{-1}, use Assumptions IA.2.1 and Lemma IA.3.9 to obtain

$$
\begin{aligned}
\beta'V_R^{-1}\beta &= \beta'V_e^{-1}\beta - \beta'V_e^{-1}\beta(V_f^{-1} + \beta'V_e^{-1}\beta)^{-1}\beta'V_e^{-1}\beta \\
&= V_f^{-1} + o(1),
\end{aligned}
$$

$$
\begin{aligned}
\beta'V_R^{-1}e_{t+1} &= \beta'V_e^{-1}e_{t+1} - \beta'V_e^{-1}\beta(V_f^{-1} + \beta'V_e^{-1}\beta)^{-1}\beta'V_e^{-1}e_{t+1} \\
&= O_p(N^{-\frac{1}{2}}) + O(1) \times [O(1) + O(N)]^{-1} \times O_p(N^{\frac{1}{2}}) \\
&= O_p(N^{-\frac{1}{2}}).
\end{aligned}
$$

These two results imply that

$$
\begin{aligned}
&- (\beta\lambda)'V_R^{-1}(R_{t+1} - \mathbb{E}(R_{t+1})) - \frac{1}{2}(\beta\lambda)'V_R^{-1}\beta\lambda \\
&\quad = -\lambda'V_f^{-1}(f_{t+1} - \mathbb{E}(f_{t+1})) - \frac{1}{2}\lambda'V_f^{-1}\lambda + o_p(1)
\end{aligned}
$$

and therefore

$$\hat{M}^{\beta}_{\text{exp},t+1} - M^{\beta}_{\text{exp},t+1} \xrightarrow{p} 0 \quad \text{as} \quad N \to \infty.$$

Independence of f_t and e_t implies that the covariance between $M^{\beta}_{\text{exp},t+1}$ and $M^{a}_{\text{exp},t+1}$ is zero. The same remains true asymptotically for the projected versions, $\hat{M}^{a}_{\text{exp},t+1}$ and $\hat{M}^{\beta}_{\text{exp},t+1}$, thanks to the asymptotic-in-N equivalences proven above. □

Proof of an extended version of Proposition 2

The following proposition extends Proposition 2 to the case in which we correct a misspecified candidate factor model to obtain an admissible SDF under the assumptions of the APT. The correction includes the systematic risk factors missing in the candidate factor model and compensation for unsystematic risk. This case is described in Section 2.3 of the manuscript.

Proposition IA.4.1 (Feasible Admissible SDF Constructed by Correcting a Candidate Factor Model)**.** *Under Assumptions 2.1 and 2.2 of the APT, the assumptions of Proposition 3, and the assumption that the factors f^{can}_{t+1} and f^{mis}_{t+1} and unsystematic shocks e_{t+1} are jointly Gaussian, the SDF*

$$M_{\text{exp},t+1} = M^{a}_{\text{exp},t+1} \times M^{\beta,\text{can}}_{\text{exp},t+1} \times M^{\beta,\text{mis}}_{\text{exp},t+1} \quad \textit{with} \tag{IA1}$$

$$M^{a}_{\text{exp},t+1} = \exp\left(-a'V_e^{-1}e_{t+1} - \frac{1}{2}a'V_e^{-1}a\right),$$

$$M^{\beta,\text{can}}_{\text{exp},t+1} = \frac{1}{R_f} \times \exp\left(-\lambda^{\text{can}\prime}V^{-1}_{f^{\text{can}}}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})) - \frac{1}{2}\lambda^{\text{can}\prime}V^{-1}_{f^{\text{can}}}\lambda^{\text{can}}\right),$$

$$M^{\beta,\text{mis}}_{\text{exp},t+1} = \exp\left(-\lambda^{\text{mis}\prime}V^{-1}_{f^{\text{mis}}}(f^{\text{mis}}_{t+1} - \mathbb{E}(f^{\text{mis}}_{t+1})) - \frac{1}{2}\lambda^{\text{mis}\prime}V^{-1}_{f^{\text{mis}}}\lambda^{\text{mis}})\right),$$

is admissible. Furthermore, under Assumptions IA.2.1, IA.2.2, IA.2.3, and IA.2.4, as $N \to \infty$, the following results hold

$$\hat{M}^{a}_{\text{exp},t+1} - M^{a}_{\text{exp},t+1} \xrightarrow{p} 0, \quad \hat{M}^{\beta,\text{mis}}_{\text{exp},t+1} - M^{\beta,\text{mis}}_{\text{exp},t+1} \xrightarrow{p} 0, \quad \text{cov}(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1}, \hat{M}^{a}_{\text{exp},t+1}) \to 0,$$

where

$$\hat{M}^{a}_{\text{exp},t+1} = \exp\Big(-a'V_R^{-1}(R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2}a'V_R^{-1}a\Big),$$

$$\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1} = \exp\Big(-(\beta^{\text{mis}}\lambda^{\text{mis}})'V_R^{-1}(R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2}(\beta^{\text{mis}}\lambda^{\text{mis}})'V_R^{-1}\beta^{\text{mis}}\lambda^{\text{mis}}\Big),$$

implying that

$$\hat{M}_{\text{exp},t+1} = M^{\beta,\text{can}}_{\text{exp},t+1} \times \hat{M}^{\beta,\text{mis}}_{\text{exp},t+1} \times \hat{M}^{a}_{\text{exp},t+1} \tag{IA2}$$

is a feasible positive admissible SDF, when $N \to \infty$.

Proof: First, we prove that the exponential SDF specified in equation (IA1) is admissible. To this end, we use a guess-and-verify method. We guess that the admissible exponential SDF is

$$M_{\exp,t+1} = \exp\left[\mu_+ + b_+^{\mathrm{can}\prime}(f_{t+1}^{\mathrm{can}} - \mathbb{E}(f_{t+1}^{\mathrm{can}})) + b_+^{\mathrm{mis}\prime}(f_{t+1}^{\mathrm{mis}} - \mathbb{E}(f_{t+1}^{\mathrm{mis}})) + c_+' e_{t+1}\right] \quad \text{(IA3)}$$

with unknown vectors b_+^{can}, b_+^{mis}, and c_+, as well as an unknown scalar μ_+.

To identify the unknowns and verify our guess we use the following $K^{\mathrm{can}} + K^{\mathrm{mis}} + N + 1$ equations, which are the implications of the Law of One Price:

$$\begin{aligned} -\mathrm{cov}(M_{\exp,t+1}, f_{t+1}^{\mathrm{can}}) \times R_f &= \lambda^{\mathrm{can}}, \\ -\mathrm{cov}(M_{\exp,t+1}, f_{t+1}^{\mathrm{mis}}) \times R_f &= \lambda^{\mathrm{mis}}, \\ \mathbb{E}(M_{\exp,t+1}(R_{t+1} - R_f 1_N)) &= 0_N, \\ \mathbb{E}(M_{\exp,t+1}) &= \frac{1}{R_f}. \end{aligned}$$

The first K^{can} equations imply that

$$-\mathbb{E}(M_{\exp,t+1}(f_{t+1}^{\mathrm{can}} - \mathbb{E}(f_{t+1}^{\mathrm{can}}))) = \mathbb{E}(M_{\exp,t+1}) \times \lambda^{\mathrm{can}},$$

which, along with Lemma IA.3.1, gives:

$$b_+^{\mathrm{can}} = -V_{f^{\mathrm{can}}}^{-1} \lambda^{\mathrm{can}}.$$

Similarly, the next K^{mis} equations imply that

$$-\mathbb{E}(M_{\exp,t+1}(f_{t+1}^{\mathrm{mis}} - \mathbb{E}(f_{t+1}^{\mathrm{mis}}))) = \mathbb{E}(M_{\exp,t+1}) \times \lambda^{\mathrm{mis}},$$

which, along with Lemma IA.3.1, leads to:

$$b_+^{\mathrm{mis}} = -V_{f^{\mathrm{mis}}}^{-1} \lambda^{\mathrm{mis}}.$$

Then, we use the next N equations and Lemma IA.3.1 to obtain

$$\begin{aligned} 0_N &= \mathbb{E}(M_{\exp,t+1}(R_{t+1} - R_f 1_N)) \\ &= \mathbb{E}(M_{\exp,t+1}(a + \beta^{\mathrm{mis}}\lambda^{\mathrm{mis}} + \beta^{\mathrm{can}}\lambda^{\mathrm{can}} + \beta^{\mathrm{can}}(f_{t+1}^{\mathrm{can}} - \mathbb{E}(f_{t+1}^{\mathrm{can}})) \\ &\quad + \beta^{\mathrm{mis}}(f_{t+1}^{\mathrm{mis}} - \mathbb{E}(f_{t+1}^{\mathrm{mis}})) + e_{t+1})) \\ &= (a + \beta^{\mathrm{mis}}\lambda^{\mathrm{mis}} + \beta^{\mathrm{can}}\lambda^{\mathrm{can}})\mathbb{E}(M_{\exp,t+1}) + \mathbb{E}(M_{\exp,t+1}e_{t+1}) \\ &\quad + \mathbb{E}(M_{\exp,t+1}\beta^{\mathrm{can}}(f_{t+1}^{\mathrm{can}} - \mathbb{E}(f_{t+1}^{\mathrm{can}}))) + \mathbb{E}(M_{\exp,t+1}\beta^{\mathrm{mis}}(f_{t+1}^{\mathrm{mis}} - \mathbb{E}(f_{t+1}^{\mathrm{mis}}))) \\ &= (a + \beta^{\mathrm{mis}}\lambda^{\mathrm{mis}} + \beta^{\mathrm{can}}\lambda^{\mathrm{can}})\mathbb{E}(M_{\exp,t+1}) + V_e c_+ \mathbb{E}(M_{\exp,t+1}) \\ &\quad - \beta^{\mathrm{can}}\lambda^{\mathrm{can}}\mathbb{E}(M_{\exp,t+1}) - \beta^{\mathrm{mis}}\lambda^{\mathrm{mis}}\mathbb{E}(M_{\exp,t+1}) \\ &= (a + V_e c_+)\mathbb{E}(M_{\exp,t+1}). \end{aligned}$$

As a result,

$$c_+ = -V_e^{-1}a.$$

Finally, the last identifying condition implies

$$\begin{aligned}
R_f^{-1} &= \mathbb{E}(M_{\exp,t+1}) \\
&= \mathbb{E}(\exp[\mu_+ + b_+^{\text{can}\prime}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})) + b_+^{\text{mis}\prime}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) + c_+' e_{t+1}]) \\
&= \exp\left[\mu_+ + b_+^{\text{can}\prime} V_{f^{\text{can}}} b_+^{\text{can}}/2 + b_+^{\text{mis}\prime} V_{f^{\text{mis}}} b_+^{\text{mis}}/2 + c_+' V_e c_+/2\right].
\end{aligned}$$

Thus,

$$\exp(\mu_+) = R_f^{-1} \times \exp\big[-\lambda^{\text{can}\prime} V_{f^{\text{can}}}^{-1} \lambda^{\text{can}}/2 - \lambda^{\text{mis}\prime} V_{f^{\text{mis}}}^{-1} \lambda^{\text{mis}}/2 - a' V_e^{-1} a/2\big].$$

We substitute b_+^{can}, b_+^{mis}, c_+, and μ_+ in equation (IA3) and obtain the exponential SDF given in equation (IA1). Thus, we have successfully verified our guess.

Next, we prove that, as $N \to \infty$, the feasible SDF given in equation (IA2) recovers the exponential SDF specified in equation (IA1). We start by analyzing the exponent of $M^a_{\exp,t+1}$. First, we note that

$$\begin{aligned}
-a'V_R^{-1}(R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2}a'V_R^{-1}a = &-a'V_R^{-1}\beta^{\text{can}}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})) \\
&- a'V_R^{-1}\beta^{\text{mis}}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) \\
&- a'V_R^{-1}e_{t+1} \\
&- \frac{1}{2}a'V_R^{-1}a.
\end{aligned}$$

We analyze the four right-hand-side terms one-by-one. We apply the Sherman-Morrison-Woodbury formula to V_R^{-1} and V_ε^{-1} and use Lemmas IA.3.3, IA.3.5, IA.3.6, IA.3.7, IA.3.8, IA.3.9, and the proof of Lemma IA.3.10 to obtain

$$\begin{aligned}
a'V_R^{-1}\beta^{\text{can}} &= a'V_\varepsilon{}^{-1}\beta^{\text{can}} - a'V_\varepsilon{}^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}} \\
&= a'V_\varepsilon^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}})^{-1}V_{f^{\text{can}}}^{-1} \\
&= o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times O(1) \\
&= o(N^{-1/2}), \\
a'V_R^{-1}\beta^{\text{mis}} &= a'V_\varepsilon{}^{-1}\beta^{\text{mis}} - a'V_\varepsilon{}^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{mis}} \\
&= o(N^{-\frac{1}{2}}) + o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times O(1) = \\
&= o(N^{-1/2}), \\
a'V_R^{-1}e_{t+1} &= a'V_\varepsilon{}^{-1}e_{t+1} - a'V_\varepsilon{}^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon{}^{-1}e_{t+1} \\
&= a'V_\varepsilon{}^{-1}e_{t+1} + o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times O_p(N^{\frac{1}{2}})
\end{aligned}$$

$$
\begin{aligned}
&= a'V_\varepsilon{}^{-1}e_{t+1} + o_p(1) \\
&= a'V_e^{-1}e_{t+1} - a'V_e^{-1}\beta^{\text{mis}}(V_{f^{\text{mis}}}^{-1} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}e_{t+1} + o_p(1) \\
&= a'V_e^{-1}e_{t+1} + o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times O_p(N^{\frac{1}{2}}) + o_p(1) \\
&= a'V_e^{-1}e_{t+1} + o_p(1), \\
a'V_R^{-1}a &= a'V_\varepsilon{}^{-1}a - a'V_\varepsilon{}^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon{}^{-1}a \\
&= a'V_e{}^{-1}a + o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times o(N^{\frac{1}{2}}) + o(1) \\
&= a'V_e^{-1}a + o(1).
\end{aligned}
$$

These four results imply that

$$
-a'V_R^{-1}(R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2}a'V_R^{-1}a = -a'V_e^{-1}e_{t+1} - \frac{1}{2}a'V_e^{-1}a + o_p(1),
$$

and therefore we obtain

$$
\hat{M}^a_{\exp,t+1} - M^a_{\exp,t+1} \xrightarrow{p} 0 \qquad \text{as} \quad N \to \infty.
$$

Next, we analyze the exponent of $\hat{M}^{\beta,\text{mis}}_{\exp,t+1}$:

$$
\begin{aligned}
&-(\beta^{\text{mis}}\lambda^{\text{mis}})'V_R^{-1}(R_{t+1} - \mathbb{E}(R_{t+1})) - \frac{1}{2}(\beta^{\text{mis}}\lambda^{\text{mis}})'V_R^{-1}\beta^{\text{mis}}\lambda^{\text{mis}} \\
&\quad = -(\beta^{\text{mis}}\lambda^{\text{mis}})'V_R^{-1}\beta^{\text{can}}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})) \\
&\qquad - (\beta^{\text{mis}}\lambda^{\text{mis}})'V_R^{-1}\beta^{\text{mis}}(f^{\text{mis}}_{t+1} - \mathbb{E}(f^{\text{mis}}_{t+1})) \\
&\qquad - (\beta^{\text{mis}}\lambda^{\text{mis}})'V_R^{-1}e_{t+1} \\
&\qquad - \frac{1}{2}(\beta^{\text{mis}}\lambda^{\text{mis}})'V_R^{-1}\beta^{\text{mis}}\lambda^{\text{mis}}.
\end{aligned}
$$

We apply the Sherman-Morrison-Woodbury formula and Lemmas IA.3.3, IA.3.4, IA.3.5, IA.3.9, and IA.3.10 to the first three terms above:

$$
\begin{aligned}
\beta^{\text{mis}\prime}V_R^{-1}\beta^{\text{can}} &= \beta^{\text{mis}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}} - \beta^{\text{mis}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}} \\
&= \beta^{\text{mis}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}})^{-1}V_{f^{\text{can}}}^{-1} \\
&= O(1) \times [O(1) + O(N)]^{-1} \times O(1) \\
&= O(N^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
\beta^{\text{mis}\prime}V_R^{-1}\beta^{\text{mis}} &= \beta^{\text{mis}\prime}V_\varepsilon{}^{-1}\beta^{\text{mis}} - \beta^{\text{mis}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{mis}} \\
&= (V_{f^{\text{mis}}}^{-1} + o(1)) + O(1) \times [O(1) + O(N)]^{-1} \times O(1) \\
&= V_{f^{\text{mis}}}^{-1} + o(1), \quad \text{and}
\end{aligned}
$$

$$
\beta^{\text{mis}\prime}V_R^{-1}e_{t+1} = \beta^{\text{mis}\prime}V_\varepsilon{}^{-1}e_{t+1} - \beta^{\text{mis}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon{}^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon{}^{-1}e_{t+1}
$$

$$= O_p(N^{-\frac{1}{2}}) + O(1) \times [O(1) + O(N)]^{-1} \times O_p(N^{\frac{1}{2}})$$
$$= O_p(N^{-\frac{1}{2}}).$$

These three results imply that

$$\begin{aligned}
&- (\beta^{\text{mis}}\lambda^{\text{mis}})' V_R^{-1}(R_{t+1} - \mathbb{E}(R_{t+1})) - \frac{1}{2}(\beta^{\text{mis}}\lambda^{\text{mis}})' V_R^{-1}\beta^{\text{mis}}\lambda^{\text{mis}} \\
&\quad = -\lambda^{\text{mis}\prime} V_{f^{\text{mis}}}^{-1}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) - \frac{1}{2}\lambda^{\text{mis}\prime} V_{f^{\text{mis}}}^{-1}\lambda^{\text{mis}} + o_p(1),
\end{aligned}$$

and therefore we obtain

$$\hat{M}_{\text{exp},t+1}^{\beta,\text{mis}} - M_{\text{exp},t+1}^{\beta,\text{mis}} \xrightarrow{p} 0 \quad \text{as} \quad N \to \infty.$$

Pairwise uncorrelatedness (and independence by Gaussianity) of $f_{t+1}^{\text{can}}, f_{t+1}^{\text{mis}}$, and e_{t+1} implies that the pairwise covariances between $M_{\text{exp},t+1}^{\beta,\text{can}}, M_{\text{exp},t+1}^{\beta,\text{mis}}$, and $M_{\text{exp},t+1}^{a}$ are all zero. The same remains true asymptotically for the projected versions, $\hat{M}_{\text{exp},t+1}^{a}$ and $\hat{M}_{\text{exp},t+1}^{\beta,\text{mis}}$, thanks to the asymptotic-in-N equivalences proven above. □

Proof of Proposition 3

By Chamberlain and Rothschild (1983, Theorem 4 and Corollary 2), the equations in expressions (12) hold, where $V_e > 0$ with all its eigenvalues being uniformly (i.e., as $N \to \infty$) bounded by the $(K^{\text{mis}} + 1)-$th eigenvalue of V_ε.

Finally, we need to show that $\alpha' V_\varepsilon^{-1}\alpha = O(1)$ and that the restrictions $\alpha' V_\varepsilon^{-1}\alpha \leq \tilde{\delta}_{\text{apt}}$ is asymptotically equivalent to $a' V_e^{-1} a \leq \delta_{\text{apt}}$, in the sense that one implies the other and vice versa, under Assumptions 2.1, 2.2, and IA.2.3.

We use the definition of α to express $\alpha' V_\varepsilon^{-1}\alpha$ as

$$\begin{aligned}
\alpha' V_\varepsilon^{-1}\alpha &= (a + \beta^{\text{mis}}\lambda^{\text{mis}})' V_\varepsilon^{-1}(a + \beta^{\text{mis}}\lambda^{\text{mis}}) \\
&= a' V_\varepsilon^{-1} a + \lambda^{\text{mis}\prime}\beta^{\text{mis}\prime} V_\varepsilon^{-1}\beta^{\text{mis}}\lambda^{\text{mis}} + 2a' V_\varepsilon^{-1}\beta^{\text{mis}}\lambda^{\text{mis}}.
\end{aligned}$$

Then, from Lemmas IA.3.4, IA.3.6, and IA.3.8, we obtain

$$\alpha' V_\varepsilon^{-1}\alpha = a' V_\varepsilon^{-1} a + \lambda^{\text{mis}\prime} V_{f^{\text{mis}}}^{-1}\lambda^{\text{mis}} + o(1).$$

Therefore, $\alpha' V_\varepsilon^{-1}\alpha = O(1)$ whenever $a' V_\varepsilon^{-1} a = O(1)$ and vice versa.

Remark: Proposition 3 assumes the presence of at least one omitted systematic risk factor, that is, $K^{\text{mis}} > 0$. If instead $K^{\text{mis}} = 0$, that is, all eigenvalues of V_ε are bounded, then the data-generating process of asset returns with K^{can} factors given in expression (11) satisfies the assumptions of the classical APT provided in Section 2.1.

Proof of Proposition 4

We use a guess-and-verify method to derive the SDF. We guess that the SDF has the following functional form

$$M_{t+1} = \mathbb{E}(M_{t+1}) + b^{\text{can}\prime}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})) + b^{\text{mis}\prime}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) + c' e_{t+1},$$

where b^{can} is a $K^{\text{can}} \times 1$ vector , b^{mis} is a $K^{\text{mis}} \times 1$ vector, and c is an $N \times 1$ vector. We identify the unknown vectors b^{can}, b^{mis}, and c by using the Law of One Price. Specifically, because we assume the existence of the risk-free asset, to determine the mean of the SDF we use the following condition:

$$\mathbb{E}(M_{t+1}) = \frac{1}{R_f}.$$

Next, because λ^{can} represents a vector of prices of risk of f_{t+1}^{can}, we have that

$$-\operatorname{cov}(M_{t+1}, f_{t+1}^{\text{can}}) \times R_f = \lambda^{\text{can}}.$$

These K^{can} conditions identify b^{can}:

$$b^{\text{can}} = -\frac{V_{f^{\text{can}}}^{-1} \lambda^{\text{can}}}{R_f}.$$

Similarly, λ^{mis} is the price of risk associated with factors f_{t+1}^{mis}, or equivalently,

$$-\operatorname{cov}(M_{t+1}, f_{t+1}^{\text{mis}}) \times R_f = \lambda^{\text{mis}}.$$

These K^{mis} conditions identify b^{mis}:

$$b^{\text{mis}} = -\frac{V_{f^{\text{mis}}}^{-1} \lambda^{\text{mis}}}{R_f}.$$

Finally, it must be the case that the SDF prices the N assets

$$\mathbb{E}\big(M_{t+1}(R_{t+1} - R_f 1_N)\big) = 0_N.$$

These N equations identify c:

$$c = -\frac{V_e^{-1} a}{R_f}.$$

Taken together

$$M_{t+1} = M_{t+1}^{\beta,\text{can}} + M_{t+1}^{\beta,\text{mis}} + M_{t+1}^{a},$$

where

$$M_{t+1}^{\beta,\text{can}} = \frac{1}{R_f} - \frac{\lambda^{\text{can}\prime} V_{f^{\text{can}}}^{-1}}{R_f}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})),$$

$$M_{t+1}^{\beta,\text{mis}} = -\frac{\lambda^{\text{mis}\,\prime} V_{f^{\text{mis}}}^{-1}}{R_f}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})),$$
$$M_{t+1}^{a} = -\frac{a' V_e^{-1}}{R_f} e_{t+1}.$$

Pairwise uncorrelatedness of $f_{t+1}^{\text{can}}, f_{t+1}^{\text{mis}}$, and e_{t+1} implies that the pairwise covariances between $M_{t+1}^{\beta,\text{can}}, M_{t+1}^{\beta,\text{mis}}$, and M_{t+1}^{a} are all zero. □

Proof of Proposition 5

Observe that the exposures of asset returns to the unsystematic SDF component M_{t+1}^{a} are equal to

$$\beta^a = \frac{\text{cov}(M_{t+1}^a, R_{t+1} - R_f 1_N)}{\text{var}(M_{t+1}^a)} = \frac{\text{cov}\left(-\frac{a'V_e^{-1}}{R_f} e_{t+1}, R_{t+1} - R_f 1_N\right)}{\text{var}(M_{t+1}^a)} = -\frac{a' R_f}{a' V_e^{-1} a}.$$

Thus, $\beta^{a\prime}\beta^a = R_f^2 (a'a)/(a'V_e^{-1}a)^2$, which together with the no-arbitrage restriction (5), the boundness of δ_{apt} away from zero, and the boundedness of the eigenvalues of the covariance matrix V_e, implies that $\beta^{a\prime}\beta^a = O(1)$, that is, $\beta^{a\prime}\beta^a$ is bounded. As a result, M_{t+1}^a satisfies the definition of a weak factor in the cross-section of returns on basis assets. □

Proof of Proposition 6

Below we summarize the main assumptions of the model in Merton (1987) and then analyze its equilibrium implications for the SDF and expected excess returns. For details of the model, we refer the reader to Merton (1987).

Assume that there are N firms in the economy whose end-of-period cash flows are:

$$C_i = I_i \left[\mu_i + \eta_i Y + s_i \epsilon_i\right],$$

where, for simplicity, it is assumed that there is a single random systematic factor Y with $E(Y) = 0$ and $E(Y^2) = 1$, with $E(\varepsilon_i) = E\left(\varepsilon_i \mid \varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_N, Y\right) = 0$, for $i = \{1, \ldots, N\}$, where ε_i are asset-specific shocks.[36] Here, I_i is the amount of physical investment in firm i and μ_i, η_i, and s_i represent parameters of firm i 's production technology.

Let V_i denote the equilibrium value of firm i at the beginning of the period. If R_i is the equilibrium return per dollar from investing in firm i over the period, then $R_i = C_i/V_i$, and

$$R_i = \mathbb{E}(R_i) + b_i Y + \sigma_i \epsilon_i, \tag{IA4}$$

[36] We have made the following changes to the notation used in Merton (1987) so that it is consistent with the notation in our paper. We denote an investor's risk aversion by γ instead of δ; we denote the total number of assets by N instead of n; we index individual assets by i instead of k; and we denote the unsystematic risk premium by a_i instead of λ_k.

where b_i and σ_i are functions of the parameters of firm i's production technology.

There are two additional securities in the economy, both in zero net supply: a security that is risk-free with return R_f and the $(N+1)$-th risky security, which combines the risk-free security and a forward contract with cash settlements on the factor Y. Without loss of generality, the forward price of the contract is assumed to be such that the standard deviation of the equilibrium returns on the security is unity. As a result, its return is

$$R_{N+1} = \mathbb{E}(R_{N+1}) + Y. \tag{IA5}$$

There is a sufficiently large number of investors with a sufficiently dispersed distribution of wealth so that each investor acts as a price taker. Each investor is risk averse and has mean-variance preferences over the end-of-period wealth:

$$U^j = E(R^j W^j) - \frac{\gamma^j}{2W^j}\,\mathrm{var}(R^j W^j),$$

where W^j denotes the value of the initial endowment of investor j evaluated at equilibrium prices, R^j denotes the return per dollar on investor j's optimal portfolio, and $\gamma^j > 0$ is the risk-aversion of investor j.

Investors differ in their information sets. The common part of investors' information sets includes: (i) the return on the risk-free security, (ii) the structure of securities' return given in expression (IA4), and (iii) the expected return and variance of the forward-contract security given in (IA5). However, different investors have knowledge about the parameters b_i and σ_i for different *subsets* of securities. The investors who know about security i agree on its characteristics. To simplify the analysis, investors are assumed to have identical risk aversion $\gamma^j = \gamma$ and identical initial wealth $W^j = W$.

The optimal solution to each investor's portfolio problem allows us to obtain the aggregate demand for every security. Equating this to the aggregate supply for every security leads to the equilibrium expected return for asset i (Merton, 1987, eq. (16)):

$$\mathbb{E}(R_i) = R_f + \gamma b_i b + \gamma x_i \sigma_i^2 / q_i, \quad \text{for} \quad i = \{1, \dots, N\}, \tag{IA6}$$

where x_i is the fraction of the market portfolio invested in asset i,

$$b = \sum_{i=1}^{N} x_i b_i,$$

and q_i is the fraction of investors who know about security i.

Denoting the return on the market as $R_m = \sum_{i=1}^{N} x_i R_i$, Merton (1987, eq. (24)) obtains the equilibrium expected excess return on the market:

$$\mathbb{E}(R_m) - R_f = \gamma\,\mathrm{var}(R_m) + a_m, \tag{IA7}$$

where $a_m = \sum_{i=1}^{N} x_i a_i$,

$$a_i = (1 - q_i)\Delta_i,$$
$$\Delta_i = \mathbb{E}(R_i) - R_f - b_i(\mathbb{E}(R_{N+1}) - R_f).$$

Equations (IA4) and (IA7) then imply

$$R_i - R_f = \beta_i(\mathbb{E}(R_m) - R_f) + a_i - \beta_i a_m + b_i Y + \sigma_i \epsilon_i, \tag{IA8}$$

where β_i denotes the covariance of the return on security i with the return on the market portfolio, divided by the variance of the market return. Equation (IA8) contains Y on the right-hand side. We substitute out Y by using the definition of the market portfolio return along with equations (IA4) and (IA6), to obtain

$$R_i - R_f = a_i - \beta_i a_m + \beta_i(\mathbb{E}(R_m) - R_f) + \frac{b_i}{b}(R_m - \mathbb{E}(R_m)) + \sigma_i \epsilon_i.$$

The equilibrium process for asset returns, given by equations (2) and (25) in Merton (1987), is

$$R_i - R_f = \beta_i(\mathbb{E}(R_m) - R_f) + a_i - \beta_i a_m + b_i Y + \sigma_i \varepsilon_i. \tag{IA9}$$

We posit that the SDF M has the following form,

$$M = \xi + \chi Y + \sum_{i=1}^{N} \zeta_i \varepsilon_i,$$

where ξ, χ, and ζ_i, $i = \{1, \ldots, N\}$, are determined using the $N + 2$ equations for the Law of One Price:

$$\mathbb{E}[M] = \frac{1}{R_f}, \tag{IA10}$$
$$\mathbb{E}[M(R_{N+1} - R_f)] = 0 \tag{IA11}$$
$$\mathbb{E}[M(R_i - R_f)] = 0, \quad \text{for} \quad i = \{1, \ldots, N\}, \tag{IA12}$$

where, from (3) and (11) in Merton (1987),

$$R_{N+1} = R_f + \gamma b + Y.$$

From expression (IA10), we get

$$\xi = \frac{1}{R_f}.$$

From expression (IA11), we get

$$\chi = -\frac{\gamma b}{R_f}.$$

From expression (IA12), for each $i = \{1, \ldots, N\}$ we have

$$\xi\beta_i(\mathbb{E}(R_m) - R_f) + \xi(a_i - \beta_i a_m) + \chi\beta_i + \zeta\sigma_i = 0.$$

As a result,

$$\zeta = -\frac{1}{R_f}\frac{\beta_i(\mathbb{E}(R_m) - R_f) + a_i - \beta_i a_m - b_i\gamma b}{\sigma_i}.$$

Recalling that

$$R_m = \sum_{i=1}^{N} x_i R_i$$

and using (2) and (16) from Merton (1987), we have

$$\begin{aligned} R_m - R_f &= \sum_{i=1}^{N} x_i(\gamma b_i b + \gamma x_i \sigma_i^2/q_i) + \sum_{i=1}^{N} x_i b_i Y + \sum_{i=1}^{N} x_i \sigma_i \varepsilon_i \\ &= \gamma b^2 + \gamma \sum_{i=1}^{N} x_i^2 \sigma_i^2/q_i + bY + \sum_{i=1}^{N} x_i \sigma_i \varepsilon_i. \end{aligned}$$

From the last expression, we obtain

$$bY = (R_m - R_f) - \gamma b^2 - \gamma \sum_{i=1}^{N} x_i^2 \sigma_i^2/q_i - \sum_{i=1}^{N} x_i \sigma_i \varepsilon_i.$$

As a result, the SDF is

$$\begin{aligned} M = \frac{1}{R_f} - \frac{\gamma}{R_f}\left((R_m - R_f) - b^2\gamma - \gamma \sum_{i=1}^{N} x_i^2 \sigma_i^2/q_i - \sum_{i=1}^{N} x_i \sigma_i \varepsilon_i\right) \\ - \frac{1}{R_f}\sum_{t=1}^{N} \frac{\beta_i(\mathbb{E}(R_m) - R_f) + a_i - \beta_i a_m - b_i\gamma b}{\sigma_i}\varepsilon_i. \end{aligned}$$

Grouping together similar terms, we obtain

$$\begin{aligned} M = \frac{1}{R_f} + \frac{\gamma^2 b^2}{R_f} + \frac{\gamma^2 \sum_{i=1}^{N} x_i^2 \sigma_i^2/q_i}{R_f} - \frac{\gamma}{R_f}(R_m - R_f) \\ - \frac{1}{R_f}\sum_{i=1}^{N}\left(\frac{\beta_i(\mathbb{E}(R_m) - R_f) + a_i - \beta_i a_m - b_i\gamma b - \gamma x_i \sigma_i^2}{\sigma_i}\varepsilon_i\right). \end{aligned}$$

Finally, we use expressions (22) and (24) in Merton (1987) to simplify the loading of M on ε_i and obtain

$$-\frac{1}{R_f}\sum_{i=1}^{N}\left(\frac{\beta_i(\mathbb{E}(R_m) - R_f) + a_i - \beta_i a_m - b_i\gamma b - \gamma x_i \sigma_i^2}{\sigma_i}\right) = -\frac{1}{R_f}\sum_{i=1}^{N}\frac{a_i}{\sigma_i}.$$

Using the demeaned return on the market portfolio as a factor in the SDF, along with expressions (15), (19), and (24) in Merton (1987), we obtain

$$\begin{aligned} M &= -\frac{1}{R_f}\sum_{i=1}^{N}\left(\frac{a_i}{\sigma_i}\varepsilon_i\right) + \frac{1}{R_f} - \frac{(\mathbb{E}(R_m) - R_f)}{R_f \operatorname{var}(R_m)}(R_m - \mathbb{E}(R_m)) \\ &= \underbrace{-\frac{a' V_e^{-1}}{R_f} e_{t+1}}_{M^a} + \underbrace{\frac{1}{R_f} - \frac{(\mathbb{E}(R_m) - R_f)}{R_f \operatorname{var}(R_m)}(R_m - \mathbb{E}(R_m))}_{M^\beta}, \end{aligned} \tag{IA13}$$

where $e_i = \sigma_i \epsilon_i$ and V_e is the covariance matrix of e with σ_i^2 on its diagonal.

To characterize the limiting behavior of this economy, as $N \to \infty$, assume that $x_i \to 0$, that is the fraction of market portfolio invested in each asset i is infinitesimally small. Then, as $N \to \infty$, we have

$$\begin{aligned} \beta_i &= \frac{b_i b + x_i \sigma_i^2}{b^2 + \sum_{i=1}^{N} x_i^2 \sigma_i^2} \to \frac{b_i}{b^*}, \quad \text{where} \quad b \to b^*, \\ a_m &= \sum_{i=1}^{N} x_i a_i = \sum_{i=1}^{N} x_i (1 - q_i)\Delta_i = \sum_{i=1}^{N} \gamma x_i^2 \sigma_i^2 \frac{(1 - q_i)}{q_i} \to 0, \\ \operatorname{cov}\left(\sum_{i=1}^{N} x_i \sigma_i \varepsilon_i, \varepsilon_i\right) &= \sum_{i=1}^{N} x_i \sigma_i \to 0. \end{aligned}$$

Thus, given $N \to \infty$, we have: (i) $\beta_i \to b_i/b^*$, (ii) $a_m \to 0$, and (iii) the market return is asymptotically orthogonal to all unsystematic shocks, e_i. Making these substitutions in equations (IA9) and (IA13) gives the results in (21) and (23). □

Spanning of the SDF Components

Proposition IA.4.1 implies that, as $N \to \infty$, the log of the estimated SDF component $\log(\hat{M}^{\beta,\text{mis}}_{\exp,t+1})$ converges to a linear function of the missing systematic factors. Proposition IA.4.2 below shows how to determine whether a vector of observable variables g_t represents missing sources of systematic risk in the candidate factor model, and if so, how to estimate the prices of risk associated with these missing risk factors.

Let f_t^{mis} be the vector of true systematic risk factors that are missing in the candidate factor model. Consider the regression of $\log(\hat{M}^{\beta,\text{mis}}_{\exp,t})$ on an intercept and the vector g_t,

$$\log(\hat{M}^{\beta,\text{mis}}_{\exp,t}) = \gamma_0 + \gamma_1' g_t + u_t.$$

Denote by γ_1^{ols} the OLS-estimator of γ_1 and by R_g^2 the coefficient of determination in the corresponding regression.

Proposition IA.4.2 (Detecting Missing Systematic Factors)**.** *Under the assumptions of the extended Proposition IA.4.1 and if $g_t = Qf_t^{\text{mis}}$, for some nonsingular Q, as $N \to \infty$ we have*

$$\gamma_1^{\text{ols}} \xrightarrow{p} -(Q')^{-1}V_{f^{\text{mis}}}^{-1}\lambda^{\text{mis}} \quad \textit{and} \quad R_g^2 \xrightarrow{p} 1.$$

On the other hand, if g_t is orthogonal to f_t^{mis} then

$$\gamma_1^{\text{ols}} \xrightarrow{p} 0_{K^{\text{mis}}} \quad \textit{and} \quad R_g^2 \xrightarrow{p} 0.$$

Proof: Collect the values of the vector g_t for each t in a matrix $G = (g_1 \cdots g_T)'$. Likewise, collect the values of the vector f_t^{mis} for each t in a matrix $F^{\text{mis}} = (f_1^{\text{mis}} \cdots f_T^{\text{mis}})'$. For each t, collect the values of the systematic component $\log(\hat{M}_{\exp,t+1}^{\beta,\text{mis}})$ of the admissible SDF in a vector $\log(\hat{M}_{\exp}^{\beta,\text{mis}}) = (\log(\hat{M}_{\exp,1}^{\beta,\text{mis}}) \cdots \log(\hat{M}_{\exp,T}^{\beta,\text{mis}}))'$. Then, the R^2 of the regression of $\log(\hat{M}_{\exp,t}^{\beta,\text{mis}})$ on an intercept and the vector g_t,

$$\log(\hat{M}_{\exp,t}^{\beta,\text{mis}}) = \gamma_0 + \gamma_1' g_t + u_t,$$

is

$$R_g^2 = \frac{\gamma_1^{\text{ols}\prime} G'(I_T - 1_T 1_T'/T)G\gamma_1^{\text{ols}}}{\log(\hat{M}_{\exp}^{\beta,\text{mis}})'(I_T - 1_T 1_T'/T)\log(\hat{M}_{\exp}^{\beta,\text{mis}})},$$

where $\gamma_1^{\text{ols}} = (G'(I_T - 1_T 1_T'/T)G)^{-1}G'(I_T - 1_T 1_T'/T)\log(\hat{M}_{\exp}^{\beta,\text{mis}})$.

In Proposition IA.4.1, we have showen that

$$\log(\hat{M}_{\exp,t+1}^{\beta,\text{mis}}) \xrightarrow{p} -\lambda^{\text{mis}\prime}V_{f^{\text{mis}}}^{-1}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) - \frac{1}{2}\lambda^{\text{mis}\prime}V_{f^{\text{mis}}}^{-1}\lambda^{\text{mis}}.$$

For simplicity, we set $M_{1_T} = I_T - 1_T 1_T'/T$ and, given that $M_{1_T}1_T = 0_T$, we obtain

$$\begin{aligned}
\gamma_1^{\text{ols}} &\xrightarrow{p} -(G'M_{1_T}G)^{-1}G'M_{1_T}(F^{\text{mis}} - 1_T\mathbb{E}(f_{t+1}^{\text{mis}\prime}))V_{f^{\text{mis}}}^{-1}\lambda^{\text{mis}} \\
&= -(QF^{\text{mis}\prime}M_{1_T}F^{\text{mis}}Q')^{-1}QF^{\text{mis}\prime}M_{1_T}F^{\text{mis}}V_{f^{\text{mis}}}^{-1}\lambda^{\text{mis}} \\
&= -(Q')^{-1}V_{f^{\text{mis}}}^{-1}\lambda^{\text{mis}}.
\end{aligned}$$

The limiting behavior of the numerator of R_g^2 is as follows

$$\begin{aligned}
\gamma_1^{\text{ols}\prime}(G'M_{1_T}G)\gamma_1^{\text{ols}} &\xrightarrow{p} \lambda^{\text{mis}\prime}V_{f^{\text{mis}}}^{-1}Q^{-1}Q(F^{\text{mis}\prime}M_{1_T}F^{\text{mis}})Q'(Q')^{-1}V_{f^{\text{mis}}}^{-1}\lambda^{\text{mis}} \\
&= \lambda^{\text{mis}\prime}V_{f^{\text{mis}}}^{-1}(F^{\text{mis}\prime}M_{1_T}F^{\text{mis}})V_{f^{\text{mis}}}^{-1}\lambda^{\text{mis}}.
\end{aligned}$$

The limiting behavior of the denominator of R_g^2 is as follows

$$\log(\hat{M}_{\exp}^{\beta,\text{mis}})'(I_T - 1_T 1_T'/T)\log(\hat{M}_{\exp}^{\beta,\text{mis}})$$

$$\xrightarrow{p} \lambda^{\text{mis}\prime} V_{f^{\text{mis}}}^{-1} (F^{\text{mis}} - 1_T \mathbb{E}(f_{t+1}^{\text{mis}\prime}))' M_{1_T} (F^{\text{mis}} - 1_T \mathbb{E}(f_{t+1}^{\text{mis}\prime})) V_{f^{\text{mis}}}^{-1} \lambda^{\text{mis}}$$
$$= \lambda^{\text{mis}\prime} V_{f^{\text{mis}}}^{-1} (F^{\text{mis}\prime} M_{1_T} F^{\text{mis}}) V_{f^{\text{mis}}}^{-1} \lambda^{\text{mis}}.$$

Given that the limit of the numerator equals the limit of the denominator, $R_g^2 \xrightarrow{p} 1$.

The proof of the case of G being orthogonal to F^{mis}, that is, when $G'(I_T - 1_T 1_T'/T)F^{\text{mis}} = O_{K^{\text{mis}}}$, is straightforward, and therefore, omitted. $\square$

Along the same lines, if weak factors span the unsystematic component of the SDF $M_{\exp,t+1}^a$, Proposition IA.4.3 below shows how to determine whether a vector of observable variables h_t is a linear combination of these weak factors and if so how to estimate their prices of risk. Assume that $\log(M_{\exp,t+1}^a) = -a'V_e^{-1}e_t - a'V_e^{-1}a/2 = \gamma_{\text{weak}}' f_t^{\text{weak}}$, where f_t^{weak} is a vector of true latent K^{weak} weak factors with the identity covariance matrix $V_{f^{\text{weak}}} = I_{K^{\text{weak}}}$.

Proposition IA.4.3 (Detecting Missing Weak Factors)**.** *Consider the regression of* $\log(\hat{M}_{\exp,t}^a)$ *on an intercept and the vector* h_t*,*

$$\log(\hat{M}_{\exp,t}^a) = \gamma_0 + \gamma_1' h_t + u_t.$$

Denote by γ_1^{ols} *an OLS estimator of* γ_1 *and* R_h^2 *the coefficient of determination in the corresponding regression.*

Under the assumptions of Proposition IA.4.1 and if $h_t = Q f_t^{\text{weak}}$ *for some nonsingular* Q*, as* $N \to \infty$ *we have*

$$\hat{\gamma}_1 \xrightarrow{p} -(Q')^{-1}\gamma_{\text{weak}} \quad \textit{and} \quad R_h^2 \xrightarrow{p} 1.$$

On the other hand, if h_t *is orthogonal to* f_t^{weak} *then*

$$\hat{\gamma}_1 \xrightarrow{p} 0_{K^{\text{weak}}} \quad \textit{and} \quad R_h^2 \xrightarrow{p} 0.$$

Proof: Collect the values of the vector h_t for each t in a matrix $H = (h_1 \cdots h_T)'$. Likewise, collect the values of the vector f_t^{weak} for each t in a matrix $F^{\text{weak}} = (f_1^{\text{weak}} \cdots f_T^{\text{weak}})'$. For each t, collect the values of $\log(\hat{M}_{\exp,t}^a)$ of the admissible SDF in a vector $\log(\hat{M}_{\exp}^a) = (\log(\hat{M}_{\exp,1}^a) \cdots \log(\hat{M}_{\exp,T}^a))'$. Then, the R^2 of the regression of $\log(\hat{M}_{\exp,t}^a)$ on an intercept and the vector h_t,

$$\log(\hat{M}_{\exp,t}^a) = \gamma_0 + \gamma_1' h_t + u_t,$$

is given by

$$R_h^2 = \frac{\gamma_1^{\text{ols}\prime} H'(I_T - 1_T 1_T'/T) H \gamma_1^{\text{ols}}}{\log(\hat{M}_{\exp}^a)'(I_T - 1_T 1_T'/T)\log(\hat{M}_{\exp}^a)},$$

where $\gamma_1^{\text{ols}} = (H'(I_T - 1_T 1_T'/T)H)^{-1} H'(I_T - 1_T 1_T'/T) \log(\hat{M}^a_{\text{exp}})$.

In Proposition IA.4.1, we showed that, as $N \to \infty$,

$$\log(\hat{M}^a_{\text{exp},t+1}) - aV_e^{-1} e_{t+1} - \frac{1}{2} a' V_e^{-1} a \xrightarrow{p} 0.$$

For simplicity, we set $M_{1_T} = I_T - 1_T 1_T'/T$. Given that $M_{1_T} 1_T = 0_T$, we obtain

$$\begin{aligned}
\gamma_1^{\text{ols}} &\xrightarrow{p} -(H' M_{1_T} H)^{-1} H' M_{1_T} (F^{\text{weak}} - 1_T \mathbb{E}(f_{t+1}^{\text{weak}\,\prime}))\gamma_{\text{weak}} \\
&= -(QF^{\text{weak}\,\prime} M_{1_T} F^{\text{weak}} Q')^{-1} QF^{\text{weak}\,\prime} M_{1_T} F^{\text{weak}} \gamma_{\text{weak}} \\
&= -(Q')^{-1} \gamma_{\text{weak}}, \text{ when } N \to \infty.
\end{aligned}$$

The limiting behavior of the numerator of R_h^2 is as follows

$$\begin{aligned}
\gamma_1^{\text{ols}\,\prime} (H' M_{1_T} H) \gamma_1^{\text{ols}} &\xrightarrow{p} \gamma_{\text{weak}}' Q^{-1} Q (F^{\text{weak}\,\prime} M_{1_T} F^{\text{weak}}) Q' (Q')^{-1} \gamma_{\text{weak}} \\
&= \gamma_{\text{weak}}' (F^{\text{weak}\,\prime} M_{1_T} F^{\text{weak}}) \gamma_{\text{weak}}, \quad \text{when } N \to \infty.
\end{aligned}$$

The limiting behavior of the denominator of R_h^2 is as follows

$$\begin{aligned}
&\log(\hat{M}^a_{\text{exp}})' (I_T - 1_T 1_T'/T) \log(\hat{M}^a_{\text{exp}}) \\
&\quad \xrightarrow{p} \gamma_{\text{weak}}' (F^{\text{weak}} - 1_T \mathbb{E}(f_{t+1}^{\text{weak}\,\prime}))' M_{1_T} (F^{\text{weak}} - 1_T \mathbb{E}(f_{t+1}^{\text{weak}\,\prime})) \gamma_{\text{weak}} \\
&\quad = \gamma_{\text{weak}}' (F^{\text{weak}\,\prime} M_{1_T} F^{\text{mis}}) \gamma_{\text{weak}}, \quad \text{when } N \to \infty.
\end{aligned}$$

Given that the limit of the numerator equals the limit of the denominator, $R_h^2 \xrightarrow{p} 1$.

The proof when H is orthogonal to F^{weak}, that is, when $H'(I_T - 1_T 1_T'/T)F^{\text{weak}} = O_{K^{\text{weak}}}$, is straightforward, and therefore, omitted. □

A major strength of our approach is that we do not need to estimate the exposures of asset returns to an observable variable in order to define whether this variable represents a systematic or weak factor in the given cross-section of asset returns and quantify its price of risk. Thus, Propositions IA.4.2 and IA.4.3 complement the three-pass method of Giglio and Xiu (2021) and the supervised PCA of Giglio et al. (2021b) for describing the role of systematic and weak factors in pricing a cross-section of asset returns, and estimating the corresponding risk premia.[37]

IA.5 Weak Factors

Thanks to the assumption of the approximate factor structure of asset returns, our methodology accommodates weak factors in the unsystematic shocks e_{t+1}. The approximate factor

[37] The large-N results of Propositions IA.4.2 and IA.4.3 abstract from estimation uncertainty, unlike Giglio and Xiu (2021) and Giglio et al. (2021b), who allow for sampling variability by developing their analysis under both large N and large T.

structure implies that V_e can be non-diagonal, with the only constraint being that the maximum eigenvalue of V_e is uniformly bounded. Even though we have already mentioned that our theoretical results hold regardless of the presence of weak factors in e_{t+1}, we now prove Proposition IA.4.1 where we *explicitly* allow for weak factors in e_{t+1}. This proof strengthens the relevance of our methodology for identifying the importance of compensation for unsystematic risk, in particular, including that arising from weak factors.

Specifically, we assume now that

$$e_{t+1} = \beta^{\text{weak}} f_{t+1}^{\text{weak}} + e_{t+1}^{\text{as}}, \tag{IA14}$$

where e_{t+1}^{as} is a vector of asset-specific shocks with a diagonal covariance matrix $V_{e^{\text{as}}}$ that has the bounded maximum eigenvalue, f_{t+1}^{weak} is a vector of K^{weak} latent weak factors with the covariance matrix V_f^{weak}, and β^{weak} is the matrix of assets' exposures to the weak factors f_{t+1}^{weak}. We define weak factors in accordance with the definition of Lettau and Pelger (2020): f_{t+1}^{weak} is a weak factor, if the following condition holds $\beta^{\text{weak}\prime}\beta^{\text{weak}} \to A > 0$, if $N \to \infty$, where A is some constant matrix. This condition can be written as $\beta^{\text{weak}\prime}\beta^{\text{weak}} = O(1)$, in contrast to the condition for systematic factors $\beta^{\text{can}\prime}V_e^{-1}\beta^{\text{can}}/N = O(1)$, as indicated in Assumption IA.2.1. In practice, the condition $\beta^{\text{weak}\prime}\beta^{\text{weak}} = O(1)$ holds when either the factors f_{t+1}^{weak} affect only a subset of the asset returns or the factors f_{t+1}^{weak} affect all asset returns but only marginally.

Without loss of generality, we assume that the weak factors f_{t+1}^{weak} are orthogonal to the asset-specific shocks e_{t+1}^{as} and $V_{f^{\text{weak}}} = I_{K^{\text{weak}}}$, thus $V_e = \beta^{\text{weak}}\beta^{\text{weak}\prime} + V_{e^{\text{as}}}$.

Denote the vector of prices of unit assets' exposures to weak factors by λ^{weak}, thus compensation for the unsystematic risk includes compensation for exposure to weak factors and compensation for asset-specific shocks, $a = a^{\text{as}} + \beta^{\text{weak}}\lambda^{\text{weak}}$, such that the no-arbitrage constraint holds, that is, $a^{\text{as}\prime}V_{e^{\text{as}}}^{-1}a^{\text{as}} < \delta_{\text{apt}}^{\text{as}} < \infty$, for some constant $\delta_{\text{apt}}^{\text{as}} > 0$.

For simplicity, assume that the candidate factor model includes all systematic risk factors, that is, $K^{\text{mis}} = 0$, implying that $V_\varepsilon = V_e$.

Assumption IA.5.1. *The following assumptions explicitly incorporate weak factors in the unsystematic shocks, imposing more structure on the covariance matrix V_e:*[38]

$$N^{-1}\beta^{\text{can}\prime}V_{e^{\text{as}}}^{-1}\beta^{\text{can}} \longrightarrow D > 0, \ \ as \quad N \to \infty,$$
$$\beta^{\text{weak}\prime}V_{e^{\text{as}}}^{-1}\beta^{\text{weak}} \longrightarrow E > 0, \ \ as \quad N \to \infty,$$
$$\beta^{\text{can}\prime}V_{e^{\text{as}}}^{-1}\beta^{\text{weak}} = o(N^{\frac{1}{2}}),$$
$$\beta^{\text{can}\prime}V_{e^{\text{as}}}^{-1}a^{\text{as}} = o(N^{\frac{1}{2}}).$$

[38]Because the matrix $V_{e^{\text{as}}}^{-1}$ has uniformly bounded eigenvalues, the definition of a weak factor of Lettau and Pelger (2020) can be equivalently written as $\beta^{\text{weak}\prime}V_{e^{\text{as}}}^{-1}\beta^{\text{weak}} \longrightarrow E > 0$, as $N \to \infty$.

Lemma IA.5.1.

$$\beta^{\text{can}\,\prime} {V_e}^{-1} \beta^{\text{can}} = O(N).$$

Proof: The Sherman-Morrison-Woodbury formula applied to ${V_e}^{-1}$ leads to

$$\begin{aligned}
\beta^{\text{can}\,\prime} {V_e}^{-1} \beta^{\text{can}} &= \beta^{\text{can}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{can}} - \beta^{\text{can}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{weak}} (V_{f^{\text{weak}}}^{-1} + \beta^{\text{weak}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{weak}})^{-1} \beta^{\text{weak}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{can}} \\
&= O(N) + o(N^{\frac{1}{2}}) \times [O(1) + O(1)]^{-1} \times o(N^{\frac{1}{2}}) \\
&= O(N) + o(N) = O(N).
\end{aligned}$$

□

Lemma IA.5.2.

$$\beta^{\text{weak}\,\prime} {V_e}^{-1} \beta^{\text{can}} = o(N^{\frac{1}{2}}).$$

Proof: The Sherman-Morrison-Woodbury formula applied to ${V_e}^{-1}$, leads to

$$\begin{aligned}
\beta^{\text{weak}\,\prime} {V_e}^{-1} \beta^{\text{can}} &= \beta^{\text{weak}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{can}} - \beta^{\text{weak}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{weak}} (V_{f^{\text{weak}}}^{-1} + \beta^{\text{weak}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{weak}})^{-1} \beta^{\text{weak}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{can}} \\
&= V_{f^{\text{weak}}}^{-1} (V_{f^{\text{weak}}}^{-1} + \beta^{\text{weak}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{weak}})^{-1} \beta^{\text{weak}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{can}} \\
&= O(1) \times [O(1) + O(1)]^{-1} \times o(N^{\frac{1}{2}}) = o(N^{\frac{1}{2}}).
\end{aligned}$$

□

Lemma IA.5.3.

$$\beta^{\text{can}\,\prime} {V_e}^{-1} a = o(N^{\frac{1}{2}}).$$

Proof: The Sherman-Morrison-Woodbury formula applied to ${V_e}^{-1}$, given $a = a^{\text{as}} + \beta^{\text{weak}} \lambda^{\text{weak}}$, leads to

$$\begin{aligned}
\beta^{\text{can}\,\prime} {V_e}^{-1} a^{\text{as}} &= \beta^{\text{can}\,\prime} V_{e^{\text{as}}}^{-1} a^{\text{as}} - \beta^{\text{can}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{weak}} (V_{f^{\text{mis}}}^{-1} + \beta^{\text{weak}\,\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{weak}})^{-1} \beta^{\text{weak}\,\prime} V_{e^{\text{as}}}^{-1} a^{\text{as}} \\
&= o(N^{\frac{1}{2}}) + o(N^{\frac{1}{2}}) \times [O(1) + O(1)]^{-1} \times O(1) \\
&= o(N^{\frac{1}{2}}).
\end{aligned}$$

Thus, $\beta^{\text{can}\,\prime} {V_e}^{-1} \beta^{\text{weak}} \lambda^{\text{weak}} = o(N^{\frac{1}{2}})$ by Lemma IA.5.2. □

Lemma IA.5.4. *Let* e^{as} *be an* $N \times 1$ *random vector with zero mean and covariance matrix* $V_{e^{\text{as}}}$.

$$\beta^{\text{can}\,\prime} {V_e}^{-1} e^{\text{as}} = O_p(N^{\frac{1}{2}}).$$

Proof: For any random variable X with a finite second moment, we have that $X = O_p((\mathbb{E}(X^2))^{\frac{1}{2}})$. If $X = \beta^{\text{can}\prime} V_{e^{\text{as}}}^{-1} e^{\text{as}}$, then

$$\mathbb{E}(\beta^{\text{can}\prime} V_{e^{\text{as}}}^{-1} e^{\text{as}} e^{\text{as}\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{can}}) = \beta^{\text{can}\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{can}} = O(N),$$

and therefore, $\beta^{\text{can}\prime} V_{e^{\text{as}}}^{-1} e^{\text{as}} = O_p(N^{\frac{1}{2}})$. Similarly, we can show that $\beta^{\text{weak}\prime} V_{e^{\text{as}}}^{-1} e^{\text{as}} = O_p(1)$. Finally, we apply the Sherman-Morrison-Woodbury formula to $V_e{}^{-1}$ and obtain

$$\begin{aligned}\beta^{\text{can}\prime} V_e{}^{-1} e^{\text{as}} &= \beta^{\text{can}\prime} V_{e^{\text{as}}}^{-1} e^{\text{as}} - \beta^{\text{can}\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{weak}} (V_{f^{\text{weak}}}^{-1} + \beta^{\text{weak}\prime} V_{e^{\text{as}}}^{-1} \beta^{\text{weak}})^{-1} \beta^{\text{weak}\prime} V_{e^{\text{as}}}^{-1} e^{\text{as}} \\ &= O_p(N^{\frac{1}{2}}) + o(N^{\frac{1}{2}}) \times [O(1) + O(1)]^{-1} \times O_p(1) = O_p(N^{\frac{1}{2}}). \qquad \square\end{aligned}$$

We now generalize Proposition IA.4.1 for the case in which the unsystematic shocks explicitly include weak factors, that is, when (IA14) holds. For this, the only result that we need to prove is the presence of weak factors does not have implications for the limiting behavior of $\hat{M}^a_{\exp,t+1}$ and $M^a_{\exp,t+1}$.

Proposition IA.5.4 (Properties of $\hat{M}^a_{\exp,t+1}$, when Shocks e_{t+1} Include Weak Factors)**.** *Under Assumptions 2.1 and 2.2, the assumption that returns R_{t+1} are Gaussian, Assumptions IA.5.1, and the assumption that $K^{\text{mis}} = 0$, we have*

$$\hat{M}^a_{\exp,t+1} - M^a_{\exp,t+1} \xrightarrow{p} 0 \quad \text{as} \quad N \to \infty.$$

Proof: Recall that

$$\begin{aligned}\hat{M}^a_{\exp,t+1} &= \exp\Big[-aV_R^{-1}(R_{t+1} - \mathbb{E}(R_{t+1})) - \frac{1}{2}a'V_R^{-1}a\Big] \text{ and} \\ M^a_{\exp,t+1} &= \exp\Big[-a'V_e{}^{-1}e_{t+1} - \frac{1}{2}a'V_e{}^{-1}a\Big].\end{aligned}$$

The exponent of $\hat{M}^a_{\exp,t+1}$, given that $K^{\text{mis}} = 0$, is

$$\begin{aligned}&-a'V_R^{-1}(R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2}a'V_R^{-1}a = \\ &-a'V_R^{-1}\beta^{\text{can}}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})) - a'V_R^{-1}e_{t+1} - \frac{1}{2}a'V_R^{-1}a.\end{aligned}$$

We analyze the three terms of the exponent of $\hat{M}^a_{\exp,t+1}$ one-by-one. We use Lemmas IA.5.1 and IA.5.3 and apply the Sherman-Morrison-Woodbury formula to V_R^{-1}:

$$\begin{aligned}a'V_R^{-1}\beta^{\text{can}} &= a'V_e^{-1}\beta^{\text{can}} - a'V_e^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_e^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_e^{-1}\beta^{\text{can}} \\ &= a'V_e^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_e^{-1}\beta^{\text{can}})^{-1}V_{f^{\text{can}}}^{-1} \\ &= o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times O(1)\end{aligned}$$

$$= o(N^{-\frac{1}{2}}).$$

Next, by Lemmas IA.5.1, IA.5.3, and IA.5.4, and by taking into account that $e_{t+1} = \beta^{\text{weak}} f^{\text{weak}}_{t+1} + e^{\text{as}}_t$ and $a = a^{\text{as}} + \beta^{\text{weak}} \lambda^{\text{weak}}$, we obtain

$$\begin{aligned} a' V_R^{-1} e_{t+1} &= a' V_e^{-1} e_{t+1} - a' V_e^{-1} \beta^{\text{can}} (V^{-1}_{f^{\text{can}}} + \beta^{\text{can}\prime} V_e^{-1} \beta^{\text{can}})^{-1} \beta^{\text{can}\prime} V_e^{-1} e_{t+1} \\ &= a' V_e^{-1} e_{t+1} + o_p(N^{\frac{1}{2}}) [O(1) + O(N)]^{-1} O_p(N^{\frac{1}{2}}) = a' V_e^{-1} e_{t+1} + o_p(1). \end{aligned}$$

Finally, by Lemmas IA.5.1 and IA.5.3,

$$\begin{aligned} a' V_R^{-1} a &= a' V_e^{-1} a - a' V_e^{-1} \beta^{\text{can}} (V^{-1}_{f^{\text{can}}} + \beta^{\text{can}\prime} V_e^{-1} \beta^{\text{can}})^{-1} \beta^{\text{can}\prime} V_e^{-1} a \\ &= a' V_e^{-1} a + o(N^{\frac{1}{2}}) \times [O(1) + O(N)]^{-1} \times o(N^{\frac{1}{2}}) \\ &= a' V_e^{-1} a + o(1). \end{aligned}$$

Putting these results together, we obtain

$$-a' V_R^{-1} (R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2} a' V_R^{-1} a = -a' V_e^{-1} e_{t+1} - \frac{1}{2} a' V_e^{-1} a + o_p(1),$$

implying that

$$\hat{M}^a_{\exp,t+1} - M^a_{\exp,t+1} \xrightarrow{p} 0 \quad \text{as} \quad N \to \infty. \qquad \square$$

Remark: Proposition IA.5.4 shows that our methodology is still valid if expected excess returns include compensation for exposure to weak factors that are present in unsystematic shocks e_{t+1}. This is an important result because of the challenges associated with identifying weak factors. For example, it is well known that weak factors cannot be estimated consistently (Lettau and Pelger, 2020). Our methodology does not require estimating weak factors but can accurately characterize the importance of unsystematic risk that includes weak factors. This makes our approach compelling.

IA.6 Estimation

We start this section by discussing the identification conditions that fix the rotation of risk factors. These identification conditions do not have any implications for the SDF but allow us to estimate the model of asset returns. Next, we show how to estimate the model of asset returns. In the empirical analysis, we use an observable time-varying risk-free rate R_{ft} in place of R_f.

IA.6.1 Identification conditions

In a candidate model, the loadings of asset returns on the missing factors, and the missing factors themselves, are unique up to a rotation. Similarly, identifying the loadings of asset returns on the latent factors in the APT model is unique up to a rotation. Thus, at the estimation stage, we need to impose identification conditions. These identification conditions affect the interpretation of latent factors but not the estimated admissible SDF.

Below, we detail the identification strategy, which we use to correct a candidate model of asset returns. The difference between identifying missing factors in the candidate factor model and identifying latent factors of the APT model is only because of the presence of observable factors in the candidate model. Thus, the identification strategy for the APT model is equivalent to that described below but in which $K^{\text{can}} = 0$ and $K^{\text{mis}} = K$.

We follow the identification strategy of Bai and Li (2012) and adapt it to the case in which a model has K^{can} observable and K^{mis} latent risk factors. Denote F^{can} a matrix $T \times K^{\text{can}}$ that collects candidate factors column by column. Denote by F^{mis} a matrix $T \times K^{\text{mis}}$ that collects missing factors column by column. Combine these matrices in a $T \times (K^{\text{can}} + K^{\text{mis}})$ matrix $F = [F^{\text{can}},\ F^{\text{mis}}]$. Note that the rotation of this matrix is defined by a squared invertible matrix of a dimension $(K^{\text{can}} + K^{\text{mis}}) \times (K^{\text{can}} + K^{\text{mis}})$, and therefore, the rotation is pinned down by $(K^{\text{can}} + K^{\text{mis}})^2$ parameters.

At the estimation stage, we impose the following $(K^{\text{can}} + K^{\text{mis}})^2$ identification conditions to fix the rotation:

- The first K^{can} columns of the rotation matrix are fixed because F^{can} includes only observable factors. This is equivalent to $K^{\text{can}} \times (K^{\text{can}} + K^{\text{mis}})$ restrictions being already imposed.
- $V_{f^{\text{mis}}} = I_{K^{\text{mis}}}$ introduces $K^{\text{mis}} \times (K^{\text{mis}} + 1)/2$ restrictions.
- $\beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}}$ is a diagonal matrix that is equivalent to imposing $(K^{\text{mis}} - 1) \times K^{\text{mis}}/2$ restrictions. We also introduce an order restriction that requires that the diagonal elements of the matrix $\beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}}$ follow in descending order. In addition, we require the eigenvectors of $\beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}}$ to have positive means to identify the latent factors uniquely, rather than up to a sign.
- Candidate factors f_{t+1}^{can} are uncorrelated with missing factors f_{t+1}^{mis}. This requirement is equivalent to imposing $K^{\text{can}} \times K^{\text{mis}}$ additional restrictions.

IA.6.2 Constrained Maximum-Likelihood (ML) Estimator

This section describes how to estimate the candidate factor model and its required correction to obtain an admissible SDF. The underlying problem is described in Section 2.3.

To simplify exposition, introduce the following notation

$$\bar{R} = T^{-1}\sum_{t=1}^{T} R_t, \qquad \bar{R}_f = T^{-1}\sum_{t=1}^{T} R_{ft}, \qquad \bar{f}^{\text{can}} = T^{-1}\sum_{t=1}^{T} f_t^{\text{can}},$$
$$\hat{Q}_{f^{\text{can}}} = T^{-1}\sum_{t=1}^{T} f_t^{\text{can}} f_t^{\text{can}\prime}, \quad \text{and} \quad \hat{Q}_{Rf^{\text{can}}} = \frac{1}{T}\sum_{t=1}^{T}(R_t - R_{ft-1}1_N) f_t^{\text{can}\prime}.$$

For this section only, we use the notation $\hat{}$ to denote an estimator.

Without loss of generality, assume that the candidate factors f_{t+1}^{can} are tradable factors in the form of excess returns on investment strategies (if any candidate factor is not tradable, we use its factor-mimicking portfolio, as in Breeden et al. (1989)).

For a generic vector Θ that collects all the unknown parameters of the corrected model of asset returns, $\Theta = (a',\ vech(V_e)',\ vech(V_{f^{\text{can}}})',\ vec(\beta^{\text{can}})',\ \lambda^{\text{can}\prime},\ vec(\beta^{\text{mis}})',\ \lambda^{\text{mis}\prime})$, denote $L(\Theta)$ the (up to a constant) Gaussian joint likelihood of the vector of asset returns in excess of the risk-free rate, $R_{t+1} - R_{ft}1_N$, and observable factors f_{t+1}^{can} scaled by the number of time-series observations T

$$\begin{aligned}\log(L(\Theta)) = &-\frac{1}{2}\log(|V_\varepsilon|) - \frac{1}{2}\log(|V_f^{\text{can}}|) - \frac{1}{2T}\sum_{t=0}^{T-1}\varepsilon_{t+1}' V_\varepsilon^{-1}\varepsilon_{t+1} \\ &- \frac{1}{2T}\sum_{t=0}^{T-1}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}}))' V_{f^{\text{can}}}^{-1}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})),\end{aligned} \tag{IA15}$$

where $\varepsilon_{t+1} = R_{t+1} - R_{ft}1_N - a - \beta^{\text{mis}}\lambda^{\text{mis}} - \beta^{\text{can}}\lambda^{\text{can}} - \beta^{\text{can}}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}}))$ and V_ε is given in (12).

We maximize this log-likelihood function (IA15) subject to the no-arbitrage restriction (5). Without loss of generality, we replace the no-arbitrage restriction (5) with the expression

$$a' V_\varepsilon^{-1} a \le \delta_{\text{apt}}$$

that is, replacing V_e with V_ε, is more convenient when deriving the first-order conditions.

We use the Karush-Kuhn-Tucker (KKT) multiplier method to solve the resulting constrained optimization problem,

$$\hat{\Theta} = \text{argmax}\ \log(L(\Theta)) \quad \text{subject to} \quad a' V_\varepsilon^{-1} a \le \delta_{\text{apt}}, \tag{IA16}$$

and denote the KKT multiplier by $\kappa/2$.

The optimization problem for estimating the parameters of the admissible SDF implied by the APT model of asset returns is identical to that formulated in expression (IA16), in which there are no candidate factors, $K^{\text{can}} = 0$, and the missing factors f_{t+1}^{mis} are replaced with latent factors f_{t+1}. Correspondingly, the parameters characterizing the missing factors f_{t+1}^{mis}, such as β^{mis}, λ^{mis}, and K^{mis}, are replaced with the parameters characterizing the latent factors f_{t+1}, which are β, λ, and K, respectively.

Proposition IA.6.5 (Constrained ML Estimator)**.** *Suppose that the assumptions of Proposition IA.4.1 hold. Assume that the number K^{mis} of missing factors in the candidate model and the no-arbitrage bound δ_{apt} are known and that the sample covariance matrix $\hat{V}_{f^{\text{can}}}$ of candidate factors is nonsingular. Then the estimators of λ^{can} and $V_{f^{\text{can}}}$ coincide with the sample mean and sample covariance of the candidate factors f_t^{can}:*

$$\hat{\lambda}^{\text{can}} = \bar{f}^{\text{can}},$$
$$\hat{V}_f^{\text{can}} = \hat{Q}_{f^{\text{can}}} - \bar{f}^{\text{can}} \bar{f}^{\text{can}\prime}.$$

The estimators $\hat{\beta}^{\text{mis}}$ and $\hat{V}_e$ of β^{mis} and V_e do not admit a closed-form solution.

(i) If the optimal value of the Karush-Kuhn-Tucker multiplier $\hat{\kappa}$ is greater than zero, the estimators of β^{can}, λ^{mis}, and a, are

$$\begin{aligned} \operatorname{vec}(\hat{\beta}^{\text{can}}) &= \left(\hat{Q}_{f^{\text{can}}} \otimes I_N - \bar{f}^{\text{can}} \bar{f}^{\text{can}\prime} \otimes \hat{G}\right)^{-1} \\ &\qquad \times \operatorname{vec}\left(\hat{Q}_{Rf^{\text{can}}} - \hat{G}(\bar{R} - \bar{R}_f 1_N)\bar{f}^{\text{can}\prime}\right) \qquad \text{(IA17)} \\ \hat{\lambda}^{\text{mis}} &= (\hat{\beta}^{\text{mis}\prime} \hat{V}_\varepsilon^{-1} \hat{\beta}^{\text{mis}})^{-1} \hat{\beta}^{\text{mis}\prime} \hat{V}_\varepsilon^{-1} \left(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}} \hat{\lambda}^{\text{can}}\right), \quad \textit{and} \\ \hat{a} &= \frac{1}{\hat{\kappa}+1} \left(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}} \hat{\lambda}^{\text{can}} - \hat{\beta}^{\text{mis}} \hat{\lambda}^{\text{mis}}\right), \end{aligned}$$

where

$$\hat{\kappa} = \left(\frac{(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}} \hat{\lambda}^{\text{can}} - \hat{\beta}^{\text{mis}} \hat{\lambda}^{\text{mis}})' V_\varepsilon^{-1} (\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}} \hat{\lambda}^{\text{can}} - \hat{\beta}^{\text{mis}} \hat{\lambda}^{\text{mis}})}{\delta_{\text{apt}}} - 1\right)^{1/2},$$

$$\begin{aligned} \hat{G} &= \frac{1}{(\hat{\kappa}+1)} I_N + \frac{\hat{\kappa}}{(\hat{\kappa}+1)} \hat{\beta}^{\text{mis}} (\hat{\beta}^{\text{mis}\prime}\, \hat{V}_\varepsilon^{-1}\, \hat{\beta}^{\text{mis}})^{-1} \hat{\beta}^{\text{mis}\prime}\, \hat{V}_\varepsilon^{-1}, \quad \textit{and} \\ \hat{V}_\varepsilon &= \hat{\beta}^{\text{mis}} \hat{\beta}^{\text{mis}\prime} + \hat{V}_e. \end{aligned}$$

(ii) If the optimal value of the Karush-Kuhn-Tucker multiplier satisfies $\hat{\kappa} = 0$, it is possible to estimate the vector α

$$\hat{\alpha} = \bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}} \hat{\lambda}^{\text{can}}$$

but not its components, a and $\beta^{\text{mis}} \lambda^{\text{mis}}$. The estimator of $\operatorname{vec}(\beta^{\text{can}})$ is given by expression (IA17) with $\hat{\kappa} = 0$.

Proof: The Lagrangian for our optimization problem is

$$L_p(\Theta) = -\frac{\kappa}{2}(a'V_\varepsilon^{-1}a - \delta_{\text{apt}}) - \frac{1}{2}\log(|V_\varepsilon|) - \frac{1}{2}\log(|V_{f^{\text{can}}}|) - \frac{1}{2T}\sum_{t=0}^{T-1}\varepsilon_{t+1}'V_\varepsilon^{-1}\varepsilon_{t+1}$$
$$- \frac{1}{2T}\sum_{t=0}^{T-1}(f_{t+1}^{\text{can}} - \mathbb{E}(f_t^{\text{can}}))'V_{f^{\text{can}}}^{-1}(f_{t+1}^{\text{can}} - \mathbb{E}(f_t^{\text{can}})). \tag{IA18}$$

Recall that the candidate factors f_t^{can} represent excess returns on tradable investment strategies, that is, $\mathbb{E}(f_t^{\text{can}}) = \lambda^{\text{can}}$. The first-order condition for λ^{can} results in

$$\hat{\lambda}^{\text{can}} = \frac{1}{T}\sum_{t=1}^{T} f_t^{\text{can}}.$$

Similarly, the first-order condition for V_f^{can} gives

$$\hat{V}_{f^{\text{can}}} = \frac{1}{T}\sum_{t=1}^{T}(f_t^{\text{can}} - \hat{\lambda}^{\text{can}})(f_t^{\text{can}} - \hat{\lambda}^{\text{can}})'.$$

Next, we consider two cases, $\hat{\kappa} > 0$ and $\hat{\kappa} = 0$.

First, suppose that $\hat{\kappa} > 0$, and therefore $a'V_\varepsilon^{-1}a = \delta_{\text{apt}}$. We differentiate the Lagrangian in equation (IA18) with respect to λ^{mis} and a and obtain the following $K^{\text{mis}} + N$ first-order conditions:

$$\begin{pmatrix} \hat{\beta}^{\text{mis}\prime}\hat{V}_\varepsilon^{-1} \\ I_N \end{pmatrix}\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\hat{\lambda}^{\text{can}}\Big) = \begin{pmatrix} \hat{\beta}^{\text{mis}\prime}\hat{V}_\varepsilon^{-1}\hat{\beta}^{\text{mis}} & \hat{\beta}^{\text{mis}\prime}\hat{V}_\varepsilon^{-1} \\ \hat{\beta}^{\text{mis}\prime} & (1+\hat{\kappa})I_N \end{pmatrix}\begin{pmatrix} \hat{\lambda}^{\text{mis}} \\ \hat{a} \end{pmatrix}.$$

The matrix premultiplying the vector $(\hat{\lambda}^{\text{mis}\prime}, \hat{a}')'$ is nonsingular when the no-arbitrage restriction binds, implying that $\hat{\lambda}^{\text{mis}}$ and $\hat{a}$ are identified separately :

$$\hat{\lambda}^{\text{mis}} = (\hat{\beta}^{\text{mis}\prime}\hat{V}_\varepsilon^{-1}\hat{\beta}^{\text{mis}})^{-1}\hat{\beta}^{\text{mis}\prime}\hat{V}_\varepsilon^{-1}\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\hat{\lambda}^{\text{can}}\Big), \tag{IA19}$$

$$\hat{a} = \frac{1}{\hat{\kappa}+1}\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\hat{\lambda}^{\text{can}} - \hat{\beta}^{\text{mis}}\hat{\lambda}^{\text{mis}}\Big). \tag{IA20}$$

Next, we use equation (IA20) and the binding no-arbitrage restriction $a'V_\varepsilon^{-1}a = \delta_{\text{apt}}$ to obtain

$$\hat{\kappa} = \Big(\frac{(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\hat{\lambda}^{\text{can}} - \hat{\beta}^{\text{mis}}\hat{\lambda}^{\text{mis}})'\hat{V}_\varepsilon^{-1}(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\hat{\lambda}^{\text{can}} - \hat{\beta}^{\text{mis}}\hat{\lambda}^{\text{mis}})}{\delta_{\text{apt}}} - 1\Big)^{1/2}. \tag{IA21}$$

Finally, we consider the first-order condition with respect to the generic (i, j)th element of β^{can}, denoted by β_{ij}^{can} with $1 \le i \le N$, $1 \le j \le K^{\text{can}}$, and obtain

$$-\frac{1}{T}\sum_{t=1}^{T}\Big(R_t - R_{ft}1_N - \hat{\beta}^{\text{mis}}\hat{\lambda}^{\text{mis}} - \hat{a} - \hat{\beta}^{\text{can}}f_t^{\text{can}}\Big)'\hat{V}_\varepsilon^{-1}\Big(-\frac{\partial\beta^{\text{can}}}{\partial\beta_{ij}^{\text{can}}}|_{\beta^{\text{can}}=\hat{\beta}^{\text{can}}}f_t^{\text{can}}\Big) = 0,$$

which can be rearranged by stacking together the first-order conditions as

$$\hat{Q}_{Rf^{\text{can}}} - (\hat{a} + \hat{\beta}^{\text{mis}}\hat{\lambda}^{\text{mis}})\bar{f}^{\text{can}\prime} - \hat{\beta}^{\text{can}}\hat{Q}_{f^{\text{can}}} = 0_{N\times K^{\text{can}}}. \tag{IA22}$$

Next, we define

$$\hat{G} = \frac{1}{(\hat{\kappa}+1)} I_N + \frac{\hat{\kappa}}{(\hat{\kappa}+1)} \hat{\beta}^{\text{mis}}(\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1}\hat{\beta}^{\text{mis}})^{-1}\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1},$$

and use the formulas (IA19) and (IA20) to rewrite equation (IA22) as follows

$$\hat{\beta}^{\text{can}}\hat{Q}_{f^{\text{can}}} - \hat{G}\hat{\beta}^{\text{can}}\bar{f}^{\text{can}}\bar{f}^{\text{can}\prime} = \hat{Q}_{Rf^{\text{can}}} - G(\bar{R} - \bar{R}_f 1_N)\bar{f}^{\text{can}\prime}.$$

Then, we take the vec operator and solve for $\hat{\beta}^{\text{can}}$ to obtain

$$\text{vec}(\hat{\beta}^{\text{can}}) = \left(\hat{Q}_{f^{\text{can}}} \otimes I_N - \bar{f}^{\text{can}}\bar{f}^{\text{can}\prime} \otimes \hat{G}\right)^{-1} \times \text{vec}\left(\hat{Q}_{Rf^{\text{can}}} - G(\bar{R} - \bar{R}_f 1_N)\bar{f}^{\text{can}\prime}\right) \tag{IA23}$$

The solution for $\hat{\beta}^{\text{can}}$ exists because the matrix $\left(\hat{Q}_{f^{\text{can}}} \otimes I_N - \bar{f}^{\text{can}}\bar{f}^{\text{can}\prime} \otimes \hat{G}\right)$ is nonsingular. We note that

$$\hat{Q}_{f^{\text{can}}} \otimes I_N - \bar{f}^{\text{can}}\,\bar{f}^{\text{can}\prime} \otimes \hat{G} = \hat{V}_{f^{\text{can}}} \otimes I_N + \bar{f}^{\text{can}}\,\bar{f}^{\text{can}\prime} \otimes (I_N - \hat{G}),$$

where $\hat{V}_{f^{\text{can}}}$, being a covariance matrix, is positive-definite, and $\bar{f}^{\text{can}}\,\bar{f}^{\text{can}\prime} \otimes (I_N - \hat{G})$ is positive semi-definite, because

$$\begin{aligned}
I_N - \hat{G} &= I_N - \frac{1}{(\hat{\kappa}+1)} I_N - \left(\frac{\hat{\kappa}}{1+\hat{\kappa}}\right)\hat{\beta}^{\text{mis}}(\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1}\hat{\beta}^{\text{mis}})^{-1}\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1} \\
&= \left(\frac{\hat{\kappa}}{1+\hat{\kappa}}\right)(I_N - \hat{\beta}^{\text{mis}}(\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1}\hat{\beta}^{\text{mis}})^{-1}\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1}) \\
&= \left(\frac{\hat{\kappa}}{1+\hat{\kappa}}\right)\hat{V}_{\varepsilon}(\hat{V}_{\varepsilon}^{-1} - \hat{V}_{\varepsilon}^{-1}\hat{\beta}^{\text{mis}}(\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1}\hat{\beta}^{\text{mis}})^{-1}\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1}) \\
&= \left(\frac{\hat{\kappa}}{1+\hat{\kappa}}\right)\hat{V}_{\varepsilon}(\hat{V}_{\varepsilon}^{-1})^{\frac{1}{2}}(I_N - (\hat{V}_{\varepsilon}^{-1})^{\frac{1}{2}}\hat{\beta}^{\text{mis}}(\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1}\hat{\beta}^{\text{mis}})^{-1}\hat{\beta}^{\text{mis}\prime}(\hat{V}_{\varepsilon}^{-1})^{\frac{1}{2}})(\hat{V}_{\varepsilon}^{-1})^{\frac{1}{2}}
\end{aligned}$$

is the product of the positive-definite matrices $I_N - (\hat{V}_{\varepsilon}^{-1})^{\frac{1}{2}}\hat{\beta}^{\text{mis}}(\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1}\hat{\beta}^{\text{mis}})^{-1}\hat{\beta}^{\text{mis}\prime}(\hat{V}_{\varepsilon}^{-1})^{\frac{1}{2}}$ (projection matrix), $\hat{V}_{\varepsilon}$, and $(\hat{V}_{\varepsilon}^{-1})^{\frac{1}{2}}$. Note that $\hat{\lambda}^{\text{mis}}$, $\hat{a}$, $\hat{\kappa}$ are functions of $\hat{\beta}^{\text{mis}}$, $\hat{V}_e$, and $\hat{\beta}^{\text{can}}$:

$$\hat{\lambda}^{\text{mis}} = \hat{\lambda}^{\text{mis}}(\hat{\beta}^{\text{mis}}, \hat{\beta}^{\text{can}}, \hat{V}_e,), \quad \hat{a} = \hat{a}(\hat{\beta}^{\text{mis}}, \hat{\beta}^{\text{can}}, \hat{V}_e), \quad \hat{\kappa} = \hat{\kappa}(\hat{\beta}^{\text{mis}}, \hat{\beta}^{\text{can}}, \hat{V}_e). \tag{IA24}$$

However, we cannot obtain the explicit representation of $\hat{\lambda}^{\text{mis}}$, $\hat{a}$, and $\hat{\kappa}$ in terms of fewer parameters, for example, only $\hat{\beta}^{\text{mis}}$ and $\hat{V}_e$. This is because substituting $\hat{\kappa}$ into expression (IA23) for $\hat{\beta}^{\text{can}}$ creates a fixed-point problem for $\hat{\beta}^{\text{can}}$.

Because a fixed-point problem slows down substantially the optimization routine, we do not use the closed-form solution (IA23) for $\hat{\beta}^{\text{can}}$ and instead substitute expressions (IA24) into $L_p(\Theta)$ to obtain the concentrated log-likelihood function, which is a function of only β^{mis}, β^{can}, and V_e. We maximize the concentrated log-likelihood numerically, thereby obtaining the estimates of β^{mis}, β^{can}, and V_e, which also imply the optimal values of the other parameters. Finally, we verify if equation (IA23) holds, thereby checking convergence of our optimization algorithm.

If equation (IA21) implies that $\hat{\kappa} < 0$ then we ignore all the obtained above results and move to the next case of $\hat{\kappa} = 0$.

Consider the second case, in which the Karush-Kuhn-Tucker multiplier is zero: $\hat{\kappa} = 0$. In this case, a feasible solution to the optimization problem satisfies $a'V_\varepsilon^{-1}a < \delta_{\text{apt}}$.

The first-order conditions with respect to λ^{mis} and a imply the following singular system of $K^{\text{mis}} + N$ equations

$$\begin{pmatrix} \hat{\beta}^{\text{mis}\prime}\hat{V}_\varepsilon^{-1} \\ I_N \end{pmatrix}\left(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\hat{\lambda}^{\text{can}}\right) = \begin{pmatrix} \hat{\beta}^{\text{mis}\prime}\hat{V}_\varepsilon^{-1}\hat{\beta}^{\text{mis}} & \hat{\beta}^{\text{mis}\prime}\hat{V}_\varepsilon^{-1} \\ \hat{\beta}^{\text{mis}} & I_N \end{pmatrix}\begin{pmatrix} \hat{\lambda}^{\text{mis}} \\ \hat{a} \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} \hat{\beta}^{\text{mis}\prime}\hat{V}_\varepsilon^{-1}\hat{\beta}^{\text{mis}} & \hat{\beta}^{\text{mis}\prime}\hat{V}_\varepsilon^{-1} \\ \hat{\beta}^{\text{mis}} & I_N \end{pmatrix}$$

is of dimension $(N+K^{\text{mis}})\times(N+K^{\text{mis}})$ but of rank N, and therefore it is noninvertible. As a result, we cannot identify separately a and λ^{mis}, implying that if $\hat{\kappa} = 0$, we can only identify the sum $a + \beta^{\text{mis}}\lambda^{\text{mis}}$ but not its two components separately. All the other parameters of the vector Θ are identified separately, and their expressions follow from differentiating the Lagrangian (IA18) and solving the resulting first-order conditions. For instance, the formula for $\hat{\beta}^{\text{can}}$ follows by setting $\hat{G} = I_N$ into (IA17).

When both cases, $\hat{\kappa} > 0$ and $\hat{\kappa} = 0$, are feasible, we choose the one under which the log-likelihood $L_p(\Theta)$ is larger. $\square$

IA.7 The SDF with Nonorthogonal Components

In the main text of the manuscript, we assumed that the candidate risk factors f_{t+1}^{can} are orthogonal to the missing sources of systematic risk f_{t+1}^{mis} and unsystematic shocks e_{t+1}. This assumption is without loss of generality because if the (observable) systematic risk factors f_{t+1}^{mis} that the candidate model omits are correlated with f_{t+1}^{can}, there exists an *observationally equivalent* representation of the SDF M_{t+1}, such that the factors f_{t+1}^{can} are orthogonal to some latent systematic risk factors (residuals from an orthogonal projection of omitted observable risk factors onto the candidate factors). Thus, the assumption of orthogonality affects the interpretation of the missing factors but not the admissibility of the pricing kernel.

In particular,

$$
\begin{aligned}
M_{t+1} &= \frac{1}{R_f} + b^{\text{can}\prime}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})) + b^{\text{mis}\prime}(f^{\text{mis}}_{t+1} - \mathbb{E}(f^{\text{mis}}_{t+1})) + c' e_{t+1} \\
&= \frac{1}{R_f} + \tilde{b}^{\text{can}\prime}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})) + b^{\text{mis}\prime}(\tilde{f}^{\text{mis}}_{t+1} - \mathbb{E}(\tilde{f}^{\text{mis}}_{t+1})) + c' e_{t+1},
\end{aligned}
$$

where $Q = \text{cov}(f^{\text{can}}_{t+1},\ f^{\text{mis}\prime}_{t+1})$ is a $K^{\text{can}} \times K^{\text{mis}}$ matrix of covariances and

$$
\begin{aligned}
\tilde{b}^{\text{can}} &= b^{\text{can}} + V^{-1}_{f^{\text{can}}} Q b^{\text{mis}}, \\
\tilde{f}^{\text{mis}}_{t+1} - \mathbb{E}(\tilde{f}^{\text{mis}}_{t+1}) &= (f^{\text{mis}}_{t+1} - \mathbb{E}(f^{\text{mis}}_{t+1})) - Q' V_{f^{\text{can}}}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})).
\end{aligned}
$$

Notice that by construction $\text{cov}(f^{\text{can}}_{t+1}, \tilde{f}^{\text{mis}\prime}_{t+1})$ is a $K^{\text{can}} \times K^{\text{mis}}$ matrix of zeros, because $\tilde{f}^{\text{mis}}_{t+1}$ represent the linear-projection residual from projecting $f^{\text{mis}}_{t+1} - \mathbb{E}(f^{\text{mis}}_{t+1})$ on $f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})$.

We now show how, starting from a candidate factor model with factors f^{can}_{t+1} that are correlated with the missing systematic factors f^{mis}_{t+1}, we can construct an admissible SDF.

Proposition IA.7.6 (SDF: Correlated case)**.** *Under Assumptions 2.1 and 2.2 of the APT, there exists an admissible SDF of the form*

$$
\begin{aligned}
M_{t+1} &= \frac{1}{R_f} + b^{\text{can}\prime}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})) + b^{\text{mis}\prime}(f^{\text{mis}}_{t+1} - \mathbb{E}(f^{\text{mis}}_{t+1})) + c' e_{t+1}, \quad \textit{where} \\
b^{\text{can}\prime} &= \left(-\frac{\lambda^{\text{can}\prime} V^{-1}_{f^{\text{can}}}}{R_f} + \frac{\lambda^{\text{mis}\prime} V^{-1}_{f^{\text{mis}}}}{R_f} Q' V^{-1}_{f^{\text{can}}} \right) \times \left(I_{K^{\text{can}}} - Q V^{-1}_{f^{\text{mis}}} Q' V^{-1}_{f^{\text{can}}} \right)^{-1}, \\
b^{\text{mis}\prime} &= \left(-\frac{\lambda^{\text{mis}} V^{-1}_{f^{\text{mis}}}}{R_f} + \frac{\lambda^{\text{can}\prime} V^{-1}_{f^{\text{can}}}}{R_f} Q V^{-1}_{f^{\text{mis}}} \right) \times \left(I_{K^{\text{mis}}} - Q' V^{-1}_{f^{\text{can}}} Q V^{-1}_{f^{\text{mis}}} \right)^{-1}, \\
c' &= -\frac{a V_e^{-1}}{R_f}.
\end{aligned}
$$

Proof: We guess that the SDF has the following functional form

$$
M_{t+1} = \mathbb{E}(M_{t+1}) + b^{\text{can}\prime}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})) + b^{\text{mis}\prime}(f^{\text{mis}}_{t+1} - \mathbb{E}(f^{\text{mis}}_{t+1})) + c' e_{t+1},
$$

where b^{can} is a $K^{\text{can}} \times 1$ vector, b^{mis} is a $K^{\text{mis}} \times 1$ vector, and c is an $N \times 1$ vector. We identify the unknown vectors b^{can}, b^{mis}, and c by using the Law of One Price. Specifically, because we assume the existence of the risk-free asset, to determine the mean of the SDF, we use the condition

$$
\mathbb{E}(M_{t+1}) = \frac{1}{R_f}.
$$

Next, because λ^{can} represents a vector of prices of risk of f^{can}_{t+1} we have that

$$
-\,\text{cov}(M_{t+1}, f^{\text{can}}_{t+1}) \times R_f = \lambda^{\text{can}}.
$$

These K^{can} conditions identify b^{can}:

$$b^{\text{can}\prime} = -\frac{1}{R_f}\lambda^{\text{can}\prime}V_{f^{\text{can}}}^{-1} - b^{\text{mis}\prime}Q'V_{f^{\text{can}}}^{-1}. \tag{IA25}$$

Similarly, λ^{mis} is the price of risk associated with factors f_{t+1}^{mis}, or equivalently,

$$-\operatorname{cov}(M_{t+1}, f_{t+1}^{\text{mis}}) \times R_f = \lambda^{\text{mis}}.$$

These K^{mis} conditions identify b^{mis}:

$$b^{\text{mis}\prime} = -\frac{\lambda^{\text{mis}\prime}V_{f^{\text{mis}}}^{-1}}{R_f} - b^{\text{can}\prime}QV_{f^{\text{mis}}}^{-1}. \tag{IA26}$$

Putting together expressions (IA25) and (IA26), we obtain

$$b^{\text{can}\prime} = \left(-\frac{\lambda^{\text{can}\prime}V_{f^{\text{can}}}^{-1}}{R_f} + \frac{\lambda^{\text{mis}\prime}V_{f^{\text{mis}}}^{-1}}{R_f}Q'V_{f^{\text{can}}}^{-1}\right) \times \left(I_{K^{\text{can}}} - QV_{f^{\text{mis}}}^{-1}Q'V_{f^{\text{can}}}^{-1}\right)^{-1},$$

$$b^{\text{mis}\prime} = \left(-\frac{\lambda^{\text{mis}}V_{f^{\text{mis}}}^{-1}}{R_f} + \frac{\lambda^{\text{can}\prime}V_{f^{\text{can}}}^{-1}}{R_f}QV_{f^{\text{mis}}}^{-1}\right) \times \left(I_{K^{\text{mis}}} - Q'V_{f^{\text{can}}}^{-1}QV_{f^{\text{mis}}}^{-1}\right)^{-1}.$$

Finally, it must be the case that the SDF prices the N basis assets:

$$\mathbb{E}(M_{t+1}(R_{t+1} - R_f 1_N)) = 0_N.$$

These N equations identify c. Given expressions (IA25) and (IA26), we obtain

$$c' = -\frac{aV_e^{-1}}{R_f}. \qquad \square$$

Next, we provide a non-negative SDF.

Proposition IA.7.7 (Nonnegative SDF: Correlated case)**.** *Under Assumptions 2.1 and 2.2 of the APT and the assumption that returns R_{t+1} are Gaussian, there exists an admissible SDF $M_{\exp,t+1}$*

$$M_{\exp,t+1} = M_{\exp,t+1}^{\beta,\text{can}} \times M_{\exp,t+1}^{a} \times M_{\exp,t+1}^{\beta,\text{mis}} \quad \textit{where}$$

$$M_{\exp,t+1}^{\beta,\text{can}} = \frac{1}{R_f}\exp\left(b_+^{\text{can}\prime}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})) - \frac{1}{2}b_+^{\text{can}\prime}V_{f^{\text{can}}}b_+^{\text{can}} - \frac{1}{2}b_+^{\text{can}\prime}Qb_+^{\text{mis}}\right)$$

$$M_{\exp,t+1}^{\beta,\text{mis}} = \exp\left(b_+^{\text{mis}\prime}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) - \frac{1}{2}b_+^{\text{mis}\prime}V_{f^{\text{mis}}}b_+^{\text{mis}} - \frac{1}{2}b_+^{\text{can}\prime}Qb_+^{\text{mis}}\right)$$

$$M_{\exp,t+1}^{a} = \exp\left(-a'V_e^{-1}e_{t+1} - \frac{1}{2}a'V_e^{-1}a\right), \quad \textit{where}$$

$$b_+^{\text{can}\prime} = (-\lambda^{\text{can}\prime}V_{f^{\text{can}}} + \lambda^{\text{mis}\prime}V_{f^{\text{mis}}}^{-1}Q'V_{f^{\text{can}}}^{-1}) \times (I_{K^{\text{can}}} - QV_{f^{\text{mis}}}^{-1}Q'V_{f^{\text{can}}}^{-1})^{-1},$$

$$b_+^{\text{mis}\prime} = (-\lambda^{\text{mis}\prime}V_{f^{\text{mis}}} + \lambda^{\text{can}\prime}V_{f^{\text{can}}}^{-1}QV_{f^{\text{mis}}}^{-1}) \times (I_{K^{\text{mis}}} - Q'V_{f^{\text{can}}}^{-1}QV_{f^{\text{mis}}}^{-1})^{-1}.$$

Proof: We use a guess-and-verify method to derive a nonnegative SDF. We guess that the SDF has the following functional form

$$M_{\exp,t+1} = \exp\left[\mu_+ + b_+^{\text{can}\prime}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})) + b_+^{\text{mis}\prime}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) + c_+' e_{t+1}\right]$$

with unknown vectors b_+^{can}, b_+^{mis}, and c_+, as well as an unknown scalar μ_+. To identify the unknowns and verify our guess, we use the following $K^{\text{can}} + K^{\text{mis}} + N + 1$ equations, which are implications of the Law of One Price:

$$\begin{aligned} -\operatorname{cov}(M_{\exp,t+1}, f_{t+1}^{\text{can}}) \times R_f &= \lambda^{\text{can}}, \\ -\operatorname{cov}(M_{\exp,t+1}, f_{t+1}^{\text{mis}}) \times R_f &= \lambda^{\text{mis}}, \\ \mathbb{E}(M_{\exp,t+1}(R_{t+1} - R_f 1_N)) &= 0_N \\ \mathbb{E}(M_{\exp,t+1}) &= R_f^{-1}. \end{aligned}$$

The first K^{can} equations imply that

$$-\mathbb{E}(M_{\exp,t+1}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}}))) = \mathbb{E}(M_{\exp,t+1}) \times \lambda^{\text{can}},$$

which. along with Lemma IA.3.1, give

$$V_{f^{\text{can}}}(b_+^{\text{can}} + V_{f^{\text{can}}}^{-1} Q b_+^{\text{mis}}) = -\lambda^{\text{can}}. \tag{IA27}$$

Similarly, the next K^{mis} equations imply that

$$-\mathbb{E}(M_{\exp,t+1}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}}))) = \mathbb{E}(M_{\exp,t+1}) \times \lambda^{\text{mis}},$$

which, along with Lemma IA.3.1, lead to:

$$V_{f^{\text{mis}}}(b_+^{\text{mis}} + V_{f^{\text{mis}}}^{-1} Q' b_+^{\text{mis}}) = -\lambda^{\text{mis}}. \tag{IA28}$$

From expressions (IA27) and (IA28), we obtain

$$\begin{aligned} b_+^{\text{can}\prime} &= (-\lambda^{\text{can}\prime} V_{f^{\text{can}}} + \lambda^{\text{mis}\prime} V_{f^{\text{mis}}}^{-1} Q' V_{f^{\text{can}}}^{-1}) \times (I_{K^{\text{can}}} - Q V_{f^{\text{mis}}}^{-1} Q' V_{f^{\text{can}}}^{-1})^{-1}, \\ b_+^{\text{mis}\prime} &= (-\lambda^{\text{mis}\prime} V_{f^{\text{mis}}} + \lambda^{\text{can}\prime} V_{f^{\text{can}}}^{-1} Q V_{f^{\text{mis}}}^{-1}) \times (I_{K^{\text{mis}}} - Q' V_{f^{\text{can}}}^{-1} Q V_{f^{\text{mis}}}^{-1})^{-1}. \end{aligned}$$

Next, we use the condition that the SDF prices the N basis assets and Lemma IA.3.1 to derive:

$$c_+' = -a' V_e^{-1}.$$

Finally, the last identification condition implies

$$\begin{aligned} &\frac{1}{R_f} = \mathbb{E}(M_{\exp,t+1}) \\ &= \mathbb{E}(\exp(\mu_+ + b_+^{\text{can}\prime}(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})) + b_+^{\text{mis}\prime}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) + c_+' e_{t+1})) \end{aligned}$$

$$= \exp\left(\mu_+ + (b_+^{\text{can}} + V_{f^{\text{can}}}^{-1} Q b_+^{\text{mis}})' V_{f^{\text{can}}} (b_+^{\text{can}} + V_{f^{\text{can}}}^{-1} Q b_+^{\text{mis}})/2 + b_+^{\text{mis}\prime} V_{f^{\text{mis}}} b_+^{\text{mis}}/2 + c_+' V_e c_+/2\right)$$
$$= \exp(\mu_+ + b_+^{\text{can}\prime} V_{f^{\text{can}}} b_+^{\text{can}}/2 + b_+^{\text{mis}\prime} V_{f^{\text{mis}}} b_+^{\text{mis}}/2 + a' V_e^{-1} a/2 + b_+^{\text{can}\prime} Q b_+^{\text{mis}}).$$

In the last equation, we use $V_{f^{\text{mis}}} = Q' V_{f^{\text{can}}}^{-1} Q + V_{\tilde{f}^{\text{mis}}}$, where f_{t+1}^{can} and $\tilde{f}_{t+1}^{\text{mis}}$ are orthogonal, and $c' = -a' V_e^{-1}$. As a result,

$$\exp(\mu_+) = R_f^{-1} \times \exp(-b_+^{\text{can}\prime} V_{f^{\text{can}}} b_+^{\text{can}}/2 - b_+^{\text{mis}\prime} V_{f^{\text{mis}}} b_+^{\text{mis}}/2 - a' V_e^{-1} a/2 - b_+^{\text{can}\prime} Q b_+^{\text{mis}}). \quad \square$$

Next, let us introduce the projection version of the SDF $M_{\exp,t+1}$. First, note that it is convenient to express $M_{\exp,t+1}$ as

$$\begin{aligned} M_{\exp,t+1} &= \frac{1}{R_f} \times \exp\left(m_{t+1} - \frac{1}{2} m\right) \\ &= \frac{1}{R_f} \times \exp\left(m_{t+1}^{\beta,\text{can}} + m_{t+1}^{\beta,\text{mis}} + m_{t+1}^{a} - \frac{1}{2} m^{\beta,\text{can}} - \frac{1}{2} m^{\beta,\text{mis}} - \frac{1}{2} m^{a}\right), \end{aligned} \tag{IA29}$$

where

$$\begin{aligned} m_{t+1} &= m_{t+1}^{\beta,\text{can}} + m_{t+1}^{\beta,\text{mis}} + m_{t+1}^{a}, \\ m &= m^{\beta,\text{can}} + m^{\beta,\text{mis}} + m^{a}, \\ m_{t+1}^{\beta,\text{can}} &= b_+^{\text{can}\prime} (f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})), \\ m_{t+1}^{\beta,\text{mis}} &= b_+^{\text{mis}\prime} (f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})), \\ m_{t+1}^{a} &= -a' V_e^{-1} e_{t+1}, \\ m^{\beta,\text{can}} &= b_+^{\text{can}\prime} V_{f^{\text{can}}} b_+^{\text{can}} + b_+^{\text{can}\prime} Q b_+^{\text{mis}}, \\ m^{\beta,\text{mis}} &= b_+^{\text{mis}\prime} V_{f^{\text{mis}}} b_+^{\text{mis}} + b_+^{\text{can}\prime} Q b_+^{\text{mis}}, \\ m^{a} &= a' V_e^{-1} a. \end{aligned}$$

Second, set $X_{t+1} = R_{t+1} - R_f 1_N - \mu$ with $\mu = \mathbb{E}(R_{t+1} - R_f 1_N)$, $f_t = (f_t^{\text{can}\prime}, f_t^{\text{mis}\prime})'$, and $\beta = (\beta^{\text{can}}, \beta^{\text{mis}})$ and notice that $V_R = \beta V \beta' + V_e$ with $V = \begin{pmatrix} V_{f^{\text{can}}} & Q \\ Q' & V_{f^{\text{mis}}} \end{pmatrix}$.

Finally, define the projected non-negative SDF as

$$\begin{aligned} \hat{M}_{\exp,t+1} &= \frac{1}{R_f} \times \exp\left(\hat{m}_{t+1} - \frac{1}{2}\hat{m}\right), \quad \text{where} \\ \hat{m}_{t+1} &= \mathbb{E}(m_{t+1} X_{t+1}') \mathbb{E}(X_{t+1} X_{t+1}')^{-1} X_{t+1}, \\ \hat{m} &= \frac{1}{2} Var(\hat{m}_{t+1}). \end{aligned} \tag{IA30}$$

Thus,

$$\hat{m}_{t+1} = \hat{m}_{t+1}^{\beta,\text{can}} + \hat{m}_{t+1}^{\beta,\text{mis}} + \hat{m}_{t+1}^{a} \quad \text{and} \quad \hat{m} = \hat{m}^{\beta,\text{can}} + \hat{m}^{\beta,\text{mis}} + \hat{m}^{a}, \quad \text{where}$$

$$\hat{m}^{\beta,\text{can}}_{t+1} = b^{\text{can}\,\prime}_{+}(V_{f^{\text{can}}}\beta^{\text{can}\,\prime} + Q\beta^{\text{mis}\,\prime})V_R^{-1}X_{t+1} = b^{\text{can}\,\prime}_{+}(V_{f^{\text{can}}},\ Q)\beta' V_R^{-1}X_{t+1},$$
$$\hat{m}^{\beta,\text{mis}}_{t+1} = b^{\text{mis}\,\prime}_{+}(Q'\beta^{\text{can}\,\prime} + V_{f^{\text{mis}}}\beta^{\text{mis}\,\prime})V_R^{-1}X_{t+1} = b^{\text{mis}\,\prime}_{+}(Q',\ V_f^{\text{mis}})\beta' V_R^{-1}X_{t+1},$$
$$\hat{m}^{a}_{t+1} = c'_{+}V_e V_R^{-1}X_{t+1},$$
$$\hat{m}^{\beta,\text{can}} = b^{\text{can}\,\prime}_{+}(V_{f^{\text{can}}},Q)\beta' V_R^{-1}\beta(V_{f^{\text{can}}},Q)' b^{\text{can}}_{+} + b^{\text{can}\,\prime}_{+}(V_{f^{\text{can}}},Q)\beta' V_R^{-1}\beta(Q',V_{f^{\text{mis}}})' b^{\text{mis}}_{+},$$
$$\hat{m}^{\beta,\text{mis}} = b^{\text{mis}\,\prime}_{+}(Q',V_{f^{\text{mis}}})\beta' V_R^{-1}\beta(Q',V_{f^{\text{mis}}})' b^{\text{mis}}_{+} + b^{\text{can}\,\prime}_{+}(V_{f^{\text{can}}},Q)\beta' V_R^{-1}\beta(Q',V_{f^{\text{mis}}})' b^{\text{mis}}_{+},$$
$$\hat{m}^{a} = c'_{+}V_e V_R^{-1}V_e c_{+}.$$

Proposition IA.7.8 (Asymptotic Properties of the SDF Projections: Correlated Case). *Under the assumptions of Proposition IA.4.1, as $N \to \infty$, $M_{\exp,t+1}$ and $\hat{M}_{\exp,t+1}$ of (IA29) and (IA30) satisfy*

$$\hat{M}_{\exp,t+1} - M_{\exp,t+1} \xrightarrow{p} 0.$$

Proof: We have, as $N \to \infty$,

$$c'_{+}V_e V_R^{-1}e_{t+1} - c'_{+}e_{t+1} \xrightarrow{p} 0,$$
$$\beta' V_e V_R^{-1}c_{+} \longrightarrow 0_{K^{\text{can}}+K^{\text{mis}}},$$
$$\beta' V_R^{-1}e_{t+1} \xrightarrow{p} 0_{K^{\text{can}}+K^{\text{mis}}},$$
$$\beta' V_R^{-1}\beta \longrightarrow V_f^{-1},$$

and therefore,

$$\begin{aligned}\hat{m}^{\beta,\text{can}}_{t+1} &= b^{\text{can}\,\prime}_{+}(V_{f^{\text{can}}},\quad Q)\beta' V_R^{-1}X_{t+1} \xrightarrow{p} b^{\text{can}\,\prime}_{+}(V_{f^{\text{can}}},\quad Q)V^{-1}(f_{t+1} - \mathbb{E}(f_{t+1}))\\ &= b^{\text{can}\,\prime}_{+}(I_{K^{\text{can}}},\quad O_{K^{\text{can}}})(f_{t+1} - \mathbb{E}(f_{t+1})) = b^{\text{can}\,\prime}_{+}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})),\end{aligned} \tag{IA31}$$

and

$$\begin{aligned}\hat{m}^{\beta,\text{mis}}_{t+1} &= b^{\text{mis}\,\prime}_{+}(Q,\quad V_{f^{\text{mis}}})\beta' V_R^{-1}X_{t+1} \xrightarrow{p} b^{\text{mis}\,\prime}_{+}(Q',\quad V_{f^{\text{can}}})V^{-1}(f_{t+1} - \mathbb{E}(f_{t+1}))\\ &= b^{\text{mis}\,\prime}_{+}(O_{K^{\text{mis}}},\quad I_{K^{\text{mis}}})(f_{t+1} - \mathbb{E}(f_{t+1})) = b^{\text{mis}\,\prime}_{+}(f^{\text{mis}}_{t+1} - \mathbb{E}(f^{\text{mis}}_{t+1})).\end{aligned} \tag{IA32}$$

Given that $\beta' V_e V_R^{-1}c_{+} \longrightarrow 0_{K^{\text{can}}+K^{\text{mis}}}$ and $c'_{+}V_e V_R^{-1}e_{t+1} - c'_{+}e_{t+1} \xrightarrow{p} 0$, as $N \to \infty$, then

$$c'_{+}V_e V_R^{-1}X_{t+1} - c'_{+}e_{t+1} \xrightarrow{p} 0, \quad \text{as} \quad N \to \infty. \tag{IA33}$$

As a result, expressions (IA31), (IA32), and (IA33) imply that

$$\hat{m}_{t+1} - m_{t+1} \xrightarrow{p} 0, \quad \text{as} \quad N \to \infty, \tag{IA34}$$

Similarly, the result that, as $N \to \infty$, $\beta' V_R^{-1}\beta \longrightarrow V_f^{-1}$, implies that

$$\hat{m}^{\beta\,\text{can}} \longrightarrow b^{\text{can}\,\prime}_{+}V_{f^{\text{can}}}b^{\text{can}}_{+} + b^{\text{can}\,\prime}_{+}Qb^{\text{mis}}_{+} = m^{\beta,\text{can}}, \quad \text{as} \quad N \to \infty, \quad \text{and} \tag{IA35}$$
$$\hat{m}^{\beta\,\text{mis}} \longrightarrow b^{\text{mis}\,\prime}_{+}V_{f^{\text{mis}}}b^{\text{mis}}_{+} + b^{\text{can}\,\prime}_{+}Qb^{\text{mis}}_{+} = m^{\beta,\text{mis}}, \quad \text{as} \quad N \to \infty. \tag{IA36}$$

Notice that

$$\hat{m}^a = c_+' V_e V_R^{-1} V_e c_+ = a' V_e^{-1} V_e V_R^{-1} V_e V_e^{-1} a = a' V_R^{-1} a$$

and recall that the proof of Proposition IA.4.1 shows that $a' V_R^{-1} a - a' V_e^{-1} a \longrightarrow 0$, as $N \to \infty$, which implies that

$$\hat{m}^a - m^a \longrightarrow 0, \quad \text{as} \quad N \to \infty, \tag{IA37}$$

Expressions (IA35), (IA36), and (IA37) imply that

$$\hat{m} - m \longrightarrow 0, \quad \text{as} \quad N \to \infty. \tag{IA38}$$

From the results in expressions (IA34) and (IA38), we obtain

$$\hat{M}_{\exp,t+1} - M_{\exp,t+1} \xrightarrow{p} 0, \quad \text{as} \quad N \to \infty. \qquad \square$$

Remark: Note that because f_{t+1}^{can} are observable factors, in empirical work, we may use the exact component $m_{t+1}^{\beta,\text{can}} + m^{\beta,\text{can}}$ rather than its projected counterpart $\hat{m}_{t+1}^{\beta,\text{can}} + \hat{m}^{\beta,\text{can}}$.

IA.8 Data description

To examine which economic variables may explain variation in the SDF, we collect the returns on a set of 457 trading strategies and 103 macroeconomic and financial indicators.

The set of trading strategies includes:

- 205 strategies from Chen and Zimmermann (2022).
- 153 strategies in the Global Factor Dataset from Jensen et al. (2022).
- 55 strategies from Kozak et al. (2020).
- 35 strategies from Bryzgalova et al. (2023). The sources of these strategies are specified in their Internet Appendix. Their dataset includes 34 trading strategies, but we consider two versions of the size strategy, one from Fama and French (1993) and the other from Fama and French (2015).
- We add the following nine strategies:
  - Industry-adjusted value, momentum, and profitability factors; intra-industry value, momentum, and profitability factors; profitable-minus-unprofitable factor from Novy-Marx (2013), available from `http://rnm.simon.rochester.edu`.
  - Expected-growth factor of Hou et al. (2021), available from `https://global-q.org/index.html`.

  - Up-minus-down (UMD) factor from the AQR data library, available from https://www.aqr.com/Insights/Datasets.

The set of macroeconomic and financial indicators includes:

- 53 variables constructed from 17 variables from Bryzgalova et al. (2023). Below we explain how we get to 53 variables.
  - For indices of financial uncertainty, real uncertainty, and macroeconomic uncertainty, we consider time horizons of 1, 3, and 12 months. We use these variables in levels and consider their AR(1) innovations, for a total of 18 variables.
  - For the investor-sentiment measures of Baker and Wurgler (2006) and Huang et al. (2015), labeled as BW_INV_SENT and HJTZ_INV_SENT, respectively, we consider both the orthogonalized and non-orthogonalized versions. We use these variables in levels and consider AR(1) innovations of these variables for a total of 8 variables.
  - For other persistent variables, such as the term spread (TERM), change in the difference between a 10-year Treasury bond yield and a 3-month Treasury bill yield (DELTA_SLOPE), credit spread (CREDIT), dividend yield (DIV), price-earnings ratio (PE), unemployment rate (UNRATE), the growth rate of industrial production (IND_PROD), the monthly growth rate of the Producer Price Index for Crude Petroleum (OIL), we look at both levels and first-order differences, for a total of 16 variables.
  - Real per capita consumption growth on nondurable goods and services separately and jointly. We also include the 3-year consumption growth (nondurable goods and services) and its AR(1) innovations, for a total of 5 variables.
  - Inflation, computed as the log-difference in the price index for both nondurable goods and services and its AR(1) innovations, for a total of 2 variables.
  - The level of the intermediary-capital ratio and its innovations, for a total of 2 variables.
  - The level of the aggregate liquidity factor and its innovations.
- The first 3 principal components and their VAR(1) innovations for the 279 macroeconomic variables from Jurado et al. (2015), for a total of 6 variables.
- The first eight principal components and their VAR(1) innovations for the 128 macroeconomic variables from the FRED-MD dataset of McCracken and Ng (2015), gives a total of 16 variables. We obtain these macro variables from https://research.stlouisfed.org/econ/mccracken/fred-databases and use the data vintage for

February 2021. We exclude four variables, ACOGNO, ANDENOx, TWEXAFEGSMTHx, and UMCSENT, which have missing observations at the start of the sample.

- Consumer sentiment and its first-order differences.
- The market-dislocation index of Pasquariello (2014), its first-order differences, and AR(1) innovations.
- The disagreement index of Huang et al. (2021) and its first-order differences.
- The Chicago Board Options Exchange (CBOE) volatility index (VIX) available on the website of the CBOE, its first-order differences, and AR(1) innovations.
- The U.S. economic policy uncertainty index (EPU) of Baker et al. (2016) and the equity market volatility (EMV) tracker of Baker et al. (2019), which are available from `www.policyuncertainty.com`. For both indices, we also consider their first-order differences and AR(1) innovations.
- The U.S. business-confidence index, the U.S. consumer-confidence index, and the U.S. composite leading indicator from the OECD library.
- The coincident economic-activity index and its first-order differences from `https://fred.stlouisfed.org/series/USPHCI`.
- The NBER recession index from `https://fred.stlouisfed.org/series/USREC`.
- The TED spread from `https://fred.stlouisfed.org/series/TEDRATE`.
- The effective federal funds rate and the real federal funds rate from `https://fred.stlouisfed.org/series/FEDFUNDS`.
- The credit-spread index (Gilchrist and Zakrajšek, 2012) and its first order differences.
- The Chicago Fed National Financial Condition Index from `https://fred.stlouisfed.org/series/NFCI`.

IA.9 Additional Tables

This section contains additional tables supporting the interpretation of our empirical results described in the manuscript.

Table IA.1: The Admissible SDF under the APT and observable variables
This table reports the explanatory power of the selected variables for the unsystematic (Panel A) and systematic (Panel B) components of the SDF implied by the APT model of asset returns.

	$R^2(\%)$	p-value
Panel A: $\log(M^{a}_{\text{exp},t+1})$		
NBER recession indicator	0.18	0.27
Intermediary constraints (He et al., 2017)	2.71	0.00
Sentiment index (Baker and Wurgler, 2006)	2.60	0.00
Sentiment index (Huang et al., 2015)	3.41	0.00
Shocks in credit spread (Gilchrist and Zakrajšek, 2012)	1.92	0.00
Shocks in VIX	2.24	0.00
Panel B: $\log(M^{\beta}_{\text{exp},t+1})$		
NBER recession indicator	0.76	0.02
Chicago Fed National Financial Condition Index	2.64	0.00
Intermediary constraints (He et al., 2017)	55.15	0.00
Shocks in aggregate liquidity (Pástor and Stambaugh, 2003)	11.12	0.00
Shocks in credit spread (Gilchrist and Zakrajšek, 2012)	13.79	0.00
Shocks in dividend yield (Campbell, 1996)	40.39	0.00
Shocks in financial uncertainty (Jurado et al., 2015)	11.09	0.00
Shocks in VIX	54.32	0.00
TED spread	4.39	0.00

Table IA.2: What explains the systematic SDF component?
This table shows the explanatory power of returns on five trading strategies for the systematic SDF component, $\log(\hat{M}^{\beta}_{\text{exp},t+1})$, obtained when estimating the APT model of asset returns.

	Estimate	Std. error	t-statistic	p-value
Intercept	0.01	0.00	13.73	0.00
MKT	−3.25	0.02	−171.09	0.00
Sales-to-Market	−0.49	0.02	−24.62	0.00
Dollar Volume	0.77	0.03	26.27	0.00
Bid-Ask Spread	0.20	0.01	20.55	0.00
Zero Trade Days	0.21	0.02	9.18	0.00
$R^2 = 0.9905$, $R^2_{adj} = 0.9904$				

IA.10 Additional Figures

Figure IA.1: Spanning the unsystematic SDF component

The blue curve shows the R^2 (left axis) of 325 regressions of $\log(\hat{M}^a_{\exp,t+1})$ on the returns of trading strategies that are available for the entire sample. The first regression includes the return on the trading strategy that explains the most variation in $\log(\hat{M}^a_{\exp,t+1})$; each subsequent regression includes the return on an extra trading strategy that adds the most to explain the variation in $\log(\hat{M}^a_{\exp,t+1})$. The red curve shows the value of the Bayesian Information Criteria (BIC) (right axis) associated with these regressions. The minimal BIC$= -552.86$ is for the regression with 39 explanatory variables; the associated $R^2 = 66.45\%$.

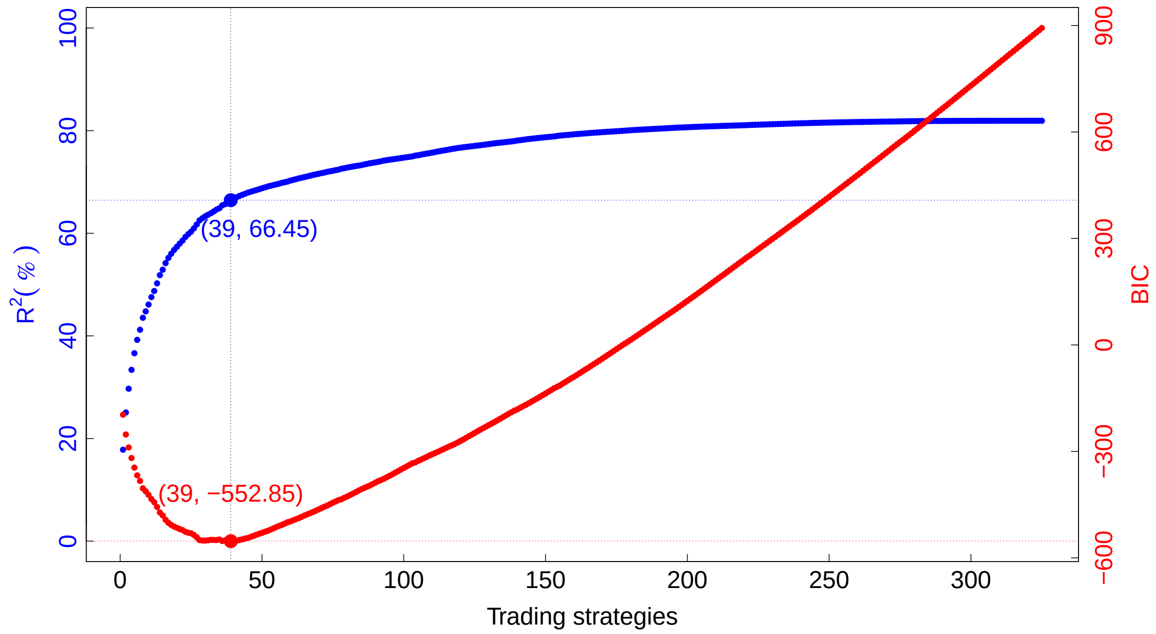

Figure IA.2: Spanning the systematic SDF component

The blue curve shows the R^2 (left axis) for 325 regressions of $\log(\hat{M}^{\beta}_{\text{exp},t+1})$ on returns of trading strategies. The first regression includes the return on the trading strategy that explains the most variation in $\log(\hat{M}^{\beta}_{\text{exp},t+1})$; each subsequent regression includes the return on another trading strategy that adds the most to explain the variation in $\log(\hat{M}^{\beta}_{\text{exp},t+1})$. The red curve shows the BIC values (right axis) associated with these regressions. The minimal BIC$= -4285.57$ is for the regression with 54 explanatory variables; the associated $R^2 = 99.73\%$.

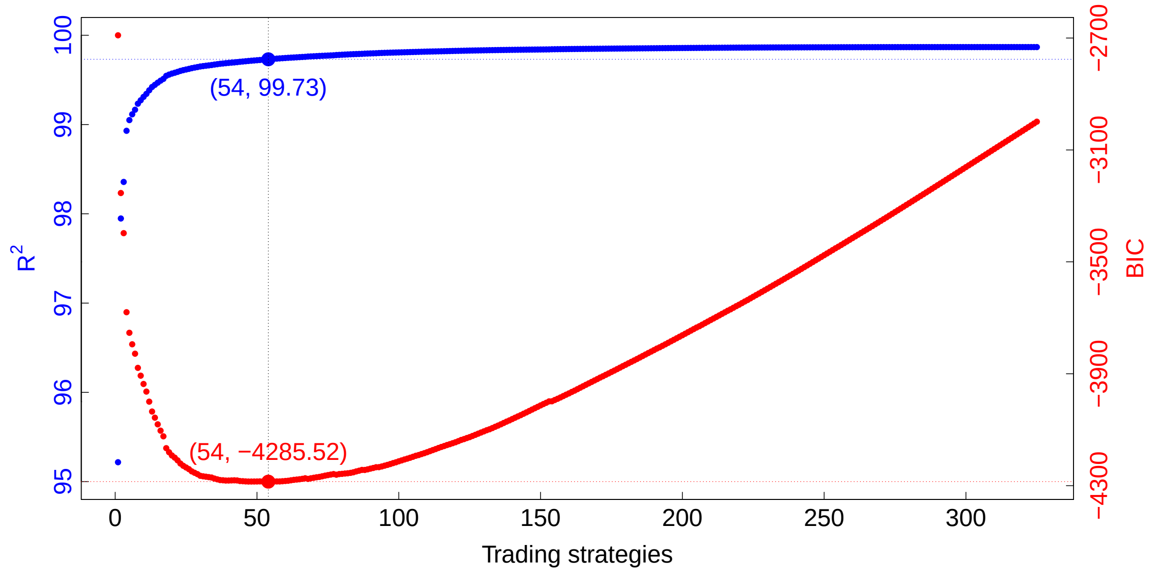

Figure IA.3: Correcting the CAPM

This figure illustrates how the HJ distance changes with K^{mis} and δ_{apt}, when the candidate model includes only the market return as a systematic factor. The top panel shows the estimation results based on cross-validation with 10 folds. The bottom panel shows the in-sample results.

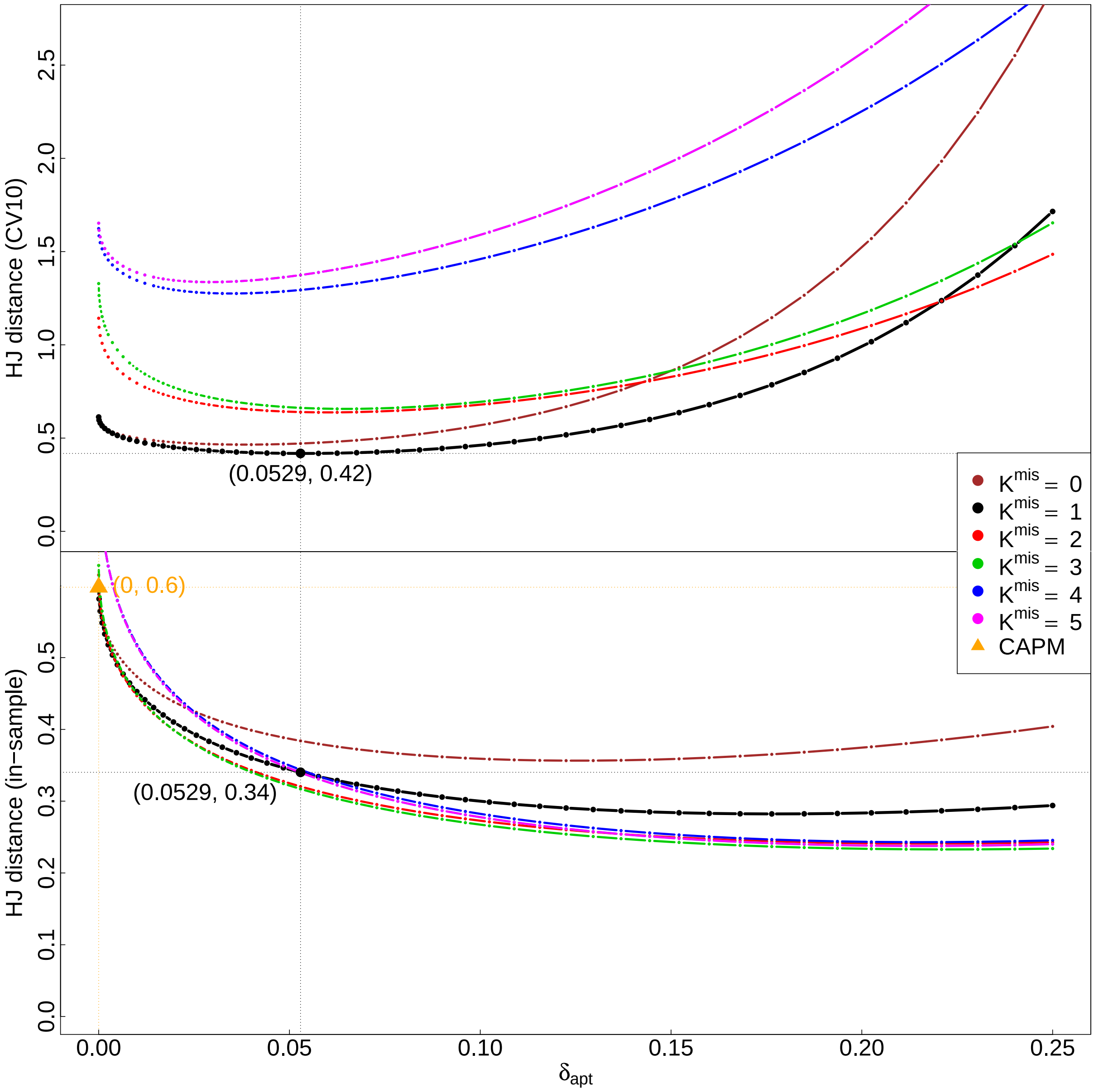

Figure IA.4: Time-series of SDF and its components for the corrected CAPM

This figure has four panels, which show the dynamics of the admissible SDF, $\hat{M}_{\text{exp},t+1}$ and its three components: the unsystematic component $\hat{M}^{a}_{\text{exp},t+1}$, the component $\hat{M}^{\beta,\text{can}}_{\text{exp},t+1}$ corresponding to the candidate model with the market factor, and the missing systematic component $\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1}$. Gray bars indicate NBER recession periods.

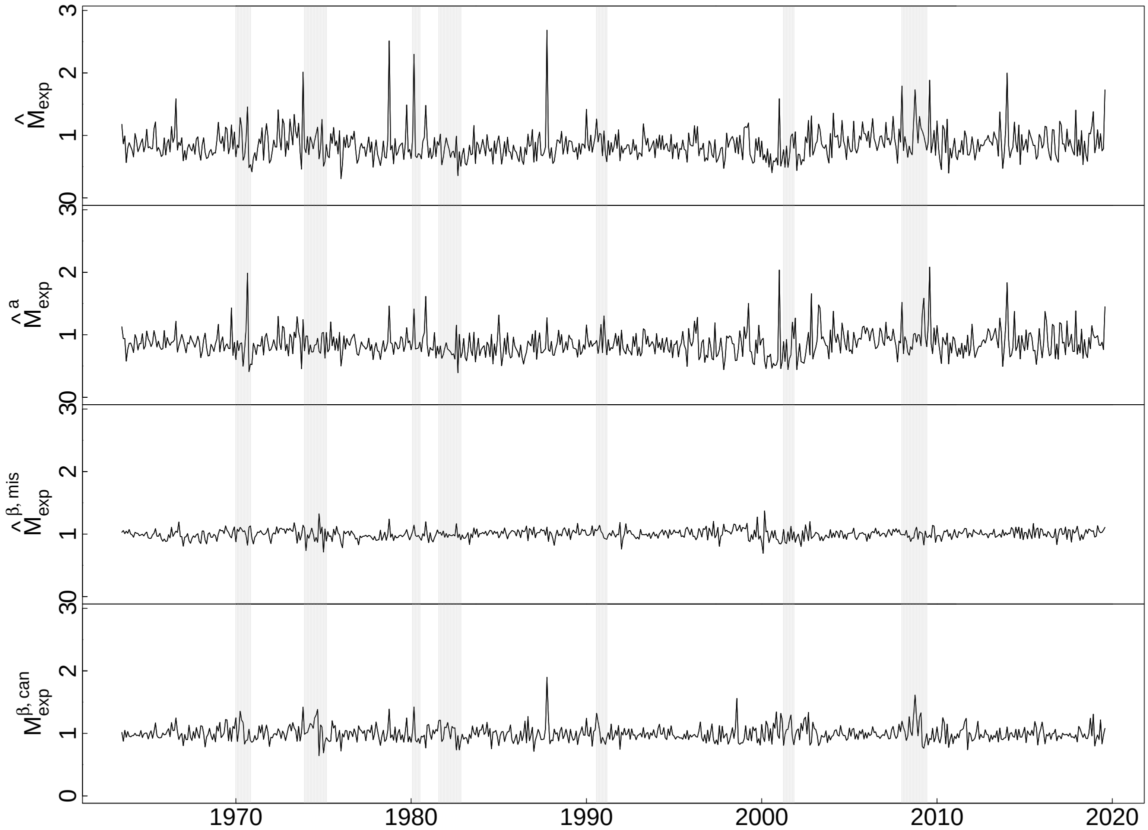

Figure IA.5: Pricing errors in the candidate and corrected CAPM

This plot displays the pricing errors in the candidate and corrected CAPM models. The red dots indicate the pricing errors for the 202 basis assets using the candidate CAPM model. The blue dots indicate the pricing errors using the corrected CAPM model.

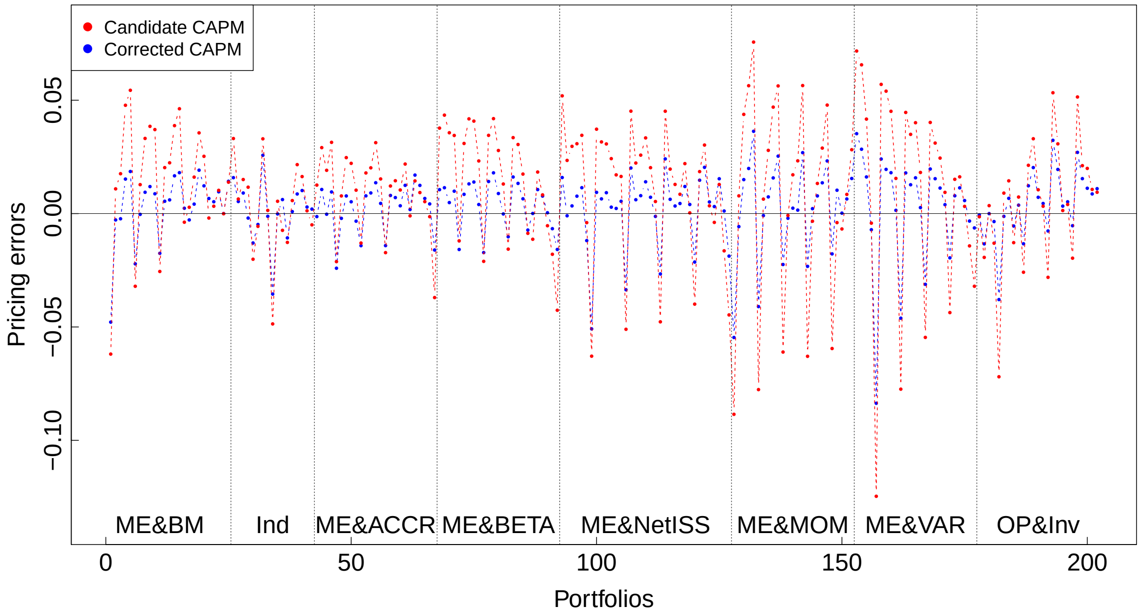

Figure IA.6: Correcting the C-CAPM

This figure illustrates how the HJ distance changes with K^{mis} and δ_{apt}, when the candidate model includes only the return on the consumption-mimicking portfolio of Breeden et al. (1989). The top panel shows the estimation results based on cross-validation with 10 folds. The bottom panel shows the in-sample results.

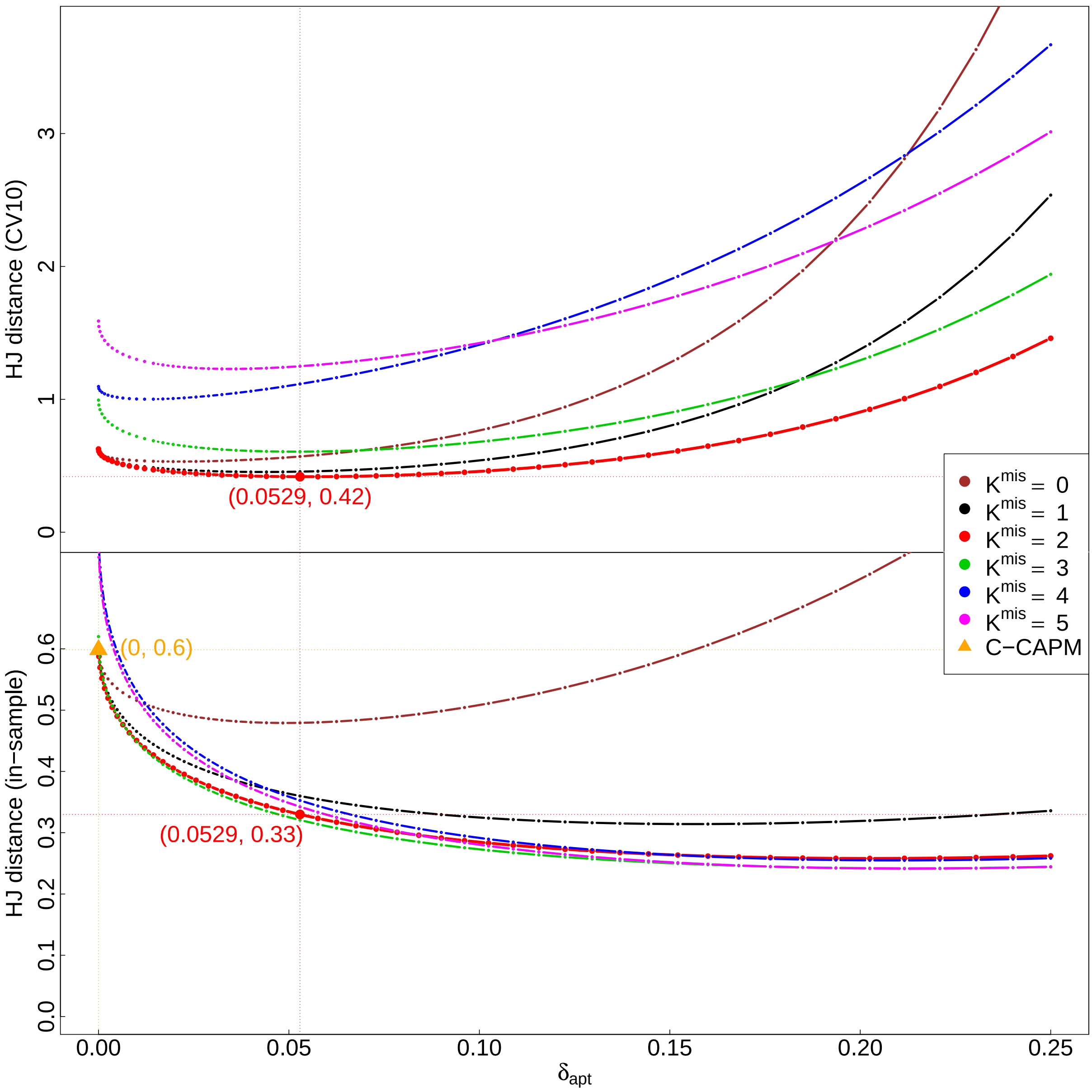

Figure IA.7: Time-series of SDF and its components for the corrected C-CAPM

This figure has four panels, which show the dynamics of the admissible SDF $\hat{M}_{\text{exp},t+1}$ and its three components: the unsystematic component $\hat{M}^{a}_{\text{exp},t+1}$, the component $\hat{M}^{\beta,\text{can}}_{\text{exp},t+1}$ corresponding to the candidate model with the consumption mimicking portfolio as the sole factor, and the missing systematic component $\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1}$. Gray bars indicate NBER recession periods.

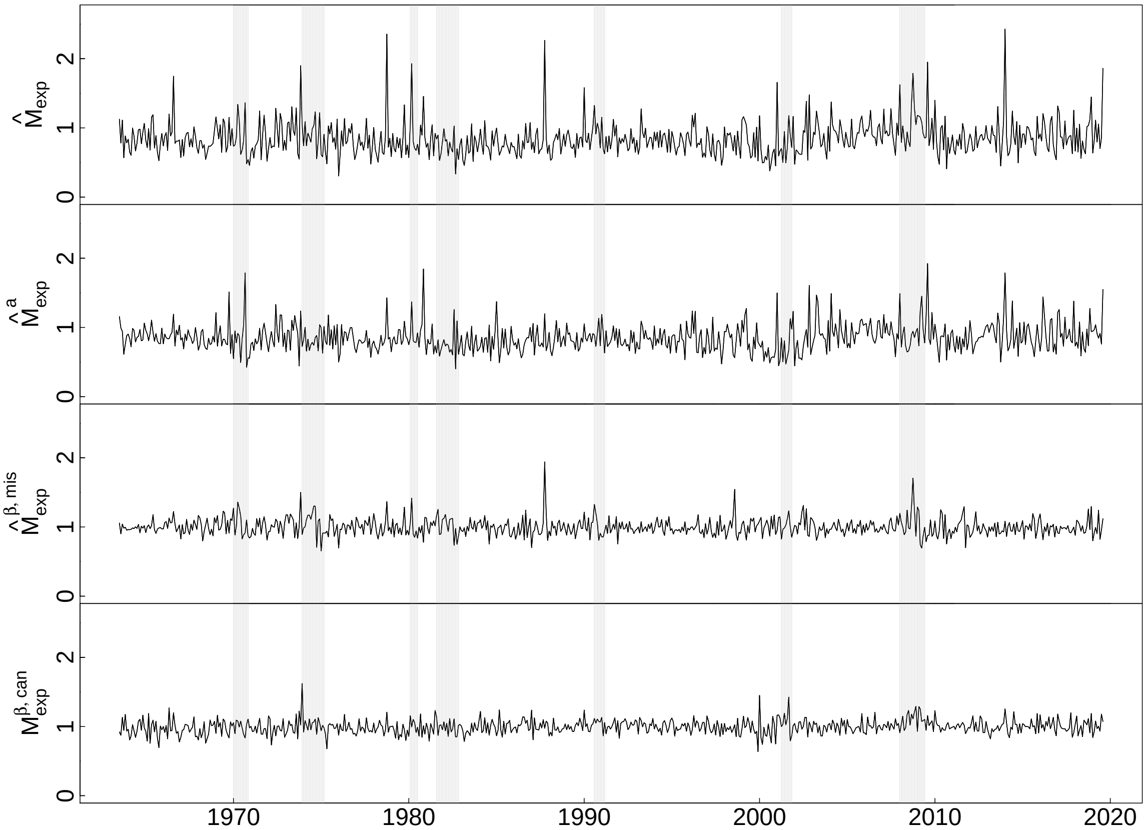

Figure IA.8: Correcting the FF3

This figure illustrates how the HJ distance changes with K^{mis} and δ_{apt}, when the candidate model is the three-factor model of Fama and French (1993). The top panel shows the estimation results based on cross-validation with 10 folds. The bottom panel shows the in-sample results.

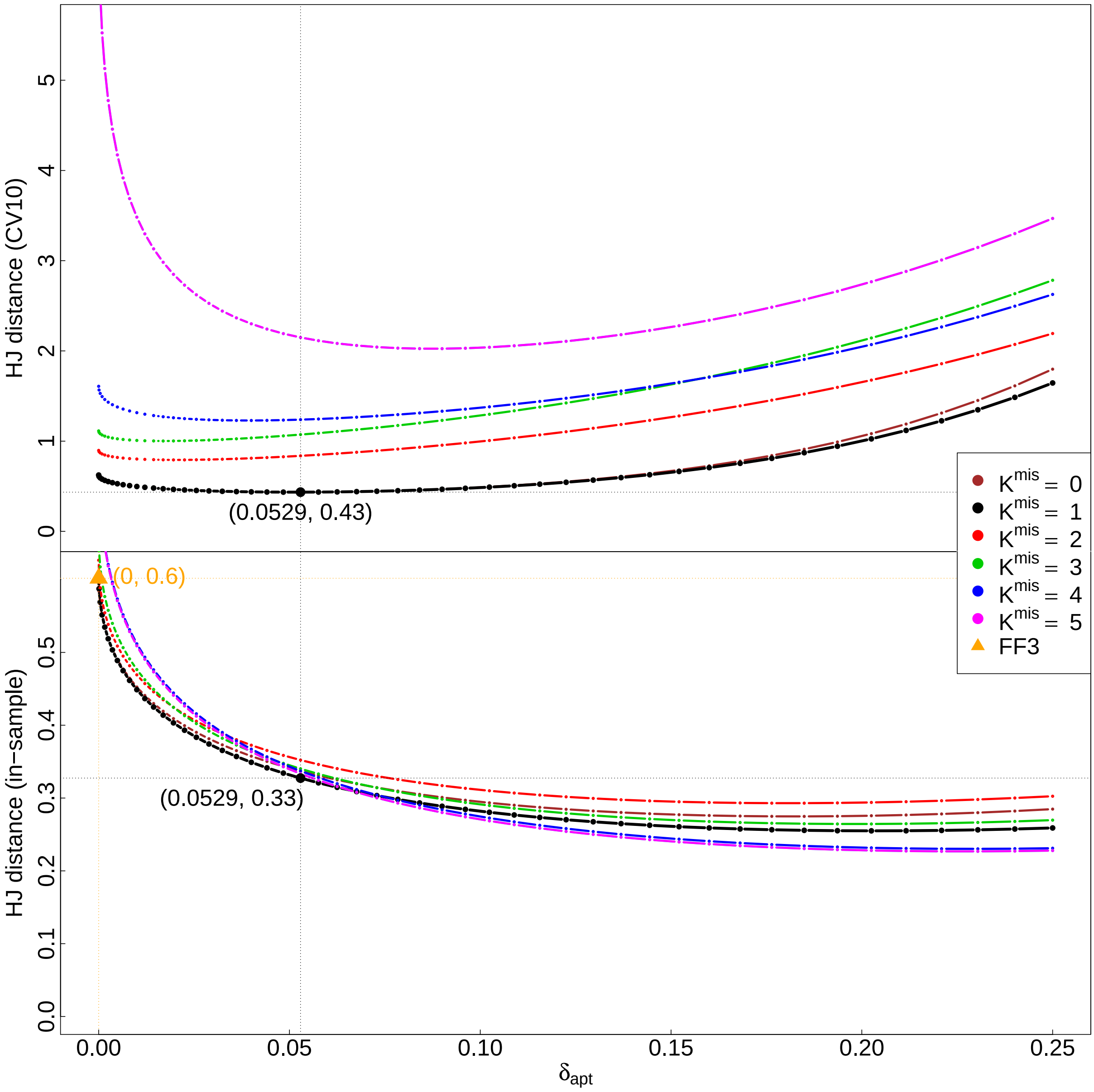

Figure IA.9: Time-series of SDF and its components for the corrected FF3 model

This figure has four panels, which show the dynamics of the admissible SDF, $\hat{M}_{\exp,t+1}$ and its three components: the unsystematic component $\hat{M}^{a}_{\exp,t+1}$, the component $\hat{M}^{\beta,\mathrm{can}}_{\exp,t+1}$ corresponding to the candidate FF3 model, and the missing systematic component $\hat{M}^{\beta,\mathrm{mis}}_{\exp,t+1}$. Gray bars indicate NBER recession periods.

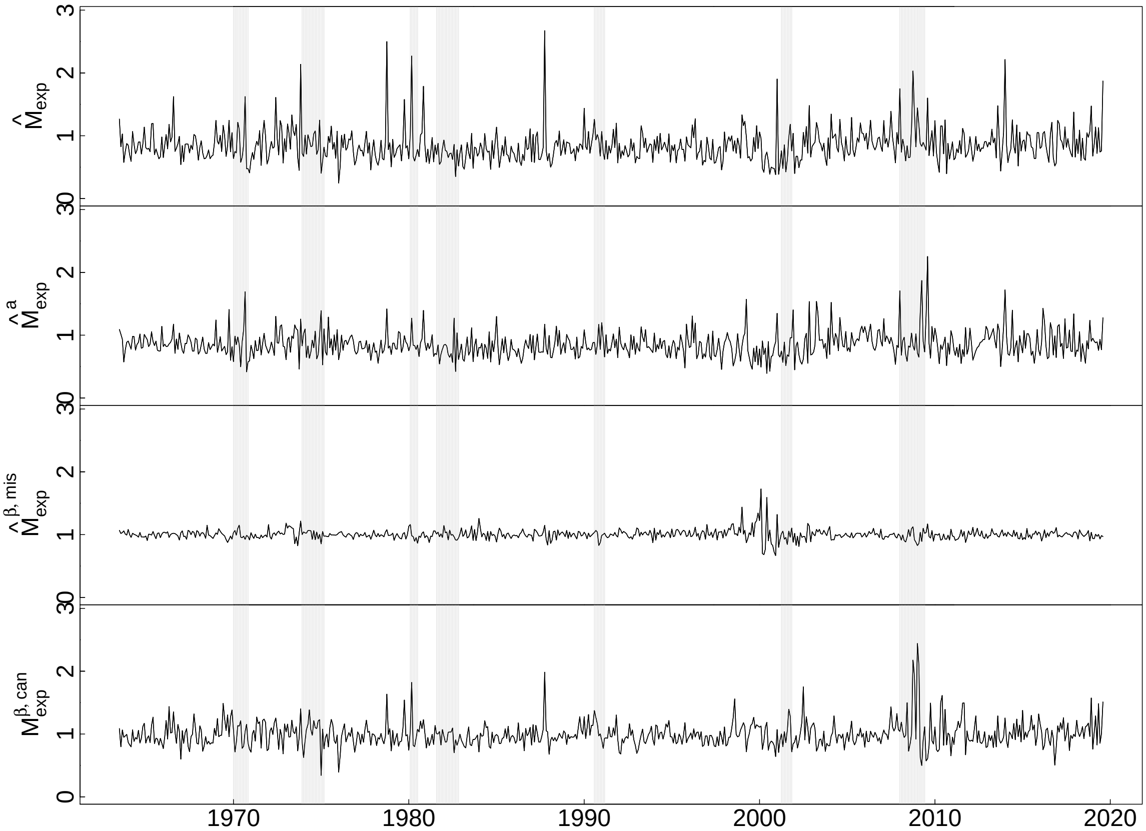

Figure IA.10: Pricing errors in the candidate and corrected FF3 model

This plot shows the pricing errors in the candidate and corrected FF3 models. The red dots indicate the pricing errors for the 202 basis assets using the candidate FF3 model. The blue dots indicate the pricing errors using the corrected FF3 model.