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Recursive Preferences and Ambiguity Attitudes

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Abstract

We illustrate the strong implications of recursivity, a standard assumption in dynamic environments, on attitudes toward uncertainty. In intertemporal consumption choice problems, recursivity always implies constant absolute ambiguity aversion (CAAA) when applying the standard dynamic extension of monotonicity. Our analysis also yields a functional equation called "generalized rectangularity", as it generalizes the standard notion of rectangularity for recursive maxmin preferences to general certainty equivalents. Our results highlight that if uncertainty aversion is modeled as a form of convexity of preferences, recursivity limits us to only recursive variational preferences.

Keywords: Dynamic choice, recursive utility, uncertainty aversion, absolute attitudes, generalized rectangularity.

JEL classification: C61, D81.

1 Introduction

Recursive preferences are a key tool for dynamic economic models. In dynamic models with strategic interaction, they have recently been used to study repeated games

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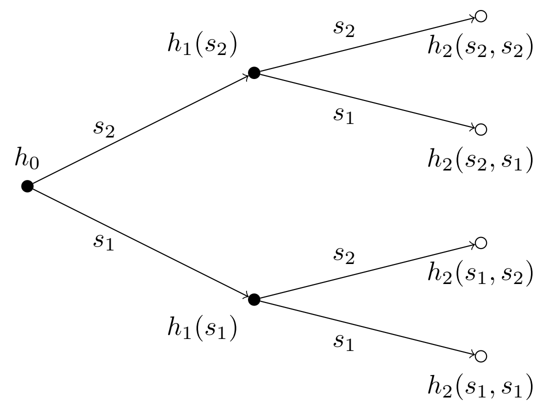

Figure 1: Illustration of a consumption program.

(Kochov and Song (2021)) and Bayesian Persuasion (Pahlke (2022)). They are the workhorse in macroeconomics and finance (e.g., see the literature review in Backus et al. (2004)) for studying a variety of different problems, ranging from optimal taxation to asset pricing. Recursivity entails several restrictions on dynamic choice behavior, among which one of the key assumptions is a notion of time or dynamic consistency, i.e., that at every time period the decision maker will carry out the plan of actions that was determined to be optimal ex-ante. The assumption of recursivity provides analytical tractability in that it permits the use of well-known tools from dynamic programming.

In this paper, we show that recursivity has strong restrictions on attitudes toward uncertainty, i.e., how uncertainty attitudes change when individuals become better off overall. We focus on a major class of dynamic choice problems, which we refer to as intertemporal consumption choice problems. These problems take place over long horizons, and the source of utility is a consumption stream. Figure 1 offers a graphical representation of a (stochastic) consumption stream. At every period, a shock $s \in S$ is realized, and the total sequence of shocks determines the consumption level at any given time period. For such problems the implications of recursivity for ambiguity attitudes depend on the notion of monotonicity that is employed. Under the standard notion of monotonicity adopted in the literature (see e.g. Epstein and Schneider (2003b) and Maccheroni et al. (2006b)), our first result (Theorem 2 and Corollary 1) shows that recursive preferences *always* satisfy constant absolute ambiguity aversion (CAAA). As a byproduct, we obtain a generalized "rectangularity" condition for recursive preferences that satisfy this notion of recursivity. Similarly to rectangularity from Epstein and Schneider (2003a, 2003b), generalized rectangularity characterizes "generalized beliefs"—as modeled by certainty equivalents—that are dynamically consistent. Our generalized "rectangularity" condition for an ex-ante and one-step-ahead certainty equivalent can be written as:

$$I_0(\xi) = \beta I_{+1}\left(I_0\left(\frac{1}{\beta}\xi^1\right)\right), \tag{1}$$

for every random variable ξ, where I_0 and I_{+1} are certainty equivalents reflecting ex-ante and one-step-ahead beliefs, ξ^1 denotes a shift operator applied to the random variable ξ, and $\beta \in (0,1)$ is a discount factor. Condition (1) can be seen as a generalized law of iterated expectations in a dynamic setting with intertemporal consumption. The condition has a simple interpretation. Beliefs over an entire sequence of states can be decomposed into beliefs over the next-period state and conditional beliefs over the entire sequence of states. The above functional equation highlights that in order to elicit the ex-ante certainty equivalent I_0—which can be interpreted as describing beliefs about the entire realizations of shocks—it is enough to elicit the one-step-ahead certainty equivalent. Our Theorem 3 provides the formalization of the previous observation: given the one-step ahead certainty equivalent I_{+1} generalized rectangularity allows the analyst to recover I_0.[1] In the special case of recursive maxmin expected utility (MEU) prefererences condition (1) is shown to be equivalent to the sets of ex-ante beliefs $\mathcal{P}$ and one-step-ahead beliefs $\mathcal{L}$ be connected by the relationship

$$\min_{P\in\mathcal{P}} P(A) = \min_{\ell\in\mathcal{L}}\left[\sum_{s\in S}\ell(s)\min_{P\in\mathcal{P}} P\left(A_s\right)\right]$$

which is equivalent to the notion of rectangularity introduced by Epstein and Schneider (2003a).[2] Another important implication is that if one assumes that preferences, on top of recursivity, satisfy the notion of uncertainty aversion introduced by Gilboa and Schmeidler (1989) we have that preferences must admit a variational representation in the sense of Maccheroni et al. (2006a). More specifically, following Cerreia-Vioglio et al. (2011), suppose that there are a utility index u and a quasiconvex function G such that one has the recursive representation

$$V(h) = u(h_0) + \beta \inf_{\ell\in\Delta(S)} G\left(\sum_{s\in S}\ell(s)\left[V\circ h^{s,1}\right],\ell\right),$$

[1] If the functional equation does not admit an analytic solution, I_0 can be determined by means of the numerical methods we develop.

[2] See Equation 2.4 in their paper.

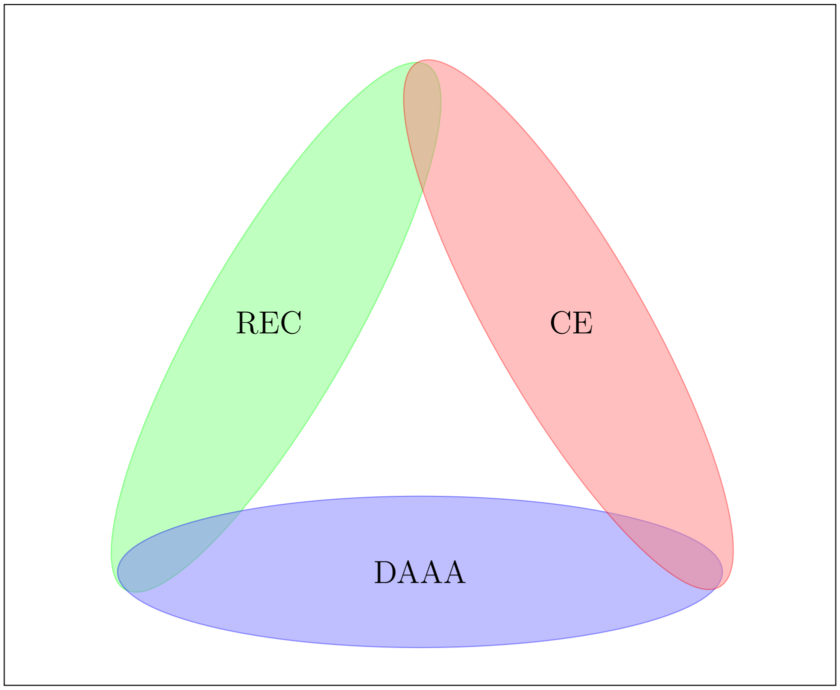

Figure 2: A decision-theoretic trilemma: no preferences satisfy recursivity (REC), non-trivial decreasing absolute ambiguity aversion (DAAA) and admit a representation by an ex-ante certainty equivalent (CE).

our results imply that preferences the aggregator G has the form

$$G(t, \ell) = t + c(\ell),$$

where $c : \Delta(S) \to [0, \infty]$ is a convex function. To appreciate the relevance of these results, observe that constant absolute ambiguity aversion is known to be inconsistent with experimental evidence, as shown for example in Baillon and Placido (2019).[3] Our results therefore suggest, broadly speaking, the following modeling trade-off: one has to choose between tractability—as reflected by recursivity (REC), experimental validity, represented by non-trivial decreasing absolute ambiguity aversion (DAAA), and preferences that admit a representation by means of an ex-ante certainty equivalent (CE). This trilemma is depicted in Figure 2. Because of this problem, for intertemporal consumption choice problems we propose a different notion of monotonicity, which

[3]See for example p. 325 of that paper: "Our findings seem to encourage the use of ambiguity models that are flexible enough to accommodate changes in ambiguity attitudes at increased utility levels."

we refer to as state-time monotonicity (Axiom I.2). This notion is a basic consistency principle which requires that an uncertain consumption plan is preferred to another, whenever such a ranking holds jointly at any possible state of the world and at any possible time period. State-time monotonicity allows us to keep DAAA and REC together while dropping CE.

Finally, building on Epstein (1992), we also consider a different major formalization of a dynamic choice problem: sequential choice problems. Sequential choice problems take place over short intervals of time during which consumption plans can be taken to be fixed, and the source of utility is terminal wealth as opposed to the latter case in which it is given by a consumption stream. Sequential choice problems are typically employed when dealing with updating rules (see e.g., Pires (2002), Klibanoff and Hanany (2007)). In this setting, the implications of recursivity are more nuanced. Savochkin et al. (2022), inter alia, show that CAAA is implied by recursivity when preferences admit a smooth ambiguity representation. We show that in general recursivity imposes no restriction on uncertainty attitudes for sequential choice problems. We provide an example of recursive preferences that can allow for unrestricted uncertainty attitudes. However, our examples feature probabilistic sophistication, leaving the door open as to whether the result holds for more general classes of preferences that are not probabilistically sophisticated.

1.1 Organization of the paper

Section 2 introduces the notation and the main choice-theoretic objects used in the paper. The implications for intertemporal consumption choice problems in terms of ambiguity attitudes are studied in Section 2.3. Section 4.3 discusses the case of sequential choice problems. Section 5 concludes with a discussion of the results in light of the existing literature. All the proofs can be found in the Appendix.

2 Framework

2.1 Static choice problems and mathematical preliminaries

Let Ω be a nonempty set of *states of the world* and $\mathcal{G}$ an algebra of *events* over it. By X we denote a convex subset of a vector space, interpreted as a set of *consequences.*

A function $f : \Omega \to X$ is said to be a (simple) *act* if it is $\mathcal{G}$-measurable and $f(\Omega)$ is finite; the set of acts is denoted by $\mathbf{F}$. As usual we identify X as a subset of $\mathbf{F}$. We denote by $\succsim$ a binary relation over $\mathbf{F}$, by $\sim$ and $\succ$ its symmetric and asymmetric parts, respectively. A function $V : \mathbf{F} \to \mathbb{R}$ represents $\succsim$ if

$$f \succsim g \iff V(f) \geq V(g)$$

for all $f, g \in \mathbf{F}$. Fix $K \subseteq \mathbb{R}$, we denote by $B_0(K, \Omega, \mathcal{G})$ the set of bounded simple $\mathcal{G}$-measurable functions taking values in K. We equip this space with the supnorm, $\|\cdot\|_\infty$, and denote by $B(K, \Omega, \mathcal{G})$ its closure (i.e., the space of bounded $\mathcal{G}$-measurable functions). We set $B_0(\Omega, \mathcal{G}) := B_0(\mathbb{R}, \Omega, \mathcal{G})$ and $B(\Omega, \mathcal{G}) := B(\mathbb{R}, \Omega, \mathcal{G})$. For all $A \in \mathcal{G}$, we denote by $\mathbf{1}_A$ its indicator function. Fix $K \subseteq \mathbb{R}$, a functional $I : B(K, \Omega, \mathcal{G}) \to \mathbb{R}$ is said to be *normalized* if $I(k\mathbf{1}_\Omega) = k$ for all $k \in K$. We say that I is *monotone* if $I(\xi) \geq I(\xi')$ whenever $\xi \geq \xi'$, for all $\xi, \xi' \in B(K, \Omega, \mathcal{G})$. If I is both monotone and normalized, then it is said to be a *certainty equivalent*. Moreover, we say that I is *translation invariant* if $I(\xi + k\mathbf{1}_\Omega) = I(\xi) + k$, for all $\xi \in B(K, \Omega, \mathcal{G})$ and all $k \in K$ such that $\xi + k \in B(K, \Omega, \mathcal{G})$.

Given a nonempty set Y and an algebra $\mathcal{A}$ over it, we denote by $\Delta(Y)$ the set of finitely additive probability measures over it. For any nonempty set A, we will denote by $A^\infty := \prod_{t=1}^\infty A^t$ its countably infinite Cartesian product and by 2^A its power set.

2.1.1 Ambiguity attitudes

In the next sections, we will focus on dynamic choice problems. Building upon the work of Bommier et al. (2017) we will show how the combination of the classical notions of *monotonicity* and *stationarity* of preferences will impose restrictions on the decision maker's attitudes toward uncertainty. Before going into the dynamic setting, we discuss the notion of *constant absolute ambiguity aversion* as introduced by Grant and Polak (2013).

Definition 1. *A binary relation $\succsim$ on $\mathbf{F}$ exhibits* constant absolute ambiguity aversion (CAAA) *if for all $f \in \mathbf{F}$, $x, y, z \in X$, and $\alpha \in (0, 1)$,*

$$\alpha f + (1-\alpha)x \succsim \alpha y + (1-\alpha)x \Longrightarrow \alpha f + (1-\alpha)z \succsim \alpha y + (1-\alpha)z.$$

In words, constant absolute ambiguity aversion requires that whenever an uncertain alternative is preferred to a sure outcome, "adding" the same certain alternative to both does not invert the preference. Absolute ambiguity attitudes have been thoroughly studied by Xue (2020) and Cerreia-Vioglio et al. (2019), in terms of utility and wealth, respectively. In the context of rational preferences, constant absolute ambiguity aversion is the same as requiring the certainty equivalent being translation invariant.

Proposition 1. *Consider a binary relation $\succsim$ on $\mathbf{F}$. Suppose there exist an affine function such that $u : X \to \mathbb{R}$ and a translation invariant certainty equivalent $I : B_0(u(X), \Omega, \mathcal{G}) \to \mathbb{R}$ such that $f \mapsto I(u(f))$ represents $\succsim$. The following are equivalent*

(i) $\succsim$ *satisfies constant absolute ambiguity aversion.*

(ii) I *is translation invariant.*

Proof. The proof is routine and hence it is omitted; the reader can consult Xue (2020). □

This proposition highlights how the constant absolute ambiguity aversion of preferences is translated to the representing certainty equivalent. Many notable models of decision-making under uncertainty satisfy constant absolute ambiguity aversion. Among others, maxmin (Gilboa and Schmeidler (1989)), α-maxmin (Ghirardato et al. (2004)), and variational (Maccheroni et al. (2006a)) models exhibit constant absolute ambiguity aversion. In their experimental work Baillon and Placido (2019) provide experimental evidence calling for the use of ambiguity models that can accommodate decreasing aversion toward ambiguity, rather than constant absolute ambiguity aversion.

2.2 Intertemporal consumption choice problems

Our focus will be on dynamic choice problems. Here, we formally present the setting we adopt which is analogous to Strzalecki (2013), but with an infinite horizon (see also Section A.1 of Bommier et al. (2017)). More specifically, we will adopt a stationary IID ambiguity setting as the one introduced by Epstein and Schneider (2003a). Let S be a finite set representing the states of the world to be realized in each period.[4] We assume that S has at least three elements and let $\Sigma := 2^S$ be the associated algebra

[4]In finance and macroeconomics S is also interpreted as a set of *shocks*.

of events. The full state space is denoted by $\Omega := S^\infty$, with a state $\omega \in \Omega$ specifying a complete history $(s_1, s_2, \ldots)$. In each period $t > 0$, the individual knows the partial history $s^t := (s_1, \ldots, s_t)$. The evolution of such information can be represented by a filtration $(\mathcal{G}_t)_{t=0}^\infty$ on Ω where $\mathcal{G}_0 := \{\emptyset, \Omega\}$ and $\mathcal{G}_t := \Sigma^t \times \{\emptyset, S\}^\infty$ for all $t > 0$. Let $\mathcal{G} = \sigma\left(\bigcup_{t=0}^\infty \mathcal{G}_t\right)$ for all $t > 0$, that is, the smallest sigma-algebra generated by the union of the sigma-algebras of the filtration $(\mathcal{G}_t)_{t=0}^\infty$. The relevant measurable space is $(\Omega, \mathcal{G})$.

The set of possible consumption levels is $[\underline{c}, \bar{c}]$ (with generic elements denoted by $c, c', \hat{c}$) with $\underline{c} < \bar{c}$. The entire consumption set is given by the set of lotteries over, that is $C = \Delta([\underline{c}, \bar{c}])$ (with generic elements denoted by x, y, z). We identify $[\underline{c}, \bar{c}]$ as a subset of C, looking at its elements as degenerate lotteries. A *consumption plan* is a C-valued, $(\mathcal{G}_t)_{t=0}^\infty$-adapted stochastic process, that is, a sequence $h = (h_t)_{t=0}^\infty$ such that $h_t : \Omega \to C$ is $\mathcal{G}_t$-measurable for all $t \geq 0$. The set of all consumption plans is denoted by $\mathbf{H}$ and it is endowed with the topology of pointwise convergence. We denote by $\mathbf{D} := C^\infty$ the set of all deterministic consumption plans. We identify C as a subset of $\mathbf{D}$ where each $x \in C$ is seen as the constant consumption plan that yields the lottery x in each period. For all consumption plans $h \in \mathbf{H}$ and $s \in S$ define the *conditional consumption plan* $h^s \in \mathbf{H}$ by

$$h^s(s_1, s_2, \ldots) = h(s, s_2, \ldots) = (h_0, h_1(s, s_2, \ldots), \ldots)$$

for all $(s_1, s_2, \ldots) \in \Omega$. In words, given a consumption plan $h \in \mathbf{H}$ and a state $s \in S$, the conditional consumption plan h^s is the consumption plan obtained from h when the decision maker knows that in the first period s realized. Now we can exploit conditional consumption plans to define the continuation of consumption plans. In particular, let $h = (h_0, h_1, h_2, \ldots) \in \mathbf{H}$ and $s \in S$, the *continuation of* h, denoted by $h^{s,1}$, is defined as

$$h^{s,1}(s_1, s_2, \ldots) = (h_1(s, s_2, \ldots), h_2(s, s_2, \ldots), \ldots)$$

for all $(s_1, s_2, \ldots) \in \Omega$. The continuation act $h^{s,1}$ is the consumption plan forwarding h by one period and knowing that in the first period state $s \in S$ realized. For all lotteries $x \in C$ and consumption plans $h \in \mathbf{H}$ we denote the concatenation (x, h) as,

$$(x, h)(s_1, s_2, \ldots) = (x, h(s_2, s_3, \ldots))$$

for all $(s_1, s_2, \ldots) \in \Omega$. Likewise, let $s \in S$ and $\xi \in B(K, \Omega, \mathcal{G})$ where $K \subseteq \mathbb{R}$ is an interval, define $\xi^{s,1} \in B(K, \Omega, \mathcal{G})$ as

$$\xi^{s,1}(s_1, s_2, s_3, \ldots) = \xi(s, s_2, s_3, \ldots)$$

for all $(s_1, s_2, s_3, \ldots) \in \Omega$. For all $h \in \mathbf{H}$, we will denote by h^1 the mapping $s \mapsto h^{s,1}$, ξ^1 is defined analogously.

To ease notation, since we will consider preferences that are dynamically consistent, here we only consider an ex-ante preference modeled by a binary relation $\succsim$ on $\mathbf{H}$.[5] Observe that in this setting the set of consumption plans $\mathbf{H}$ can be seen as a subset of acts $\mathbf{F} \subseteq \mathbf{D}^{\Omega}$. Indeed an act here can be seen as a mapping from states into consumption streams

$$\omega \mapsto h(\omega) = (h_0, h_1(\omega), \ldots) \in \mathbf{D}.$$

We study a product space and not a general filtration for several reasons. First, it is a standard setting in the decision-theoretic literature (see Strzalecki (2013) or Bommier et al. (2017)). Second, it is the natural setting to study attitudes toward uncertainty. With a general filtration, attitudes toward uncertainty will depend on changing beliefs.[6]

2.3 Recursive preferences

We consider now the intertemporal consumption choice setting described in Section 2.2. The primitive is a preference relation (i.e., a preorder) $\succsim$ on the set of consumption plans $\mathbf{H}$. We consider preferences $\succsim$ that admit a (separable) recursive representation as follows. Given $u : C \to \mathbb{R}$, we define $U : \mathbf{D} \to \mathbb{R}$ as

$$U(d) := \sum_{t=0}^{\infty} \beta^t u(d_t)$$

for all $d \in \mathbf{D}$.

[5]Otherwise one would have to state all the axioms for the collection of preferences $(\succsim_{s^t})_{s^t}$ for every possible sequence $s^t \in S^t$, $t = 1, \ldots$. Under the assumption of recursivity, it is not needed to consider this richer framework. More precisely, if preferences admit the recursive representation given in Definition 2 then it is possible to define conditional preferences $(\succsim_{s^t})_{s^t}$, $t = 1, \ldots$ that satisfy the traditional notion of dynamic consistency.

[6]On this point, see the discussion in Strzalecki (2013) (pp. 1048-1049).

Definition 2. *A preference relation* $\succsim$ *admits a* (separable) recursive representation *if there exists a tuple* (V, I_{+1}, u, β) *such that* $V : \mathbf{H} \to \mathbb{R}$ *represents* $\succsim$ *and*

$$V(h) = u(h_0) + \beta I_{+1}\left(V \circ h^1\right), \tag{2}$$

where $u : C \to \mathbb{R}$ *is an affine utility function,* $\beta \in (0,1)$*,* $I_{+1} : B(U(\mathbf{D}), S, \Sigma) \to \mathbb{R}$ *is a certainty equivalent, and* $V \circ h^1 : s \mapsto V\left(h^{s,1}\right)$*.*

In the context of the previous definition, we refer to I_{+1} as a *one-step ahead certainty equivalent* and to $\succsim$ as a *recursive preference relation.* The axiomatic characterization of recursive preferences is well understood in the literature (see for example de Castro and Galvao (2022), section 4 or Sarver (2018), Appendix A.1, for similar axiomatizations). However, here we provide a novel axiomatization based on a weakening of the standard axiom of monotonicity used in the literature.

3 Axioms

Let $\succsim$ represent the decision maker's preferences on $\mathbf{H}$. Next we state several properties (axioms) of the preference relation, which will be discussed and used to characterize recursive preferences.

Axiom I.1 (Weak order)**.** $\succsim$ is total and transitive.

We introduce the following notion of monotonicity.

Axiom I.2 (State-time monotonicity)**.** For all $h, g \in \mathbf{H}$

$$[\forall \omega \in \Omega, t \geq 0,\ h_t(\omega) \succsim g_t(\omega)] \implies h \succsim g.$$

State-time monotonicity can be seen as a minimal consistency requirement. If a decision maker is asked to compare two uncertain consumption streams h and g and h weakly dominates g (according to the DM's preferences) in each period and each state, then h should be preferred to g.
A further interpretation of state-time monotonicity is to think of the *relevant* state space as being the set of all state-time pairs (ω, t). Seeing each part of this combination as a node on our event tree, if the consumption level of h is higher than that of g, then h should be preferred to g. Figure 1 offers a graphical description of

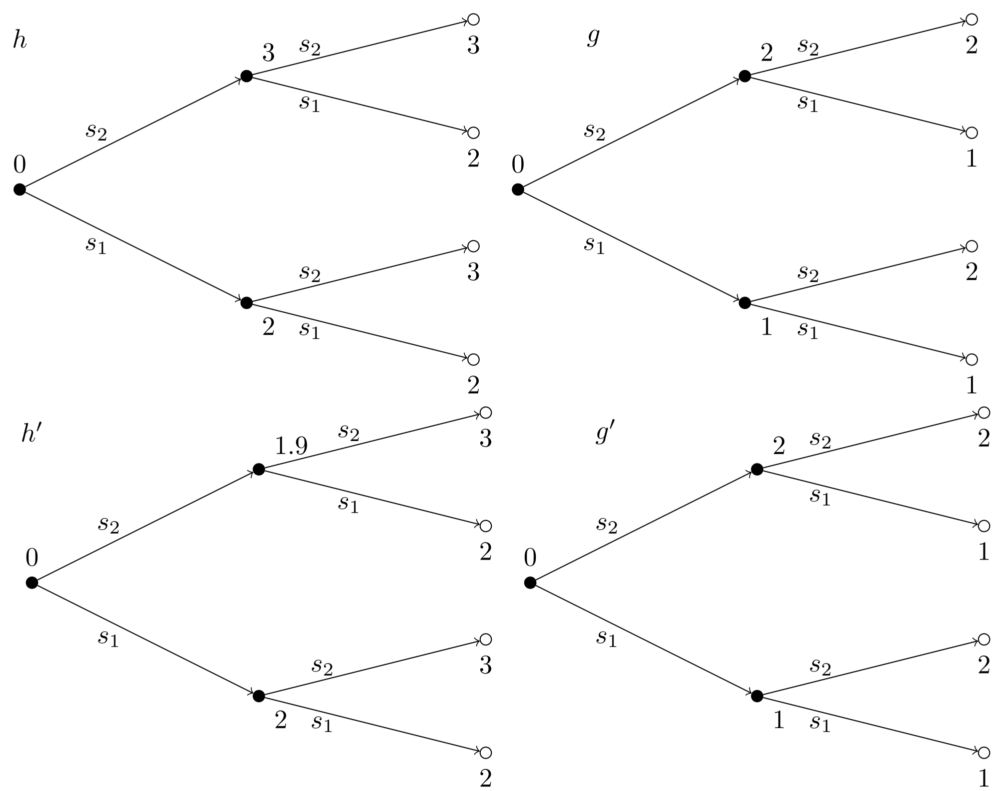

Figure 3: Illustration of state-time monotonicity

state-time monotonicity. The act h pays better than g at every possible node, while h' pays better than g' at every possible node except at the node s_1. By state-time monotonicity h should be preferred to g but h' need not be preferred to g'. As we shall discuss, a stronger notion of monotonicity is typically adopted in the literature (e.g., see Epstein and Schneider (2003b), Maccheroni et al. (2006b), Bastianello and Faro (2022)).

Axiom I.3 (State-monotonicity)**.** For all $h, g \in \mathbf{H}$,

$$[\forall \omega \in \Omega,\ (h_t(\omega))_{t=0}^{\infty} \succsim (g_t(\omega))_{t=0}^{\infty}] \implies h \succsim g.$$

State-monotonicity's interpretation is analogous to that of state-time monotonicity, but when the *relevant* state space is just Ω. One may expect state-monotonicity to be a stronger requirement than state-time monotonicity, this indeed the case when preferences over deterministic consumption streams are time separable. Bommier et al. (2017) show that state-monotonicity is equivalent to the translation invariance of the one-step ahead certainty equivalent I_{+1}. However, their characterization is silent about ambiguity attitudes. In our setting, their result can be seen as implying that $\succsim$ has to satisfy constant absolute ambiguity aversion when restricted to one-step ahead acts.

The next two axioms are a technical and a monotonicity requirement on preferences.

Axiom I.4 (Continuity)**.** For all $h \in \mathbf{H}$, the sets $\{g \in \mathbf{H} \mid g \succsim h\}$ and $\{g \in \mathbf{H} \mid h \succsim g\}$ are closed in $\mathbf{H}$.

Axiom I.5 (Monotonicity on levels of consumption)**.** For all $c, \hat{c} \in [\underline{c}, \bar{c}]$ such that $c \geq \hat{c}$, we have $c \succsim \hat{c}$. Moreover, $\bar{c} \succ \underline{c}$.

Then, we will also consider two classical axioms in the literature on discounting and dynamic choice, namely *time separability* and *stationarity.*

Axiom I.6 (Time separability)**.** For all $x, y, x', y' \in C$ and $d, d' \in \mathbf{D}$, $(x, y, d) \sim (x', y', d)$ if and only if $(x, y, d') \sim (x', y', d')$.

Consider two deterministic consumption plans that yield identical outcomes from the third period onward. Time separability requires that their ranking does not depend on the common continuation.

Axiom I.7 (Stationarity)**.** For all $x \in C$ and $h, g \in \mathbf{H}$, $h \succsim g$ if and only if $(x, h) \succsim (x, g)$.

Stationarity expresses Koopmans's idea that "the passage of time does not have an effect on preferences." It is well known from Koopmans (1972) that, together with time separability and continuity, stationarity implies that the intertemporal utility index is additive. Thus, the ranking of consumption in any given subset of time periods is independent of consumption levels in all other time periods.

In order to talk about ambiguity attitudes we will have to impose some restrictions on the curvature of the utility function that describes consumer's preferences over deterministic alternatives. The next axiom will guarantee the affinity of such utility function, allowing us to identify the translation invariance of our certainty equivalents as an expression of constant absolute ambiguity aversion.

Axiom I.8 (Independence for Deterministic Prospects)**.** For all $d, d', d'' \in \mathbf{D}$ and $\alpha \in (0, 1)$,

$$d \sim d' \Longrightarrow \alpha d + (1-\alpha)d'' \sim \alpha d' + (1-\alpha)d''.$$

Before presenting the next axiom, we will first define the concept of a one-step-ahead consumption plan.

Definition 3. *A consumption plan $h \in \mathbf{H}$ is said to be* one-step-ahead *if h_t is $\mathcal{G}_1$-measurable for all $t \geq 2$.*

We will refer to such plans as one-step-ahead consumption plans. In words, one-step ahead consumption plans resolve all the uncertainty at $t = 1$ and pay a stream of consumption in $\mathbf{D}$ thereafter.

Axiom I.9 (One-step-ahead equivalence)**.** For all $h \in \mathbf{H}$, there exists a one-step-ahead consumption plan $h^{+1} \in \mathbf{H}$ such that $h_0 = h_0^{+1}$, $h \sim h^{+1}$, and

$$(h_1^{+1}(\omega), \ldots, h_t^{+1}(\omega), \ldots) \sim (h_1(\omega), \ldots, h_t(\omega), \ldots)$$

for all $\omega \in \Omega$.

This axiom basically requires the existence of a one-step ahead certainty equivalent. We will show how the combination of state-time monotonicity and one-step-ahead equivalence yields *dynamic consistency.* More formally, they imply the following

Axiom I.10 (Dynamic Consistency)**.** For all $h, g \in \mathbf{H}$ with $h_0 = g_0$,

$$(\forall s \in S, h^s \succsim g^s) \Longrightarrow h \succsim g,$$

if in addition $h^s \succ g^s$ for some $s \in S$, then we have $h \succ g$.

This axiom embeds both a property of recursivity and state independence allowing to combine time consistency and consequentialism in dynamic frameworks (see Johnsen and Donaldson (1985)).

4 Main results

In the following sections we provide two axiomatizations for recursive preferences and their implications. First, we retrieve a recursive representation for preferences satisfying the notion of *state-time monotonicity* presented and discussed in the previous section. Second, we show a result analogous to Proposition 4 of Bommier et al. (2017), highlighting how in our setting the combination of *state-monotonicity* and *stationarity* imposes a restriction on the decision maker's behavioral attitudes towards ambiguity, specifically the preferences will satisfy constant absolute ambiguity aversion. Third, we show how even though the combination of these two axioms leads to such a restriction it also provides some benefit. In particular, it is shown how under the stronger monotonicity assumption the ex-ante and one-step-ahead certainty equivalents involved in the representation are related by a condition that we call *generalized rectangularity*. This condition provides a functional equation that is shown to have a unique solution by means of Blackwell's sufficient conditions for contractions. Moreover, we show that generalized rectangularity includes the classic rectangularity condition of Epstein and Schneider (2003b) and we discuss its relation to the *no-gain condition* of Maccheroni et al. (2006b).

4.1 Recursive representations and generalized rectangularity

The following result shows that in our setting these axioms characterize recursive preferences of the form presented in Definition 2.

Theorem 1. *Let $\succsim$ be a binary relation on $\mathbf{H}$. The following are equivalent,*

(i) $\succsim$ satisfies axioms I.1, I.2, I.4-I.9.

(ii) $\succsim$ *admits a recursive representation.*

The main feature of the representation theorem stated above is that it relies on state-time monotonicity (Axiom I.2). In particular, in the proof of Theorem 1, we demonstrate that the combination of state-time monotonicity and one-step-ahead equivalence axioms implies a standard notion of dynamic consistency (recalled in Axiom I.10). This means that our result breaks down dynamic consistency into two more fundamental axioms, one of which is a more intuitive requirement of monotonicity. In the following example we provide a preference relation that admits a recursive representation, but that does not satisfy state-monotonicity.

Example 1. Assume that $S = \{H, T\}$. Consider recursive preferences with the smooth ambiguity certainty equivalent

$$I_{+1} : \xi \mapsto \phi^{-1}\left(\frac{1}{2}\phi\left(\mathbb{E}_{P^1}[\xi]\right) + \frac{1}{2}\phi\left(\mathbb{E}_{P^2}[\xi]\right)\right),$$

where $P^1(H) = P^1(T) = \frac{1}{2}$ and $P^2(H) = \frac{2}{3} = 1 - P^2(T)$. Assume $\phi(\cdot) = \sqrt{\cdot}$, $u(x) = \mathbb{E}_x[\cdot]$ and $\beta = 0.1$. Let $h = (1, f, \ldots, f, \ldots)$ and $g = (0, 10 + f, f, \ldots, f, \ldots)$ where $f : \Omega \to C$ satisfies $f(H, H, \ldots) = 1$ and equals 0 otherwise. Observe that

$$\sum_{t=0}^{\infty} \beta^t h_t(\omega) = \sum_{t=0}^{\infty} \beta^t g_t(\omega),$$

for every $\omega \in \Omega$. Yet we also have

$$\begin{aligned} V(h) &= 1 + 0.1\left(\frac{1}{2}\sqrt{\frac{1}{0.9}\cdot\frac{1}{2}} + \frac{1}{2}\sqrt{\frac{1}{0.9}\cdot\frac{2}{3}}\right)^2 \\ &\approx 1.0645 \\ &< 1.0648 \\ &\approx 0.1\left(\frac{1}{2}\sqrt{10 + \frac{1}{0.9}\cdot\frac{1}{2}} + \frac{1}{2}\sqrt{10 + \frac{1}{0.9}\cdot\frac{2}{3}}\right)^2 = V(g). \end{aligned}$$

thus implying a violation of state-monotonicity. Notice however that h is neither better nor worse than g according to state-time monotonicity.

The next result shows that under state-monotonicity, $\succsim$ satisfies constant absolute ambiguity aversion. In particular, preferences can be represented by means of an ex-ante certainty equivalent I_0 that is translation invariant. Moreover, I_0 is linked to

I_{+1} through an equation which we refer to as generalized rectangularity as it is a generalization of the rectangularity of MEU preferences from Epstein and Schneider (2003a).

Theorem 2. *Let* $\succsim$ *be a binary relation on* $\mathbf{H}$*. The following are equivalent,*

(i) $\succsim$ *admits the representation* (2) *and satisfies Axiom I.3.*

(ii) $\succsim$ *admits a recursive representation with* I_{+1} *translation invariant and there exists a translation invariant certainty equivalent* $I_0 : B\left(U(\mathbf{D}), \Omega, \mathcal{G}\right) \to \mathbb{R}$ *such that* $\succsim$ *is represented by*

$$h \mapsto I_0\left(\sum_{t=0}^{\infty} \beta^t u(h_t)\right).$$

Moreover, I_0 *satisfies*

$$I_0(\xi) = \beta I_{+1}\left(I_0\left(\frac{1}{\beta}\xi^1\right)\right), \tag{3}$$

where $I_0(\xi^1) : s \in S \mapsto I_0\left(\xi^{s,1}\right)$ *for all* $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$*.*

Outline of the proof. To show the "only if" part, we use state-monotonicity and continuity for the existence of a certainty equivalent I_0 such that the mapping

$$h \mapsto I_0\left(\sum_{t=0}^{\infty} \beta^t u(h_t)\right) \tag{4}$$

represents $\succsim$. We show that I_0 must satisfy translation invariance by stationarity. Using translation invariance, (4) and (2), we show that I_0 and I_{+1} are related to each other by (3), which implies that I_{+1} is also translation invariant. The details are in the Appendix. □

As a consequence, we observe immediately that under state-monotonicity recursive preferences satisfy constant absolute ambiguity aversion.

Corollary 1. *Suppose that* $\succsim$ *admits the representation* (2) *and satisfies Axiom I.3. Then* $\succsim$ *exhibits constant absolute ambiguity aversion.*

Proof. The statement follows immediately by applying Theorem 2 and Proposition 1. □

A few comments are in order. First, as previously discussed, this result highlights an important trade-off in modeling. If one desires to represent preferences through ex-ante certainty equivalents, it becomes impossible to simultaneously have recursive preferences that abide by decreasing absolute ambiguity aversion. Our suggestion is that to align with this evidence, state-time monotonicity should be adopted as the standard for monotonicity, instead of state monotonicity. This approach accommodates more versatile ambiguity attitudes while retaining a degree of tractability. On the other hand, we also show allowing for state-monotonicity—or better, for constant absolute ambiguity aversion—yields a useful condition that we call *generalized rectangularity* (3). This condition follows the idea of seeing rectangularity as a generalized form of law of iterated expectations. The primary distinction from a conventional law of iterated expectations is its dependence on the degree of impatience. However, this dependence vanishes when the certainty equivalents are positively homogeneous. As mentioned in the introduction, this functional equation enables us to solve for the ex-ante certainty equivalent I_0 using the one-step-ahead certainty equivalent I_{+1}. The following result demonstrates that, with a fixed I_{+1}, there exists a unique I_0 that satisfies equation (3). Notably, a standard numerical procedure based on the contraction mapping theorem can determine such I_0. The interpretation is that given a one-step-ahead certainty equivalent, under recursivity and monotonicity one can recover uniquely the ex-ante certainty equivalent.

Theorem 3. *Fix a translation invariant one-step-ahead certainty equivalent* $I_{+1} : B(U(\mathbf{D}), S, \Sigma) \to \mathbb{R}$*. There exists a unique translation invariant certainty equivalent* $I_0^* : B(U(\mathbf{D}), \Omega, \mathcal{G}) \to \mathbb{R}$ *such that*

$$I_0^*(\xi) = \beta I_{+1}\left(I_0^*\left(\frac{1}{\beta}\xi^1\right)\right)$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$*. Moreover,* I_0^* *is globally attracting.*

Outline of the proof. Using the fact that I_{+1} is translation invariant, this result follows by applying a version of Blackwell's contraction mapping theorem to a suitably chosen operator. The complete proof is available in the Appendix. □

In general, however, one should not expect an explicit solution of this functional equation. For example, assume that

$$I_{+1}(\xi) = \int \xi dv_{+1} \text{ for all } \xi \in B(U(\mathbf{D}), S, \Sigma),$$

where $v_{+1} : \Sigma \to [0,1]$ is a capacity that is neither convex nor concave. In this case, there need not be a capacity $v_0 : \mathcal{G} \to [0,1]$ such that generalized rectangularity holds, which in this case is equivalent to

$$\int \xi dv_0 = \int \int \xi^1 dv_0 dv_{+1},$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$.[7] Nevertheless, our finding still enables an analyst interested in retrieving I_0 to do so using numerical techniques. To illustrate, in the Choquet case one can still find a certainty equivalent I_0^* that satisfies

$$I_0^*(\xi) = \beta \int I_0^* \left(\frac{1}{\beta} \xi^1 \right) dv_{+1}.$$

4.2 Generalized rectangularity for MEU and variational preferences

In this subsection we show how generalized rectangularity behaves in the case of maxmin and variational preferences. In particular, we show how it is equivalent to the original rectangularity from Epstein and Schneider (2003a), and Epstein and Schneider (2003b) for recursive MEU preferences and its connection with the no-gain condition introduced by Maccheroni et al. (2006b). Before providing the results we will need some more notation. We recall that $\mathcal{G} = \sigma \left(\bigcup_{t=1}^{\infty} \mathcal{G}_t \right)$. For all $A \in \mathcal{G}$ and $s \in S$, let

$$A_s = \left\{ (s_t)_2^{\infty} : (s, (s_t)_2^{\infty}) \in A \right\}.$$

Further, given $P \in \Delta(\Omega)$ and $s \in S$, P_{+1} denotes the marginal over the first coordinate while P_s denotes the marginal over the cylinder set $\{s^{\infty} \in \Omega : s_1 = s\}$. For our present specification rectangularity takes the following form. Let $\mathcal{L} \subseteq \Delta(S)$ and $\mathcal{P} \subseteq \Delta(\Omega)$. We say that $\mathcal{P}$ is *$\mathcal{L}$-rectangular* if $P \in \mathcal{L}$ if and only if there exist $\ell \in \mathcal{L}$ and $\{Q^s \in \mathcal{P} : s \in S\}$ such that

$$P(A) = \sum_{s \in S} \ell(s) Q^s(A_s)$$

for all $A \in \mathcal{G}$. This is the classic notion of rectangularity as introduced by Epstein and Schneider (2003a). We have the following characterization.

[7] We refer to Zimper (2011) and Dominiak (2013) for cases in which it does or does not hold.

Corollary 2 (Rectangularity for MEU[8]). *Suppose that* $\mathcal{P} \subseteq \Delta(\Omega)$ *and* $\mathcal{L} \subseteq \Delta(S)$ *are convex and weak* compact sets. If*

$$I_{+1}(\xi) = \min_{P \in \mathcal{P}} \mathbb{E}_P[\xi] \quad \textit{for all } \xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$$

and

$$I_0(\xi) = \min_{\ell \in \mathcal{L}} \mathbb{E}_\ell[\xi] \quad \textit{for all } \xi \in B(U(\mathbf{D}), S, \Sigma),$$

then the following are equivalent

1. I_{+1} *and* I_0 *satisfy* (3);
2. $\mathcal{P}$ *and* $\mathcal{L}$ *satisfy*
$$\min_{P \in \mathcal{P}} P(A) = \min_{\ell \in L} \left[\sum_{s \in S} \ell(s) \min_{P \in \mathcal{P}} P(A_s) \right] \quad \textit{for all } A \in \mathcal{G}; \tag{5}$$
3. $\mathcal{P}$ *is* $\mathcal{L}$*-rectangular.*

Proof. See the Appendix. □

A further major implication is that under state-monotonicity, the only recursive preferences that satisfy uncertainty aversion are variational. We say that $c : \Delta(\Omega) \to [0, \infty]$ is *grounded* if its infimum value is zero. In addition, we say that c is a *cost function* if it is convex, grounded, and lower semicontinuous.

Corollary 3. *Suppose that* $\succsim$ *admits the representation* (2) *and satisfies*

$$h \sim g \Rightarrow \alpha h + (1-\alpha) g \succsim h, \tag{6}$$

for all $h, g \in \mathbf{H}$ *and* $\alpha \in (0,1)$. *Then* $\succsim$ *satisfies state-monotonicity if and only if there exist cost functions* $c_{+1} : \Delta(S) \to [0, \infty]$, $c_0 : \Delta(\Omega) \to [0, \infty]$ *such that for all* $h \in \mathbf{H}$,

$$V(h) = u(h_0) + \beta \min_{\ell \in \Delta(S)} \left\{ \mathbb{E}_\ell \left[V \circ h^1 \right] + c_{+1}(\ell) \right\}$$

and

$$I_0 \left(\sum_{t=0}^{\infty} \beta^t u(h_t) \right) = \min_{P \in \Delta(\Omega)} \left\{ \mathbb{E}_P \left[\sum_{t=0}^{\infty} \beta^t u(h_t) \right] + c_0(P) \right\}.$$

Moreover, a sufficient condition for (3) *is given by*

$$c_0(P) = \beta \left[\sum_{s \in S} P_{+1}(s) c_0(P_s) + c_{+1}(P_{+1}) \right], \tag{7}$$

for all $P \in \Delta(\Omega)$.

[8] See also Epstein and Schneider (2003a) equations 2.4 and 2.6.

Outline of the proof. Because of (6) and state-monotonicity, I_{+1} is translation invariant and quasi-concave. Likewise, there exists I_0 that is translation invariant and quasi-concave. Hence by standard results the certainty equivalents I_{+1} and I_0 have the desired variational representation. It then follows that generalized rectangularity is implied by (7). See the Appendix for the full proof. □

Observe that (7) is reminiscent of the no-gain condition in Maccheroni et al. (2006b). In this setting, the no-gain condition is sufficient only because we do not have unbounded utility. We discuss more technical points related to the necessity of (7) in the Appendix. In particular, under weak technical requirements the inequality

$$c_0(P) \leq \beta \left[\sum_{s \in S} P_{+1}(s) c_0(P_s) + c_{+1}(P_{+1}) \right], \tag{8}$$

is implied by generalized rectangularity. This result further illustrates how our generalized rectangularity subsumes the major characterizations of recursive beliefs.

4.3 Sequential choice

As mentioned, we follow Epstein's terminology (see Epstein (1992)) and therefore distinguish sequential choice problems from intertemporal consumption choice problems. The former models situations taking place over short intervals of time during which consumption/savings plans can be taken to be fixed. The source of utility is terminal wealth rather than a consumption sequence. In the latter, consumption decisions take place over a long period of time, and the source of utility is given by a consumption stream. In sequential choice problems, the decision maker (DM) has to choose a bet ex-ante and at an interim information stage, where the DM can bet based on partial resolution of uncertainty (see for example Section 1.1 in Hanany and Klibanoff (2009) for an example). Subsequently, a state realizes and payment occurs depending on the DM's choice. Figure 4 offers a graphical representation of a sequential choice problem. This setting builds upon the one on static choice problems presented above. In particular, there is a finite partition Π of Ω, and $\mathcal{G} := \sigma(\Pi)$ is the σ-algebra generated by the partition Π. Given $\omega \in \Omega$, we denote by $\Pi(\omega)$ the unique element of Π such that $\omega \in \Pi(\omega)$. The set of acts is denoted by $\mathbf{F}$ and the DM's preferences are expressed as unconditional and conditional total preorders over $\mathbf{F}$. Such preferences are denoted by $\left\langle \succsim, (\succsim_E)_{E \in \Pi} \right\rangle$. The binary relation $\succsim$ represents the ex-ante preferences of the DM while $\succsim_E$ models the DM's preferences conditional on the event $E \in \Pi$. We introduce a set of basic rationality axioms for $\succsim$.

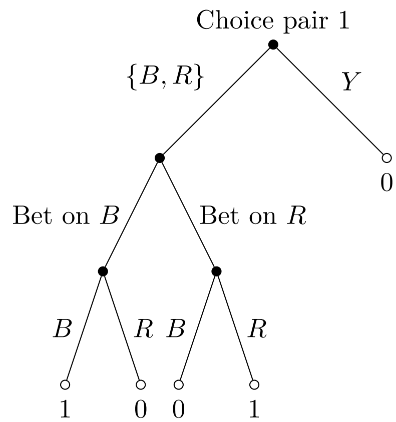

Figure 4: An Ellsberg type sequential choice problem

Axiom S.1 (Monotonicity)**.** For all $f, g \in \mathbf{F}$,

$$[\forall \omega \in \Omega, f(\omega) \succsim g(\omega)] \implies f \succsim g,$$

and if $f(\omega) \succ g(\omega)$ for some $\omega \in \Omega$, then $f \succ g$.

Axiom S.2 (Mixture continuity)**.** For all $f, g, h \in \mathbf{F}$ the sets

$$\{\lambda \in [0,1] : \lambda f + (1-\lambda)g \succsim h\} \text{ and } \{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda)g\},$$

are closed.

Axiom S.3 (Risk independence)**.** For all $x, y, z \in X$ and $\alpha \in (0,1)$,

$$x \succsim y \iff \alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z,$$

The next two axioms refer to preferences $\left\langle \succsim, (\succsim_E)_{E \in \Pi} \right\rangle$ and link them to recursivity.

Axiom S.4 (Dynamic consistency (DC))**.** For all $f, g \in \mathbf{F}$,

$$[\forall E \in \Pi, f \succsim_E g] \Rightarrow f \succsim g.$$

For all $f, g \in \mathbf{F}$ and all $E \in \Pi$, denote by fEg the act equal to f when restricted to E, and equal to g otherwise.

Axiom S.5 (Consequentialism (C))**.** For all $E \in \Pi$ and all $f, g \in \mathbf{F}$,

$$f \sim_E fEg.$$

We call *recursive* those preferences that satisfy axioms S.1-S.5. The term recursivity is justified from the fact that such axioms characterize the following representation.

Theorem 4. *Let $\succsim$ be a binary relation on $\mathbf{F}$. The following are equivalent,*

(i) $\succsim$ satisfies axioms S.1-S.5.

(ii) There exist an affine function $u : X \to \mathbb{R}$ and certainty equivalents $I : B_0(u(X), \Omega, \mathcal{G}) \to \mathbb{R}$, and $\bar{I}(\omega, \cdot) : B_0(u(X), \Omega, \mathcal{G}) \to \mathbb{R}$ for all $\omega \in \Omega$ such that

(a) $I(u(f)) = I(\bar{I}(\cdot, u(f)))$ for all $f \in \mathbf{F}$,

(b) $\bar{I}\left(\omega, u(f)1_{\Pi(\omega)} + u(g)1_{\Pi(\omega)^c}\right) = \bar{I}(\omega, u(f))$ for all $\omega \in \Omega$ and all $f, g \in \mathbf{F}$,

(c) $f \mapsto I(u(f))$ represents $\succsim$,

(d) for all $E \in \Pi$ and $\omega \in E$ the mapping $f \mapsto I(\omega, u(f))$ represents $\succsim_E$.

Proof. We omit the proof of this classic result. A proof can be found for example in Cerreia-Vioglio et al. (2022), Proposition 8. □

In this setting, however, constant absolute ambiguity aversion is not implied by recursivity, as shown in the next example.

Example 2. Consider $\succsim$ represented by

$$I(u(f)) = \phi^{-1}\left(\int \phi(u(f))d\mu\right),$$

for all $f \in \mathbf{F}$ and $\succsim_E$ is represented by $I_E(u(f)) = \phi^{-1}\left(\int \phi(u(f))d\mu_E\right)$. In this case DC and C are satisfied for all invertible ϕ even when ϕ is not an exponential, such as in the CRRA case. Indeed,

$$\begin{aligned} I(I_E(u(f))) &= I\left(\phi^{-1}\left(\int \phi(u(f))d\mu_E\right)\right) \\ &= \phi^{-1}\left(\int\left(\int \phi(u(f))d\mu_E\right)d\mu\right) \\ &= \phi^{-1}\left(\int \phi(u(f))d\mu\right). \end{aligned}$$

by the law of iterated expectations. Therefore, the function ϕ can have any shape, thus allowing for arbitrary attitudes toward uncertainty. For example, if $\phi(x) = x^\rho$ for $0 < \rho < 1$, then $\succsim$ will not satisfy CAAA over $\mathbf{F}$.

However, as shown by Savochkin et al. (2022) CAAA is implied by recursivity when $\succsim$ belongs to the smooth ambiguity class, i.e., when preferences admit the representation

$$V : f \mapsto \phi^{-1}\left(\int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega))d\mu\right) d\pi(\mu)\right),$$

for some concave and differentiable function ϕ. Their results imply that ϕ has to be either linear or such that $\phi(x) = -e^{-\theta x}$ for some $\theta > 0$. Our work emphasizes that the key behavioral restriction entailing constant absolute ambiguity aversion is the following

$$[\forall E \in \Pi,\ f \sim_E g] \Rightarrow [\forall E \in \Pi,\ fEh \sim gEh],$$

which is a joint implication of Axioms S.4 and S.5. In their proof, step 3a (p. 23 of their working paper), such a property is used to derive a functional equation that implies that ϕ is an exponential, therefore satisfying CAAA.

5 Discussion

5.1 Related literature

Bommier et al. (2017) is the theoretical work closest to the present paper. Similar to our approach, they examine recursive preferences that satisfy state-monotonicity, which they refer to as just monotonicity. However, there are notable differences between their study and ours. They do not investigate the implications of state-monotonicity for general ambiguity attitudes as we do. Additionally, our work diverges from theirs in terms of methodology; we do not employ their techniques based on the theory of functional equations to derive translation invariance of the one-step ahead certainty equivalent I_{+1}. Furthermore, we assume time-separable preferences, whereas they do not. In a related work, Li et al. (2023) investigate various forms of monotonicity when preferences are defined over matrices. Their notion of outcome monotonicity is analogous to our concept of state-time monotonicity in the current setting. They demonstrate that the various forms of monotonicity they consider hold jointly if and only if preferences can be represented by discounted expected utility.

So far, we have discussed how ambiguity attitudes evolve when a decision maker becomes better off in terms of utility. Alternatively, one may want to predict changes in ambiguity attitude when the decision maker becomes better off in terms of wealth.

This approach requires to account for risk attitudes, as shown by Cerreia-Vioglio et al. (2019). We leave this to future research. For the moment, we observe that our results still apply using Cerreia-Vioglio et al.'s methodology under the assumption of risk neutrality.

Savochkin et al. (2022) consider a setting of sequential choice and characterize recursive smooth ambiguity preferences. While the major source of appeal of the smooth ambiguity model is that it need not satisfy constant ambiguity aversion (either absolute or relative), they show that CAAA is necessary under the assumption of recursivity. Further, they derive a condition for the decision maker's beliefs that ensures recursivity.

Baillon and Placido (2019) and Berger and Bosetti (2020) provide experimental evidence on non-constant ambiguity aversion. In particular, Baillon and Placido's results call for the use of ambiguity models that can accommodate decreasing aversion toward ambiguity. In the context of intertemporal consumption choice problems, we proposed a novel notion of monotonicity to address this point.

5.2 Concluding remarks

Our paper loosely suggest a result of the following type: one cannot have at the same time

1. preferences that are recursive and hence tractable;
2. a representation of preferences by means of an ex-ante certainty equivalent; and
3. (strictly) decreasing absolute ambiguity aversion, as consistent with the experimental literature.

Under the assumption of points (1) and (2) above, we have provided a full characterization of recursive preferences, establishing in particular a generalized notion of rectangularity of beliefs. At the same time for intertemporal consumption choice problems—the main focus of the applied literature—we suggest a novel notion of monotonicity, namely state-time monotonicity. State-time monotonicity is compatible with discarding point (2) from the list above and allows for the examination of tractable preferences in a dynamic setting while accommodating realistic uncertainty attitudes. The only minor trade-off from a decision-theoretic standpoint is the absence of an ex-ante certainty equivalent to represent preferences.

Appendix

Proofs of the main results

We provide the proof in several steps starting with the derivation of discounted utility representation $d \mapsto \sum_{t=0}^{\infty} \beta^t u(d_t)$ for $\succsim$ over the set of deterministic processes $\mathbf{D}$. We detail the proof to make our paper as self-contained and explicit as possible. In particular, we follow the approach of Bastianello and Faro (2022). Consider the following axioms:

(P1) (Continuity) For all compact sets $K \subseteq C$ and all $d \in \mathbf{D}$, the sets $\{d' \in K^\infty | d \succsim d'\}$ and $\{d' \in K^\infty | d' \succsim d\}$ are closed in the product topology over K^∞.

(P2) (Sensitivity) There exist $x, y \in C$, $d \in \mathbf{D}$ such that $(x, d) \succ (y, d)$.

(P3) (Stationarity) For all $x \in C$ and $d, d' \in \mathbf{D}$, $d \succsim d'$ if and only if $(x, d) \succsim (x, d')$.

(P4) (Time separability) For all $x, y, x', y' \in C$ and $d, d' \in \mathbf{D}$, $(x, y, d) \sim (x', y', d)$ if and only if $(x, y, d') \sim (x', y', d')$.

(P5) (Monotonicity) Let $d, d' \in \mathbf{D}$. If $d_t \succsim d'_t$ for all $t \geq 0$, then $d_t \succsim d'_t$; if moreover $d_t \succ d'_t$ for some $t \geq 0$ then $d \succ d'$.

Proposition 2 (Bastianello and Faro (2022))**.** *A preference relation $\succsim$ over $\mathbf{D}$ satisfies P.1-P.5 if and only if there exists a continuous function $u : C \to \mathbb{R}$ and a discount factor $\beta \in (0, 1)$ such that $\succsim$ is represented by*

$$U : d \mapsto \sum_{t=0}^{\infty} \beta^t u(d_t).$$

Proof. See Proposition 5 in Bastianello and Faro (2022). □

Lemma 1. *Axioms I.1, I.2, I.4, I.5, and I.7 imply **(P2)***

Proof. The proof is analogous to the Lemma 5 in Kochov (2015); we report it here for the sake of completeness. Suppose that $(x, d) \sim (y, d)$ for all $x, y \in C$ and all $d \in \mathbf{D}$, then by stationarity $(z, x, d) \sim (z', x, d) \sim (z', y, d)$ for all $z, z', x, y \in C$ and $d \in \mathbf{D}$. Repeating this argument we have that $d \sim d'$ for all $d, d' \in \mathbf{D}$ differ in at most finitely many points. Let $d = (x_0, x_1, \ldots)$ and $d' = (y_0, y_1, \ldots)$ in $\mathbf{D}$ and define $d^t = (x_0, \ldots, x_{t-1}, y_t, y_{t+1})$. The previous argument shows that $d^t \sim d$ for all $t \geq 0$

and $(d^t)_{t=1}^{\infty}$ converges to d'. By continuity and completeness of $\succsim$ we have $d \sim d'$ because $d, d' \in \mathbf{D}$ were arbitrarily chosen and we have a contradiction of Axiom I.5 since $\bar{c} > \underline{c}$. $\square$

Lemma 2 (Bastianello and Faro (2022))**.** *Axioms I.1, I.4, I.6, and I.7 imply* ***(P5)****.*

Proof. See Lemma A.2 in Bastianello and Faro (2022). $\square$

Lemma 3. *A preference relation $\succsim$ over $\mathbf{D}$ axioms I.1, I.2, and I.4-I.8 if and only if there exists a continuous and affine function $u : C \to \mathbb{R}$ and a discount factor $\beta \in (0,1)$ such that $\succsim$ is represented by*

$$U : d \mapsto \sum_{t=0}^{\infty} \beta^t u(d_t).$$

Proof. By Lemmas 1 and 2 we have that axioms I.1, I.2, and I.4-I.8 imply **(P2)** and **(P5)**. It is immediately observable that all the others **(P1)**, **(P3)**, **(P4)** are directly implied by axioms I.1, I.2, and I.4-I.8. Therefore, by Proposition 2 we have that there exists a continuous function $u : C \to \mathbb{R}$ and a discount factor $\beta \in (0,1)$ such that $U : d \mapsto \sum_{t=0}^{\infty} \beta^t u(d_t)$ represents $\succsim$. Now notice that when restricted to C the preference relation $\succsim$ satisfies all the hypotheses of Theorem 8 in Herstein and Milnor (1953); therefore $\succsim$ admits an affine utility representation v. Since v must be cardinally unique it follows that u must be a positive affine transformation of v and as such it must be affine. $\square$

Proof of Theorem 1. $[(ii) \Rightarrow (i)]$. Checking that the axioms are necessary for the representation is routine, except for the one-step-ahead equivalence and state-time monotonicity, whose necessity we now show. To show that, take any $h \in \mathbf{H}$ and let $(c^s)_{s \in S} \in C^S$ be such that $U(c^s, c^s, \ldots) = (V(h^{s,1}), V(h^{s,1}), \ldots)$ for all $s \in S$. Define $h_0^{+1} = h_0$ and $h_t^{+1} = f$ for all $t \geq 0$ where $f : S \to C$ with $f(s) = c^s$ for all $s \in S$. Observe that by construction we have $h_0 = h_0^{+1}$ and

$$\left(h_1^{+1}(\omega), \ldots, h_t^{+1}(\omega), \ldots\right) \sim (h_1(\omega), \ldots, h_t(\omega), \ldots)$$

for all $\omega \in \Omega$. It follows that $V(h) = V(h^{+1})$ which implies $h \sim h^{+1}$ as desired. Turning to state-time monotonicity, suppose that

$$(h_t(\omega), \ldots, h_t(\omega), \ldots) \succsim (g_t(\omega), \ldots, g_t(\omega), \ldots)$$

for all $\omega \in \Omega$ and $t \geq 0$. This implies that we can find one-step ahead acts such that $h^{+1} \sim h$, $g^{+1} \sim g$ and

$$\left(h_t^{+1}(\omega), \ldots, h_t^{+1}(\omega), \ldots\right) \succsim \left(g_t^{+1}(\omega), \ldots, g_t^{+1}(\omega), \ldots\right),$$

for every $t \geq 1$. By monotonicity of I_{+1} we obtain $V(h^{s,1}) \geq V(g^{s,1})$ for every $s \in S$ which by monotonicity of I_{+1} implies that $V(h) \geq V(g)$ as desired.

$[(i) \Rightarrow (ii)]$. First observe that by state-time monotonicity and the one-step-ahead equivalence axioms we have that for every $h, g \in \mathbf{H}$ such that $h_0 = g_0$

$$(\forall s \in S,\ h^s \succsim g^s) \implies h \succsim g. \tag{9}$$

Further, by Lemma 3 there exists an affine function $u : C \to \mathbb{R}$ such that

$$U : d \mapsto \sum_{t=0}^{\infty} \beta^t u(d_t),$$

represents $\succsim$ on $\mathbf{D}$ with $\beta \in (0,1)$. Fix $h \in \mathbf{H}$. Then, by state-time monotonicity and monotonicity on consumption levels we have that the sets

$$\{c \in [\underline{c}, \bar{c}] : (h_0, c, \ldots, c, \ldots) \succsim h\},$$

and

$$\{c \in [\underline{c}, \bar{c}] : h \succsim (h_0, c, \ldots, c, \ldots)\},$$

are non-empty. Furthermore, by the continuity they are both closed. Since $\succsim$ is a weak order, it holds

$$\{c \in [\underline{c}, \bar{c}] : (h_0, c, \ldots, c, \ldots) \succsim h\} \cup \{c \in [\underline{c}, \bar{c}] : h \succsim (h_0, c, \ldots, c, \ldots)\} = [\underline{c}, \bar{c}].$$

Therefore, since $[\underline{c}, \bar{c}]$ is connected, we must have that

$$\{c \in [\underline{c}, \bar{c}] : (h_0, c, \ldots, c, \ldots) \succsim h\} \cap \{c \in [\underline{c}, \bar{c}] : h \succsim (h_0, c, \ldots, c, \ldots)\} \neq \varnothing.$$

which implies that there exists $c_h \in [\underline{c}, \bar{c}]$ such that $(h_0, c_h, \ldots, c_h, \ldots) \sim h$. Thus we can define the map

$$V(h) = u(h_0) + \beta U((c_h, \ldots, c_h, \ldots))$$

for all $h \in \mathbf{H}$ and V represents $\succsim$. Now define $I_{+1} : B(U(\mathbf{D}), S, \Sigma) \to \mathbb{R}$ as $I_{+1}(V(h^1)) = U((c_h, \ldots, c_h, \ldots))$. It is straightforward to verify that I_{+1} is well-defined and normalized.[9] Now we prove that I_{+1} is also monotone. Suppose that

[9]Notice in particular, that the following holds true

$$B(U(\mathbf{D}), S, \Sigma) = \left\{V(h^1) : h \in H\right\}.$$

$\xi \geq \xi'$ for some $\xi, \hat{\xi} \in B(U(\mathbf{D}), S, \Sigma)$. Given that I_{+1} is independent of the first period 0, we can assume without loss of generality that there exist $h, \hat{h} \in \mathbf{H}$ such that $h_0 = \hat{h}_0$ and $\xi = V(h^1)$, $\hat{\xi} = V(\hat{h}^1)$. Since V represents $\succsim$, we have that $h^{s,1} \succsim \hat{h}^{s,1}$ for all $s \in S$. But then, by (9) we have that $(h_0, h^1) \succsim (h_0, \hat{h}^1)$, that is $h \succsim \hat{h}$. Therefore we obtain $V(h) \geq V(\hat{h})$ which implies that $U(c_h) \geq U(c_{\hat{h}})$, thus delivering the desired monotonicity of I_{+1}. Hence, we obtain the desired recursive representation

$$V : h \mapsto u(h_0) + \beta I_{+1}(V(h^1)).$$

$\square$

Proof of Theorem 2. $[(i) \Rightarrow (ii)]$. By the representation (2) we have that for all $d \in \mathbf{D}$

$$U(d) = \sum_{t=0}^{\infty} \beta^t u(d_t).$$

Now notice that, by continuity and state-monotonicity, for all $h \in \mathbf{H}$ there exists $d^h \in \mathbf{D}$ such that $h \sim d^h$ (one can use similar arguments as Lemma 8 in Kochov (2015)). Therefore V satisfies

$$V(h) = U(d^h),$$

for every $h \in \mathbf{H}$. Let $I_0 : B(U(\mathbf{D}), \Omega, \mathcal{G}) \to \mathbb{R}$ be defined as $I_0(\xi) = U(d^h)$ for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G}) \to \mathbb{R}$ with $\xi = \sum_{t=0}^{\infty} \beta^t u(h_t)$ for some $h \in \mathbf{H}$. The functional I_0 is a well-defined monotone certainty equivalent because of continuity and state-monotonicity. Now we prove that I_0 must be translation invariant. By contradiction, assume without loss of generality that there exists $k \in u(C)$ and $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ such that

$$I_0(\xi + k) > I_0(\xi) + k. \tag{10}$$

Now take $c \in [\underline{c}, \bar{c}]$ such that $k = \frac{u(c)}{\beta}$ and $h \in \mathbf{H}$ with $\xi = \sum_{t=0}^{\infty} \beta^t u(h_t)$. Observe that we have

$$\beta I_0 \left(\frac{u(c)}{\beta} + \sum_{t=0}^{\infty} \beta^t u(h_t) \right) = \beta \left(\frac{u(c)}{\beta} + u(h_0) + \beta I_{+1}(V(h^1)) \right) = V(d^{(c,h)}).$$

Stationarity implies that

$$V(d^{(c,h)}) = V(c, d^h) = u(c) + \beta V(d^h)$$

which contradicts (10) since we have that

$$u(c) + \beta V(d^h) = \beta \left(\frac{u(c)}{\beta} + I_0(\xi) \right) < \beta \left(I_0 \left(\xi + k \right) \right) = V(d^{(c,h)}).$$

We can therefore conclude that I_0 is translation invariant. Now we prove that I_0 and I_{+1} must satisfy condition (3). To this end fix $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ with $\xi = \sum_{t=1}^{\infty} \beta^t u(h_t)$ for some $h \in \mathbf{H}$. Since $\succsim$ admits a recursive representation and $I_0(\xi) = V(h)$, we have that

$$I_0\left(\sum_{t=0}^{\infty} \beta^t u(h_t)\right) = u(h_0) + \beta I_{+1}\left(V \circ h^1\right).$$

Now using the translation invariance of I_0, we have that

$$I_0\left(\sum_{t=1}^{\infty} \beta^t u(h_t)\right) = \beta I_{+1}\left(V \circ h^1\right). \tag{11}$$

Recall that $\xi = \sum_{t=1}^{\infty} \beta^t u(h_t)$ and observing that by (11) we must have

$$V \circ h^1 = I_0\left(\sum_{t=0}^{\infty} \beta^t u(h_{t+1})\right) = I_0\left(\frac{1}{\beta}\xi^1\right),$$

we obtain

$$I_0(\xi) = \beta I_{+1}\left(I_0\left(\frac{1}{\beta}\xi^1\right)\right) \tag{12}$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$.

Finally, translation invariance of I_{+1} is implied by generalized rectangularity (12). Indeed, suppose by contradiction that $I_{+1}(\xi + k\mathbf{1}_\Omega) \neq I_{+1}(\xi) + k$ for some $\xi \in B(U(\mathbf{D}), S, \Sigma)$ and $k \in U(\mathbf{D})$ with $\xi + k\mathbf{1}_S \in B(U(\mathbf{D}), S, \Sigma)$. Then it is routine to find $\varphi \in B(U(\mathbf{D}), \Omega, \mathcal{G}), w \in U(\mathbf{D})$ such that $\xi = I_0\left(\frac{1}{\beta}\varphi^1\right)$ and $k = \frac{w}{\beta}$. Clearly by translation invariance of I_0, we must have that $\xi + k = I_0\left(\frac{1}{\beta}\left(\varphi^1 + w\mathbf{1}_\Omega\right)\right)$. Thanks to these observations, applying generalized rectangularity we have

$$\begin{aligned}
I_0\left(\varphi + w\mathbf{1}_\Omega\right) &= \beta I_{+1}\left(I_0\left(\frac{1}{\beta}\left(\varphi^1 + w\mathbf{1}_\Omega\right)\right)\right) \\
&= \beta I_{+1}\left(\xi + k\mathbf{1}_S\right) \\
&\neq \beta I_{+1}\left(\xi\right) + \beta k \\
&= \beta I_{+1}\left(I_0\left(\frac{1}{\beta}\varphi^1\right)\right) + \beta k \\
&= I_0(\varphi) + w
\end{aligned}$$

contradicting the translation invariance of I_0.

$[(ii) \Rightarrow (i)]$ Since I_{+1} is translation invariant this part of the proof follows from Proposition 4 (case 1) in Bommier et al. (2017).

$\square$

Proof of Theorem 3. We start defining the following set

$$B_{ce}(U(\mathbf{D}),\Omega,\mathcal{G}) := \{I : B(U(\mathbf{D}),\Omega,\mathcal{G}) \to \mathbb{R} \mid I \text{ is a translation invariant certainty equivalent}\}.$$

Each $I \in B_{ce}(U(\mathbf{D}),\Omega,\mathcal{G})$ is a translation invariant certainty equivalent and as such it is 1-Lipschitz. More importantly, notice that

$$U(\mathbf{D}) \subseteq \left[\frac{u(\underline{c})}{1-\beta}, \frac{u(\overline{c})}{1-\beta}\right].$$

Since each $I \in B_{ce}(U(\mathbf{D}),\Omega,\mathcal{G})$ is normalized and monotone, we have that

$$\|I\|_\infty \leq \frac{u(\overline{c})}{1-\beta}.$$

Thus, we can endow $B_{ce}(U(\mathbf{D}),\Omega,\mathcal{G})$ with the sup-metric d_∞, defined as

$$d_\infty(I,I') = \|I - I'\|_\infty$$

for all $I, I' \in B_{ce}(U(\mathbf{D}),\Omega,\mathcal{G})$. Now we verify that $B_{ce}(U(\mathbf{D}),\Omega,\mathcal{G})$ is a complete metric space with respect to d_∞. To this end, fix a Cauchy sequence $(I_n)_{n\in\mathbb{N}}$ in $B_{ce}(U(\mathbf{D}),\Omega,\mathcal{G})$ and define $I \in \mathbb{R}^{B(U(\mathbf{D}),\Omega,\mathcal{G})}$ as $I(\xi) = \lim_{n\to\infty} I_n(\xi)$, I is well-defined since for all $\xi \in B(U(\mathbf{D}),\Omega,\mathcal{G})$ we have that $(I_n(\xi))_{n\in\mathbb{N}}$ is a Cauchy sequence in the set of real numbers. For all $k \in U(\mathbf{D})$ we have that $I_n(k) = k$ and hence $I(k) = \lim_{n\to\infty} k = k$, thus I is normalized. Monotonicity and translation invariance follow analogously. In particular, if $\xi \geq \xi'$, then $I_n(\xi) \geq I_n(\xi')$ for all $n \in \mathbb{N}$ and thus $I(\xi) \geq I(\xi')$. For translation invariance, fix $\xi \in B(U(\mathbf{D}),\Omega,\mathcal{G})$ and $k \in U(\mathbf{D})$ with $\xi + k \in B(U(\mathbf{D}),\Omega,\mathcal{G})$. Then, it follows that

$$I(\xi + k) = \lim_{n\in\mathbb{N}} I_n(\xi) + k = I(\xi) + k.$$

Thus, $I \in B_{ce}(U(\mathbf{D}),\Omega,\mathcal{G})$. To conclude we need to show that $I_n \xrightarrow{d_\infty} I$. Let $\varepsilon > 0$ and $\xi \in B(U(\mathbf{D}),\Omega,\mathcal{G})$. Then there exists N_ε, such that $d_\infty(I_n, I_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$. This yields

$$|I(\xi) - I_n(\xi)| = \lim_{m\to\infty} |I_m(\xi) - I_n(\xi)| < \varepsilon$$

for all $n \geq N_\varepsilon$. Since ξ was chosen arbitrarily $d_\infty(I, I_n) < \varepsilon$. Thus, we have that $(B_{ce}(U(\mathbf{D}),\Omega,\mathcal{G}), d)$ is a complete metric space. Now we define the following map,

$$\begin{aligned} T :&\mathbb{R}^{B(U(\mathbf{D}),\Omega,\mathcal{G})} \to \mathbb{R}^{B(U(\mathbf{D}),\Omega,\mathcal{G})} \\ &I \mapsto \beta I_{+1}\left(I\left(\frac{\xi^1}{\beta}\right)\right) \qquad \text{for all } \xi \in \mathbb{R}^{B(U(\mathbf{D}),\Omega,\mathcal{G})}. \end{aligned}$$

First observe that because of translation invariance of I_{+1}, we have

$$T(I+k) \le T(I) + \beta k$$

for all $k \in U(\mathbf{D})$ and all $I \in B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$.[10] Thus, following the same steps of Blackwell's contraction Theorem (Blackwell (1965)), it follows that T is a contraction mapping when restricted to $B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$ with respect to the metric d_∞, i.e.,

$$d_\infty \left(T(I), T(I')\right) \le \beta d_\infty \left(I, I'\right)$$

for all $I, I' \in B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$.

Moreover, notice that T is a self-map when restricted to $B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$. To see this, notice that if $\xi \ge \xi'$, then $\xi^1 \ge \xi'^1$ and hence, given that I_{+1} and I are monotone

$$\beta I_{+1}\left(I\left(\frac{\xi^1}{\beta}\right)\right) \ge \beta I_{+1}\left(I\left(\frac{\xi'^1}{\beta}\right)\right).$$

Let $k \in U(\mathbf{D})$, then $k^1 = k$ and hence

$$\beta I_{+1}\left(I\left(\frac{k^1}{\beta}\right)\right) = \beta I_{+1}\left(\frac{k}{\beta}\right) = k$$

thus $T(I)(k) = k$. To verify translation invariance, suppose $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ and $k \in U(\mathbf{D})$ with $\xi + k \in B(U(\mathbf{D}), \Omega, \mathcal{G})$. Then, $(\xi + k)^1 = \xi^1 + k$ and since I_{+1} and I are translation invariant, we have,

$$\begin{aligned}
\beta I_{+1}\left(I\left(\frac{\xi^1 + k}{\beta}\right)\right) &= \beta I_{+1}\left(I\left(\frac{\xi^1}{\beta}\right)\right) + \frac{k}{\beta} \\
&= \beta I_{+1}\left(I\left(\frac{\xi^1}{\beta}\right)\right) + k.
\end{aligned}$$

Thus, $T(I)(\xi+k) = T(I)(\xi)+k$. Therefore, we have that T is indeed a self-map when restricted to $B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$. Hence the result follows by applying the contraction mapping theorem. □

[10] To be completely rigorous notice that there exists a unique translation invariant, monotone, and normalized extension of I_{+1} to the *tube* $B(U(\mathbf{D}), \Omega, \mathcal{G}) + \mathbb{R}$. Thus, the reader can think of I_{+1} as this unique extension to verify the Blackwell discounting condition.

Generalized rectangularity, MEU, and variational preferences

Generalized rectangularity and MEU

In what follows we denote by $\mathcal{P} \subseteq \Delta(\Omega)$ a weak* compact and convex set of probability measures on $\mathcal{G}$. Moreover, we denote by $\mathcal{P}_\star$ the *lower probability* of $\mathcal{P}$ defined as

$$\begin{aligned} \mathcal{P}_\star : &\mathcal{G} \to [0,1] \\ &A \mapsto \min_{P \in \mathcal{P}} P(A). \end{aligned}$$

Fix also $\mathcal{L} \subseteq \Delta(S)$. We start recalling the following characterization of rectangularity.

Proposition 3. *$\mathcal{P}$ is $\mathcal{L}$-rectangular if and only if*

$$\min_{\ell \in \mathcal{L}} \left[\sum_{s \in S} \ell(s) \mathcal{P}_\star(A_s) \right] = \min_{P \in \mathcal{P}} P(A). \tag{13}$$

for all $A \in \mathcal{G}$

For the sake of completeness we provide the proof of this characterization.

Lemma 4. *Suppose $\mathcal{P}$ is $\mathcal{L}$-rectangular, then*

$$\min_{\ell \in \mathcal{L}} \left[\sum_{s \in S} \ell(s) \mathcal{P}_\star(A_s) \right] = \min_{P \in \mathcal{P}} P(A)$$

for all $A \in \mathcal{G}$.

Proof. Let $Q \in \mathcal{P}$, since $\mathcal{P}$ is $\mathcal{L}$-rectangular, there exists $\ell_Q \in \mathcal{L}$ and a collection $\{Q^s \in \mathcal{P} : s \in S\}$ such that $Q(A) = \sum_{s \in S} \ell_Q(s) Q^s(A_s)$ for all $A \in \mathcal{G}$. This implies that for all $A \in \mathcal{G}$,

$$Q(A) = \sum_{s \in S} \ell_P(s) Q^s(A_s) \geq \min_{\ell \in \mathcal{L}} \sum_{s \in S} \ell(s) \left[\min_{P \in \mathcal{P}} P(A_s) \right]$$

where the last inequality follows from observing that $\{Q^s \in \mathcal{P} : s \in S\} \subseteq \mathcal{P}$. Since $Q \in \mathcal{P}$ was chosen arbitrarily,

$$\min_{P \in \mathcal{P}} P(A) \geq \min_{\ell \in \mathcal{L}} \sum_{s \in S} \ell(s) \left[\min_{P \in \mathcal{P}} P(A_s) \right]$$

for all $A \in \mathcal{G}$. Conversely, fix $\{Q_s \in \mathcal{P} : s \in S\}$ and $\ell \in \mathcal{L}$. Since $\mathcal{P}$ is convex we have that $A \mapsto \sum_{s\in S} \ell(s)Q^s(A_s)$ belongs to $\mathcal{P}$. This yields that

$$\sum_{s\in S} \ell(s)Q^s(A_s) \geq \min_{P\in\mathcal{P}} P(A) \tag{14}$$

for all $A \in \mathcal{G}$. Now define $\mathcal{P}^s_\star := \{P^s_\star \in \mathcal{P} : s \in S\}$ where each $P^s_\star$ is such that $P^s_\star(A_s) = \min_{P\in\mathcal{P}} P(A_s)$, for all $A \in \mathcal{G}$ and all $s \in S$. Then, using $\mathcal{P}^s_\star$ as collection, (14) becomes

$$\sum_{s\in S} \ell(s)P^s_\star(A_s) \geq \min_{P\in\mathcal{P}} P(A)$$

and, since $\ell \in \mathcal{L}$ was chosen arbitrarily,

$$\min_{\ell\in\mathcal{L}} \sum_{s\in S} \ell(s) \left[\min_{P\in\mathcal{P}} P(A_s)\right] = \min_{\ell\in\mathcal{L}} \sum_{s\in S} \ell(s)P^s_\star(A_s) \geq \min_{P\in\mathcal{P}} P(A)$$

for all $A \in \mathcal{G}$. Connecting the two inequalities the claim follows. □

Lemma 5. *If* $\min_{\ell\in\mathcal{L}} \left[\sum_{s\in S} \ell(s)\mathcal{P}_\star(A_s)\right] = \min_{P\in\mathcal{P}} P(A)$ *for all* $A \in \mathcal{G}$*, then*

$$\min_{\ell\in\mathcal{L}} \mathbb{E}_\ell \left[\sum_{s\in S} \mathbf{1}_{\{s\}} \min_{Q^s\in\mathcal{P}} \mathbb{E}_{Q^s}[\xi]\right] = \min_{P\in\mathcal{P}} \mathbb{E}_P[\xi]$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$.

Proof. Indeed, Let $\xi = \sum_{i=1}^n \xi_i \mathbf{1}_{A_i}$ for some $\mathcal{G}$-measurable partition and some $(\xi_i)_{i=1}^n \in U(\mathbf{D})^n_+$. Since all A_i's are disjoint we have that

$$\begin{aligned}
\min_{\ell\in\mathcal{L}} \sum_{s\in S} \ell(s) \min_{P\in\mathcal{P}} \left[\sum_{i=1}^n \xi_i P_s(A_{i,s})\right] &= \min_{\ell\in\mathcal{L}} \sum_{s\in S} \ell(s) \sum_{i=1}^n \xi_i \min_{P\in\mathcal{P}} [P_s(A_{i,s})] \\
&= \sum_{i=1}^n \xi_i \left[\min_{\ell\in\mathcal{L}} \sum_{s\in S} \ell(s) \min_{P\in\mathcal{P}} P_s(A_{i,s})\right] \\
&= \sum_{i=1}^n \xi_i \min_{P\in\mathcal{P}} P(A_i) \\
&= \min_{P\in\mathcal{P}} \sum_{i=1}^n \xi_i P(A_i).
\end{aligned}$$

Then, using continuity and the monotone convergence theorem the claim follows. □

Now denote by $\mathrm{rect}_{\mathcal{L}}(\mathcal{P})$ the $\mathcal{L}$-rectangular hull of $\mathcal{P}$, i.e.,

$$\mathrm{rect}_{\mathcal{L}}(\mathcal{P}) = \left\{\sum_{s\in S} \ell(s)Q^s(\cdot_s) : \ell \in \mathcal{L} \text{ and } Q^s \in \mathcal{P}\right\}.$$

Clearly $\mathcal{P}$ is $\mathcal{L}$-rectangular if and only if $\mathcal{P} = \mathrm{rect}_{\mathcal{L}}(\mathcal{P})$.

Lemma 6. *If* $\min_{\ell\in\mathcal{L}} [\sum_{s\in S} \ell(s)\mathcal{P}_\star(A_s)] = \min_{P\in\mathcal{P}} P(A)$ *for all* $A \in \mathcal{G}$*, then* $\mathcal{P}$ *is* $\mathcal{L}$*-rectangular.*

The following proof is analogous to Lemma 1 in Amarante and Siniscalchi (2019).

Proof. By Lemma 5 we have

$$\min_{Q\in\mathrm{rect}_{\mathcal{L}}(\mathcal{P})} \mathbb{E}_Q[\xi] = \min_{P\in\mathcal{P}} \mathbb{E}_P[\xi] \tag{15}$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$. Then, by Theorem 7.52 in Aliprantis and Border (2006) equality (15) yields $\mathcal{P} = \mathrm{rect}_{\mathcal{L}}(\mathcal{P})$, proving that $\mathcal{P}$ is $\mathcal{L}$-rectangular. □

Proof of Proposition 3. It follows from Lemmas 4 and 6. □

The proof of Corollary 2 is now quite straightforward.

Proof of Corollary 2. Suppose that I_0 and I_{+1} satisfy (3). Then, since both I_0 and I_{+1} are positively homogeneous, it follows that generalized rectangularity is equivalent to

$$\begin{aligned}
\min_{P\in\mathcal{P}} \mathbb{E}_P[\xi] &= I_0(\xi) \\
&= I_{+1}\left(I_0(\xi^1)\right) \\
&= \min_{\ell\in\mathcal{L}} \mathbb{E}_\ell\left[\min_{P\in\mathcal{P}} \mathbb{E}_P\left[\xi^1\right]\right] \\
&= \min_{\ell\in\mathcal{L}} \sum_{s\in S} \ell(s) \min_{P^s\in\mathcal{P}} \mathbb{E}_{P^s}\left[\xi^1\right]
\end{aligned}$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$. This concludes the proof in light of Proposition 3. □

Generalized rectangularity and variational preferences

Proof of Corollary 3. By Theorem 2 there exist I_0 and I_{+1} that satisfy translation invariance. Observe that by (6) I_{+1} and I_0 are quasi-concave, and so by Lemma 25 in Maccheroni et al. (2006a) they are both concave. By Theorem 3 in the same paper we obtain the desired variational representation. In particular, there exist $c_0 : \Delta(\Omega) \to [0, \infty]$ and $c_{+1} : \Delta(S) \to [0, \infty]$ such that

$$I_0 = \min_{P\in\Delta(\Omega)} \{\mathbb{E}_P[\cdot] + c_0(P)\} \text{ and } I_{+1} = \min_{\ell\in\Delta(S)} \{\mathbb{E}_\ell[\cdot] + c_{+1}(\ell)\}.$$

To prove that the no-gain condition implies generalized rectangularity (3), fix $\xi \in B_0(U(\mathbf{D}), \Omega, \mathcal{G})$ with $\xi = \sum_{i=1}^n \xi_i 1_{A_i}$ for some $\mathcal{G}$-measurable partition $(A_i)_{i=1}^n$ and $(\xi_i)_{i=1}^n \in U(\mathbf{D})_+^n$. Then, we have that

$$\begin{aligned}
\beta I_{+1}\left(I_0\left(\frac{1}{\beta}\xi^1\right)\right) &= \beta \min_{\ell\in\Delta(S)} \left\{\sum_{s\in S} \ell(s) \min_{P_s\in\Delta(\Omega)} \left\{\sum_{i=1}^n P_s(A_{i,s})\frac{\xi_i}{\beta} + c_0(P_s)\right\} + c_{+1}(\ell)\right\} \\
&= \min_{\ell\in\Delta(S)} \left\{\sum_{s\in S} \ell(s) \min_{P_s\in\Delta(\Omega)} \left\{\sum_{i=1}^n P_s(A_{i,s})\xi_i + \beta c_0(P_s)\right\} + \beta c_{+1}(\ell)\right\} \\
&= \min_{\ell\in\Delta(S)} \min_{(P_s)_{s\in S}\in(\Delta(\Omega))^S} \left\{\sum_{s\in S}\sum_{i=1}^n \ell(s)P_s(A_{i,s})\xi_i + \beta\sum_{s\in S}\ell(s)c_0(P_s) + \beta c_{+1}(\ell)\right\} \\
&= \min_{P\in\Delta(\Omega)} \left\{\sum_{s\in S}\sum_{i=1}^n P_{+1}(s)P_s(A_{i,s})\xi_i + \beta\sum_{s\in S}P_{+1}(s)c_0(P_s) + \beta c_{+1}(P_{+1})\right\} \\
&= \min_{P\in\Delta(\Omega)} \left\{\sum_{i=1}^n P(A_i)\xi_i + c_0(P)\right\} \\
&= I_0(\xi).
\end{aligned}$$

where the last equality is implied by (7). Generalized rectangularity (3) for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ then holds by continuity and monotone convergence theorem. $\square$

The following proposition highlights a further connection between our generalized rectangularity and the no-gain condition.

Proposition 4. *Suppose that for all $P \in \Delta(\Omega)$,*

$$c_0(P) = \sup_{h\in\mathbf{H}} \left\{U(d^h) - \mathbb{E}_P\left[\sum_{t=0}^\infty \beta^t u(h_t)\right]\right\}.$$

Then generalized rectangularity (3) *implies the inequality*

$$c_0(P) \leq \beta\left[\sum_{s\in S} P_{+1}(s)c_0(P_s) + c_{+1}(P_{+1})\right]$$

for all $P \in \Delta(\Omega)$. Conversely, (7) *implies generalized rectangularity* (3)*.*

Proof. We have that

$$\begin{aligned}
\min_{P\in\Delta(\Omega)} &\left\{\sum_{s\in S}\sum_{i=1}^n P_{+1}(s)P_s(A_{i,s})\xi_i + \beta\sum_{s\in S}P_{+1}(s)c_{+1}(P_s) + \beta c_{+1}(\ell)\right\} \\
&= \min_{P\in\Delta(\Omega)} \left\{\sum_{i=1}^n P(A_i)\xi_i + c_0(P)\right\},
\end{aligned}$$

for all $\xi \in B_0(U(\mathbf{D}), \Omega, \mathcal{G})$. By Theorem 3 in Maccheroni et al. (2006a) we have that

$$c_0(P) \leq \beta \left[\sum_{s \in S} P_{+1}(s) c_0(P_s) + c_{+1}(P_{+1}) \right].$$

The other side of the claim was already shown in the proof of Corollary 3. □

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