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Optimal additional voluntary contribution in DC pension schemes to manage inadequacy risk

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Abstract

In defined contribution pension schemes the member bears the investment risk and her main concern is to obtain an inadequate fund at retirement. To address inadequacy risk, flexibility is often given to the member to pay additional voluntary contributions (AVCs) into the fund. In many countries the AVC schemes allow members of the workplace pension plan to increase the amount of retirement benefits by paying extra contributions. In this paper, we define a target-based optimization problem where the member of an AVC scheme can choose at any time the investment strategy and the additional voluntary contributions to the fund. In setting the problem, the member faces a trade-off between the importance given to the stability of payments during the accumulation phase and the achievement of the desired annuity at retirement. We derive closed-form solutions via dynamic programming and prove that (i) the optimal fund never reaches the target final fund, (ii) the optimal amount invested in the risky asset is always positive, and (iii) the optimal AVC is always higher than the target one. We run numerical simulations to allow for different member's preferences, and perform sensitivity analysis to check the controls' robustness.

Keywords. Stochastic optimal control, dynamic programming, defined contribution pension scheme, additional voluntary contributions, target-based approach, net replacement ratio.

JEL classification: C61, D81, G11.

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1 Introduction

Due to the ageing population problem that is threatening the sustainability of public pension systems all around the world, it is well known that in future decades the pension provision of most countries will rely also on the so-called second pillar, i.e. private and occupational pension funds. In the last decades, there has been an evident shift from defined benefit (DB) to defined contribution (DC) pension schemes in all pension fund systems. The difference between a DB and a DC scheme is the way the financial risk is treated: it is borne by the sponsor (the employer) in a DB scheme, while it is borne by the member in a DC scheme. As a direct consequence, in DC pension schemes the pension rate received by the member will depend essentially on the performance of financial markets experienced during the accumulation phase: periods of high investment performance will lead to high pension rates and vice versa. Thus, the major drawback of DC pension schemes consists in the high degree of uncertainty regarding the level of pension benefits. To put this concept into simple words: *"In essence, a system that defines a set level of contributions cannot define the level of benefits received"*, see Knox (1993).

The large inequality among members of DC pension schemes of different cohorts is well documented in the analysis by Antolin (2009). He finds that *"Two individuals with identical work histories could end up with markedly different retirement incomes just because they happen to retire in two different years with different market conditions"*. Indeed, he finds that the replacement ratio[1] hypothetically enjoyed by retirees of DC pension plans with the same working and contribution history (5% contribution for 40 years service) but retiring in different years between 1940 and 2008 varies remarkably, the range being as broad as $3\% - 80\%$ for Japan, and $18\% - 55\%$ for the U.S.A. The impact of market conditions on retirement savings accumulated in DC pension plans can be particularly dramatic in the case of bad market performance just prior to retirement. For instance, according to Antolin (2009), due to the 2008 market crash the replacement ratio could have suffered an approximate 10% drop from 2007 to 2008: a retiree in U.S.A. in 2007 aged 65 could have enjoyed a replacement ratio of about 24%, while the same individual retiring in 2008 would have obtained only about 15%.

The analysis conducted by Antolin (2009) concludes that four measures could be adopted in order to deliver adequate retirement savings from DC pension funds: (i) increasing contributions,

[1] The replacement ratio, also called retirement rate, is the ratio between the pension income and the final wage.

(ii) postponing retirement, (iii) adopting conservative investment strategies in the period prior to retirement, (iv) managing inflation risk in the decumulation phase with appropriate index-linked annuities.

The possibility of increasing periodically the contribution rate has recently been considered also by the regulator, under the name *additional voluntary contributions* (AVCs), also due to the higher levels of uncertainty experienced during the COVID-19 pandemics. According to OECD (2021), a number of countries allow the possibility for workers to make AVCs into the pension fund. In Australia, employees have no obligation to contribute to a plan but can make voluntary contributions on top of the employer's contributions. In New Zeland, workers in the "Kiwisaver Plan" are required to contribute with a minimum of 3% but have the option to pay a higher contribution ($4\%, 6\%, 8\%$ or 10% of the wage). In Poland, the "Employee Capital Plans (PPK)" requires a minimum 2% contribution for employees and 1.5% for employers, with the possibility for both of making additional contributions of up to 2.5% and 2%, respectively. The OECD analysis highlights the role of the AVCs during the pandemics, when some countries provided flexibility on mandatory contributions. Due to this flexibility, in some cases the effective contributions in 2021 have been higher than in 2020, thanks also to the additional contributions. For example, according to official data from the Inland Revenue Department of New Zeland,[2] in 2021 there was a (small) reduction in employees deductions, partially offset by a (small) increase in their voluntary contributions. Also in the UK workers are allowed to make AVCs. For example, the UK Government reports the financial position of its Civil Service Additional Voluntary Contribution Scheme (CSAVCS),[3] which provides for civil service employees to make AVCs to increase their pension entitlements or to increase life cover. Making AVCs is even proposed in the House of Commons Staff Pension Schemes as a way for workers in the Parliament to boost their pensions.[4] Nkeki (2017) highlights that in Nigeria, the Pension Reform in 2014 allows for AVC on top of the compulsory minimum contribution of 8% by workers. In Ireland, AVC to the occupational pensions is one of the possibilities for workers to set aside savings under the third pillar.[5] Ombuki

[2]See Inland Revenue NZ database at https://www.ird.govt.nz/about-us/tax-statistics/kiwisaver/datasets.

[3]See the Civil Superannuation annual report and account 2021-22 at https://www.gov.uk/government/publications/civil-superannuation-annual-account-2021-to-2022/civil-superannuation-annual-report-and-account-2021-22-html

[4]See the House of Commons Staff Pension Plan guidebook at https://www.parliament.uk/globalassets/documents/commons-resources/Staff-handbook/chapter-27-pensions.pdf

[5]See https://assets.gov.ie/200479/05ab1b26-da17-4e86-b1e7-653de0d4791b.pdf

& Oteki (2019) investigated the key determinants of voluntary contributions in Kenya, and the survey indicated an inverse relation between income and AVCs: members in the lowest earning brackets participated more in AVCs than those on higher earning brackets, the plausible reason being that those earning less not only contributed less but also tended to have lower contributions by their employers, and therefore felt the need to make AVCs. The publication also revealed that insufficient income and lack of information were the main reasons for not making AVCs.

Apparently, the actuarial literature on AVCs in a DC pension scheme is scarce. The typical way to deal with the investment risk of DC pension funds has been the choice of the investment strategy only. Indeed, the literature on the accumulation phase of DC pension schemes is full of examples of optimal investment strategies to attack the investment risk.[6]

In the mentioned papers, the contribution paid into the fund is assumed to be a fixed proportion of the wage, in line with the essence of a DC pension scheme. In other words, in the optimization problems mainly solved in the literature, the contribution rate is not a control variable. To the best of our knowledge, the only exception is Nkeki (2017) in a jump-diffusion model with constant relative risk aversion utility function. A few papers find the optimal investment strategy with constant absolute risk aversion (CARA) utility function in a DC scheme modelling the AVCs as a stochastic state variable rather than a control variable, see Akpanibah & Oghen'Oro (2018) and Chinyere, Gbarayorks & Inamete (2022). They find, intuitively, that the payment of AVCs reduces the investment in the risky asset.

The recommendations made in the analysis by Antolin (2009) indicate that paying higher contributions could be a way to reduce the risk of inadequate retirement income. The indications provided by OECD (2021) during the Covid-19 pandemics go in the same direction. The analysis of the possibility of paying additional contributions seems to be an unexplored area of research in the actuarial community.

To fill in this gap in the literature, in this paper we solve an optimization problem in a DC pension scheme with two control variables: the investment strategy and the AVC paid by the

[6] See, among many others, Boulier, Huang & Taillard (2001), Haberman & Vigna (2002), Deelstra, Grasselli & Koehl (2003), Devolder, Bosch Princep & Dominguez Fabian (2003), Battocchio & Menoncin (2004), Cairns, Blake & Dowd (2006), Xiao, Zhai & Qin (2007), Gao (2008), Di Giacinto, Federico & Gozzi (2011) for the maximization of expected utility, and He & Liang (2013), Yao, Yang & Chen (2013), Yao, Lai, Ma & Jian (2014), Vigna (2014), Menoncin & Vigna (2017), Menoncin & Vigna (2020), Guan & Liang (2015) and Wu, Zhang & Chen (2015) for mean-variance criteria.

member on top of the employer's contribution. The optimization problem is defined reflecting two conflicting needs of the member: reaching an adequate replacement ratio at retirement coupled with the desire for stable contribution payments over time. We solve the problem via dynamic programming and provide closed-form solutions. At theoretical level, we prove that the optimal final fund never reaches the target final fund, the optimal amount invested in the risky asset is always positive, and the optimal AVC is always higher than the target one. Numerical simulations show the behaviour of the optimal policies over time. We find that the desired net replacement ratio can be approached very closely by paying appropriate AVCs into the pension fund. The contributions' stability can also be achieved, but at the price of a lower net replacement ratio. As expected, a longer accumulation phase as well as higher contributions paid by the employer help better managing the trade-off faced by the worker.

The remainder of the paper is structured as follows. In Section 2, we define the financial market. In Section 3, we set the optimization problem. In Section 4, we derive the analytical solution and in Section 5 we prove the additional theoretical results. Section 6 is devoted to numerical simulations and Section 7 concludes.

2 The model

The financial market is described by a Black and Scholes framework with a riskless asset and a risky asset. The risk is described by a Brownian Motion $W(t)$ defined on a complete filtered-probability space $\{\Omega, \mathcal{F}(t), \mathcal{P}\}$, with $\mathcal{F}(t)$ being the filtration generated by $W(t)$. The price of the riskless asset $G(t)$ evolves according to the following deterministic ordinary differential (ODE) equation:

$$dG(t) = rG(t)dt \tag{1}$$

where $r \geq 0$. The price of the risky asset $S(t)$ evolves according to a Geometric Brownian Motion with constant drift $\lambda > r$ and volatility $\sigma > 0$:

$$dS(t) = \lambda S(t)dt + \sigma S(t)dW(t) \tag{2}$$

The member joins the fund at time 0 aged χ, works for the next T years and then retires at time T. There is no risk of unemployment, and her wage $w(t)$ grows exponentially at a constant rate $g \geq 0$:

$$dw(t) = gw(t)dt \tag{3}$$

starting from an initial salary $w(0) = w_0$. At any time $t \in [0, T]$, the member chooses (i) the proportion $y(t)$ of the portfolio wealth invested in the risky asset, and (ii) the additional voluntary contribution (AVC) $c(t)$. The employer contributes to the fund with a fixed proportion $\gamma \in (0, 1)$ of the wage $w(t)$. Therefore, the value of the fund $X(t)$ evolves according to the following stochastic differential equation (SDE), with initial condition $x_0 \geq 0$:

$$\begin{cases} dX(t) = \{X(t)\,[y(t)(\lambda - r) + r] + \gamma w(t) + c(t)\}\,dt + X(t)y(t)\sigma dW(t) \\ X(0) = x_0 \end{cases} \tag{4}$$

The control process $u(t)$ consists in a couple of two stochastic processes:

$$\{u(t)\}_{t\in[0,T]} = \left(\{y(t)\}_{t\in[0,T]}, \{c(t)\}_{t\in[0,T]}\right) \tag{5}$$

3 The optimization problem

The preferences of the member are driven by two conflicting objectives. On the one hand, the member desires stability for the contribution inflow, i.e. she has a target for the ideal AVC, say a fraction $\eta \in (0, 1)$ of the wage, and would like to pay AVCs that do not deviate too much from η times the wage. On the other hand, she pays additional contributions because she aims at achieving a given targeted pension rate at retirement upon conversion of the final fund $X(T)$ into lifetime annuity. The targeted pension rate b_T at retirement time T is defined as a fraction $\alpha \in (0, 1)$ of the last wage, in order to obtain a net replacement ratio equal to α. Therefore, the continuously experienced running loss $L(t, c(t))$ and the final loss $\Phi(T, X(T))$ are defined, respectively, as:

$$L(t, c(t)) = v\,(\eta w(t) - c(t))^2 \tag{6}$$

$$\Phi(T, X(T)) = (b_T a_{\chi+T} - X(T))^2 \tag{7}$$

where $b_T = \alpha w(T)$, $a_{\chi+T}$ is the price of the unitary lifetime annuity for a policyholder aged $\chi + T$ and $v > 0$ is a weight that gives relative importance to the running loss with respect to the terminal one (see Remark 1). The loss experienced by the member is composed by two parts: an immediate, continuous loss experienced at t whenever the voluntary contribution paid $c(t)$ is different from $\eta w(t)$, plus a final loss at time T, if the income provided by the annuity that can be purchased with the final fund $X(T)$ differs from b_T. The total expected loss at time 0 with wealth x_0 is:

$$\mathbb{E}_{0,x_0}\left[\int_0^T e^{-\rho s} L(s, c(s))ds + e^{-\rho T}\Phi(T, X(T))\right] \tag{8}$$

where $\rho > 0$ is the subjective inter-temporal discount rate.

Remark 1. *Notice that the weight $v > 0$ plays a crucial role in displaying the member's preferences. The higher v, the higher the willingness to keep the additional contributions close to the target AVC, i.e. the higher the need for stability in the contributions payment, and the lower the importance given to the achievement of the desired net replacement ratio. Vice versa, the lower v, the higher the importance given to the achievement of the desired net replacement ratio, and the lower the desire for contributions' stability. The trade-off between the two conflicting objectives of contributions' stability and adequate net replacement ratio is solved by an appropriate choice of v, which is then crucial for the optimization outcomes, as we will see in Section 6. In the remainder of the paper, we will refer to v as the weight given to the contributions' stability.*

The member wishes to minimize the total expected loss from 0 to T, choosing amongst all possible investment and AVC strategies in the set of admissible strategies $\mathcal{U}$, defined as the set of $\mathbb{R}^2$-valued stochastic processes $u = \{u_s\}_{s\in[0,T]}$ that are Markov control processes, $\mathcal{F}_t$-adapted and such that the SDE (4) has a unique strong solution.

The use of quadratic loss functions is quite common in the literature on pension funds: in the context of DB pension funds some examples include Haberman & Sung (1994), Owadally & Haberman (2004), Josa-Fombellida & Rincón-Zapatero (2008), Josa-Fombellida & Rincón-Zapatero (2010), Josa-Fombellida & Rincón-Zapatero (2012). In the context of DC schemes, see, among others, Cairns (2000), Vigna & Haberman (2001), Haberman & Vigna (2002), Gerrard, Haberman & Vigna (2004), Gerrard, Haberman & Vigna (2006), Emms (2010), Gerrard, Højgaard & Vigna (2010). Moreover, the quadratic target-based loss function has been shown to be a particular case of the celebrated mean-variance approach, see Zhou & Li (2000).

We solve the problem using the dynamic programming principle, hence we define the criterion function $J_{t,x}(u(.))$: $\mathcal{U} \rightarrow \mathbb{R}$:

$$J_{t,x}(u(.)) = J_{t,x}(y(.), c(.)) = \mathbb{E}_{t,x}\left[\int_t^T e^{-\rho s} L(s, c(s))ds + e^{-\rho T}\Phi(T, X(T))\right] \tag{9}$$

and define the value function $V(t, x)$ as:

$$V(t, x) = \inf_{u(.)\in\mathcal{U}} J_{t,x}\left(u(.)\right) \tag{10}$$

We aim to find $V(t, x)$ and the associated optimal control $\{u^*(t)\} = (\{y^*(t)\,\{c^*(t)\}\})$ such that:

$$V(t, x) = J_{t,x}(u^*(.)) = J_{t,x}(y^*(.), c^*(.)) \tag{11}$$

4 The analytical solution

The problem of minimizing (9) is a stochastic optimal control problem, which we solve using dynamic programming, see for instance Yong & Zhou (1999), Björk (1998).

The value function $V(t, x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) Equation:

$$\inf_{u(.)\in\mathcal{U}} \left[e^{-\rho t} L(t, c(t)) + \mathcal{L}^u V(t, x)\right] = 0 \tag{12}$$

with the boundary condition:

$$V(T, x) = e^{-\rho T}\Phi(T, x) \tag{13}$$

where $\mathcal{L}^u V(t, x)$ is the linear infinitesimal operator applied to $V(t, x)$, given by:

$$\mathcal{L}^u V(t, x) = V_t + \left[x\left(y(\lambda - r) + r\right) + \gamma w(t) + c\right] V_x + \frac{1}{2}x^2 y^2 \sigma^2 V_{xx} \tag{14}$$

where V_t, V_x and V_{xx} stand for $\dfrac{\partial V(t,x)}{\partial t}$, $\dfrac{\partial V(t,x)}{\partial x}$ and $\dfrac{\partial^2 V(t,x)}{\partial x^2}$ respectively.

To solve Equation (12), we first fix an arbitrary point (t, x) and see it as a static optimization in $\mathbb{R}^2$. We lighten notation in (12) by denoting the term in square brackets by $\Psi(t, x, y, c)$, i.e.:

$$\Psi(t,x,y,c) = e^{-\rho t} v\left(\eta w(t) - c\right)^2 + V_t + \left[x\left(y(\lambda - r) + r\right) + \gamma w(t) + c\right] V_x + \frac{1}{2} x^2 y^2 \sigma^2 V_{xx} \tag{15}$$

So that the minimization, in y and c, is written as:

$$\inf_{(y,c)} \Psi(t,x,y,c) = 0 \tag{16}$$

We find the minimizers y^* and c^* in Equation (16) using necessary First Order Conditions (FOCs):

$$\frac{\partial \Psi(t,x,y^*,c)}{\partial y} = 0 \Rightarrow \quad y^* = -\left(\frac{\lambda - r}{x\sigma^2}\right) \frac{V_x}{V_{xx}} \tag{17}$$

$$\frac{\partial \Psi(t,x,y,c^*)}{\partial c} = 0 \Rightarrow \quad c^* = \eta w(t) - \frac{1}{2v} e^{\rho t} V_x \tag{18}$$

which depend on the partial derivatives of $V(t,x)$. The sufficient Second Order Condition (SOC) for the Hessian matrix is checked afterwards.

Replacing the expressions of y^* and c^* given by (17) and (18) back into the HJB Equation (12) leads to the following partial differential equation (PDE):

$$V_t + \left[xr + (\gamma + \eta) w(t)\right] V_x - \left[\frac{e^{\rho t}}{4v} + \frac{\beta^2}{2V_{xx}}\right] V_x^2 = 0 \tag{19}$$

where β is the Sharpe Ratio:

$$\beta = \frac{\lambda - r}{\sigma} \tag{20}$$

To solve the PDE we use an *ansatz*. Since the loss function is quadratic, it is reasonable to assume that $V(t,x)$ is in the form:

$$V(t,x) = e^{-\rho t}[A(t)x^2 + B(t)x + C(t)] \tag{21}$$

In such case, the boundary condition of Equation (13) becomes:

$$V(T, X(T)) = e^{-\rho T} \Phi(T, X(T)) = e^{-\rho T} [b_T \ddot{a}_{\chi+T} - X(T)]^2$$

that implies

$$A(T) = 1 \qquad B(T) = -2b_T \ddot{a}_{\chi+T} \qquad C(T) = b_T^2 \ddot{a}_{\chi+T}^2 \tag{22}$$

The partial derivatives of the *ansatz* are given by:

$$V_t = e^{-\rho t}\left[\left(A'(t) - \rho A(t)\right)x^2 + \left(B'(t) - \rho B(t)\right)x + \left(C'(t) - \rho C(t)\right)\right] \tag{23}$$

$$V_x = e^{-\rho t}\left[2A(t)x + B(t)\right] \tag{24}$$

$$V_{xx} = 2e^{-\rho t}A(t) \tag{25}$$

Replacing the partial derivatives (23), (24) and (25) into the PDE (19) yields

$$\begin{aligned}
&e^{-\rho t}\left[\left(A'(t) - \rho A(t)\right)x^2 + \left(B'(t) - \rho B(t)\right)x + \left(C'(t) - \rho C(t)\right)\right] + \\
&+ \left[xr + (\gamma+\eta)w(t)\right]e^{-\rho t}\left[2A(t)x + B(t)\right] - \left[\frac{e^{\rho t}}{4v} + \frac{\beta^2}{2\left(2e^{-\rho t}A(t)\right)}\right]\left[e^{-\rho t}\left(2A(t)x + B(t)\right)\right]^2 = 0
\end{aligned} \tag{26}$$

which holds $\forall (t,x) \in [0,T] \times \mathbb{R}$. Combining it with the boundary conditions of Equation (22), it leads to a system of three initial value problems (IVPs) that needs to be verified if the *ansatz* formulation is to work:

$$\begin{cases} A'(t) = \left(\rho - 2r + \beta^2\right)A(t) + \dfrac{A^2(t)}{v} \\ A(T) = 1 \end{cases} \tag{27}$$

$$\begin{cases} B'(t) = \left(\rho - r + \beta^2 + \dfrac{A(t)}{v}\right)B(t) - 2(\gamma+\eta)w(t)A(t) \\ B(T) = -2b_T \ddot{a}_{\chi+T} \end{cases} \tag{28}$$

$$\begin{cases} C'(t) = \rho C(t) - (\gamma+\eta)w(t)B(t) + \left(\dfrac{1}{4v} + \dfrac{\beta^2}{4A(t)}\right) B^2(t) \\ C(T) = b_T^2 \ddot{a}_{\chi+T}^2 \end{cases} \tag{29}$$

Notice that Equations (17) and (18) for the optimal control depend only on the partial derivatives V_x and V_{xx}, given by (24) and (25), neither of which depends on $C(t)$. Therefore, in the following we provide solutions only for $A(t)$ and $B(t)$.

It turns out that a crucial quantity is δ defined as follows:

$$\delta = 2r - \rho - \beta^2 \tag{30}$$

From now on, in the remaining of the paper, we will assume that $\delta \neq 0$, the case $\delta = 0$ leading to similar results (see Footnote 7). The differential equation in the IVP (27) is a Bernoulli type. The solution is given by:[7]

$$A(t) = \frac{v\delta e^{\delta(T-t)}}{e^{\delta(T-t)} + v\delta - 1} \tag{31}$$

It is easy to show that[8]

$$A(t) > 0 \qquad \forall \delta \in \mathbb{R}, \quad \forall t \in [0,T] \tag{32}$$

Noting from (3) that the wage at time t is given by

$$w(t) = w_0 e^{gt}$$

the solution of the IVP in Equation (28) for $B(t)$ is:

$$B(t) = \frac{-2v\delta e^{\delta(T-t)}}{e^{\delta(T-t)} + v\delta - 1} \left[\frac{(\gamma+\eta)w_0}{g-r} \left(e^{gt} - e^{gT - r(T-t)} \right) + b_T \ddot{a}_{\chi+T} e^{-r(T-t)} \right] \tag{33}$$

[7]If $\delta = 0$, $A(t) = v(v + T - t)^{-1}$. Notice that in this case $A(t) > 0$. Furthermore, if $\delta = 0$, Equations (34), (35), (36) and (38) still hold, while Equations (37) and (39) should be rewritten accordingly.

[8]In all cases, we have: $v > 0$. If $\delta = 0$, see Footnote 7. If $\delta \neq 0$, if $\delta > 0$ (< 0), both the numerator and the denominator are > 0 (< 0).

Note that $B(t)$ can be written as:

$$B(t) = -2A(t)\left[\frac{(\gamma+\eta)w_0}{g-r}\left(e^{gt} - e^{gT-r(T-t)}\right) + b_T\ddot{a}_{\chi+T}e^{-r(T-t)}\right] \tag{34}$$

4.1 Optimal Control

To find the optimal control, first simplify the expressions for y^* and b^*, given by (17) and (18), using Equations (20), (24) and (25):

$$y^*(t,x) = -\frac{\beta}{\sigma}\left[1 + \frac{B(t)}{2A(t)x}\right] = -\frac{\beta}{\sigma x}\left[x - h(t)\right] \tag{35}$$

$$c^*(t,x) = \eta w(t) - \frac{1}{2v}\left[2A(t)x + B(t)\right] = \eta w(t) - \frac{A(t)}{v}\left[x - h(t)\right] \tag{36}$$

where

$$h(t) = -\frac{B(t)}{2A(t)} = \frac{(\gamma+\eta)w_0}{g-r}\left(e^{gt} - e^{gT-r(T-t)}\right) + b_T\ddot{a}_{\chi+T}e^{-r(T-t)} \tag{37}$$

Therefore, we obtain the expressions for the optimal control process:

$$y^*(t,x) = -\frac{\beta}{\sigma}\left\{1 - \frac{1}{x}\left[\frac{(\gamma+\eta)w_0}{g-r}\left(e^{gt} - e^{gT-r(T-t)}\right) + b_T\ddot{a}_{\chi+T}e^{-r(T-t)}\right]\right\} \tag{38}$$

$$c^*(t,x) = \eta w_0 e^{gt} - \frac{\delta e^{\delta(T-t)}}{e^{\delta(T-t)} + v\delta - 1}\left[x - \frac{(\gamma+\eta)w_0}{g-r}\left(e^{gt} - e^{gT-r(T-t)}\right) - b_T\ddot{a}_{\chi+T}e^{-r(T-t)}\right] \tag{39}$$

4.2 Second Order Conditions

Now we check the sign of Hessian of $\Psi(t,x,y,c)$ given by Equation (15). The partial derivatives of $\Psi(y,c)$ are:

$$\frac{\partial^2\Psi(y,c)}{\partial c\partial y} = 0 \qquad \frac{\partial^2\Psi(y,c)}{\partial y^2} = x^2\sigma^2 V_{xx} \qquad \frac{\partial^2\Psi(y,c)}{\partial^2 c} = 2ve^{-\rho t}$$

Therefore, given a generic vector $u = (y,c) \in \mathbb{R}^2$, the Hessian Matrix of $\Psi(y,c)$ is:

$$H_\Psi(y,c) = \begin{bmatrix} x^2\sigma^2 V_{xx} & 0 \\ 0 & 2ve^{-\rho t} \end{bmatrix}$$

Due to (25) and (32), we have $V_{xx} > 0 \quad \forall t \in [0,T]$. Hence we see that $H_\Psi(y,c)$ is positive definite for every point $(t,x) \in [0,T] \times \mathbb{R}$, which implies that $\Psi(y,c)$ is strictly convex. Therefore, the FOCs become sufficient to guarantee that the unique stationary point (y^*, c^*) is indeed the unique point of global minimum.

5 Some theoretical results

In this section, we prove three theoretical results that shed some light on the nature of the optimal investment in the risky asset, and on the relationship between the optimal contribution and the optimal final fund versus the targeted contribution and the targeted replacement ratio.

The first result is that the optimal amount invested in the risky asset is always positive. The second result is that the optimal contribution paid into the fund is always higher than the targeted contribution. The third result is that the final fund under optimal control is always lower than the targeted final fund. These results are due to the quadratic nature of the loss functions and are in line with previous works on DC pension schemes, see, among others, Gerrard et al. (2004), Menoncin & Vigna (2017).

The theoretical results, which are proved in Proposition 5.2, are based upon the following assumption.

Assumption 5.1. *Let the financial-labour market and the member's preferences be described as in Sections 2 and 3. Then, we say that this assumption is satisfied if*

$$b_T \ddot{a}_{\chi+T} > x_0 e^{rT} + \int_0^T (\gamma + \eta) w(s) e^{r(T-s)} ds \tag{40}$$

Remark 2. *Notice that condition* (40) *is satisfied if the target fund chosen (the l.h.s. of* (40)*) is larger than the final fund at time T that one would have by investing the initial fund and the future flows (i.e. the contributions paid by the employer and the target contributions paid by the employee) in the riskless asset (the r.h.s. of* (40)*). In other words, the target chosen by the* *member must be "large enough": if the target was equal or lower than such threshold, it would be possible to reach or exceed it at time T just by choosing the trivial strategy of investing 100% in the riskless asset without the need to take any further risk. Therefore, Assumption 5.1 is necessary for the optimization problem to be interesting.*

We are now ready to prove the theoretical results mentioned above.

Proposition 5.2. *Let the financial-labour market and the member's preferences be described as in Sections 2 and 3, and let Assumption 5.1 hold. Then:*

1. *the optimal amount invested in the risky asset at any time $t \in [0, T]$ is strictly positive, i.e. $X^*(t)y^*(t) > 0$;*
2. *the optimal contribution paid into the fund at any time $t \in [0, T]$ is strictly larger than the targeted contribution, i.e. $c^*(t) > \eta w(t)$;*
3. *the final fund under optimal control is strictly lower than the targeted final fund, i.e. $X^*(T) < b_T \ddot{a}_{\chi+T}$.*

Proof. Due to (32), (35) and (36), for claims 1. and 2. it suffices to show that

$$X^*(t) - h(t) < 0 \tag{41}$$

where $h(t)$ is given by (37). Let us define the new process $Z(t)$ as:

$$Z(t) = X^*(t) - h(t) \tag{42}$$

It is useful to rewrite $h(t)$ as follows:

$$h(t) = Ee^{gt} + He^{-r(T-t)} \tag{43}$$

where

$$E = \frac{(\eta + \gamma)w_0}{g - r} \qquad F = b_T \ddot{a}_{\chi+T} \qquad H = F - Ee^{gT} \tag{44}$$

so that

$$h'(t) = Ege^{gt} + Hre^{-r(T-t)} \tag{45}$$

It turns out that

$$dZ(t) = dX^*(t) - h'(t)dt = \mu_Z(t,\cdot)dt + \sigma_Z(t,\cdot)dW(t) \tag{46}$$

where it remains to calculate the drift $\mu_Z(t,\cdot)$ and the volatility $\sigma_Z(t,\cdot)$.[9] Due to (35), from (4) and (46) we can easily find the volatility

$$\sigma_Z(t,\cdot) = xy^*\sigma = -\beta\left[x - h(t)\right] = -\beta Z(t) \tag{47}$$

Using (35), (36), (43), (45) and (44), the drift is given by

$$\begin{aligned}
\mu_Z(t,\cdot) &= xy^*(\lambda - r) + rx + \gamma w(t) + c^* - h'(t) = \\
&= -\beta^2\left[x - h(t)\right] + rx + \gamma w(t) + \eta w(t) - \frac{A(t)}{v}\left[x - h(t)\right] - Ege^{gt} - Hre^{-r(T-t)} = \\
&= \left[x - h(t)\right]\left(-\beta^2 - \frac{A(t)}{v}\right) + rx + (\gamma+\eta)w_0e^{gt} - Ege^{gt} - Hre^{-r(T-t)} = \\
&= \left[x - h(t)\right]\left(-\beta^2 - \frac{A(t)}{v}\right) + rx - Hre^{-r(T-t)} + e^{gt}\left[(\gamma+\eta)w_0 - \frac{(\gamma+\eta)w_0g}{g-r}\right] = \\
&= \left[x - h(t)\right]\left(-\beta^2 - \frac{A(t)}{v}\right) + rx - Hre^{-r(T-t)} + e^{gt}\left[-r\frac{(\gamma+\eta)w_0}{g-r}\right] = \\
&= \left[x - h(t)\right]\left(-\beta^2 - \frac{A(t)}{v}\right) + rx - Hre^{-r(T-t)} - rEe^{gt} = \\
&= \left[x - h(t)\right]\left(-\beta^2 - \frac{A(t)}{v}\right) + r\left(x - He^{-r(T-t)} - Ee^{gt}\right) = \\
&= \left[x - h(t)\right]\left(-\beta^2 - \frac{A(t)}{v}\right) + r\left[x - h(t)\right] = \left[x - h(t)\right]\left(r - \beta^2 - \frac{A(t)}{v}\right)
\end{aligned} \tag{48}$$

[9] In what follows, we will sometimes write x in the place of $X^*(t)$, y^* in the place of $y^*(t, X^*(t))$ and c^* in the place of $c^*(t, X^*(t))$.

Therefore the process $Z(t)$ follows the dynamics

$$dZ(t) = \left(r - \beta^2 - \frac{A(t)}{v}\right) Z(t)dt - \beta Z(t)dW(t) \tag{49}$$

yielding[10]

$$Z(t) = Z(0)\exp\left[\int_0^t \left(r - \frac{3}{2}\beta^2 - \frac{A(s)}{v}\right) ds - \int_0^t \beta dW(s)\right] \tag{50}$$

Observe now that

$$Z(0) = x_0 - h(0) = x_0 - (\gamma+\eta)w_0\frac{1-e^{(g-r)T}}{g-r} - b_T\ddot{a}_{\chi+T}e^{-rT} =$$
$$x_0 + \int_0^T (\gamma+\eta)w_0e^{(g-r)s}ds - b_T\ddot{a}_{\chi+T}e^{-rT} = x_0 + \int_0^T (\gamma+\eta)w(s)e^{-rs}ds - b_T\ddot{a}_{\chi+T}e^{-rT} < 0 \tag{51}$$

where the inequality is due to Assumption 5.1. Inequality (51), coupled with (50), implies

$$Z(t) < 0 \qquad \forall t \in [0,T] \tag{52}$$

that proves claims 1. and 2. As for claim 3., notice that from (52) we have

$$0 > Z(T) = X^*(T) - h(T) = X^*(T) - b_T\ddot{a}_{\chi+T} \;\Rightarrow\; X^*(T) < b_T\ddot{a}_{\chi+T} \tag{53}$$

that is what we needed to prove. □

6 Numerical simulations

In order to investigate quantities of interest to the member, we have run Monte Carlo simulations (1000 simulations for each combination of parameters) with monthly discretisation for the risky asset, and in each scenario we have adopted the optimal investment and contribution strategies derived above. We have run simulations first in the unconstrained case, adopting the optimal investment and contribution strategies provided by Equations (38) and (39), and then in the

[10] By application of Ito's lemma to $f(t,z) = \ln z$.

financial constrained case, where the proportion invested in the risky asset is constrained to lie in the range $[0, 1]$. In the unconstrained case, we perform a sensitivity analysis with respect to the weight v given to the contributions' stability (see Remark 1), while in the financial constrained case the sensitivity analysis is performed also with respect to the time horizon T, the wage growth g, the net replacement ratio target α and the contribution rate paid by the employer γ. In all scenarios tested (base case and sensitivity analysis), Assumption 5.1 is satisfied.

6.1 Base case

The parameters for the base case are[11]

- initial fund: $x_0 = 1$
- time horizon: $T = 30$ years
- financial market parameters: $r = 3\%$, $\lambda = 8\%$, and $\sigma = 15\%$, which imply $\beta = \frac{1}{3}$
- age at retirement: 65; unitary annuity price at age 65: $a_{65} = 16.86$
- subjective discount rate: $\rho = 3\%$
- initial wage: $w_0 = 12,000$
- annual wage growth: $g = 3.5\%$
- contribution rate paid by the employer: $\gamma = 2\%$
- weight given to contributions' stability: $v = 10$
- net replacement ratio target: $\alpha = 30\%$
- additional voluntary contribution rate target: $\eta = 5\%$

[11] The parameters that shape the member's preferences (v, α and η) are in line with those chosen by Antolin (2009). The parameters of the financial market are as in Menoncin & Vigna (2020). The annuity price is calculated using the Italian projected mortality table IP55 with discount rate $i = 2\%$. The wage growth is in line with Antolin (2009) and with the OECD statistics of the average wage growth observed in Australia, U.S.A. and United Kingdom over a 30 years period, see $https://stats.oecd.org/index.aspx?DataSetCode = AV_A N_W AGE$.

6.2 Unconstrained case

As mentioned above, for the unconstrained case, we report the simulation results in the base case of Section 6.1, and when changing only the value of v (leaving the other parameters as in the base case). Figure 1 reports in the base case some statistics of: (i) evolution of the fund under optimal control, (ii) optimal investment strategy, (iii) net replacement ratio achievable at retirement, (iv) distribution of the optimal AVC and (v) optimal AVC rate (intended as the ratio between the AVC and the wage). The same statistics for the case $v = 1$ are reported in Figure 2, and for the case $v = 100$ in Figure 3. To facilitate the comparison across different values of the weight v, the scale of the corresponding graphs remains unchanged.

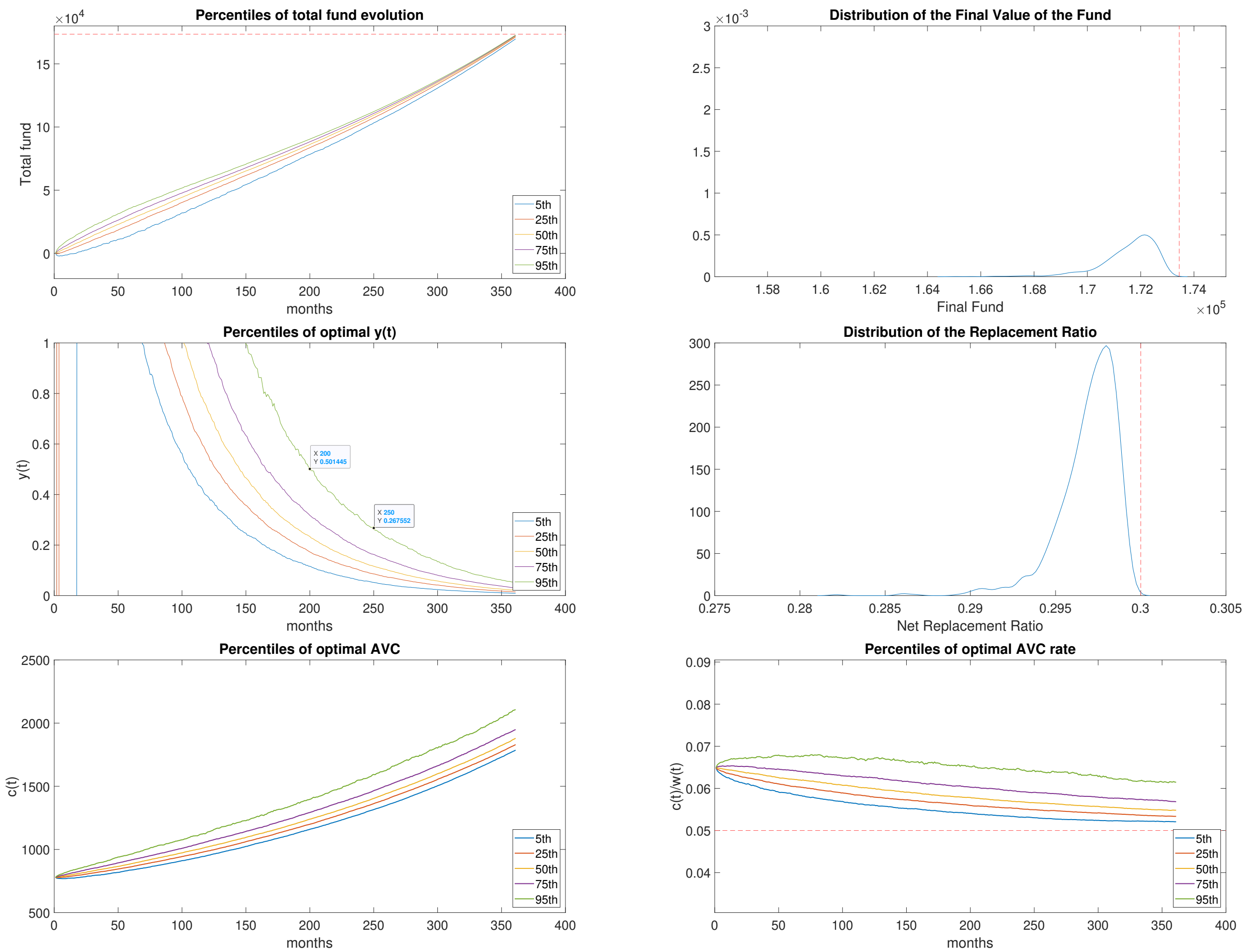

Figure 1: Simulation results in the base case (Section 6.1). Top-left: percentiles of fund's evolution (red dotted line: target fund). Top-right: distribution of final wealth (red dotted line: target fund). Middle-left: percentiles of optimal investment strategy. Middle-right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution).

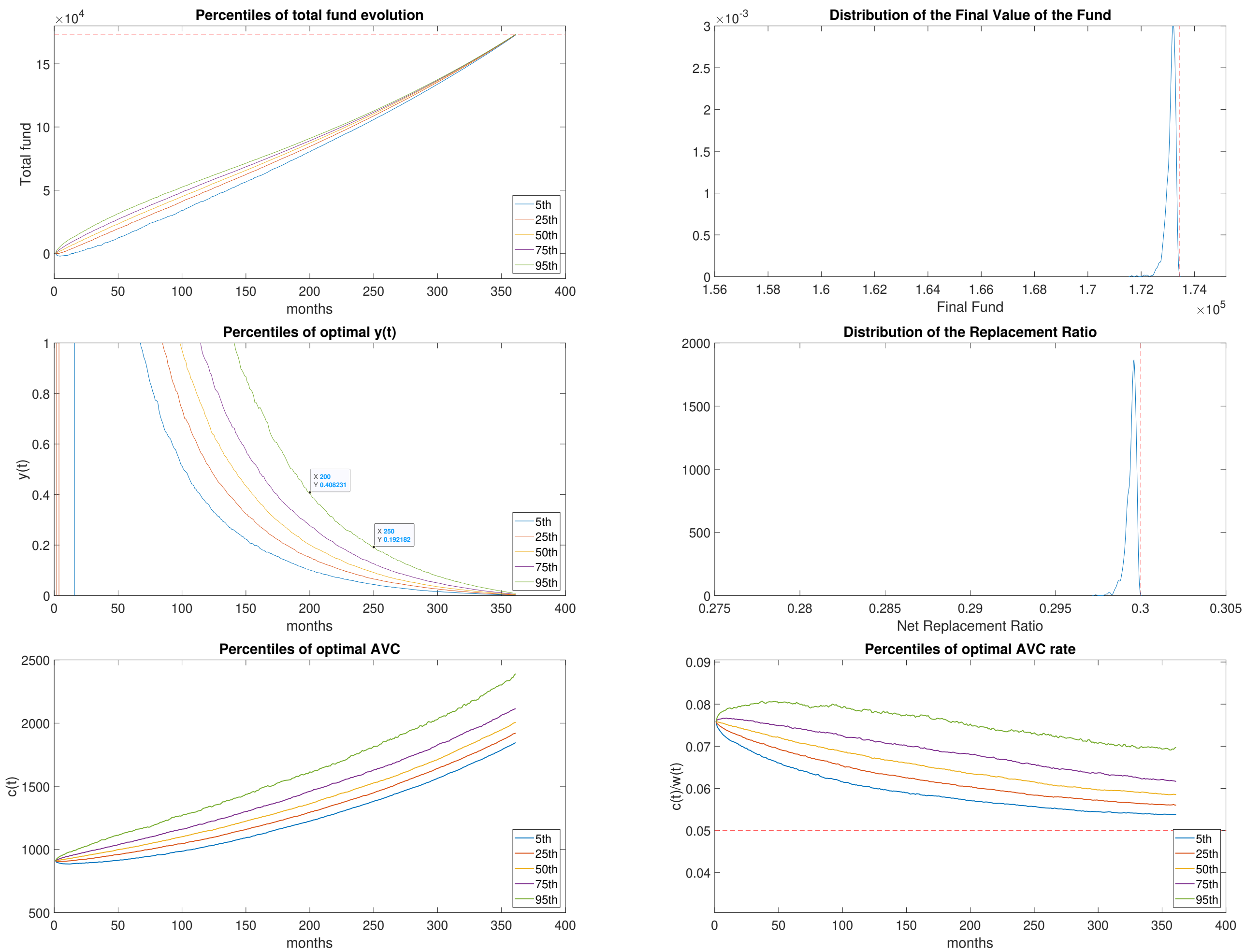

Figure 2: Simulation results with $v = 1$ and other parameters as in Section 6.1. Top-left: percentiles of fund's evolution (red dotted line: target fund). Top-right: distribution of final wealth (red dotted line: target fund). Middle-left: percentiles of optimal investment strategy. Middle-right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution).

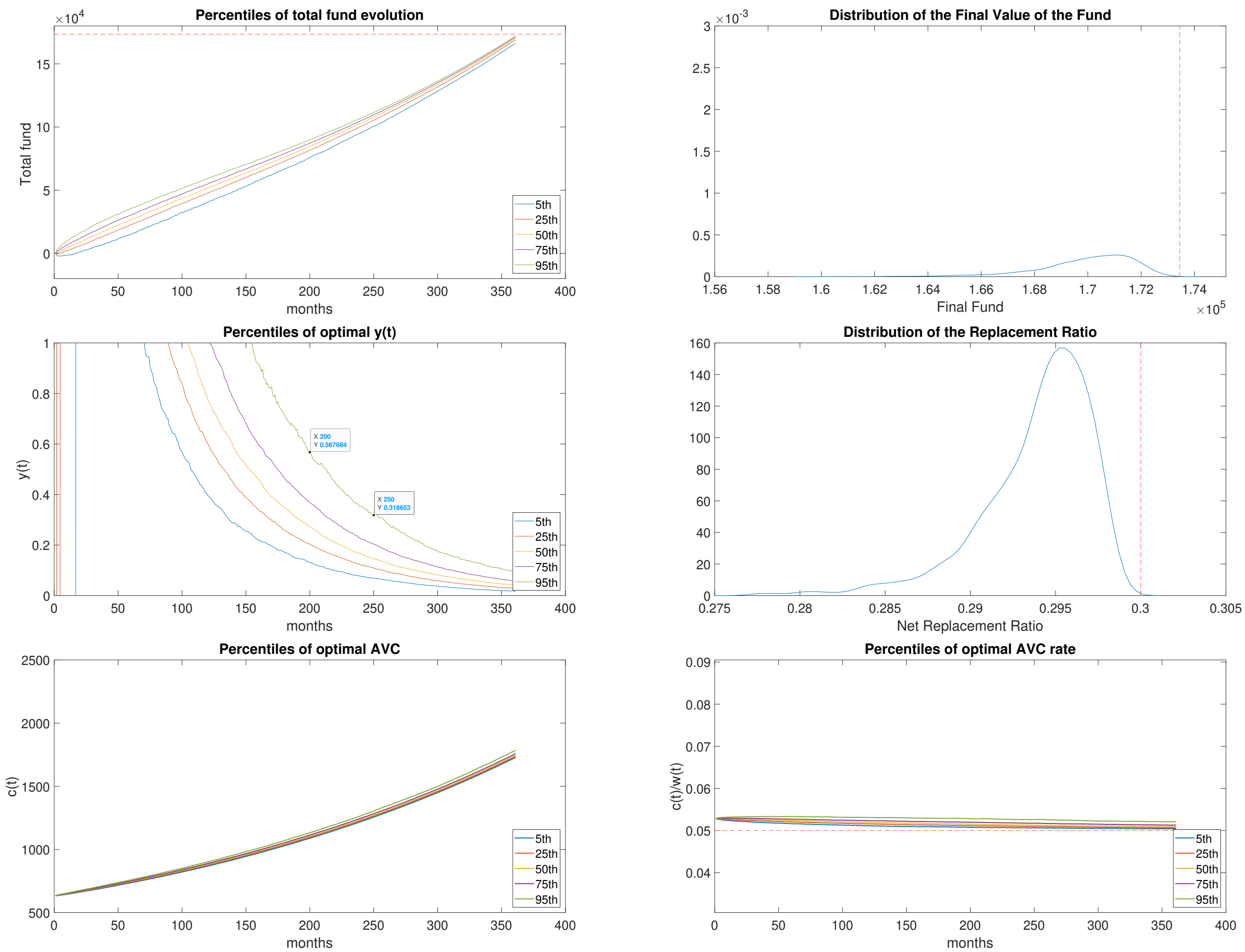

Figure 3: Simulation results with $v = 100$ and other parameters as in Section 6.1. Top-left: percentiles of fund's evolution (red dotted line: target fund). Top-right: distribution of final wealth (red dotted line: target fund). Middle-left: percentiles of optimal investment strategy. Middle-right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution).

We observe the following:

- As a general result, the AVC rate is slightly decreasing over time.
- As expected, the higher the weight v given to the contributions' stability, the less disperse

the AVC rate above the target, and the more disperse the distribution of the final fund and of the net replacement ratio on the left of the target. And vice versa.

- To be more precise, when $v = 10$, in 99% of the cases, the net replacement ratio obtainable at retirement is between 29% and 29.91%, against the 30% target; the AVC rate lies always between 5% and 7%, and after 10 years in 50% of the cases it lies between 5% and 6%. When $v = 1$, in 99.70% of the cases, the net replacement ratio obtainable at retirement is between 29.80% and 29.99%, the AVC rate lies always between 5% and 8%, and after 10 years in 75% of the cases it lies between 5% and 7%. Finally, when $v = 100$, in 99% of the cases, the net replacement ratio obtainable at retirement is between 28% and 29.85% and the AVC rate lies always between 5% and 5.5%.

- In all cases, the optimal share to be invested in the risky asset $y^*(t)$ in the first years is often negative, implying short-selling of the risky asset, and after a certain number of years becomes positive and above 1, implying borrowing the riskless asset to invest in the risky one.

- To be more precise, in the base case $v = 10$, after 12 months the optimal investment strategy is negative in 8.1% of the scenarios, and the 95^{th} percentile of $y^*(t)$ remains higher than 100% for the first 148 months, arriving at 5.3% at the time of retirement. For $v = 1$, after 12 months the optimal investment strategy is negative in 7.4% of the cases, and the 95^{th} percentile of $y^*(t)$ remains higher than 100% for the first 138 months, arriving at 0.87% at retirement. For $v = 100$, after 12 months the optimal strategy is negative in 8.4% of the cases, and the 95^{th} percentile of $y^*(t)$ remains higher than 100% for the first 153 months, arriving at 9.6% at retirement.

- As expected from Proposition 5.2, the final fund is always lower than the target fund, and the optimal AVC is always larger than the target contribution.

6.3 Constrained case

The optimal investment strategy that requires both borrowing and short-selling is likely to be forbidden in the practice. For this reason, we analyze the so-called *clipped* investment strategy (see Forsyth & Vetzal, 2023) defined as follows

$$\hat{y}(t) = \max\{0, \min\{y^*(t), 1\}\} \tag{54}$$

Clipped investment strategies of the same type, also called "cut-shares" (see Menoncin & Vigna, 2017), were applied e.g. by Forsyth & Vetzal (2023), and by Gerrard et al. (2006) and Vigna (2014) in the context of DC pension schemes, and proved to be satisfactory: with respect to the unrestricted case the effect on the final results turned out to be negligible and the controls resulted to be more stable over time. We remark that solving the same optimization problem imposing constraints on the control variables turns out to be remarkably complex (see Di Giacinto et al., 2011).

6.3.1 Sensitivity with respect to v

Similarly to what done in the unconstrained case, Figure 4 reports in the base case some statistics of: (i) evolution of the fund under optimal control, (ii) optimal investment strategy, (iii) net replacement ratio achievable at retirement, (iv) distribution of the optimal AVC and (v) optimal AVC rate. The same statistics for the case $v = 1$ are reported in Figure 5 and for the case $v = 100$ in Figure 6. To facilitate the comparison across different values of the weight v, the scale of the corresponding graphs remains unchanged. Notice, however, that the scale is different from the unconstrained one because, as expected, the simulation results turns out to be quite different.

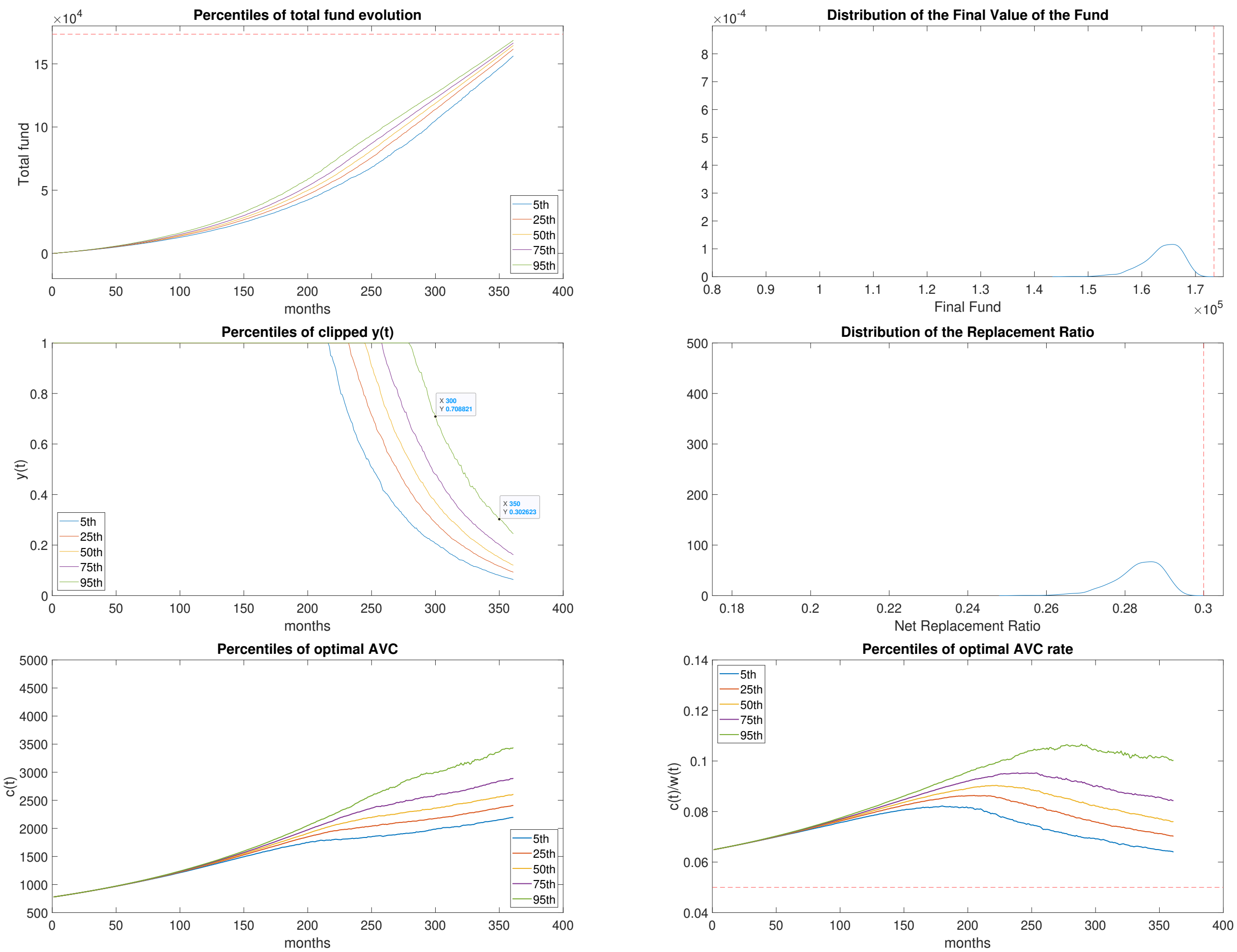

Figure 4: Simulation results in the base case (Section 6.1). Top-left: percentiles of fund's evolution (red dotted line: target fund). Top-right: distribution of final wealth (red dotted line: target fund). Middle-left: percentiles of clipped investment strategy. Middle-right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution).

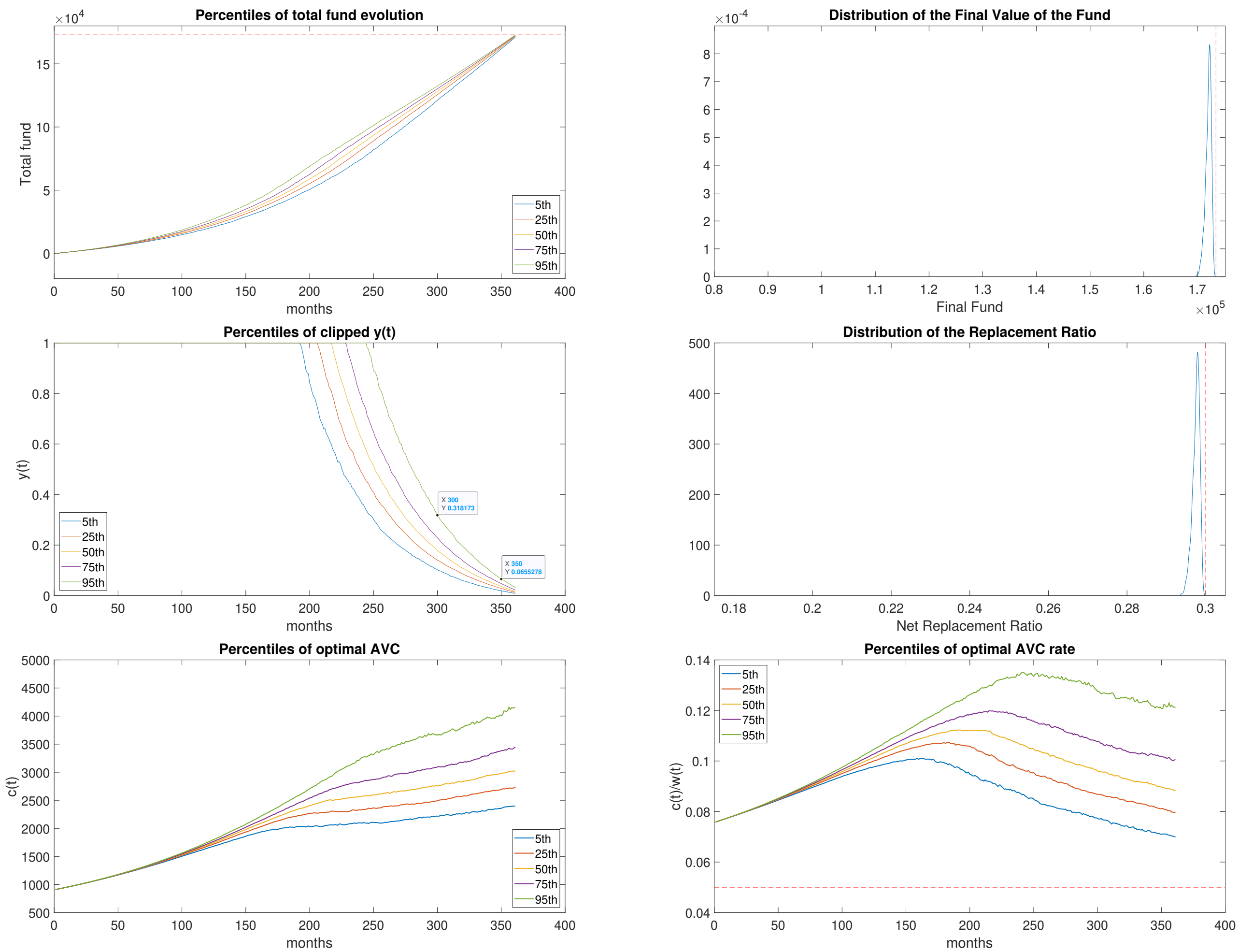

Figure 5: Simulation results with $v = 1$ and other parameters as in Section 6.1. Top-left: percentiles of fund's evolution (red dotted line: target fund). Top-right: distribution of final wealth (red dotted line: target fund). Middle-left: percentiles of clipped investment strategy. Middle-right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution).

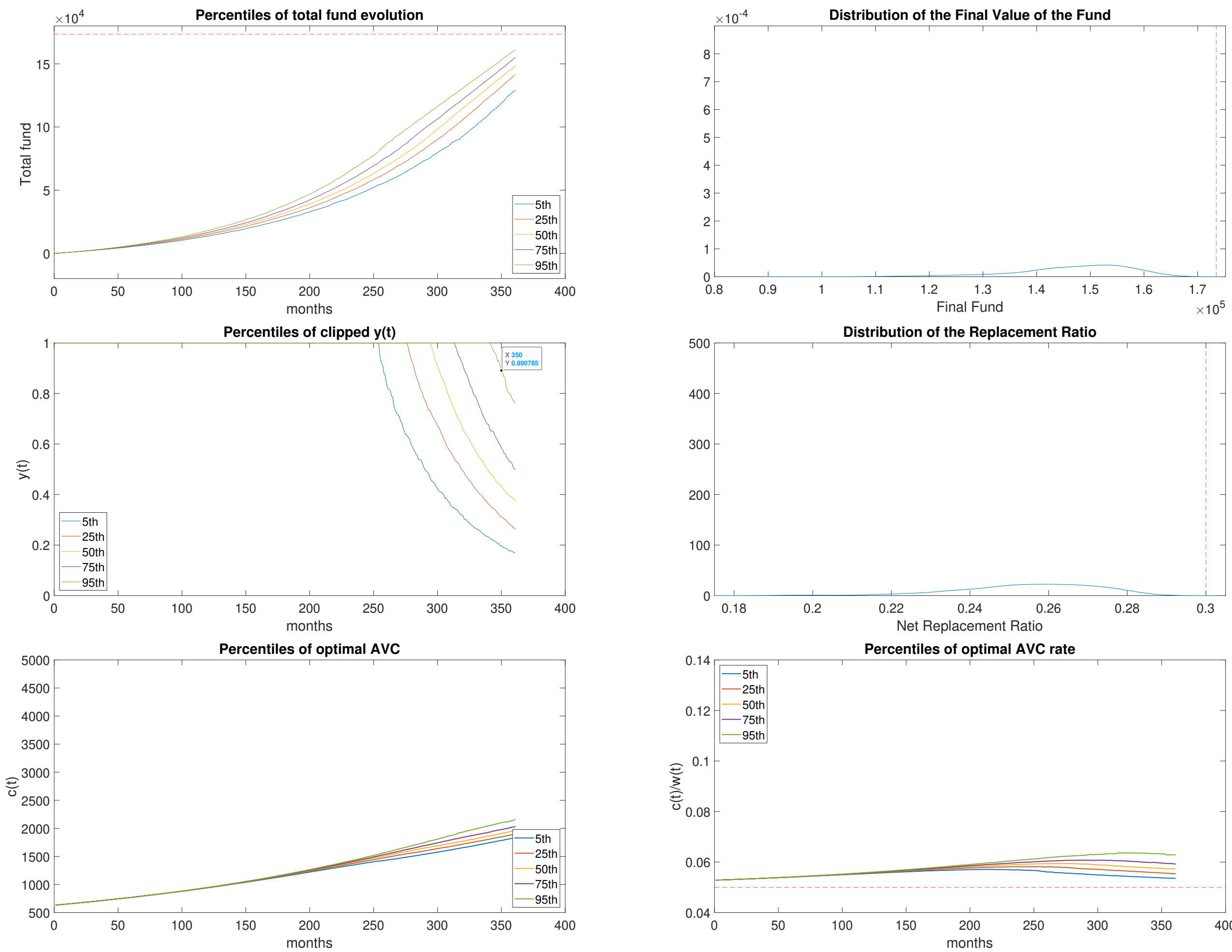

Figure 6: Simulation results with $v = 100$ and other parameters as in Section 6.1. Top-left: percentiles of fund's evolution (red dotted line: target fund). Top-right: distribution of final wealth (red dotted line: target fund). Middle-left: percentiles of clipped investment strategy. Middle-right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution).

We observe the following:

- The AVC rate is increasing in the first 15-20 years, while it is decreasing in the remaining years, with the increasing and decreasing trends being more pronounced with small values of v.

- As in the unconstrained case and as expected, the higher the weight v given to the contributions' stability, the more concentrated the AVC rate close to the target, and the more disperse the distribution of the final fund and of the net replacement ratio on the left of the target. And vice versa. The difference with the unconstrained case consists in the level of AVC, final fund and net replacement ratio, being clearly more unfavourable for the member in the clipped strategy.

- To be more precise, when $v = 10$, in 99% of the cases, the net replacement ratio obtainable at retirement is between 26% and 29.40%; the AVC rate lies always between 6% and 11%. When $v = 1$, in 99.80% of the cases, the net replacement ratio obtainable at retirement is between 29.25% and 29.93% and the AVC rate lies always between 7% and 14%, and in more than 50% of the cases it lies above 9% after the fifth year and almost until retirement time T. Finally, when $v = 100$, in 99% of the cases, the net replacement ratio obtainable at retirement is between 19% and 28.30% and the AVC rate lies always between 5.5% and 6.5%.

- For $v = 1$ the percentiles of the investment strategy are closer to each other with respect to the base case, indicating a lower variability of the investment strategy. Conversely, for $v = 100$ the percentiles become wider with respect to the case $v = 10$, showing a higher variability of the clipped strategy.

- Clearly, at any time $t \in [0, T]$ the clipped proportion to be invested in the risky asset $\hat{y}(t)$ is bounded between 0 and 1. In all cases, it is equal to 1 for 20-25 years, and then it gradually reduces towards lower values. When v is lower, the investment strategy is less risky, and vice versa. This is due to the fact that with a low value of v, less weight is given to the contributions' stability, and therefore the payment of higher contributions compensates the investment in the risky asset. Opposite, when v is higher, the optimal contribution keeps close to the target and it is necessary to adopt more aggressive investment strategies to reach the replacement ratio target. These results are in line with those found by Akpanibah & Oghen'Oro (2018) and Chinyere et al. (2022).

- To be more precise, in the base case $v = 10$, the 95^{th} percentile of the clipped investment strategy $\hat{y}(t)$ remains capped at 100% for the first 279 months, decreasing to 24.8% at the time of retirement. For $v = 1$, the 95^{th} percentile of the clipped strategy $\hat{y}(t)$ remains capped at 100% for the first 244 months, decreasing to 3.2% at retirement. For $v = 100$, the 95^{th} percentile of the clipped strategy $\hat{y}(t)$ remains capped at 100% for the first 341 months, decreasing to 76.2% at retirement.

6.3.2 Further sensitivity analysis

We have performed sensitivity analysis with respect to other parameters, changing only one parameter at a time, and leaving all the others as in the base case (Section 6.1). The parameters stressed are: (i) the time horizon T, considering $T = 20$ and $T = 40$, reported in Figure 7 in the Appendix; (ii) the wage growth g, considering $g = 2\%$ and $g = 5\%$, reported in Figure 8 in the Appendix; (iii) the net replacement ratio target α, considering $\alpha = 20\%$ and $\alpha = 40\%$, reported in Figure 9 in the Appendix; (iv) the contribution rate paid by the employer γ, considering $\gamma = 1\%$ and $\gamma = 3\%$, reported in Figure 10 in the Appendix. For each case, we report only the distribution of the net replacement ratio (on the left in each figure) and the percentiles of the AVC rate (on the right in each figure), that are the quantities of main interest to the pension fund's member (the other statistics are available upon request).

We observe the following:

- When the time horizon decreases ($T = 20$), the shorter period makes more difficult the achievement of the targets. As expected, the comparison of Figure 4 versus Figure 7 shows that the optimal AVC rate increases, ranging between 11% and 18%, and the net replacement ratio achievable worsens, lying mainly between 22% and 28%; in addition, the clipped investment strategy becomes more aggressive. On the opposite, when the time horizon increases ($T = 40$) there is more time to achieve the targets. As a consequence, the optimal AVC rate decreases considerably, ranging between 5% and 6%, and the net replacement ratio achievable improves remarkably, lying mainly between 29% and 30%; in addition, the investment strategy becomes less aggressive. This result shows the great importance of the time horizon in the achievement of the desired net replacement ratio at retirement, coupled with a significant stability of the additional contributions to be paid by the employee during the accumulation phase.

- When the wage growth decreases ($g = 2\%$), the fund target to be achieved decreases, while the financial market parameters remain unchanged. This makes easier the achievement of the target while maintaining stability of the contributions. Therefore, as expected, the comparison of Figure 4 versus Figure 8 shows that the optimal AVC rate decreases, ranging between 5% and 8%, and the distribution of the net replacement ratio achievable improves, lying mainly between 28% and 30%; in addition, the investment strategy becomes less aggressive. On the opposite, when the wage growth increases ($g = 5\%$), the optimal AVC rate increases (ranging between 7% and 13%), and the distribution of the net replacement ratio achievable worsens, lying mainly between 26% and 29%, in addition, the investment strategy becomes more aggressive. This result is in line with OECD (2021), Table 4.4, which finds that *"For low earners (with half of average worker earnings), the average net replacement rate across OECD countries is* 74.4% *while it is* 54.9% *for high earners (*200% *of average worker earnings).*

- When the net replacement ratio target α decreases ($\alpha = 20\%$), the situation is similar to that of the reduced wage growth: the target fund decreases, everything else remaining the same. Therefore, as expected, the comparison of Figure 4 versus Figure 9 shows that the optimal AVC rate decreases, ranging between 5% and 6.5%, the distribution of the net replacement ratio achievable improves, lying mainly between 19% and 20%, and the investment strategy becomes less aggressive. On the opposite, when the net replacement ratio increases ($\alpha = 40\%$), the optimal AVC rate increases (ranging between 7% and 15%), the distribution of the net replacement ratio achievable worsens, lying mainly between 35% and 39%, and the investment strategy becomes more aggressive.

- When the contribution rate paid by the employer reduces ($\gamma = 1\%$) it is slightly more difficult to reach the target. Therefore, as expected, the comparison of Figure 4 versus Figure 10 shows that the optimal AVC rate slightly increases, ranging between 7% and 11%, and the investment strategy becomes slightly more aggressive. On the other hand, the distribution of the net replacement ratio achievable remains almost unchanged with respect to the base case. Therefore, the final target seems to be achieved as in the base case, but at the price of paying higher additional contribution and investing slightly more aggressively to compensate the reduction in the contribution paid by the employer. On the opposite, when the employer's contribution increases ($\gamma = 3\%$), the optimal AVC rate slightly decreases, the distribution of the net replacement ratio achievable remains almost unchanged and the investment strategy becomes slightly less aggressive.

6.4 Discussion

The simulation results considered in Section 6.3 show that the joint application of the clipped investment strategy (54) and the optimal AVC (39) provide outcomes that are very sensitive to the choice of the parameters. The main conclusions are:

- The choice of the weight v given to the contributions' stability is crucial for the outcome distribution: a higher v, that reflects the desire of contributions that remain close to the target contribution rate of 5% over time, results in AVC rates that lie always very close to each other and very close to the 5% target, at the price of (i) remarkably aggressive and variable investment strategies (total investment in the risky asset for more than 20 years followed by a gradual shift to the riskless asset, but with a substantial share invested in the risky asset until retirement) (ii) a very disperse net replacement ratio well below the desired one. Vice versa, a low value of v gives high priority to the achievement of the desired replacement ratio, which is almost reached in nearly all cases with less aggressive and less variable investment strategies, at the cost of paying considerably variable and significantly high AVC rates, that are far from the desired 5% level. An intermediate value of v provides intermediate results for the investment strategy, the optimal AVC rate and the net replacement ratio achieved.

- Keeping the value of v fixed, increasing the time horizon T dramatically improves the situation, and provides higher net replacement ratios achieved, lower and more stable AVC rates and less aggressive and less volatile investment strategies, that is an obvious result. The extent of improvement is considerable when the time horizon is increased by one third (in the case considered, 10 years out of 30).

- Also a reduction of the annual wage growth g results in an improvement of the situation, with the net replacement ratio that can be almost achieved in nearly all cases without making the investment strategy more risky or increasing the AVC rates. This would be perhaps the case of low earners, who would probably benefit more than high earners from the application of this model.

- Expectedly, also a reduction of the net replacement ratio target implies a remarkable improvement of the outcomes, with final net replacement ratios rather close to the desired one, stable AVC rates quite close to 5% and investment strategies close to the lifestyle strategy largely adopted in the U.K. in DC pension schemes (whereby the fund is entirely invested into equities until 10 years prior to retirement, and is then gradually switched into riskless assets, by moving every year 10% from equities into bonds, see Cairns et al., 2006).

- Finally, an increase of the percentage γ of wage to be paid into the fund by the employer clearly improves the situation under all points of view, by slightly reducing the AVCs paid by the member, slightly improving the net replacement ratio achievable and making the investment strategy slightly less risky.

7 Conclusive remarks

In this paper, we provide a flexible analytical optimization tool that could help the member of a DC pension scheme in making her decisions during the accumulation phase of the fund. In particular, she can first set a desired replacement ratio which can be obtained at retirement via the achievement of a target annuity, and then determine throughout the investment period her optimal investment strategy and optimal additional voluntary contributions. For the latter, she defines a contribution rate target to be pursuit at each point in time, which is driven by her budgetary conditions and reflects her desire for contributions' stability. The problem has been tackled and solved using the tools of stochastic optimal control theory, in a Black and Scholes financial market. Closed-form solutions have been derived for the optimal investment strategy and the optimal AVC. We prove that the optimal fund never reaches the target final fund, the optimal amount invested in the risky asset is always positive, and the optimal AVC is always higher than the target one. Numerical Monte Carlo simulations using the unconstrained optimal investment strategy and a clipped constrained investment strategy have been performed, both in a base case scenario and by performing sensitivity analysis on the key parameters.

From the numerical analysis, it turns out that as dictated by the theory the chosen annuity target is never reached, but the replacement ratio obtained can approach very closely the desired one, and the additional voluntary contributions play an important role in doing so. We find that the key parameter is the weight given to the contributions' stability. The higher this weight, the closer the additional voluntary contributions are to the desired contribution rate, at the expense of (i) reaching a lower net replacement ratio at retirement, (ii) needing to take a more aggressive and more volatile investment strategy across all the investment phase. Conversely, the lower the weight given to the contributions' stability, the higher the net replacement ratio achievable, the lower the need to invest in the risky asset, the lower the variability of the investment strategy at the price of higher deviations from the targeted contribution during the accumulation phase. The numerical analysis shows that a 5% contribution rate target is appropriate to reach a 30% net replacement ratio target. This result is in line with Antolin (2009), who finds that a contribution rate of $5-15\%$ of the wage is adequate to reach a replacement ratio of $25-70\%$. Furthermore, the sensitivity analysis shows that the longer the accumulation phase, the lower the wage growth rate and the higher the contribution rate paid into the fund by the employer, the easier it is for the member to reach a satisfactory net replacement ratio respecting the contributions' stability. These are expected and intuitive results.

The main messages of this paper are (i) the trade-off between two conflicting desires of the member - i.e. to respect her current budgetary conditions via contributions' stability and to reach a satisfactory annuity level at retirement - can be tackled by choosing an appropriate weight given to the contributions' stability in the implementation of this model; (ii) the joint achievement of both desires can be facilitated by a longer membership in the pension fund and by a higher contribution rate paid by the employer; (iii) low earners characterized by a lower annual wage growth seem to benefit by the implementation of this optimization tool better than high earners with a higher wage growth (this result is in line with OECD, 2021).

This study integrates and complements the analysis made by Antolin (2009), and his relevant policy recommendation of paying higher contributions into the pension fund in order to deliver adequate retirement income. An analytical tool that facilitates the implementation of those crucial policy recommendations could be of practical help to the investment manager of defined contribution pension schemes.

Declarations

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Appendix

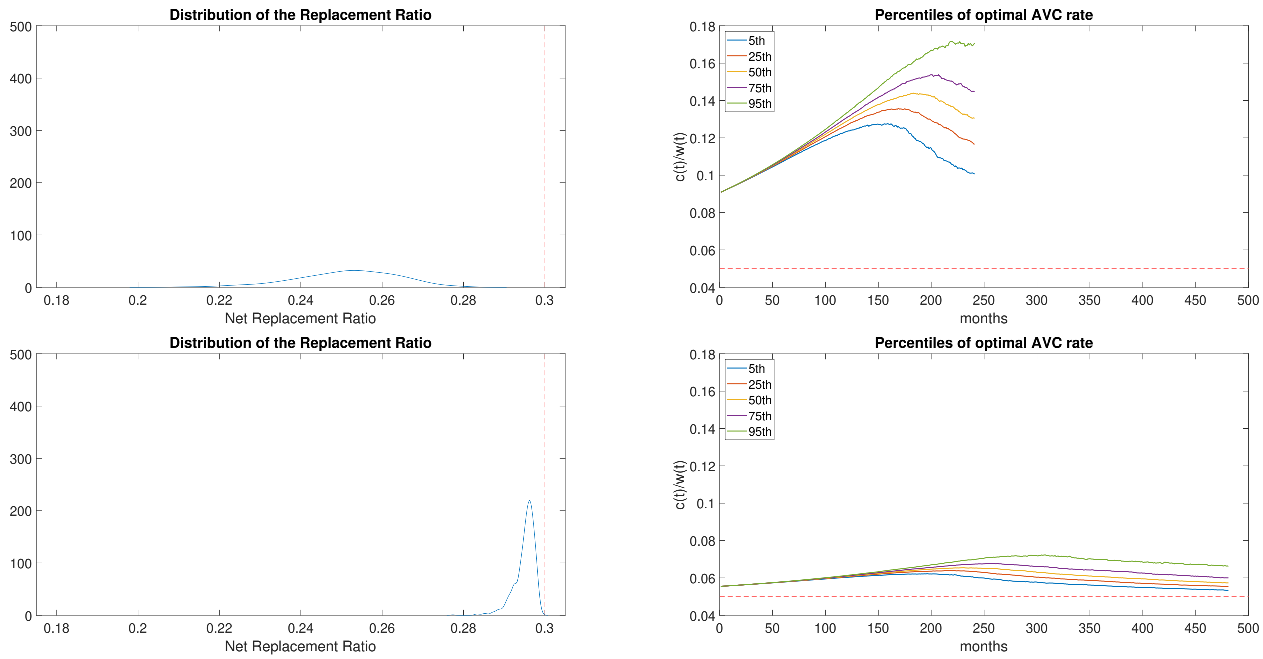

Figure 7: Simulation results with $T = 20, 40$ and other parameters as in Section 6.1. Top graphs: $T = 20$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate). Bottom graphs: $T = 40$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate).

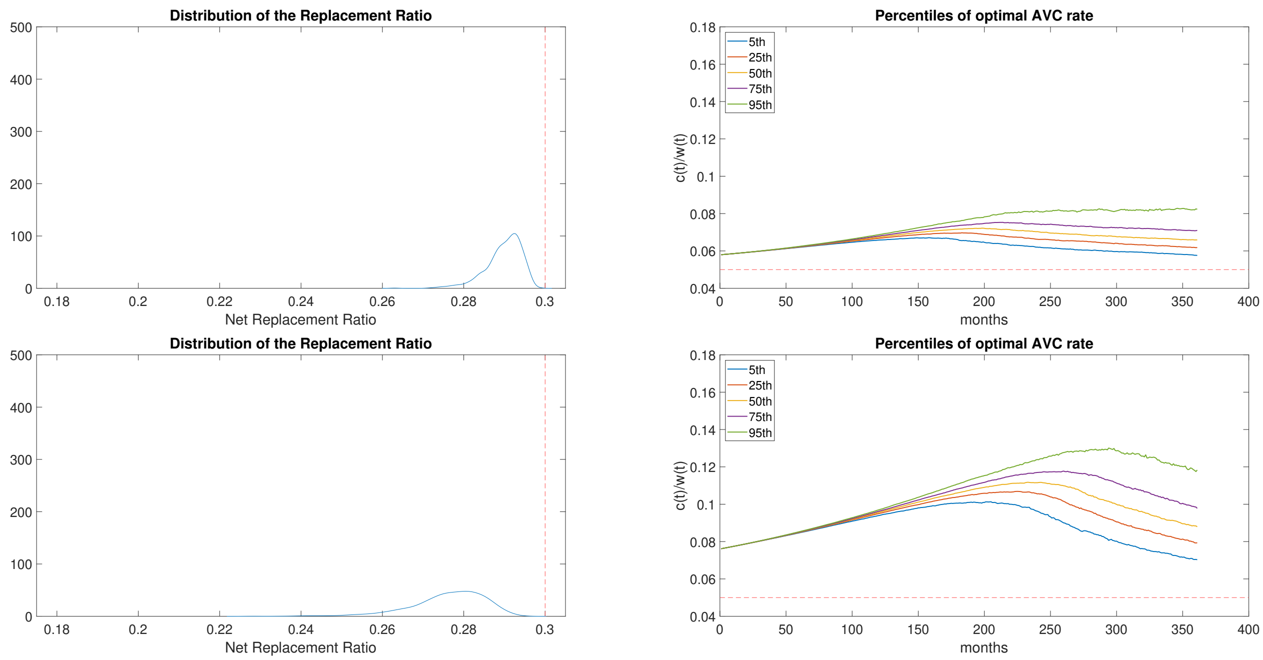

Figure 8: Simulation results with $g = 2\%, 5\%$ and other parameters as in Section 6.1. Top graphs: $g = 2\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate). Bottom graphs: $g = 5\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate).

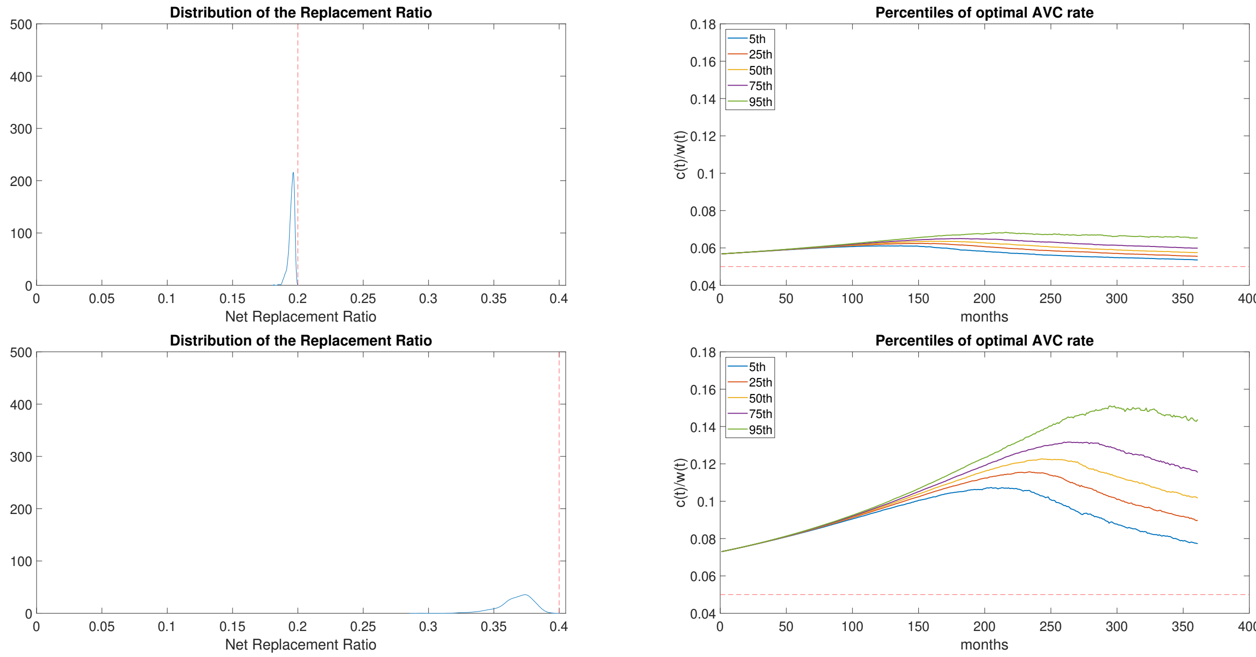

Figure 9: Simulation results with $\alpha = 20\%, 40\%$ and other parameters as in Section 6.1. Top graphs: $\alpha = 20\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate). Bottom graphs: $\alpha = 40\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate).

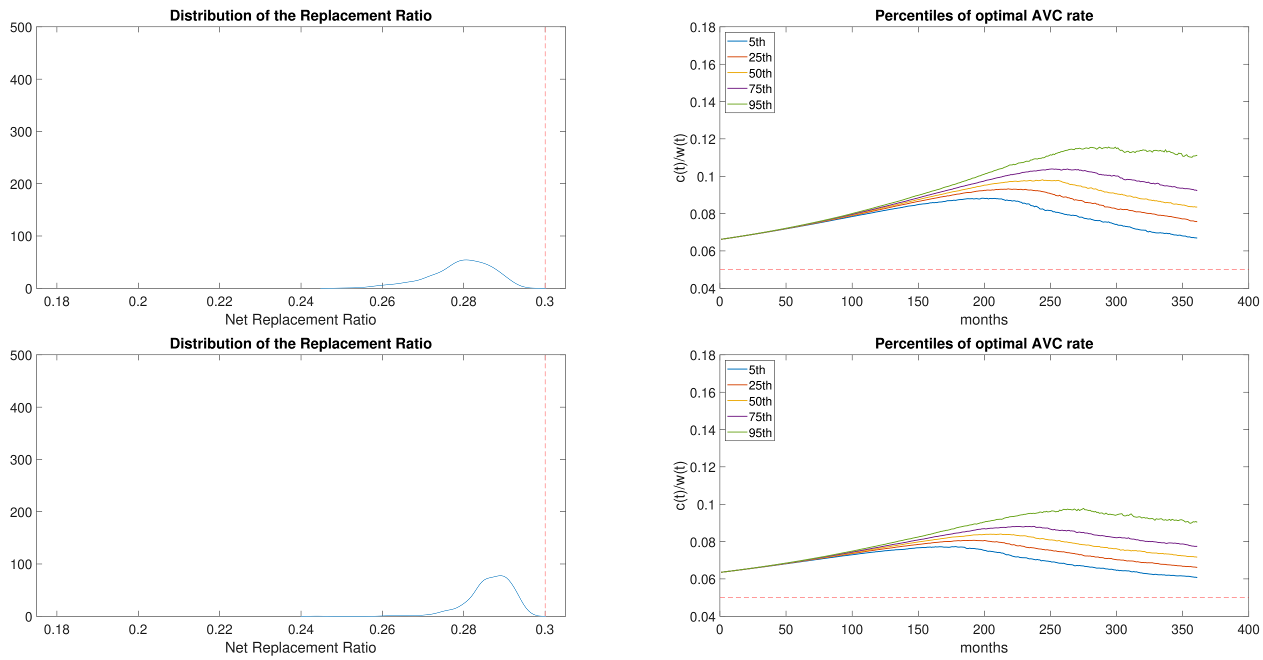

Figure 10: Simulation results with $\gamma = 1\%, 3\%$ and other parameters as in Section 6.1. Top graphs: $\gamma = 1\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate). Bottom graphs: $\gamma = 3\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate).