ISSN 2279-9362

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No. 711
December 2023

Carlo Alberto Notebooks

www.carloalberto.org/research/working-papers

Delegation with Endogenous States*

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December 6, 2023

Abstract

We present a model of delegation with moral hazard. A principal delegates a decision to an agent, who affects the distribution of the state of the world by exerting costly and unobservable effort. The principal faces a trade-off between (i) granting the agent discretion, so he can adapt the decision to the state and (ii) limiting the agent's discretion, to induce him to exert effort. Our model is flexible on how effort affects the state distribution, thus capturing several distinct economic environments. Optimal delegation takes one of four simple forms, all commonly used in practice: floors, ceilings, floor-ceilings or gaps.

JEL: C70, C78, D82

KEYWORDS: delegation, moral hazard, endogenous state, floors, ceilings, caps, gaps

*We thank Ricardo Alonso, Edoardo Grillo, Leandro Gorno, Fei Li and Daniel Monte for valuable suggestions and various seminar audiences for useful comments. Dino Gerardi and Ignacio Monzón are grateful to the Italian Ministry of Research for financial support (grant PRIN 2022P85F94).

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1. Introduction

In large organizations, tasks are too diverse and complex for those with decision authority to handle all of them. CEOs, top managers, and high level government officials all have limited time, so they must delegate tasks to agents. Delegating effectively is not straightforward. Authorities want agents to exert effort and to react to the information that the agents receive while performing the task.

Consider for instance a CEO who tasks a manager with opening a branch in a new location. The harder the manager works, the larger the probability that he attracts more customers. After finding customers, the manager decides the number of employees for the new branch. Similarly, consider a mayor who tasks a bureaucrat with developing a new project. By working hard, the bureaucrat may develop innovative projects. After learning the quality of the project, the bureaucrat decides how much to spend on it. Finally, consider a CEO who tasks a manager with forming a new team of employees. Potential employees differ in their skillset: while some have soft skills, others have hard skills. The harder the manager works, the more likely that the composition is balanced. Once the composition of the team is set, the manager allocates employees to one of two jobs: customer care or product development.

The examples above highlight some key features of the problem of assigning tasks in large organizations. First, the distribution of the state of the world is endogenous. The agent (the manager or the bureaucrat) affects the distribution of the state (the number of customers, the quality of the project or the composition of the team) by exerting costly and unobservable effort. This introduces a moral hazard component in the relationship between the principal (CEO or mayor) and the agent. Second, while performing his task, the agent learns the unknown state of the world, so he has superior information. Third, the agent chooses an action (the number of employees in the new branch, the amount to spend in the project, the allocation of employees to jobs). Since the agent's action is observable, the principal can ex-ante restrict him. The principal could grant the agent full discretion or instead limit the agent's actions.

We study how moral hazard shapes optimal delegation. How much flexibility should the principal grant the agent? On the one hand, the principal wants to grant the agent flexibility, so the (informed) agent can adapt the action to the state. On the other hand, she may have to limit discretion to induce the agent to exert effort. The CEO may choose to set a lower bound on the number of employees in the new branch. In this way, the manager must work hard in order to avoid being associated to a major failure: a large branch with low sales.

We characterize optimal delegation when the state is endogenous. Optimal delegation sets take one of four simple forms, all commonly used in real life. They can be *floors* (or *ceilings*): the agent can only choose actions above (or below) a given threshold. Optimal delegation sets can also be *gaps*: the agent cannot choose intermediate actions and must instead choose among extreme options. Finally, they can also be *floor-ceilings*: the agent can only choose intermediate actions. The key determinant of the shape of optimal delegation sets is how effort affects the distribution of the state of the world.

We present a model of delegation with an endogenous state distribution. A principal offers the agent a delegation set that limits the actions the agent can choose. Then, the agent selects an unobservable and costly level of effort. The level of effort determines the distribution of the state of the world. This captures that although the agent can affect the state, he cannot fully control it. Even when the manager works hard, he may still attract few customers. The agent observes the state of the world and chooses an action from the delegation set.

We assume that conditional on the state, both parties have common preferences over actions (described by quadratic loss). After the manager exerts effort and finds customers, both the CEO and the manager prefer the size of the branch to correspond to the number of customers. As the principal does not internalize the agent's cost of effort, there is a conflict between the parties, even with common preferences over actions. Thus, our model isolates the effects of moral hazard on optimal delegation. Any limits that the principal places on the agent result solely from moral hazard. Finally, in line with the literature of delegation, we assume that there are no monetary transfers between the parties. Oftentimes, in large organizations, it is not possible to condition monetary transfers on every decision of the agent and every state of the world (especially so in the public sector).

We assume that the state of the world is distributed uniformly and that the agent's level of effort affects its support. Both the lower and the upper bounds of the support are linear in the agent's effort level. We allow both the lower bound and the upper bound to be either decreasing or increasing in effort. We thus split the analysis into four different cases. Each case reflects a different economic environment. We describe each environment in detail below.

We divide the principal's problem into two steps. First, we fix an arbitrary level of effort and find the optimal delegation set to implement such effort. The optimal delegation set limits the agent's discretion, so it introduces some ex-post inefficiency. In the second step, the principal chooses the effort level to implement taking into account (i) the inefficiency that results from optimal delegation and (ii) the direct effect of effort on the principal's payoff. The main difficulty in the analysis lies on the first part: the identification of optimal delegation sets, which is the main focus of this paper.

In the first environment, *upwards support*, both the lower and upper bounds of the support increase with the agent's effort level. The distribution associated to a higher effort first-order stochastically dominates that of a lower one. This captures environments where a higher effort improves the productivity of a project. The number of customers of the new branch depends both on the manager's effort (he needs to develop his sales pitch and contact potential customers) and on exogenous factors (the unknown preferences of the customers in the new location). By working hard, the manager makes it more likely that the branch attracts many customers. Both the CEO and the manager agree that branches with more customers should be assigned more employees. Under upwards support, optimal delegation takes the form of a floor. The CEO induces the manager to exert effort by preventing him from hiring few employees. The second environment, *downwards support* is exactly the mirror image of upwards support: both the lower and upper bounds of the support decrease with the agent's effort level. Thus, optimal delegation takes the form of a ceiling.

In the third environment, *shrinking support*, the lower bound increases and the upper bound decreases with effort. The sequence of supports induced by different effort levels is like Matryoshka dolls: the support of a lower effort level contains the support of a higher one. This captures environments where there is a desired state of the world. The CEO who tasks a manager with forming a new team has an ideal team composition: she wants it to be as close as possible to 50% soft skilled and 50% hard skilled. The agent must exert care to achieve states close to the desired one. By interviewing many candidates and screening them carefully, the manager makes it more likely that the composition is close to 50%. After forming a team, the manager learns (through daily interaction) the skillset of each employee, the manager assigns each employee to a job. Both the manager and the CEO agree that employees with soft skills should work in customer care and those with hard skills should work in product development. Under shrinking support, optimal delegation takes the form of a floor-ceiling. The CEO sets both a lower and an upper bound on the fraction of employees that the agent can assign to each job. The CEO induces effort by only allowing for fractions that are optimal under balanced team compositions.

In the final environment, *expanding support*, the lower bound decreases and the upper bound increases with effort. Like in the previous environment, the supports induced by successive levels of effort are nested. Differently from it, a higher level of effort expands the support: any state that is possible with a certain effort is also feasible with a higher one. The environment of expanding support captures economic situations where a higher level of effort is associated to innovative projects, which are intrinsically more uncertain. The bureaucrat tasked with developing a new project may devote his time to adapting an existing one. Alternatively, he may try to develop a better project from scratch. Developing a brand new project is risky: its quality may be higher or lower than the safe quality from adapting an existing project. Both the mayor and the bureaucrat agree that higher quality projects should receive larger funding. Under expanding support, optimal delegation takes the form of a gap.[1] The mayor allows the bureaucrat to spend either large amounts (which are appropriate for high quality projects) or to not fund the project (which is better for low quality ones). Intuitively, the mayor discourages low effort levels by preventing the agent from choosing intermediate levels of funding

[1] We characterize optimal delegation sets using the first order approach. This approach is always valid in the environments of upwards, downwards and shrinking support. However, this approach may fail under expanding support. We provide sufficient conditions for the first order approach to hold in this case.

A common feature in all four environments is that the principal induces the agent to exert effort by limiting his discretion. One would expect that in order to induce a larger level of effort, the principal would have to reduce the agent's discretion. However, this is not necessarily the case. In our model, both the delegation set and the (endogenous) support affect the agent's incentives to exert effort. The interaction of these two forces leads to a potential non-monotonicity of discretion. Under upwards, downwards and expanding support it can happen that the principal grants the agent more discretion when she intends to induce a larger effort level.

We present a tractable model of delegation of complex tasks in large organizations. We show the optimality of many simple delegation sets commonly used in practice. Previous work has linked these simple forms of delegation to either misalignment of preferences over actions or to costly information acquisition. In our model, the parties' preferences over actions are aligned and the agent is always informed about the state. Differently from previous work, the distribution of the state of the world is endogenous in our model. Thus, our paper provides a new rationale for the emergence of floors, ceilings, floor-ceilings and gaps.

1.1 Related literature

An extensive literature, started by the seminal work by Holmström [1977, 1984], studies delegation when the state of the world is exogenous. Holmström proves the existence of optimal delegation sets and characterizes the optimal delegation set within the class of interval delegation sets.[2] Melumad and Shibano [1991] identify the optimal delegation set when the distribution of the state is uniform and preferences are quadratic —see also Martimort and Semenov [2006]. Alonso and Matouschek [2008] assume that the parties' payoffs are single-peaked and symmetric. They allow for an arbitrary state distribution and characterize the optimal delegation set. Alonso and Matouschek [2008] provide necessary and sufficient conditions for the optimality of specific delegation sets, such as centralization and intervals. The optimality of interval delegation is the focus of Amador

[2] Holmström [1977] also constructs an example in which the restriction to interval delegation sets is without loss of generality: the principal does not benefit from choosing non-convex delegation sets.

and Bagwell [2013]. Amador and Bagwell allow for a more general class of preferences and provide necessary and sufficient conditions for the optimality of interval delegation. Differently from these papers, we assume that the distribution of the state is endogenous and that the parties preferences over actions are aligned. We take advantage of the literature's standard workhorse model: in our paper, preferences are quadratic and, for each effort level, the state is uniformly distributed.

In most of the literature —and also in our paper— the parties' preferences (for a given state) and the distribution of the state are common knowledge, the state is unidimensional, and the principal solves a static problem. Armstrong [1995] and Frankel [2014] introduce uncertainty over the agent's preferences. Frankel [2014, 2016] and Kleiner [2023] study delegation with multidimensional states and decisions. Hu and Li [2023] show the optimality of interval delegation when the principal is uncertain about the distribution of the state. Alonso and Matouschek [2007] present a dynamic model of delegation and show that the principal's commitment power arises endogenously. Finally, Guo [2016] studies the problem of delegating experimentation.[3]

A notable exception to the assumption that the distribution of the state of the world is exogenous is present in Armstrong and Vickers [2010]. In their model, a principal delegates the choice of a project to an agent. The state of the world consists of a set of available projects. The agent is privately informed about which projects are available. The parties' preferences over projects are misaligned. The agent can implement a project only if (i) it is available and (ii) it belongs to the delegation set. Otherwise, the status quo prevails. In Armstrong and Vickers's benchmark model, the distribution of the state is exogenous. Armstrong and Vickers present an extension where the agent, through costly effort, affects the probability that a project materializes. Conditional on the project materializing, its probability distribution is exogenous. The analysis and findings in our paper differ significantly from those in Armstrong and Vickers [2010] as in our framework the support changes with the effort level and our model is flexible in how effort affects the state distribution.

[3] A different strand of the literature studies the trade-off between flexibility and commitment in models of consumption-savings —Amador, Werning, and Angeletos [2006]— and fiscal policy —Halac and Yared [2014, 2018].

Szalay [2005] presents a model with a different form of moral hazard in delegation. In Szalay's model, the state of the world is exogenous and the agent invests in information acquisition: the higher the effort level, the higher the probability that the agent observes the state. As in our model, the parties preferences over actions are aligned (and quadratic). Szalay shows that the optimal delegation set is a gap. In Szalay [2005], low effort levels are associated to no information. The principal induces effort from the agent by preventing him from choosing (intermediate) actions which are optimal under the prior. Our paper provides a novel rationale for gaps. In our model, under expanding support, low effort levels are associated to a higher likelihood of intermediate states. The principal induces effort from the agent by preventing him from choosing (intermediate) actions which are optimal for intermediate states. In a related model of information acquisition, Deimen and Szalay [2019] compare communication and delegation in an environment in which an expert must acquire information.

2. The model

2.1 Timing, actions and payoffs

First, a principal (she) selects a delegation set $D \in \mathcal{D}$. The set of all possible delegation sets $\mathcal{D}$ consists of all non-empty closed subsets of $\mathbb{R}$. Next, an agent (he) observes the delegation set D and exerts costly effort $e \geqslant 0$. The agent's level of effort affects the distribution of the state of the world ω. Finally, the agent observes the realization of the state of the world and chooses an action $a \in D$.

For any effort level e, we assume that the state of the world ω is uniformly distributed over the support $\Gamma(e) = [\alpha e, 1 + \beta e]$ where $(\alpha, \beta) \in \mathbb{R}^2$ and $(\alpha, \beta) \neq (0, 0)$. When $\beta - \alpha < 0$ we let $\bar{e} \equiv 1/(\alpha - \beta)$ be the maximum possible effort level. The family of supports $\Gamma(e) = [\alpha e, 1 + \beta e]$ captures a variety of economic environments as we allow both α and β to be either positive or negative (we describe this in detail in the next subsection).

The agent obtains utility $-(\omega - a)^2 - c(e)$. He wants the action a to match the state of the world ω and pays a cost of effort captured by the function c. We assume that c is smooth, strictly increasing, convex and satisfies $c'(0) = 0$.

The principal obtains utility $-(\omega - a)^2 + \tilde{v}(\omega)$. Similarly to the agent, the principal cares about the distance between the action a and the state ω. Differently from the agent, the principal also cares about the state of the world *per se*. The smooth function $\tilde{v}$ captures the direct payoff from the state.

In our model, any limits that the principal places on the delegation set result solely from moral hazard. In the absence of moral hazard, that is, if the distribution of the state of the world was exogenous, there would be no conflict between the principal and the agent. The principal would give the agent full discretion.[4] Thus, our model isolates the effects of moral hazard on delegation.

2.2 Preliminary analysis

We divide the principal's problem into two parts. First, we fix an arbitrary level of effort $e \geqslant 0$ (with $e \leqslant \bar{e}$ when $\beta - \alpha < 0$) and find the optimal delegation set to implement such effort. After obtaining the optimal delegation set for each possible level of effort, the principal can compare the payoffs associated to each effort level and then choose the one that gives her the highest payoff. The main difficulty in the analysis lies on the first part: the identification of optimal delegation sets. We thus focus on this part.[5]

We introduce some useful notation to study optimal delegation sets. For any delegation set D and state of the world ω, we let $a^*(\omega, D)$ denote the agent's optimal action. Formally, $a^*(\cdot, D)$ is a function that satisfies[6]

$$a^*(\omega, D) \in \arg\max_{a \in D} -(\omega - a)^2 .$$

Then, a pair (D, e) of a delegation set D and an effort level e induces the following payoffs

[4]The same would be true if the principal did not care about the state of the world per se, that is, if her utility was just $-(\omega - a)^2$.

[5]We discuss the principal's choice of level of effort to implement in Section 3.5.

[6]Since D is closed, $\max_{a \in D} -(\omega - a)^2$ exists. If D is convex, there is a unique maximizer. If D is not convex, there may be two maximizers. If so, we assume that the agent picks the maximizer to the left with probability one. This is without loss of generality as it can only happen on a zero measure set of actions.

to the agent and to the principal:

$$U_A(D,e) \equiv \frac{1}{\psi(e)} \int_{\alpha e}^{1+\beta e} -(\omega - a^*(\omega,D))^2 \, d\omega - c(e)$$
$$U_P(D,e) \equiv \frac{1}{\psi(e)} \int_{\alpha e}^{1+\beta e} -(\omega - a^*(\omega,D))^2 \, d\omega + v(e),$$

where $\psi(e) = 1 + (\beta - \alpha)e$ is the size of the support and $v(e) \equiv \frac{1}{\psi(e)} \int_{\alpha e}^{1+\beta e} \tilde{v}(\omega) d\omega$. The function $\tilde{v}$ affects the principal's payoff $U_P(D,e)$ from a pair (D,e) only through its expected value $v(e)$.[7] We define

$$\Phi(D,e) \equiv \int_{\alpha e}^{1+\beta e} -(\omega - a^*(\omega,D))^2 \, d\omega,$$

so $\Phi(D,e)/\psi(e)$ represents the parties' expected utility from the mismatch between states and actions. This allows for a compact expression of the parties payoff:

$$U_A(D,e) = \frac{\Phi(D,e)}{\psi(e)} - c(e) \quad \text{and} \quad U_P(D,e) = \frac{\Phi(D,e)}{\psi(e)} + v(e)$$

We say that a delegation set $D(e)$ is optimal *given* a level of effort e whenever

$$D(e) \in \arg\max_D \frac{\Phi(D,e)}{\psi(e)}$$
$$\text{s.t. } e \in \arg\max_{e'} U_A\left(D, e'\right).$$

Optimal delegation sets are not unique. We focus on optimal delegation sets that are *minimal*: they do not contain redundant actions.

DEFINITION. MINIMAL DELEGATION SETS. *We say that a delegation set D is minimal with respect to e if $U_A\left(D', e\right) < U_A\left(D, e\right)$ for every delegation set $D' \subset D$.*

From this point on, we restrict attention to optimal delegation sets which are minimal with respect to the level of effort they induce. Unless explicitly stated, every time that we assert that a delegation set is optimal, it means that it is also minimal.

[7] In our first step, we characterize the optimal delegation set for each effort level e. The function v does not affect this characterization, but instead has an impact on the optimal effort level to implement. See Section 3.5 for a brief discussion.

The principal's optimal delegation set for a zero effort level is $D(0) = [0, 1]$. Intuitively, with this delegation set, the agent chooses the optimal action for every state in $\Gamma(0) = [0, 1]$ and achieves his maximum possible payoff. The principal also achieves her maximum possible payoff, conditional on a zero level of effort. Thus, from now on, we only study the problem of a principal who wants to implement a strictly positive effort level.

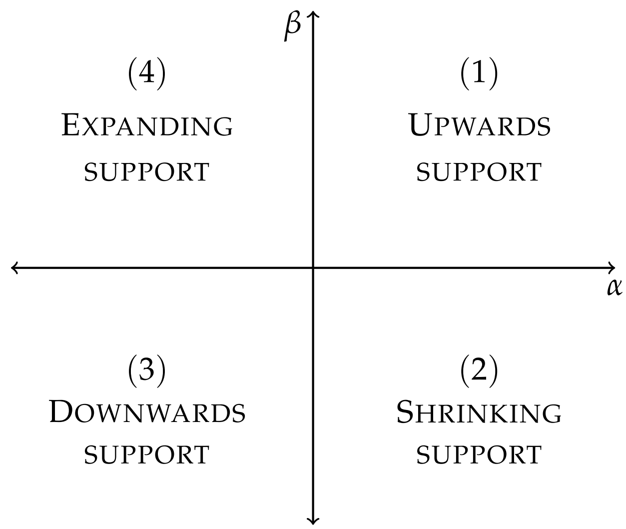


Figure 1: Four possible environments

We split our analysis into four different cases, which reflect four distinct economic environments. Figure 1 illustrates these four cases, which encompass all possible ways in which the support can change linearly with effort. The signs of α and β play a key role in reflecting how effort affects the distribution of the state. Thus, each case is characterized by a different combination of signs for α and β.

In the first case, *upwards support*, both α and β are positive. A higher effort level shifts the support upwards. The distribution associated to a higher effort first-order stochastically dominates that of a lower one. By working hard, the manager in charge of a new branch is more likely to attract more customers. In the second case, *downwards support*, both α and β are negative. This environment is exactly the opposite of upwards support: a higher effort shifts the support downwards. In the third case, *shrinking support*, α is positive and β is negative. The sequence of supports induced by a higher effort is like Matryoshka dolls: everything that is possible with a certain effort is also feasible with a lower one. By exerting care, the manager in charge of forming a new team is more likely to obtain a balanced team composition. In the fourth case, *expanding support*, α is negative and β is positive. A higher level of effort expands the support: everything that is possible with a certain effort is also feasible with a higher one. In order to develop innovative (and risky) projects the bureaucrat needs to work hard.

Before presenting our main findings, we introduce a useful intermediate result. Fix the parameters (α, β) so the support of the state for an arbitrary effort level e is $[\alpha e, 1 + \beta e]$. Assume instead that the parameters are $(-\beta, -\alpha)$, so the support is $[-\beta e, 1 - \alpha e]$. The support $[\alpha e, 1 + \beta e]$ is the mirror image of the support $[-\beta e, 1 - \alpha e]$: $\omega \in [\alpha e, 1 + \beta e]$ if and only if $1 - \omega \in [-\beta e, 1 - \alpha e]$. We show that the optimal delegation sets are also mirror images of each other. Lemma 1 presents this formally.

LEMMA 1. MIRROR IMAGE. *Fix (α, β). Suppose that D is an optimal delegation set for effort level e' when the support of the state for an arbitrary effort level e is $[\alpha e, 1 + \beta e]$. Let $M(D) = \{1 - a : a \in D\}$. If the support of the state for an arbitrary effort level e is $[-\beta e, 1 - \alpha e]$, then the delegation set $M(D)$ is optimal for effort level e'.*

See Appendix A.1 for the proof.

Therefore, it is possible to turn any problem with parameters $(-\beta, -\alpha)$ into one with parameters (α, β), and vice versa. In light of this, in our proofs we restrict attention to the case $\alpha + \beta \geqslant 0$.

3. Results

3.1 Upwards support

We first study the environment of upwards support: $(\alpha, \beta) \geqslant (0, 0)$. As the agent works harder, he increases the likelihood of higher states.

A common class of delegation sets used in real life are *floors*. A floor is a non-minimal delegation set of the form $[\underline{a}, +\infty)$: the agent can choose any action above a certain threshold $\underline{a}$. In practice, for a given effort level e, an agent who faces the floor delegation set $[\underline{a}, +\infty)$ only chooses actions in $[\underline{a}, \max\{\underline{a}, 1 + \beta e\}]$. Thus, the delegation set

$[\underline{a}, \max\{\underline{a}, 1+\beta e\}]$ is minimal with respect to e. Proposition 1 shows that the optimal delegation set $D(e)$ for an effort level e takes the form $[\underline{a}, \max\{\underline{a}, 1+\beta e\}]$ for some $\underline{a} > \alpha e$. With a slight abuse of terminology, we also refer to these delegation sets as floors.[8]

Proposition 1. Optimality of floors under upwards support. *Assume that $(\alpha, \beta) \geqslant (0,0)$. The optimal delegation set $D(e)$ for an effort level $e > 0$ takes the following form.*

1. *If $[(2\alpha+\beta)/3]\psi(e) - c'(e) \geqslant 0$, then*

$$D(e) = [\underline{a}, 1+\beta e],$$

where $\underline{a} \in (\alpha e, 1+\beta e]$ is the unique solution to

$$\alpha\left(\underline{a} - \alpha e\right)^2 - \frac{1}{3}\frac{\alpha-\beta}{\psi(e)}\left(\underline{a} - \alpha e\right)^3 = \psi(e) c'(e). \tag{1}$$

2. *If instead $[(2\alpha+\beta)/3]\psi(e) - c'(e) < 0$, then $D(e) = \{\underline{a}\}$, where*

$$\underline{a} = \frac{1}{\alpha+\beta}\left[\frac{\alpha+2\beta}{3} + \frac{2}{3}\left(\alpha^2+\beta^2+\alpha\beta\right)e + c'(e)\right] > 1+\beta e. \tag{2}$$

See Appendix A.2 for the proof.

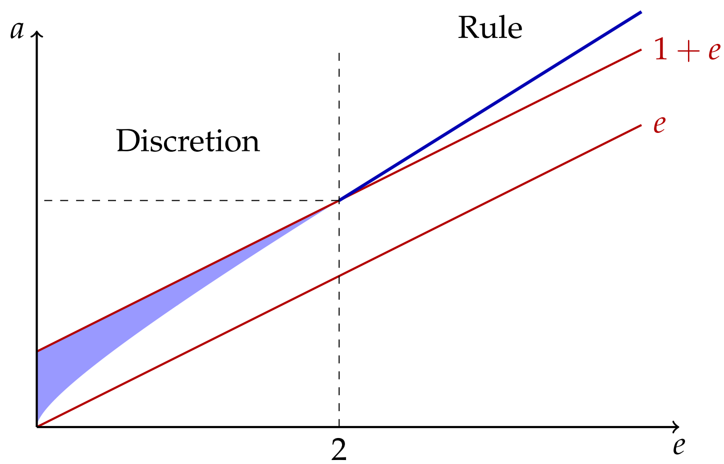
The principal induces the agent to exert effort by preventing him from choosing low actions. She offers a floor delegation set: the agent can only choose actions above a certain threshold. The degree of discretion that the principal gives to the agent depends on the comparison between $[(2\alpha+\beta)/3]\psi(e)$ and $c'(e)$. Whenever $[(2\alpha+\beta)/3]\psi(e) - c'(e) \geqslant 0$, she gives the agent some discretion. The agent can choose actions within the support, but only above a certain threshold. When instead $[(2\alpha+\beta)/3]\psi(e) - c'(e) < 0$, the principal must impose a rule. The agent has no discretion: the delegation set is a singleton that lies to the right of the support.

Figure 2 illustrates the optimal delegation set $D(e)$ for each effort level e when $(\alpha, \beta) = (1,1)$. In the symmetric environment $\alpha = \beta$, the optimal delegation set $D(e)$ takes the

[8] As we discuss at the end of this section, (standard, non-minimal) floors of the form $[\underline{a}, +\infty)$ are also optimal under upwards support.

following simpler expression:

$$D(e) = \begin{cases} \left[\alpha e + \sqrt{c'(e)/\alpha}, 1 + \alpha e\right] & \text{if } c'(e) \leqslant \alpha \\ \{1 + \alpha e + (c'(e)/\alpha - 1)/2\} & \text{if } c'(e) > \alpha \end{cases}$$



Note: In this example, $(\alpha, \beta) = (1, 1)$ and the cost of effort is $c(e) = e^2/4$. The red lines depict the lower and upper bounds of the support. The blue shaded area depicts the optimal delegation set $D(e)$ for effort levels e with $[(2\alpha + \beta)/3]\psi(e) - c'(e) \geqslant 0$, which holds if and only if $e \leqslant 2$. The thick blue line depicts the optimal delegation set $D(e)$ for effort levels $e > 2$.

Figure 2: Optimal delegation sets with upwards support

We now present the sketch of the proof of Proposition 1. In this proof, and also throughout the paper, we use the first order approach to solve for the optimal delegation set. We first study the principal's relaxed problem of maximizing her payoff subject to the agent's first-order condition holding with equality for a given level of effort e. That is, we solve $\max_D \Phi(D, e)/\psi(e)$ subject to

$$\frac{\partial U_A(D, e)}{\partial e} = \frac{1}{\psi(e)} \left[-(\beta - \alpha) \frac{\Phi(D, e)}{\psi(e)} + \Phi'(D, e) \right] - c'(e) = 0, \quad \text{where} \tag{3}$$

$$\Phi'(D, e) \equiv \frac{\partial \Phi(D, e)}{\partial e} = -\beta \left(a^*(1 + \beta e, D) - (1 + \beta e) \right)^2 + \alpha \left(a^*(\alpha e, D) - \alpha e \right)^2. \tag{4}$$

We then show that the effort level e is indeed optimal for the agent under the delegation set that solves the relaxed problem. Thus, the solution to the relaxed problem also solves the original problem.

The marginal utility of effort in equation (3) has a simple form. An agent who increases his effort level raises both the upper bound and the lower bound of the support. By raising the upper bound, his utility increases by

$$\beta\left[-\left[a^*(1+\beta e, D)-(1+\beta e)\right]^2-\Phi(D,e)/\psi(e)\right]/\psi(e),$$

which is positive as long as the payoff when the state is $\omega = 1+\beta e$ is larger than the expected payoff $\Phi(D,e)/\psi(e)$. Similarly, by raising the lower bound, his utility increases by

$$-\alpha\left[-\left[a^*(\alpha e, D)-\alpha e\right]^2-\Phi(D,e)/\psi(e)\right]/\psi(e),$$

which is *negative* as long as the payoff when the state is $\omega = \alpha e$ is larger than the expected payoff $\Phi(D,e)/\psi(e)$. The overall effect of the expected payoff $\Phi(D,e)/\psi(e)$ on the agent's incentives to work depends on the sign of $\beta-\alpha$. A higher $\Phi(D,e)/\psi(e)$ increases the agent's incentives to work if and only if $\beta-\alpha \leqslant 0$. The proof of Proposition 1 comprises two different cases, one with $\beta-\alpha \leqslant 0$ and one with $\beta-\alpha > 0$.

We describe in detail the proof for the case with $\beta-\alpha \leqslant 0$.[9] We show, by contradiction, that any delegation set that solves the relaxed problem must be a floor. To see why, take instead a delegation set D that satisfies the first order condition but is not a floor. We illustrate here our argument with a delegation set D that intersects with the support, like the one depicted in red in Figure 3 (the case with $D\cap\Gamma(e)=\varnothing$ follows a similar logic). We build a floor D' so that when the state is $\omega=\alpha e$, the agent's payoff is the same under both D and D'. Formally, $D'=\left[\alpha e+\left|a^*(\alpha e,D)-\alpha e\right|, 1+\beta e\right]$. We depict the set D' in green in Figure 3. Note that $\Phi'(\cdot,e)$ depends only on the payoffs when the state ω takes values in the boundaries of the support (either αe or $1+\beta e$) — see equation (4). Thus, $\Phi'(D',e) \geqslant \Phi'(D,e)$. Moreover, the set D' yields a larger payoff to the principal: $\Phi(D',e)/\psi(e) > \Phi(D,e)/\psi(e)$. These two facts together imply that $\partial U_A(D',e)/\partial e > \partial U_A(D,e)/\partial e = 0$. We finally construct the set D'' (in blue in Figure 3) by adding actions to the left of D' until

[9] We omit here the details for the case with $\beta-\alpha > 0$. The arguments for the case with $\beta-\alpha > 0$ differ from the ones we present below to account for the negative effect that a higher expected payoff has on the agent's incentives to work. However, the logic behind the arguments for both cases is similar.

$\partial U_A\left(D'', e\right)/\partial e = 0$ (this construction is always feasible since $\partial U_A\left([\alpha e, 1+\beta e], e\right)/\partial e = -c'(e) < 0$). The set D'' yields to the principal an even higher payoff than D' showing indeed that D cannot solve the relaxed problem.

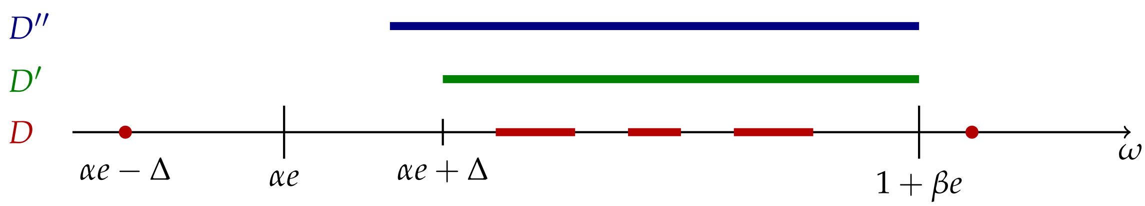


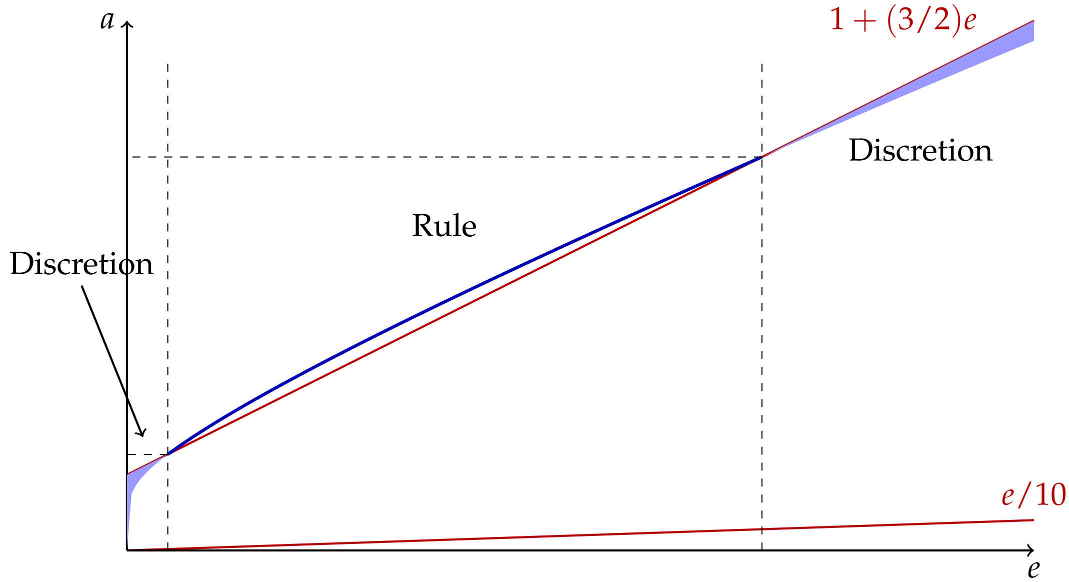
Figure 3: Optimality of a floor

The solution to the relaxed problem must be a floor, either with some discretion (as in case 1 of Proposition 1) or with no discretion at all (as in case 2). The sign of the agent's marginal utility when offered the delegation set $\{1+\beta e\}$

$$\frac{\partial U_A(\{1+\beta e\}, e)}{\partial e} = \frac{2\alpha+\beta}{3}\psi(e) - c'(e)$$

determines whether the principal grants discretion to the agent. If positive, the principal adds actions to the left of $1+\beta e$ until the first order condition is satisfied. If negative, the principal picks an action to the right of $1+\beta e$ large enough so that the first order condition is satisfied. In the final step of the proof of Proposition 1 we show that under the delegation $D(e)$, the effort level e not only satisfies the agent's first order condition, but also achieves a global maximum. Thus, the solution to the relaxed problem also solves the original problem.

Figure 2 illustrates that when $\beta - \alpha = 0$, the principal provides discretion to induce low effort levels and sets a rule for high effort levels. At first glance, this pattern may appear as a general rule: whenever the principal wants to induce a larger effort, she must reduce the agent's discretion. However, this is not the case; Figure 4 depicts an example where the level of discretion is not monotonic. For intermediate levels of effort, the principal imposes a rule: the agent must choose an action outside of the support. For all other levels of effort e (either low or high), the principal gives the agent some discretion: $D(e) \subset \Gamma(e)$.[10]



Note: In this example, $(\alpha, \beta) = (1/10, 3/2)$ and the cost of effort is $c(e) = (10/9)e^{3/2}$. The red lines depict the lower and upper bounds of the support. The blue shaded area depicts the optimal delegation set $D(e)$ for effort levels e with $[(2\alpha + \beta)/3]\psi(e) - c'(e) \geqslant 0$. The thick blue line depicts the optimal delegation set $D(e)$ for effort levels e with $[(2\alpha + \beta)/3]\psi(e) - c'(e) < 0$.

Figure 4: Optimal discretion may be non-monotonic

To see why the level of discretion need not be monotonic, recall that whether the principal grants discretion to the agent or not depends on the strength of the agent's incentives when offered the delegation set $\{1 + \beta e\}$. If $\partial U_A(\{1 + \beta e\}, e)/\partial e = [(2\alpha + \beta)/3]\, \psi(e) - c'(e) > 0$, then the principal gives the agent some discretion. When $\beta - \alpha \leqslant 0$, the marginal benefit from effort (the first term in $\partial U_A(\{1 + \beta e\}, e)/\partial e$) weakly decreases with e, since $\psi(e) = 1 + (\beta - \alpha)e$. Intuitively, the larger the effort, the smaller the size of the support and thus the lower the benefits from shifting the support to the right. The marginal cost also increases with effort. Thus, under the delegation set $\{1 + \beta e\}$, when $\beta - \alpha \leqslant 0$, a larger e reduces the agent's incentives to work. So the principal gives discretion for low effort levels and imposes a rule for high effort levels — as in Figure 2. Instead, when $\beta - \alpha > 0$ — as in the example depicted in Figure 4 — the size of the sup-

[10]Figure 4 highlights that the principal may shift back and forth between discretion and rules as the effort level increases. Optimal delegation sets, however, are monotonic in the following sense: under upwards support, the smallest action in the optimal delegation set increases with effort.

port increases with effort. Then, under the delegation set $\{1+\beta e\}$, the marginal benefit from effort e increases with e. Whenever this effect dominates the change in the marginal cost, the principal may shift from setting a rule to granting discretion when she intends to implement a larger effort level.[11]

In Proposition 1 we characterize *minimal* optimal delegation sets under upwards support. For any effort level e, they are of the form $D(e) = [\underline{a}, \max\{\underline{a}, 1+\beta e\}]$ for some $\underline{a} > \alpha e$. In addition, in the proof of Proposition 1 we show that the corresponding (non-minimal) floor $[\underline{a}, +\infty)$ also solves the principal's problem.[12]

3.2 Downwards support

Consider next the case with downwards support: $(\alpha, \beta) \leqslant (0,0)$. As the agent works harder, he increases the likelihood of low states. Under downwards support, optimal delegation sets also take a common and simple form. The principal offers a ceiling: the agent can only choose actions below a certain threshold. This result is a direct consequence of Proposition 1 and Lemma 1. The case of downwards support is the mirror image of the case of upwards support.

Corollary 1. Optimality of ceilings under downwards support. *Assume that $(\alpha, \beta) \leqslant (0,0)$. The optimal delegation set $D(e)$ for an effort level $e > 0$ takes the following form.*

1. *If $-[(\alpha+2\beta)/3]\psi(e) - c'(e) \geqslant 0$, then*

$$D(e) = [\alpha e, \overline{a}],$$

where $\overline{a} \in [\alpha e, 1+\beta e)$ is the unique solution to

$$-\beta\,(1+\beta e-\overline{a})^2 - \frac{1}{3}\frac{\alpha-\beta}{\psi(e)}\,(1+\beta e-\overline{a})^3 = \psi(e)c'(e).$$

[11] We do not have necessary and sufficient conditions for monotonicity when $\beta - \alpha > 0$. However, if $c(e) = e^{\eta}$ and $\eta \geqslant 2$, then monotonicity holds: $D(e) \subset \Gamma(e)$ if and only if e is lower than a threshold.

[12] For any realized state, the agent chooses the same action under $[\underline{a}, +\infty)$ and under $D(e)$. Then, $[\underline{a}, +\infty)$ also solves the relaxed problem. Moreover, for any arbitrary floor delegation set $[a, +\infty)$ with $a \in \mathbb{R}$, the agent's payoff is strictly concave in effort.

2. *If instead* $-[(\alpha+2\beta)/3]\psi(e) - c'(e) < 0$, *then* $D(e) = \{\overline{a}\}$, *where*

$$\overline{a} = \frac{1}{\alpha+\beta}\left[\frac{\alpha+2\beta}{3} + \frac{2}{3}\left(\alpha^2+\beta^2+\alpha\beta\right)e + c'(e)\right] < \alpha e.$$

3.3 Shrinking support

We now study the case of shrinking support: $\beta < 0 < \alpha$. As the agent exerts higher effort, the support shrinks: $\Gamma(e') \subset \Gamma(e)$ if and only if $e' > e$. The state $\alpha/(\alpha-\beta)$ belongs to the support $\Gamma(e)$ for all possible effort levels $e \in [0, \bar{e}]$ and is realized with probability 1 when the agent's exerts the maximum effort level $\bar{e} = 1/(\alpha-\beta)$. The state $\alpha/(\alpha-\beta)$ represents an ideal state of the world to the principal. As the agent exerts higher effort, he makes it more likely that states closer to $\alpha/(\alpha-\beta)$ occur.[13]

Oftentimes, delegation sets used in practice impose restrictions both on low and on high actions; thus only allowing for intermediate actions. We say that a delegation set is a *floor-ceiling* when it is of the form $[\underline{a}, \overline{a}]$, with $\underline{a} \leqslant \overline{a}$. We show in Proposition 2 that floor-ceilings are optimal with shrinking support. The principal sets both a lower bound and an upper bound on actions. Let $\tilde{e}$ be the unique solution to $(2/3)(\alpha^3-\beta^3)/(\alpha-\beta)^2 = c'(e)/\psi(e)$ and note that $\tilde{e} < \bar{e}$.[14]

PROPOSITION 2. OPTIMALITY OF FLOOR-CEILING UNDER SHRINKING SUPPORT. *Assume that* $\beta < 0 < \alpha$ *and that* $\alpha+\beta \neq 0$.[15] *The optimal delegation set* $D(e)$ *for an effort level* $e > 0$ *takes the following form.*

1. *If* $e \leqslant \tilde{e}$, *then*

$$D(e) = \left[\alpha e + \Delta, 1 + \beta e + \frac{\beta}{\alpha}\Delta\right]$$

[13] With shrinking support and $\alpha+\beta \geqslant 0$ the expectation increases and the variance decreases with effort. This is also true with upwards support and $\beta-\alpha \leqslant 0$. In spite of this similarity, the cases of upwards support and shrinking support capture distinct economic environments. With upwards support, as the agent exerts higher effort, some high states that were not possible become feasible. Instead, with shrinking support, everything that is possible with a certain effort is also feasible with a lower one.

[14] Under shrinking support, $c'(e)/\psi(e)$ strictly increases with effort e, is equal to zero when $e=0$ and equal to infinity when $e=\bar{e}$. Thus, $\tilde{e}$ exists and is unique.

[15] When $\alpha+\beta=0$, part 1 of Proposition 2 also holds. However, in this non-generic case, the principal cannot implement any effort level $e > \tilde{e}$.

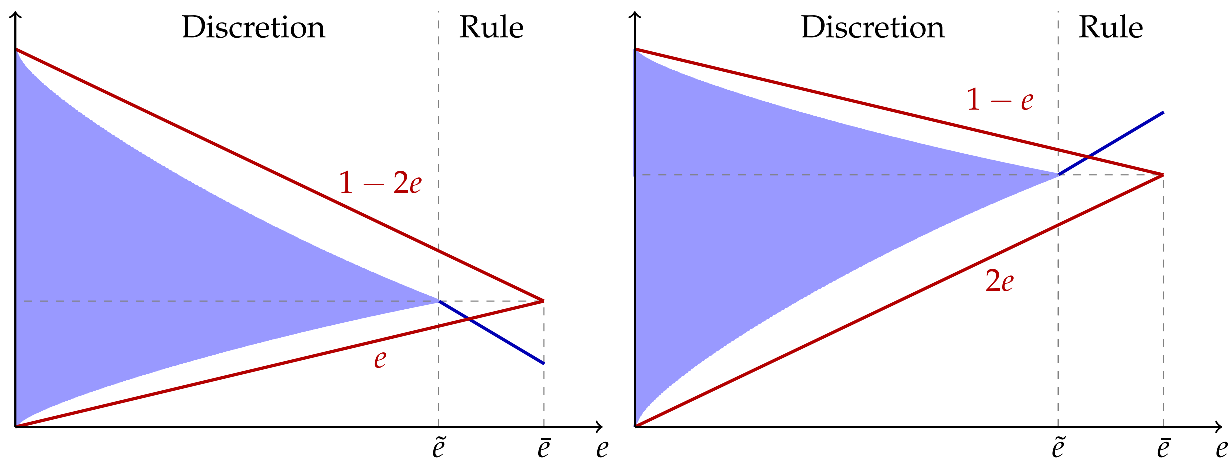
where $\Delta \in [0, \alpha\,(1/(\alpha-\beta)-e)]$ *is the unique solution to*

$$\Delta^2 \left[\alpha\psi(e) - \frac{1}{3}(\alpha-\beta)\Delta\right] = c'(e)(\psi(e))^2 \frac{\alpha^3}{\alpha^3-\beta^3}. \tag{5}$$

2. *If instead* $e > \tilde{e}$, *then*

$$D(e) = \left\{\frac{1}{\alpha+\beta}\left[\frac{\alpha+2\beta}{3} + \frac{2}{3}\left((\alpha+\beta)^2 - \alpha\beta\right)e + c'(e)\right]\right\}.$$

See Section A.3 for the proof.



(a) $(\alpha,\beta) = (1,-2)$ (b) $(\alpha,\beta) = (2,-1)$

Note: In this example the cost of effort is $c(e) = e^2/4$. The red lines depict the lower and upper bounds of the support. The blue shaded area depicts the optimal delegation set $D(e)$ for effort levels $e \leqslant \tilde{e}$. The thick blue line depicts the optimal delegation set $D(e)$ for effort levels $e > \tilde{e}$.

Figure 5: Optimal delegation sets with shrinking support

Figure 5 illustrates optimal delegation sets with shrinking support for two different parameter configurations: $(\alpha,\beta) = (1,-2)$ in Figure 5a and $(\alpha,\beta) = (2,-1)$ in Figure 5b. Whenever $e \leqslant \tilde{e}$, the principal grants the agent some discretion: he can choose from a convex set of intermediate actions. Intuitively, the principal induces effort by only allowing for actions that are optimal under intermediate realizations, which in turn are more likely for higher effort levels. The agent exerts effort to increase the probability that the state is close to available actions. When the principal wants to induce higher levels of effort, she gives the agent less discretion: the floor increases and the ceiling decreases.[16] The ratio between the decrease in the ceiling and the increase in the floor is $|\beta/\alpha|$ for any effort level $e \leqslant \tilde{e}$. This is why the ceiling decreases faster than the floor increases in Figure 5a, as $(\alpha, \beta) = (1, -2)$, while the opposite is true in Figure 5b, as $(\alpha, \beta) = (2, -1)$. The optimal delegation set becomes a singleton within the support when $e = \tilde{e}$. For effort levels higher than $\tilde{e}$, the principal gives the agent no discretion. The delegation set is a singleton that strictly decreases with effort whenever $\alpha + \beta < 0$ (as in Figure 5a), or strictly increases with effort whenever $\alpha + \beta > 0$ (as in Figure 5b).

We next present the sketch of the proof of Proposition 2. In the first step, we show, by contradiction, that any delegation set that solves the relaxed problem must be convex. We take an arbitrary non-convex delegation set D that satisfies the first order condition and let $\Delta_1 \equiv |\alpha e - a^* (\alpha e, D)|$ and $\Delta_2 \equiv |1 + \beta e - a^* (1 + \beta e, D)|$. In the proof we distinguish two cases. The first has $\Delta_1 + \Delta_2 \leqslant \psi(e)$, so it includes all delegation sets that intersect with the support. The second has $\Delta_1 + \Delta_2 > \psi(e)$, so it only contains delegation sets outside of the support with exactly two points.

To illustrate our argument for the first case, consider the delegation set D in red in Figure 6. Since $\Delta_1 + \Delta_2 \leqslant \psi(e)$, we can construct an alternative delegation set $D' = [\alpha e + \Delta_1, 1 + \beta e - \Delta_2]$ —in green in Figure 6. The delegation set D' yields a larger payoff: $\Phi(D', e)/\psi(e) > \Phi(D, e)/\psi(e)$, and, by construction, has $\Phi'(D', e) = \Phi'(D, e)$. Thus, $\partial U_A (D', e) / \partial e > \partial U_A (D, e) / \partial e = 0$.[17] We finally construct the set D'' (in blue in Figure 3) by extending D' until the first order condition holds. The set D'' yields to the principal an even higher payoff than D' showing indeed that D cannot solve the relaxed problem.

For the case with $\Delta_1 + \Delta_2 > \psi(e)$ we also construct alternative delegation sets that are convex and dominate D. The construction of these alternative sets is more involved so we omit the details here. However, the underlying logic is analogous: the principal finds it beneficial to offer actions close to the support, instead of two points outside of it.

[16]It follows from the proof of Proposition 2 that $\alpha e + \Delta$ increases and $1 + \beta e + (\beta/\alpha)\Delta$ decreases with e.

[17]The construction of D' is similar to that under upwards support —see Figure 3. Unlike the case of upwards support, in the construction here we do not include all actions up to the $1 + \beta e$. As we explain in the next paragraphs, it is indeed optimal to stop before $1 + \beta e$ and provide a ceiling.

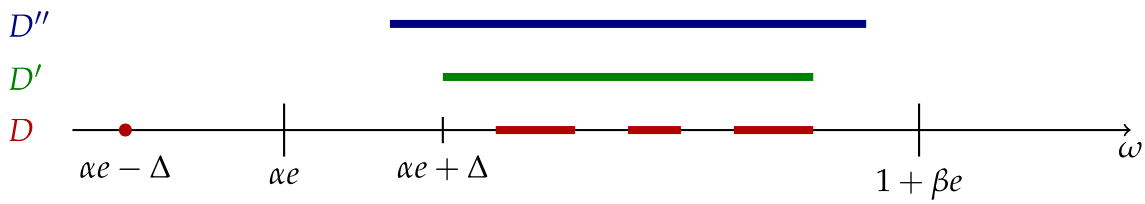


Figure 6: Floor and ceiling

Since the optimal delegation set must be convex, then whenever it is not a singleton it must be of the form $[\alpha e + \Delta_1, 1 + \beta e - \Delta_2]$. How are Δ_1 and Δ_2 related in the optimal delegation set? Consider a marginal change in Δ_1 and the corresponding change in Δ_2 (in the opposite direction) to keep $\Phi'(\cdot, e)$ constant. If the delegation set is optimal, such a change must also keep $\Phi(\cdot, e)$ constant.[18] This occurs if and only if $\Delta_2 = (-\beta/\alpha)\Delta_1$.

The solution to the relaxed problem must then either be a singleton, or a member of the family of delegation sets $[\alpha e + \Delta, 1 + \beta e + (\beta/\alpha)\Delta]$. We show that the incentives for the agent to work when the principal offers a delegation set from this family increase with Δ. They become the largest when $\Delta = \alpha\left(1/(\alpha - \beta) - e\right)$, which makes the delegation set collapse to the singleton $\{1/(\alpha - \beta)\}$. The incentives for the agent to work when the principal offers the delegation set $\{1/(\alpha - \beta)\}$ are

$$\frac{\partial U_A\left(\{1/(\alpha - \beta)\}, e\right)}{\partial e} = \frac{2}{3}\frac{\alpha^3 - \beta^3}{(\alpha - \beta)^2}\psi(e) - c'(e).$$

This expression strictly decreases with e and becomes zero when $e = \tilde{e}$. Thus, for effort levels larger than $\tilde{e}$, no delegation set with positive mass satisfies the first order condition. The principal must then set a rule: the optimal delegation set is the singleton from part 2 of Proposition 2. Instead, for effort levels lower than $\tilde{e}$, the principal grants discretion to the agent. She offers the delegation set from part 1 of Proposition 2.[19] Thus, under shrinking support, the principal grants the agent some discretion for low effort levels and

[18] Otherwise the change would provide a higher payoff and higher incentives for the agent to work. As before, by extending such delegation set, we could obtain one that also satisfies the first order condition and provides an even higher payoff.

[19] When $e < \tilde{e}$, not only the delegation set from part 1 of Proposition 2 satisfies the first order condition, but also a singleton does. We show in the proof that the singleton yields a strictly lower payoff. Moreover, in the final step of the proof we show that under the delegation $D(e)$, the agent's problem is concave. Thus, the solution to the relaxed problem also solves the original problem.

sets a rule for large effort levels. This is in contrast with the case of upwards support, where this monotonicity is not guaranteed.

3.4 Expanding support

We finally study the case of expanding support: $\alpha < 0 < \beta$. As the agent exerts higher effort, the support expands: $\Gamma(e') \subset \Gamma(e)$ if and only if $e' < e$.

Some delegation sets commonly used in practice feature a gap. The principal does not allow the agent to choose intermediate actions. A non-minimal gap has the form $D = (-\infty, \underline{a}] \cup [\overline{a}, +\infty)$, with $\underline{a} < \overline{a}$; the agent cannot choose actions in $(\underline{a}, \overline{a})$. Proposition 3 shows that, under expanding support, a (minimal) gap $D = [\min\{\alpha e, \underline{a}\}, \underline{a}] \cup [\overline{a}, \max\{1+\beta e, \overline{a}\}]$, with $\underline{a} < \overline{a}$ solves the relaxed problem. The case of expanding support differs from the previous three cases in that the solution to the relaxed problem need not solve the original problem. In Proposition 3 we also present sufficient conditions that guarantee that the solution to the relaxed problem indeed solves the principal's original problem.

PROPOSITION 3. GAPS UNDER EXPANDING SUPPORT. *Fix an effort level $e > 0$. Assume that $\alpha < 0 < \beta$ and that $\alpha + \beta \neq 0$.*[20]

1. *If $[(\beta-\alpha)/12]\psi(e) - c'(e) \geqslant 0$, then any delegation set*

$$D = [\alpha e, \alpha e + \Delta_1] \cup [1 + \beta e - \Delta_2, 1 + \beta e] \quad \textit{with}$$
$$\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_1 + \Delta_2 < \psi(e) \ \textit{ and}$$
$$\frac{\partial U_A(D,e)}{\partial e} = \frac{\beta - \alpha}{12} \frac{(\psi(e) - \Delta_1 - \Delta_2)^3}{(\psi(e))^2} - c'(e) = 0$$

   solves the relaxed problem. (And such solution exists.)

2. *If $[(\beta-\alpha)/12]\psi(e) - c'(e) < 0$, then there exists a unique solution to the relaxed problem.*

[20]The principal's problem is simpler when $\alpha + \beta = 0$. We characterize the optimal delegation set for that case in Remark 1.

The solution is of the form

$$D = \{\alpha e - \Delta_1, 1 + \beta e + \Delta_2\},$$

with $\Delta_1 > 0$, $\Delta_2 > 0$, *and* $|\Delta_2 - \Delta_1| < \psi(e)$. *Property 1 in Appendix A.4 (uniquely) pins down the pair* (Δ_1, Δ_2).

Moreover, let $D = [\min\{\alpha e, \underline{a}\}, \underline{a}] \cup [\overline{a}, \max\{1 + \beta e, \overline{a}\}]$, *with* $\underline{a} < \overline{a}$, *denote a solution to the relaxed problem from either part 1 or from part 2. Then, D also solves the original problem if at least one of the following two (sufficient) conditions hold.*

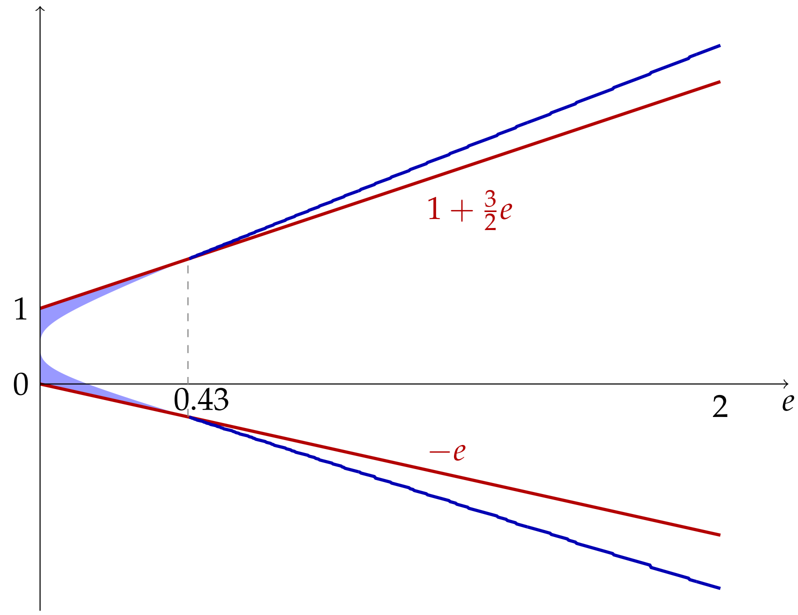
i. $0 \leqslant \underline{a} < \overline{a} \leqslant 1$

ii. $c''' \geqslant 0$ *and* $\partial U_A(D, \check{e})/\partial e > 0$, *where* $\check{e}$ *is the smallest effort level* e' *with* $(\underline{a} + \overline{a})/2 \in \Gamma(e')$.

See Appendix A.4 for the proof.

The solution to the relaxed problem features a gap: the agent cannot choose intermediate actions. Whenever $[(\beta - \alpha)/12]\, \psi(e) - c'(e) \geqslant 0$, the delegation set includes the boundaries of the support and actions close to the boundaries. In this case, the agent has some discretion. For example, when the state lies in $[\alpha e, \alpha e + \Delta_1]$, the agent can fine-tune his action to achieve his preferred outcome. When instead $[(\beta - \alpha)/12]\, \psi(e) - c'(e) < 0$, the delegation set consists of two points, one to the left and one to the right of the support. The agent's discretion is thus limited. Figure 7 illustrates the solution to the relaxed problem for each effort level when $(\alpha, \beta) = (-1, 3/2)$.

The case of expanding support differs from the other three cases in two key dimensions. First, when $[(\beta - \alpha)/12]\, \psi(e) - c'(e) > 0$, the solution to the relaxed problem is not unique. Second, and most importantly, the first order approach is not necessarily valid. The solution to the relaxed problem need not solve the principal's original problem when the effort level is sufficiently large. We have constructed examples where the solution to the relaxed problem for a given effort level e is unique. However when facing the delegation set that solves the relaxed problem, the agent chooses an effort level different from



Note: In this example, $(\alpha, \beta) = (-1, 3/2)$, the cost of effort is $c(e) = e^2/2$ and $0 < e \leqslant 2$. The red lines depict the lower and upper bounds of the support. The blue shaded area depicts an optimal delegation set $D(e)$ for effort levels e with $[(\beta - \alpha)/12]\psi(e) - c'(e) \geqslant 0$, which holds for effort levels $e \leqslant 10/23 \approx 0.43$. The thick blue line depicts the optimal delegation set $D(e)$ for effort levels $e > 10/23$. The second sufficient condition from Proposition 3 holds for any effort level e in this example.

Figure 7: Optimal delegation sets with expanding support

e. We next present the sketch of the proof of Proposition 3 and explain these two key differences.

In the proof of Proposition 3, we first show that when $[(\beta - \alpha)/12]\, \psi(e) - c'(e) \geqslant 0$, the solution to the relaxed problem must contain the boundaries of the support and thus (by minimality) fully lies within the support. Under expanding support, any delegation set D has $\Phi'(D, e) \leqslant 0$ — see equation (4). Thus, the payoff $\Phi(D, e)/\psi(e)$ from any delegation set D that satisfies the first order condition is bounded above by $-\psi(e)c'(e)/(\beta - \alpha)$ — see equation (3). This bound is tight if and only if $\{\alpha e, 1 + \beta e\} \subseteq D$. The principal can achieve this upper bound whenever the agent's marginal utility when offered the delegation set $\{\alpha e, 1 + \beta e\}$ is positive: $\partial U_A(\{\alpha e, 1 + \beta e\}, e)/\partial e = [(\beta - \alpha)/12]\psi(e) - c'(e) \geqslant 0$. When this condition holds, the principal achieves the upper bound by adding actions within the support to the set $\{\alpha e, 1 + \beta e\}$ until the first-order condition holds. The solution to the relaxed problem is not unique; any delegation set $D \supseteq \{\alpha e, 1 + \beta e\}$ that satisfies the first order condition is a solution to the relaxed problem.[21]

We next study the case with $[(\beta-\alpha)/12]\,\psi(e) - c'(e) < 0$ — part 2 of Proposition 3. The proof of this case is as follows. First, we show that any delegation set D that solves the relaxed problem must have $D \cap (\alpha e, 1+\beta e) = \varnothing$. Thus, the solution to the relaxed problem must either be a singleton or contain exactly two points. Second, we show that a singleton is always dominated by a delegation set with exactly two points. Third, we characterize the unique set $D = \{\alpha e - \Delta_1, 1+\beta e + \Delta_2\}$ that solves the relaxed problem. If D solves the relaxed problem, any marginal variations in Δ_1 and Δ_2 that keep the agent's marginal utility constant cannot increase the payoff Φ. This condition leads to a unique pair (Δ_1, Δ_2).

In the other three environments, we establish that the first order approach is valid by showing that the agent's problem is concave. In the case of upwards support, if the delegation set is a floor, the agent's payoff is concave in effort. The same is true under downwards support if the delegation set is a ceiling and also under shrinking support if the delegation set is a floor-ceiling. This is not necessarily the case under expanding support if the delegation set is a gap. Not only the agent's payoff may not be concave, but also it may achieve a maximum at an effort level different from the one the principal intends to implement. As mentioned above, we have constructed examples where the unique solution to the relaxed problem does not solve the principal's original problem.

In the last step of the proof of Proposition 3 we provide sufficient conditions for the validity of the first-order approach. Consider the first sufficient condition: $0 \leqslant \underline{a} < \overline{a} \leqslant 1$. In this case, the solution to the relaxed problem takes the form $D = [\alpha e, \underline{a}] \cup [\overline{a}, 1+\beta e]$ and comes from part 1 of Proposition 3. We build the non-minimal gap $D' = (-\infty, \underline{a}] \cup [\overline{a}, +\infty)$ and show that the agent's payoff $U_A(D', \cdot)$ is concave in effort. This is because the term $\Phi(D', \cdot)$ is constant in effort. Under D' (and therefore also under D), the agent chooses the effort level e that the principal intends to implement. This shows that D solves the original problem. Note, moreover, that the solution to the original problem is

[21] There always exist delegation sets of the class $D = [\alpha e, \alpha e + \Delta_1] \cup [1+\beta e - \Delta_2, 1+\beta e]$ that solve the relaxed problem. In part 1 of Proposition 3 we focus on this class since we can provide sufficient conditions for the validity of the first order approach for this class. Moreover, this class features a gap, as the (unique) solution in part 2.

not unique when $0 \leqslant \underline{a} < \overline{a} \leqslant 1$. Any delegation set D that solves the relaxed problem and has $\Gamma(e)\backslash D \subseteq [0,1]$ also solves the original problem.[22] Finally, although the sufficient condition $0 \leqslant \underline{a} < \overline{a} \leqslant 1$ is on endogenous variables, it is easy to find primitives such that this condition holds. In particular, it holds whenever $c'(e)$ is lower than a threshold (which does not depend on e). Thus, it holds for effort levels sufficiently small.

The second condition we provide at the end of Proposition 3 also guarantees that the solution to the relaxed problem solves the original problem. To see why, again build the non-minimal gap $D' = (-\infty, \underline{a}] \cup [\overline{a}, +\infty)$. The effort level $\check{e} < e$ is the lowest effort level such that both $\underline{a}$ and $\overline{a}$ are chosen by the agent in some states. By construction, $\partial U_A(D', \check{e})/\partial = \partial U_A(D, \check{e})/\partial e$, which is positive by assumption. The condition $c''' \geqslant 0$ guarantees that once $\partial U_A(D', \cdot)/\partial e$ becomes zero for some effort level larger than $\check{e}$, then it is negative afterwards. For any effort level lower than $\check{e}$ and for every state in the support, the agent chooses only one action, either $\underline{a}$ or $\overline{a}$. His choices are the same as if he was offered a singleton. Under a singleton delegation, set the agent's payoff is concave in effort. To summarize, there is a unique effort level that makes $\partial U_A(D', \cdot)/\partial e = 0$. This effort level is the intended effort level e. For effort levels lower than e, $\partial U_A(D', \cdot)/\partial e > 0$ and for effort levels larger than e, $\partial U_A(D', \cdot)/\partial e < 0$. Thus the agent chooses effort level e under D' (and also under D).[23]

The case of expanding support contains the symmetric environment $\alpha = -\beta$. The principal's problem is simpler in this symmetric environment. The shape of the solution to the relaxed problem is as in Proposition 3. Differently from the case with $\alpha \neq -\beta$, there exists a (symmetric) solution with a simple closed-form expression that solves the original problem of the principal. Thus, in this symmetric environment, the first-order approach holds. We summarize these findings in Remark 1.

REMARK 1. *Fix an effort level $e > 0$. Assume that $\alpha < 0 < \beta$ and that $\alpha + \beta = 0$.*

[22]Whenever $\Gamma(e)\backslash D \subseteq [0,1]$ one can build D' by adding all actions to the left of 0 and to the right of 1. Again, the term $\Phi(D', \cdot)$ is constant in effort.

[23]Condition (ii) holds every time that condition (i) holds. Moreover, there are environments in which condition (i) fails but condition (ii) holds.

1. *If* $(\beta/6)\psi(e) - c'(e) \geqslant 0$*, then the delegation set*

$$D(e) = [-\beta e, -\beta e + \Delta] \cup [1 + \beta e - \Delta, 1 + \beta e] \quad \textit{with}$$
$$\Delta = \frac{1}{2}\left[\psi(e) - \left(\frac{6c'(e)(\psi(e))^2}{\beta}\right)^{1/3}\right]$$

is optimal for the effort level e.

2. *If* $(\beta/6)\psi(e) - c'(e) < 0$*, then the optimal delegation set* $D(e)$ *for the effort level e is*

$$D(e) = \{-\beta e - \Delta, 1 + \beta e + \Delta\}, \quad \textit{with } \Delta = \frac{c'(e)}{\beta} - \frac{\psi(e)}{6}.$$

In Figure 7, the delegation set is as in part 1 of Proposition 3 for low effort levels and as in part 2 for effort levels. The sign of the expression $[(\beta - \alpha)\,\psi(e)]/12 - c'(e)$ is not necessarily monotonic in the effort level. Thus, as in the case of upwards support, the delegation set may alternate between the two shapes described in Proposition 3 as the effort level increases. A sufficient condition for monotonicity is that $c''' \geqslant 0$. If so, the delegation set has the first shape for low effort levels and the second shape (only two points) for high effort levels.

Finally, as in the rest of the paper, we present *minimal* delegation sets in Proposition 3. However, (as we pointed out in the sketch of the proof) one can replace any minimal gap $[\min\{\alpha e, \underline{a}\}, \underline{a}] \cup [\overline{a}, \max\{1 + \beta e, \overline{a}\}]$ from Proposition 3 by its corresponding non-minimal gap $(-\infty, \underline{a}] \cup [\overline{a}, +\infty)$ and the results from this proposition still hold true.

3.5 Principal's choice of effort level to implement

Under all four environments, the optimal delegation set $D(e)$ yields to the principal an expected payoff from the mismatch between states and actions equal to

$$V(e) \equiv \frac{\Phi(D(e), e)}{\psi(e)}.$$

The term $v(e) = \frac{1}{\psi(e)} \int_{\alpha e}^{1+\beta e} \tilde{v}(\omega) d\omega$ denotes the expected effect from the state on the principal's payoff. Thus, the principal solves

$$\max_{e} V(e) + v(e). \tag{6}$$

Which level of effort does the principal implement? In general, the principal's problem (6) does not admit a closed-form solution, since the optimal delegation set $D(e)$ does not have a closed-form expression either. Our characterization of optimal delegation sets allows for a numerical expression of the indirect effect $V(e)$ for each effort level e. This, together with the specification of $\tilde{v}$ (and thus of the direct effect v) allow for a numerical solution of the principal's problem (6).

The principal's problem (6) contains two terms: the direct effect of effort v and the indirect effect of effort V. The principal may benefit from a positive level of effort through the direct effect v. However, to induce any positive level of effort $e > 0$, she needs to restrict the agents' actions and accept some level of inefficiency: $V(e) < 0$. The principal then faces trade-off between the direct effect of effort v and the indirect effect of effort V.[24]

In what follows we discuss the shape of the direct effect v and the indifferent effect V. First, we explain why v is not necessarily monotonically increasing. Second, we highlight an interesting feature of V: the indirect effect V is not necessarily monotonically decreasing. The inefficiency needed to induce a higher level of effort may be lower than the one needed to induce a lower level of effort.

The shape of v depends (i) on the shape of $\tilde{v}$ and (ii) on how effort affects the distribution of the state of the world. While we do not impose any restrictions on $\tilde{v}$ in general, some assumptions on the shape of $\tilde{v}$ may be natural under different environments. These assumptions lead to an increasing v under upwards (and downwards) support. However, they do not necessarily lead to an increasing v under expanding and shrinking support.

Under upwards and expanding support, it may be reasonable to assume that $\tilde{v}$ is increasing in the state ω. The CEO prefers having more customers in the new branch and

[24]As we discuss next, v may be decreasing. If v obtains a global maximum at zero effort, the problem is not interesting. The principal can achieve her first best by granting full discretion to the agent. The agent chooses a zero level of effort and his action always matches the state.

the mayor prefers projects of higher quality. Under upwards support, if $\tilde{v}$ increases with ω, then v increases with e (this follows immediately from first order stochastic dominance). Instead, under expanding support, an increasing $\tilde{v}$ does not guarantee that v increases with e. Next, under shrinking support, it may be natural to assume that $\tilde{v}$ is single-peaked at $\alpha/(\alpha-\beta)$, which represents the ideal state for the principal. The CEO wants the team composition to be as close as possible to 50%. A single-peaked $\tilde{v}$ does not imply in general that v increases with e.[25]

The indirect effect V is decreasing in effort in a neighborhood of zero, under all four environments. Under upwards (and downwards) support, when $\beta - \alpha \geqslant 0$ the monotonicity of V extends to all levels of effort. Under expanding support, V decreases with e when $\beta = -\alpha$. When instead $\beta \neq -\alpha$, all examples suggest that V decreases with e. However, we only have a proof of monotonicity in the regions of e with $D(e) \subset \Gamma(e)$.

When instead $\beta - \alpha < 0$, the indirect effect V is not necessarily monotone. We can construct examples under upwards support, shrinking support and decreasing support (all with $\beta - \alpha < 0$) where V increases with the effort level in some region. The potential non-monotonicity results from the fact that the size of the support $\gamma(e) = 1 + (\beta - \alpha)e$ decreases with the effort level when $\beta - \alpha < 0$. To illustrate the mechanism behind this, consider the environment of shrinking support. An increase in the effort level from e to e' reduces the size of the support, so keeping the delegation set $D(e)$ constant, a higher effort level would increase the principal's payoff. However, a higher effort level also reduces the size of the optimal delegation set: $D(e') \subset D(e)$, and thus the principal's payoff decreases. The overall effect depends on how fast the optimal delegation set shrinks as the effort level increases.

4. Conclusion

We develop a novel model of delegation with moral hazard: the agent affects the distribution of the state by exerting costly effort. The principal faces a trade-off. On the one side, she wants to grant the agent flexibility so the agent can react to information. On the other

[25]However, if $\tilde{v}$ takes the shape of a quadratic loss —around $\alpha/(\alpha-\beta)$—, then v is increasing.

side, the principal may have to limit the agent's discretion in order to induce effort. We characterize optimal delegation sets. We show that the main determinant of the shape of optimal delegation sets is the way in which effort affects the distribution of the state. We show the optimality of four simple forms of delegation, all commonly used in practice: floors, ceilings, floor-ceilings and gaps.

In our model, the parties' preferences over actions are aligned. Thus, any limits to the agent's discretion result solely from the presence of moral hazard. We can introduce bias in the agent's preferences under symmetric upwards support ($\alpha = \beta > 0$) and show that optimal delegation sets are floors. However, the techniques that we develop in this paper do not extend easily to allow for bias under other environments. Thus, we leave the study of the interaction between (i) moral hazard and (ii) misalignment in preferences for further research. Further work can also develop the analysis of delegation with endogenous states beyond the quadratic-uniform framework. Our model relies on the simplifying assumptions of quadratic preferences and uniform distributions. These assumptions have the advantage of keeping the analysis tractable, while at the same time allowing for a rich variety of economic environments.

A. Appendix

A.1 Proof of Lemma 1

Note first that $\omega \in [\alpha e, 1+\beta e]$ if and only if $1-\omega \in [-\beta e, 1-\alpha e]$. Next, for any delegation set D, let its mirror delegation set $M(D)$ be defined by $M(D) = \{1-a : a \in D\}$.

Step 1. We first show that for any delegation set D and any state $\omega \in [\alpha e, 1+\beta e]$,

$$\max_{a\in D} -(\omega - a)^2 = \max_{a\in M(D)} -((1-\omega)-a)^2 . \tag{7}$$

To see why this is true, note that $-(\omega - a^*)^2 \geqslant -(\omega - a)^2$ for all $a \in D$ if and only if $-((1-\omega)-(1-a^*))^2 \geqslant -((1-\omega)-(1-a))^2$ for all $a \in D$, that is, if and only if $-((1-\omega)-(1-a^*))^2 \geqslant -((1-\omega)-a)^2$ for all $a \in M(D)$ — and that $1-a^* \in M(D)$.

Step 2. We let $\widetilde{\Phi}(D,e) = \int_{-\beta e}^{1-\alpha e} -(\omega - a^*(\omega,D))^2\, d\omega$ and show that $\widetilde{\Phi}(M(D),e) = \Phi(D,e)$. To see why, note that

$$\begin{aligned}
\widetilde{\Phi}(M(D),e) &= \int_{-\beta e}^{1-\alpha e} -(\omega - a^*(\omega, M(D)))^2\, d\omega \\
&= \int_{-(1-\alpha e)}^{\beta e} -(-\omega - a^*(-\omega, M(D)))^2\, d\omega = \int_{-(1-\alpha e)}^{\beta e} \max_{a\in M(D)} -(-\omega - a)^2\, d\omega \\
&= \int_{\alpha e}^{1+\beta e} \max_{a\in M(D)} -(-(\omega-1)-a)^2\, d\omega = \int_{\alpha e}^{1+\beta e} \max_{a\in M(D)} -(\omega-(1-a))^2\, d\omega \\
&= \int_{\alpha e}^{1+\beta e} \max_{a\in D} -(\omega-a)^2\, d\omega = \Phi(D,e).
\end{aligned}$$

Step 3. We let and $\widetilde{\psi}(e) = 1-\alpha e - (-\beta e) = \psi(e)$. Thus, when $\omega \in [-\beta e, 1-\alpha e]$, an agent who faces delegation set $M(D)$ and exerts effort level e obtains a payoff $\widetilde{\Phi}(M(D),e)/\widetilde{\psi}(e) - c(e)$. We show that D induces an effort level e' when $\omega \in [\alpha e, 1+\beta e]$ if and only if $M(D)$ induces effort level e' when $\omega \in [-\beta e, 1-\alpha e]$. To see why, note that

$$\begin{gathered}
\frac{\Phi(D,e')}{\psi(e')} - c(e') \geqslant \frac{\Phi(D,e)}{\psi(e)} - c(e) \quad \forall e \quad \text{if and only if} \\
\frac{\widetilde{\Phi}(M(D),e')}{\widetilde{\psi}(e')} - c(e') \geqslant \frac{\widetilde{\Phi}(M(D),e)}{\widetilde{\psi}(e)} - c(e) \;\; \forall e.
\end{gathered}$$

Step 4. Finally, assume towards a contradiction that $M(D)$ does not solve the principal's problem to induce effort level e' when the support for an arbitrary effort level e is $[-\beta e, 1-\alpha e]$. Then there exists a delegation set $\widehat{D}$ that induces effort level e' and with $\widetilde{\Phi}(\widehat{D},e')/\widetilde{\psi}(e') > \widetilde{\Phi}(M(D),e')/\widetilde{\psi}(e')$. But then, by step 2,

$$\frac{\Phi(M(\widehat{D}),e')}{\psi(e')} = \frac{\widetilde{\Phi}(\widehat{D},e')}{\widetilde{\psi}(e')} > \frac{\widetilde{\Phi}(M(D),e')}{\widetilde{\psi}(e')} = \frac{\Phi(D,e')}{\psi(e')}.$$

We know from step 3 that $M(\widehat{D})$ induces effort level e'. This, together with the inequality above, contradicts that D is the optimal delegation set to induce effort level e' and the support is $[\alpha e, 1+\beta e]$. ■

A.2 Proof of Proposition 1

Fix $e > 0$. We divide the proof in three parts. We first show that the delegation set from Proposition 1 solves the relaxed problem for the case with $\beta - \alpha \leqslant 0$ (part A) and for the case $\beta - \alpha > 0$ (part B). In part C we show that the solution to the relaxed problem also solves the original problem, for arbitrary $(\alpha, \beta) \geqslant (0, 0)$.

Part A. Case $\beta - \alpha \leqslant 0$. We first show that we can restrict attention to minimal floors: delegation sets of the form $D_a = [a, \max\{a, 1 + \beta e\}]$ for some $a > \alpha e$. The proof is by contradiction. Assume that the delegation set D satisfies the first order condition and is not a floor. We show next that D is dominated by a floor (in the sense that the floor also satisfies the first order condition and yields a strictly higher payoff to the principal). Define $\tilde{a} \equiv \alpha e + |a^*(\alpha e, D) - \alpha e|$ and consider the corresponding floor $D_{\tilde{a}} = [\tilde{a}, \max\{\tilde{a}, 1 + \beta e\}]$. For any state $\omega \in \Gamma(e)$, the agent obtains a weakly larger payoff under $D_{\tilde{a}}$ than under D. Moreover, this gain is strict for a set of states with positive mass. Thus, $\Phi(D_{\tilde{a}}, e) > \Phi(D, e)$ and $\Phi'(D_{\tilde{a}}, e) \geqslant \Phi'(D, e)$. If $\beta - \alpha = 0$, then $\partial U_A(D_{\tilde{a}}, e) / \partial e = \partial U_A(D, e)/\partial e = 0$. Thus, $D_{\tilde{a}}$ dominates D and this concludes the proof.

If instead that $\beta - \alpha < 0$, then $\partial U_A(D_{\tilde{a}}, e) / \partial e > \partial U_A(D, e)/\partial e = 0$. Build D' by adding actions to the left of $D_{\tilde{a}}$ until $\partial U_A(D', e) / \partial e = 0$. This is always feasible as $\partial U_A(D, e) / \partial e = -c'(e) < 0$ for any $D \supseteq \Gamma(e)$. Thus, the set D' satisfies the first order condition: $\partial U_A(D', e) / \partial e = 0$ and leads to a strictly larger payoff: $\Phi(D', e) > \Phi(D_{\tilde{a}}, e) > \Phi(D, e)$. Thus, D' dominates D.

We thus restrict attention to delegation sets of the form $D_a = [a, \max\{a, 1 + \beta e\}]$, with $a > \alpha e$. Under D_a,

$$\frac{\partial U_A(D_a, e)}{\partial e} = \begin{cases} \frac{(a-\alpha e)^2}{\psi(e)} \left(\alpha - \frac{1}{3}\frac{\alpha - \beta}{\psi(e)}(a - \alpha e)\right) - c'(e) & \text{if } a \leqslant 1 + \beta e \\ -\frac{\alpha + 2\beta}{3}\psi(e) + (\alpha + \beta)(a - \alpha e) - c'(e) & \text{if } a > 1 + \beta e \end{cases} \tag{8}$$

The expression in (8) is continuous and strictly increasing in a. Moreover, it is negative when $a = \alpha e$ and positive for a sufficiently large. Thus, there exits a unique $\underline{a}$ such that $\partial U_A(D_{\underline{a}}, e)/\partial e = 0$. The delegation set $D_{\underline{a}}$ solves the relaxed problem. Finally, if $\partial U_A(\{1+\beta e\},e)/\partial e = (2\alpha+\beta)/3\psi(e) - c'(e) \geqslant 0$, then $\underline{a} \leqslant 1+\beta e$. Otherwise, $\underline{a} > 1+\beta e$. ■

B. Case $\beta - \alpha > 0$. We show that when $\partial U_A(\{1+\beta e\},e)/\partial e \geqslant 0$, the solution to the relaxed problem is as in case 1 of Proposition 1. When instead the inequality is reversed, the solution is as in case 2 of Proposition 1.

Step 1. Assume that $\partial U_A(\{1+\beta e\},e)/\partial e \geqslant 0$. There exits a unique $\underline{a} \in (\alpha e, 1+\beta e]$ such that $\partial U_A([\underline{a},1+\beta e],e)/\partial e = 0$ — see the discussion after equation (8). We show next that $[\underline{a},1+\beta e]$ dominates any other delegation set D with $\partial U_A(D,e)/\partial e = 0$. To see this, note that

$$-\beta\left(1+\beta e - a^*\left(1+\beta e,[\underline{a},1+\beta e]\right)\right)^2 = 0 \geqslant -\beta\left(1+\beta e - a^*\left(1+\beta e, D\right)\right)^2.$$

Next, if

$$\alpha\left(\alpha e - a^*\left(\alpha e,[\underline{a},1+\beta e]\right)\right)^2 > \alpha\left(\alpha e - a^*\left(\alpha e, D\right)\right)^2$$

then $\Phi'([\underline{a},1+\beta e],e) > \Phi'(D,e)$ and as $\partial U_A([\underline{a},1+\beta e],e)/\partial e = \partial U_A(D,e)/\partial e = 0$, then $\Phi([\underline{a},1+\beta e],e) > \Phi(D,e)$. If instead

$$\alpha\left(\alpha e - a^*\left(\alpha e,[\underline{a},1+\beta e]\right)\right)^2 \leqslant \alpha\left(\alpha e - a^*\left(\alpha e, D\right)\right)^2,$$

then $\Phi([\underline{a},1+\beta e],e) > \Phi(D,e)$. Indeed, the set $[\underline{a},1+\beta e]$ is better for any $\omega \in \Gamma(e)$. Thus, $[\underline{a},1+\beta e]$ is the unique solution to the relaxed problem. ■

Step 2. Assume next that $\partial U_A(\{1+\beta e\},e)/\partial e < 0$. For any delegation set D with $D \cap \Gamma(e) \neq \varnothing$ then

$$\frac{\partial U_A(D,e)}{\partial e} \leqslant \frac{\partial U_A(\{1+\beta e\},e)}{\partial e} < 0$$

To see why the weak inequality holds, note that if $D \cap \Gamma(e) \neq \varnothing$, then 1) $\Phi(D,e) \geqslant \Phi(\{1+\beta e\},e)$ and 2) $\Phi'(D,e) \leqslant \Phi'(\{1+\beta e\},e)$.

It is then without loss to restrict attention to delegation sets D with $D \cap \Gamma(e) = \varnothing$. As D is minimal, then D is either a singleton or contains exactly two points. By contradiction, assume that the solution to the relaxed problem is $D = \{\alpha e - \Delta_1, 1+\beta e + \Delta_2\}$, with $\Delta_1 >$

$0, \Delta_2 > 0$ and

$$\alpha e < \frac{\alpha e - \Delta_1 + 1 + \beta e + \Delta_2}{2} < 1 + \beta e.$$

Note that $\partial U_A\left(\{1+\beta e+\Delta_2\}, e\right)/\partial e > \partial U_A\left(D, e\right)/\partial e = 0$. Then, there exists $\Delta' \in (0, \Delta_2)$ such that $\partial U_A(D', e)/\partial e = 0$ for $D' \equiv \{1+\beta e+\Delta'\}$. As both D' and D satisfy the first order condition and $\Phi(D, e) \geqslant \Phi(D', e)$, then $\Phi'(D', e) \leqslant \Phi'\left(D, e\right)$. That is,

$$\alpha\left(\psi(e)+\Delta'\right)^2 - \beta(\Delta')^2 \leqslant \alpha\left(\Delta_1\right)^2 - \beta\left(\Delta_2\right)^2,$$

which in turn implies that $\psi(e) + \Delta' \leqslant \Delta_1$. But then for every state $\omega \in \Gamma(e)$, the delegation set D' yields a higher payoff to the principal than the delegation set D. We have thus reached a contradiction.

We conclude that the solution to the relaxed problem must be a singleton $\{a\}$ with $a > 1 + \beta e$. The first order condition and equation (8) pin down the value of a, which is as in equation (2). ■

Part C. We finally show that the solution to the relaxed problem also solves the original problem. In doing so, we also prove the optimality of non-minimal floors.

Consider first a generic non-minimal floor $[a, +\infty)$ with $a \in \mathbb{R}$. The agent's payoff under $[a, +\infty)$ is concave in effort. In fact,

$$\frac{\partial U_A^2([a,+\infty), e')}{\partial (e')^2} = \begin{cases} -\frac{2}{3}\left(\alpha^2+\beta^2+\alpha\beta\right) - c''(e') & \text{if } 1+\beta e' < a \\ -c''(e') & \text{if } \alpha e' \geqslant a \end{cases}$$

In the remaining case of $\alpha e' < a \leqslant 1 + \beta e'$,

$$\frac{\partial U_A^2([a,+\infty), e')}{\partial (e')^2} = \overbrace{-\frac{2\left(a-\alpha e'\right)}{1+(\beta-\alpha)e'}}^{<0} \times \left[\alpha^2 + \alpha(\beta-\alpha)\frac{a-\alpha e'}{1+(\beta-\alpha)e'} + \frac{1}{3}(\beta-\alpha)^2\frac{\left(a-\alpha e'\right)^2}{\left(1+(\beta-\alpha)e'\right)^2}\right] - c''(e')$$

$$
\begin{aligned}
&< -\frac{2\left(a-\alpha e'\right)}{1+(\beta-\alpha)e'}\left[\alpha^2+\alpha(\beta-\alpha)\frac{a-\alpha e'}{1+(\beta-\alpha)e'}\right] \\
&= -\frac{2\left(a-\alpha e'\right)}{1+(\beta-\alpha)e'}\left[\alpha^2\left(1-\frac{a-\alpha e'}{1+(\beta-\alpha)e'}\right)+\alpha\beta\frac{a-\alpha e'}{1+(\beta-\alpha)e'}\right] < 0
\end{aligned}
$$

where the last inequality follows from $\frac{a-\alpha e'}{1+(\beta-\alpha)e'} \in (0,1]$.

Next, fix e and find $\underline{a} \in \mathbb{R}$ from parts A and B so that $D_{\underline{a}}$ is the unique solution to the relaxed problem. Extend $D_{\underline{a}}$ to build the non-minimal floor $[\underline{a}, +\infty)$ and note that

$$
\frac{\partial U_A([\underline{a},+\infty),e)}{\partial e} = \frac{\partial U_A(D_{\underline{a}},e)}{\partial e} = 0.
$$

This, together with the concavity of the agent's problem under non-minimal floors, imply that e is the agent's optimal effort level under $[\underline{a}, +\infty)$.[26]

Finally, the agent is indifferent between $[\underline{a}, +\infty)$ and $D_{\underline{a}}$ when the effort level is e and weakly prefers $[\underline{a}, +\infty)$ to $D_{\underline{a}}$ for any other level — since $D_{\underline{a}} \subset [\underline{a}, +\infty)$. This implies that, also under $D_{\underline{a}}$, the agent's optimal level of effort is e. Thus, $D_{\underline{a}}$ also solves the original problem. ■

A.3 Proof of Proposition 2

We present here the proof for the case with $\alpha+\beta \geqslant 0$. The proof for the case $\alpha+\beta < 0$ follows from the proof in this section and Lemma 1.

We first show (in step 1) that it is without loss of generality to restrict attention to convex sets. In step 2 we show that if the solution to the relaxed problem is not a singleton, then it must be of the form $[\alpha e+\Delta, 1+\beta e+(\beta/\alpha)\Delta)]$. We characterize the solution for effort levels $e \geqslant \tilde{e}$ in step 3 and for effort levels $e < \tilde{e}$ in step 4. Finally, in step 5 we show that the solution to the relaxed problem also solves the original problem.

Step 1. We show in this step that for any arbitrary (minimal) delegation set D which is not convex and satisfies $\partial U_A(D,e)/\partial e = 0$, there exists a (minimal) convex set D' with $\partial U_A(D',e)/\partial e = 0$ and $\Phi(D',e) > \Phi(D,e)$. Let $\Delta_1 \equiv |\alpha e - a^*(\alpha e, D)|$ and $\Delta_2 \equiv |1+\beta e - a^*(1+\beta e, D)|$.

[26] This shows that the non-minimal floor $[\underline{a}, +\infty)$ solves the original problem.

First consider the case with $\Delta_1 + \Delta_2 \leqslant \psi(e)$.[27] Build the alternative delegation set $\widetilde{D} = [\alpha e + \Delta_1, 1 + \beta e - \Delta_2]$. By construction, $\Phi'(\widetilde{D}, e) = \Phi'(D, e)$. Moreover, as D is not convex, then $\Phi(\widetilde{D}, e) > \Phi(D, e)$, so $\partial U_A(\widetilde{D}, e)/\partial e > \partial U_A(D, e)/\partial e = 0$. Next, build D' by adding actions to the left and to the right of $\widetilde{D}$ in a continuous manner until $\partial U_A(D', e)/\partial e = 0$. This is always feasible as $\partial U_A(D, e)/\partial e = -c'(e) < 0$ for any $D \supseteq \Gamma(e)$.

Next consider the case with $\Delta_1 + \Delta_2 > \psi(e)$. As D is not convex, then it contains exactly two points, outside of the support: $D = \{\alpha e - \Delta_1, 1 + \beta e + \Delta_2\}$ with $\Delta_1 > 0$ and $\Delta_2 > 0$. Moreover, $\alpha e < (1 + \beta e + \alpha e + \Delta_2 - \Delta_1)/2 < 1 + \beta e$, i.e. both actions are chosen. We divide the remainder of step 1 into three sub-steps, 1a, 1b and 1c.

Step 1a. Assume that $\Delta_1 \geqslant \Delta_2$ and that $\Delta_1 \leqslant \psi(e)$. Let $\widetilde{D} = [\alpha e + \Delta_1, 1 + \beta e]$, which is feasible because $\Delta_1 \leqslant \psi(e)$. For any realized state $\omega \in \Gamma(e)$, the agent's payoff under $\widetilde{D}$ is higher than under D. We next show that $\widetilde{D}$ also provides higher incentives:

$$\begin{aligned}\frac{\partial U_A(\widetilde{D}, e)}{\partial e} - \frac{\partial U_A(D, e)}{\partial e} &= \frac{1}{\psi(e)}\left[-(\beta - \alpha)\left[\frac{\Phi(\widetilde{D}, e)}{\psi(e)} - \frac{\Phi(D, e)}{\psi(e)}\right] + \beta\Delta_2^2\right] \\ &> \frac{1}{\psi(e)}\left[-(\beta - \alpha)\frac{\Delta_2^2}{2} + \beta\Delta_2^2\right] \geqslant 0\end{aligned}$$

To see why the strict inequality holds, note that for any $\omega \in \left[\frac{1+\beta e+\Delta_2+\alpha e-\Delta_1}{2}, 1 + \beta e\right]$, the agent's payoff under $\widetilde{D}$ is larger than under D by at least Δ_2^2 and also that $\frac{1+\beta e+\Delta_2+\alpha e-\Delta_1}{2} \leqslant \frac{1+\beta e+\alpha e}{2}$. The weak inequality holds because $\alpha + \beta \geqslant 0$. Finally, as before, build D' by extending the set $\widetilde{D}$ to the left until the first order condition holds. ■

Step 1b. Assume that $\Delta_1 \geqslant \Delta_2$ and that $\Delta_1 > \psi(e)$. Let $\widetilde{D} = \{\alpha e + \Delta_1\}$. Note that (i) $1 + \beta e < \alpha e + \Delta_1$, since $\Delta_1 > \psi(e)$ and that (ii) $\alpha e + \Delta_1 < 1 + \beta e + \Delta_2$, since D is minimal. As in step 1a, for any realized state $\omega \in \Gamma(e)$, the agent's payoff under $\widetilde{D}$ is higher than under D. We next show that $\widetilde{D}$ also provides higher incentives:

$$\frac{\partial U_A(\widetilde{D}, e)}{\partial e} - \frac{\partial U_A(D, e)}{\partial e} = \frac{1}{\psi(e)}\left[-(\beta - \alpha)\left[\frac{\Phi(\widetilde{D}, e)}{\psi(e)} - \frac{\Phi(D, e)}{\psi(e)}\right] + \beta\Delta_2^2 - \beta\left(\Delta_1 - \psi(e)\right)^2\right]$$

[27] Note that this is always the case if $D \cap \Gamma(e) \neq \varnothing$.

$$> \frac{1}{\psi(e)}\left[-(\beta-\alpha)\frac{1}{2}+\beta\right]\left[\Delta_2^2-(\Delta_1-\psi(e))^2\right] \geqslant 0$$

Now, for any $\omega \in \left[\frac{1+\beta e+\Delta_2+\alpha e-\Delta_1}{2}, 1+\beta e\right]$, the agent's payoff under $\widetilde{D}$ is larger than under D by at least $\left[\Delta_2^2-(\Delta_1-\psi(e))^2\right]$. This, together with $\frac{1+\beta e+\Delta_2+\alpha e-\Delta_1}{2} \leqslant \frac{1+\beta e+\alpha e}{2}$, imply the strict inequality. Again, build D' by extending the set $\widetilde{D}$ to the left until the first order condition holds. ■

Step 1c. Finally, assume that $\Delta_1 < \Delta_2$. Let $\widetilde{D} = \{\alpha e - \Delta_2, 1+\beta e+\Delta_1\}$. Notice that $\Phi(\widetilde{D}, e) = \Phi(D, e)$ and that $\Phi'(\widetilde{D}, e) \geqslant \Phi'(D, e)$. Thus, $\partial U_A(\widetilde{D}, e)/\partial e \geqslant \partial U_A(D, e)/\partial e = 0$. Consider the family of delegation sets $D_\eta = \{\alpha e - \eta\Delta_2, 1+\beta+\eta\Delta_1\}$ indexed by $\eta \in [0,1]$. Note that $\Phi(D_\eta, e) \geqslant \Phi(D, e)$ for any $\eta \in [0,1]$ (with strict inequality if $\eta < 1$). If there exists $\tilde{\eta} \in (0,1]$ such that $\partial U_A(D_{\tilde{\eta}}, e)/\partial e = 0$, then apply either step 1a or step 1b to $D_{\tilde{\eta}}$.[28] Otherwise, build $\widetilde{\widetilde{D}}$ by continuously adding actions that lie within the support to $D_0 = \{\alpha e, 1+\beta e\}$ until $\partial U_A(\widetilde{\widetilde{D}}, e)/\partial e = 0$. Note that $\widetilde{\widetilde{D}} \cap \Gamma(e) \neq \varnothing$, so finally apply to $\widetilde{\widetilde{D}}$ the arguments from the case with $D \cap \Gamma(e) \neq \varnothing$. ■

Step 2. Any convex set within the support is of the form $[\alpha e + \Delta_1, 1+\beta e - \Delta_2]$ with $\Delta_1 \geqslant 0$, $\Delta_2 \geqslant 0$ and $\Delta_1 + \Delta_2 \leqslant \psi(e)$. Let

$$\mathcal{S}(e) = \left\{ \begin{array}{ll} (\Delta_1, \Delta_2): & \Delta_1 \geqslant 0, \Delta_2 \geqslant 0, \Delta_1+\Delta_2 \leqslant \psi(e) \text{ and} \\ & \partial U_A\left([\alpha e+\Delta_1, 1+\beta e-\Delta_2], e\right)/\partial e = 0. \end{array} \right\}$$

The set $\mathcal{S}(e)$ indexes convex delegation sets that lie within the support and satisfy the first order condition. Whenever $S(e)$ is non-empty, the solution to the relaxed problem lies within the support.[29] In the remainder of this step we assume that $S(e)$ is non-empty.

The set $S(e)$ is closed (as it includes singletons) and $\Phi\left([\alpha e+\Delta_1, 1+\beta e-\Delta_2], e\right)$ is continuous in (Δ_1, Δ_2). Thus, $\Phi\left([\alpha e+\Delta_1, 1+\beta e-\Delta_2], e\right)$ achieves a maximum in $S(e)$. In what follows we show that if $(\Delta_1, \Delta_2) \in S(e)$, $\Delta_1+\Delta_2 < \psi(e)$ and $\Delta_2 \neq (-\beta/\alpha)\Delta_1$,

[28] Note that $\tilde{\eta} = 1$ if and only if $\alpha + \beta = 0$.

[29] Any delegation set within the support dominates a singleton outside of the support.

then

$$(\Delta_1, \Delta_2) \notin \underset{(\Delta_1', \Delta_2') \in S(e)}{\text{argmax}} \; \Phi\left(\left[\alpha e + \Delta_1', 1 + \beta e - \Delta_2'\right], e\right).$$

The proof is by contradiction. We show this for $\Delta_2 > (-\beta/\alpha)\Delta_1$ and $\Delta_1 > 0$. We omit the details for the remaining cases since they are similar.

For any set $[\alpha e + \Delta_1, 1 + \beta e - \Delta_2]$ with $(\Delta_1, \Delta_2) \in S(e)$,

$$\Phi\left([\alpha e + \Delta_1, 1 + \beta e - \Delta_2], e\right) = -\frac{1}{3}\left[(\Delta_1)^3 + (\Delta_2)^3\right]$$
$$\Phi'\left([\alpha e + \Delta_1, 1 + \beta e - \Delta_2], e\right) = -\beta(\Delta_2)^2 + \alpha(\Delta_1)^2.$$

Increase Δ_1 (marginally) and decrease Δ_2 (marginally) to keep Φ' constant. The resulting marginal variation in Φ is equal to $-\Delta_1\left(\Delta_1 + (\alpha/\beta)\Delta_2\right)$. Since $\Delta_2 > (-\beta/\alpha)\Delta_1$, this variation is strictly positive. Therefore, there exists a set $D' = [\alpha e + \Delta_1', 1 + \beta e - \Delta_2']$ with $\Phi(D', e) > \Phi([\alpha e + \Delta_1, 1 + \beta e - \Delta_2], e)$ and $\partial U_A(D', e)/\partial e > 0$. Build another set $\widetilde{D}$ by extending D' continuously until $\partial U_A(\widetilde{D}, e)/\partial e = 0$.

Step 2 implies that the solution to the relaxed problem is either (i) a singleton or (ii) a delegation set $[\alpha e + \Delta_1, 1 + \beta e - \Delta_2]$ with $(\Delta_1, \Delta_2) \in S(e)$ and $\Delta_2 = -\frac{\beta}{\alpha}\Delta_1$. We next study the family of such delegation sets.

Step 3. We show that if $e \leqslant \tilde{e}$, there exists a unique delegation set $[\alpha e + \Delta_1, 1 + \beta e - \Delta_2]$ with $(\Delta_1, \Delta_2) \in S(e)$ and $\Delta_2 = -\frac{\beta}{\alpha}\Delta_1$. Furthermore, we show that if $e > \tilde{e}$ there exists no such set.

Any delegation set $[\alpha e + \Delta_1, 1 + \beta e - \Delta_2]$ with $(\Delta_1, \Delta_2) \in S(e)$ and $\Delta_2 = -\frac{\beta}{\alpha}\Delta_1$ is of the form $D_\Delta = [\alpha e + \Delta, 1 + \beta e + (\beta/\alpha)\Delta]$ with $\Delta \in [0, \alpha(1/(\alpha - \beta) - e)]$. The first order condition $\partial U_A(D_\Delta, e)/\partial e = 0$ holds if and only if equation (5) from Proposition 2 holds, which we rewrite for convenience:

$$\Delta^2\left[\alpha\psi(e) - \frac{1}{3}(\alpha - \beta)\Delta\right] = c'(e)(\psi(e))^2 \frac{\alpha^3}{\alpha^3 - \beta^3}$$

The left hand side of this equality strictly increases with Δ in the interval $[0, \alpha(1/(\alpha - \beta) - e)]$. At $\Delta = 0$, the left hand side is lower than the right hand side. At $\Delta =$

$\alpha(1/(\alpha-\beta)-e)$, the left hand side is larger than the right hand side if and only if $(2/3)\left(\alpha^3-\beta^3\right)/(\alpha-\beta)^2 \geqslant c'(e)/\psi(e)$. The previous expression holds with equality when $e=\tilde{e}$, by the definition of $\tilde{e}$. Moreover, $c'(e)/\psi(e)$ strictly increases with e under shrinking support. Then, there exists a (unique) Δ such that $\partial U_A(D_\Delta, e)/\partial e = 0$ holds if and only if $e \leqslant \tilde{e}$.[30]

An immediate consequence of this step is that when $e \geqslant \tilde{e}$, the solution to the relaxed problem is a singleton. The expression for the singleton in Proposition 2 results directly from the first order condition.

Step 4. We now show that whenever $e < \tilde{e}$, the solution to the relaxed problem is the delegation set $D(e)$ from part 1 of Proposition 2. To do this, it is enough to show that a singleton cannot solve the relaxed problem whenever $e < \tilde{e}$.

Consider a singleton $\{x\}$ with $\partial U_A(\{x\}, e)/\partial e = 0$. If $x \geqslant \alpha/(\alpha-\beta)$, then $D(e)$ strictly dominates $\{x\}$, as $\alpha/(\alpha-\beta) \in D(e)$ and $\alpha/(\alpha-\beta) > (1+\beta e+\alpha e)/2$. Next, we show that also if $x < (1+\beta e+\alpha e)/2$, then $\{x\}$ cannot solve the relaxed problem. To see this, consider the alternative singleton set $\{1+\beta e-(x-\alpha e)\} = \{\psi(e)-x\}$. Note that $\Phi'(\{\psi(e)-x\}, e) > \Phi'(\{x\}, e)$ and also that $\Phi(\{\psi(e)-x\}, e) = \Phi(\{x\}, e)$. Thus $\partial U_A\left(\{\psi(e)-x\}, e\right)/\partial e > 0$. Then, extend the set to the left and right until the first order condition holds.

The remaining case has $(1+\beta e+\alpha e)/2 \leqslant x < \alpha/(\alpha-\beta)$. Let $\Delta_1 = x-\alpha e$ and $\Delta_2 = 1+\beta e-x$. Note that $\{x\} = [\alpha e+\Delta_1, 1+\beta e-\Delta_2]$ and that $(-\beta/\alpha)\Delta_1 < \Delta_2 < \Delta_1$. Now, as in step 2, increase (marginally) Δ_1 and decrease (marginally) Δ_2 to keep Φ' constant. By the implicit function theorem, $\partial\Delta_2/\partial\Delta_1 = (\alpha/\beta)(\Delta_1/\Delta_2)$. Note that $|\partial\Delta_2/\partial\Delta_1| > 1$ as $\Delta_1 > \Delta_2$. The increase in Δ_1 and decrease in Δ_2 results in a new delegation set that has positive mass. Moreover, as $\Delta_2 > (-\beta/\alpha)\Delta_1$, the resulting delegation set also yields a larger payoff (the argument is as in step 2). Thus, by extending the resulting delegation set to the left and right, we again obtain a new delegation set that satisfies the first order condition and yields a larger payoff than $\{x\}$. ■

Step 5. Let $e < \tilde{e}$. Fix a set $D = [\alpha e+\Delta, 1+\beta e+(\beta/\alpha)\Delta]$. Pick $\check{e} > e$ such that

[30] Note that $\Delta = \alpha(1/(\alpha-\beta)-\tilde{e})$ when the effort level is $\tilde{e}$, so the delegation D_Δ that satisfies the first order condition is the singleton $\{\alpha/(\alpha-\beta)\}$.

$\alpha\check{e} = \alpha e + \Delta$. Note that $\Gamma(\check{e}) = D$. For any $e' \geqslant \check{e}$, then $U_A(D,e') = -c(e')$. Then, the agent's payoff is strictly concave. Next, for effort levels $e' < \check{e}$,

$$\frac{\partial U_A^2(D,e')}{\partial (e')^2} = -2\underbrace{(\check{e}-e')}_{>0}\underbrace{\frac{(\alpha^3-\beta^3)}{(\psi(e'))^2}}_{>0}\left[\underbrace{\psi(\check{e})}_{>0}+\underbrace{\frac{1}{3}(\alpha-\beta)^2\frac{(\check{e}-e')^2}{\psi(e')}}_{>0}\right]-c''(e')<0$$

The agent's problem is also concave when the optimal delegation set is a singleton. The argument is the same as under upwards support. ■

A.4 Proof of Proposition 3

We present here the proof for the case with $\alpha+\beta>0$. The proof for the case $\alpha+\beta<0$ follows from the proof in this section and Lemma 1.

Step 1. In this step we prove part 1 of Proposition 3. The payoff $\Phi(D,e)/\psi(e)$ from any delegation set D that satisfies the first order condition is bounded above:

$$\frac{\Phi(D,e)}{\psi(e)} = -\frac{\psi(e)c'(e)}{\beta-\alpha} - \underbrace{\frac{\beta\left(1+\beta e-a^*(1+\beta e,D)\right)^2-\alpha\left(\alpha e-a^*(\alpha e,D)\right)^2}{\beta-\alpha}}_{\geqslant 0} \leqslant -\frac{\psi(e)c'(e)}{\beta-\alpha}$$

This upper bound is tight if and only if $\{\alpha e, 1+\beta e\}\subseteq D$. So the principal can achieve this upper bound only when she can build a delegation set D that satisfies the first order condition and has $\{\alpha e, 1+\beta e\}\subseteq D$. This is feasible whenever

$$\frac{\partial U_A\left(\{\alpha e,1+\beta e\},e\right)}{\partial e} = \frac{\beta-\alpha}{12}\psi(e)-c'(e)\geqslant 0.$$

Under this condition, the principal can build D by starting with $\{\alpha e, 1+\beta e\}$ and adding actions within the support until $\partial U_A(D,e)/\partial e = 0$. Any such set D solves the relaxed problem.

A simple way of building a solution to the relaxed problem is to consider delegation sets of the form $D = [\alpha e, \alpha e+\Delta_1]\cup[1+\beta e-\Delta_2, 1+\beta e]$ with $\Delta_1>0$, $\Delta_2>0$ and $\Delta_1+\Delta_2<\psi(e)$. Notice that when $\Delta_1=\Delta_2=0$, then $\partial U_A(D,e)/\partial e\geqslant 0$. When instead

$\Delta_1 + \Delta_2 = \psi(e)$, then $\partial U_A(D,e)/\partial e = c'(e) < 0$. Thus, there exists a set D of this form with $\partial U_A(D,e)/\partial e = 0$. ■

Step 2. We show here that if $\partial U_A(\{\alpha e, 1+\beta e\}, e)/\partial e = (\beta - \alpha)\,\psi(e)/12 - c'(e) < 0$ and D solves the relaxed problem, then D must be of the form $D = \{\alpha e - \Delta_1, 1 + \beta e + \Delta_2\}$ with $0 < \Delta_2 < \Delta_1 < \psi(e) + \Delta_2$.

We show first that if D solves the relaxed problem, then $D \cap (\alpha e, 1+\beta e) = \varnothing$. To see why, assume towards a contradiction that $D \cap (\alpha e, 1+\beta e) \neq \varnothing$ and build the alternative set $\widetilde{D} = \{\alpha e - |a^*(\alpha e, D) - \alpha e|\,, 1 + \beta e + |a^*(1+\beta e, D) - (1+\beta e)|\}$. Note that $\Phi'(\widetilde{D}, e) = \Phi'(D,e)$ and $\Phi(\widetilde{D}, e) < \Phi(D,e)$, thus $\partial U_A(\widetilde{D}, e)/\partial e > 0$. Next, move both actions in $\widetilde{D}$ towards the support until the first order condition is satisfied. The resulting delegation set D' has $\Phi'(D', e) > \Phi'(\widetilde{D}, e) = \Phi'(D,e)$. This, together with $\partial U_A(D',e)/\partial e = \partial U_A(D,e)/\partial e = 0$, imply that $\Phi(D',e) > \Phi(D,e)$.

Since it cannot contain any actions strictly within the support, a delegation set that solves the relaxed problem must either be a singleton, or contain exactly two points. We show next that it cannot be a singleton.

Assume towards a contradiction that $D = \{1 + \beta e + \Delta\}$, with $\Delta \geqslant 0$, solves the relaxed problem. (The case $D = \{\alpha e - \Delta\}$ with $\Delta \geqslant 0$ follows the same logic, so we omit it.) Build the (non-minimal) set $D_0 = \{\alpha e - (\psi(e) + \Delta)\,, 1 + \beta e + \Delta\}$. Note that D_0 induces the same payoff as D and also satisfies the first order condition. Consider next the family of delegation sets of the form $D_\varepsilon = \{\alpha e - (\psi(e) + \Delta) + \varepsilon, 1 + \beta e + \Delta\}$ with $\varepsilon \geqslant 0$ small. Consider a small variation in ε around $\varepsilon = 0$:

$$\left.\frac{\partial \frac{\partial U_A(D(\varepsilon),e)}{\partial e}}{\partial \varepsilon}\right|_{\varepsilon=0} = -2\alpha \frac{\psi(e) + \Delta}{\psi(e)} > 0$$

Thus, for some small $\tilde{\varepsilon} > 0$ we have $\partial U_A(D_{\tilde{\varepsilon}}, e)/\partial e > \partial U_A(D,e)/\partial e = 0$. Moreover, $\Phi(D_{\tilde{\varepsilon}}, e) > \Phi(D,e)$. By further moving both actions in $D_{\tilde{\varepsilon}}$ towards the support, the first order condition is eventually satisfied, with a delegation set that yields to the principal a payoff larger than that from D. We have thus reached a contradiction.

A solution to the relaxed problem must then be of the form $D = \{\alpha e - \Delta_1, 1 + \beta e + \Delta_2\}$ with $\Delta_1 \geqslant 0$, $\Delta_2 \geqslant 0$ and $|\Delta_1 - \Delta_2| < \psi(e)$. Moreover, since $\partial U_A(\{\alpha e, 1+\beta e\}, e)/\partial e < 0$, then at least one of Δ_1 and Δ_2 must be strictly positive. In the final part of this step we show that $\Delta_1 > \Delta_2 > 0$.

Assume towards a contradiction that $\Delta_2 \geqslant \Delta_1$. If $\Delta_2 > \Delta_1$, build the alternative delegation set $\widetilde{D} = \{\alpha e - \Delta_2, 1 + \beta e + \Delta_1\}$. Note that $\Phi(\widetilde{D}, e) = \Phi(D, e)$ and, since $\alpha + \beta > 0$, then also $\Phi'(\widetilde{D}, e) > \Phi'(D, e)$. Thus, $\partial U_A(\widetilde{D}, e)/\partial e > \partial U_A(D, e)/\partial e = 0$. If instead $\Delta_2 = \Delta_1$, then build $\widetilde{D}$ by increasing Δ_1 and decreasing Δ_2 (marginally in both cases) to keep Φ constant. As a result, Φ' increases and thus also in this case $\partial U_A(\widetilde{D}, e)/\partial e > \partial U_A(D, e)/\partial e = 0$. Then, proceed exactly as above — as with $D_{\tilde{\varepsilon}}$ — to reach a contradiction.

Finally, assume towards a contradiction that $\Delta_2 = 0$. Since $\partial U_A(\{\alpha e, 1 + \beta e\}, e)/\partial e < 0$, then $\Delta_1 > 0$. Increase Δ_2 and decrease Δ_1 (marginally in both cases) to keep Φ constant. The resulting marginal variation in Φ' is equal to

$$-2\alpha\Delta_1 \left(\frac{-\frac{1}{4}\left(\psi(e) + \Delta_1\right)^2}{-\frac{1}{4}\left(\psi(e) + \Delta_1\right)^2 + \Delta_1^2} \right) > 0.$$

We have thus built an alternative delegation set $\widetilde{D}$ that yields the same payoff as the original set and has $\partial U_A(\widetilde{D}, e)/\partial e > 0$. We proceed again as above to reach a contradiction. ■

Step 3. In this step we show that a solution to the relaxed problem exists and is unique. We also characterize it. Fix K large and let

$$\mathcal{S}(e) = \left\{ \begin{array}{ll} (\Delta_1, \Delta_2) : & 0 \leqslant \Delta_2 \leqslant \Delta_1 \leqslant \psi(e) + \Delta_2 \leqslant K \text{ and} \\ & \partial U_A\left(\{\alpha e - \Delta_1, 1 + \beta e + \Delta_2\}, e\right)/\partial e = 0. \end{array} \right\}$$

The set $\mathcal{S}(e)$ indexes delegation sets that (i) contain two points outside of the interior of the support and (ii) satisfy the first order condition. The set $S(e)$ is closed since we allow for $\Delta_1 = \psi(e) + \Delta_2$, i.e. we include singletons to the right of the support. We take K large enough so that (i) the (unique) singleton that satisfies the first order condition is associated to a pair $(\Delta_1, \Delta_2) \in S(e)$[31] and (ii) the payoff from this singleton exceeds

[31] When $\alpha + \beta > 0$, for each effort level there exists a unique singleton that satisfies the first order condi-

$\Phi(\{\alpha e - (K - \psi(e)), 1 + \beta e + (K - \psi(e))\}, e)$. Since $\Phi(\{\alpha e - \Delta_1, 1 + \beta e + \Delta_2\}, e)$ is continuous in (Δ_1, Δ_2), then it achieves a maximum in the compact set $S(e)$. Thus, there exists a solution to the relaxed problem.

We know from step 2 that any solution to the relaxed problem is of the form $D(e) = \{\alpha e - \Delta_1, 1 + \beta e + \Delta_2\}$. From this point on, with a slight abuse of notation, we consider $\Phi(\{\alpha e - \Delta_1, 1 + \beta e + \Delta_2\}, e)$ as a function of (Δ_1, Δ_2) and directly write $\Phi(\Delta_1, \Delta_2)$. We do the same with Φ' and with $\partial U_A(\cdot)/\partial e$. We also say that a pair (Δ_1, Δ_2) is a solution to the relaxed problem when it is associated to a solution $D(e) = \{\alpha e - \Delta_1, 1 + \beta e + \Delta_2\}$.

Any solution (Δ_1, Δ_2) to the relaxed problem must satisfy $0 < \Delta_2 < \Delta_1 < \psi(e) + \Delta_2$ (see step 2). Moreover, by construction, it must also satisfy $\psi(e) + \Delta_2 < K$. Thus, all inequalities in the definition of $S(e)$ are strict for any solution (Δ_1, Δ_2). As a result, if the pair (Δ_1, Δ_2) is a solution, any small change in Δ_1 and Δ_2 so that the first order condition still holds leads to a new pair that also lies in $S(e)$. This implies that any solution (Δ_1, Δ_2) to the relaxed problem must satisfy the following necessary condition

$$\frac{\frac{\partial \Phi(\Delta_1, \Delta_2)}{\partial \Delta_1}}{\frac{\partial \Phi(\Delta_1, \Delta_2)}{\partial \Delta_2}} = \frac{\frac{\partial(\partial U_A(\Delta_1, \Delta_2)/\partial e)}{\partial \Delta_1}}{\frac{\partial(\partial U_A(\Delta_1, \Delta_2)/\partial e)}{\partial \Delta_2}}$$

which holds if and only if

$$\frac{\frac{\partial \Phi(\Delta_1, \Delta_2)}{\partial \Delta_1}}{\frac{\partial \Phi(\Delta_1, \Delta_2)}{\partial \Delta_2}} = \frac{\frac{\partial \Phi'(\Delta_1, \Delta_2)}{\partial \Delta_1}}{\frac{\partial \Phi'(\Delta_1, \Delta_2)}{\partial \Delta_2}}.$$

This condition leads to

$$(\psi(e) + \Delta_1 + \Delta_2)^2 (\alpha \Delta_1 + \beta \Delta_2) - 4\Delta_1 \Delta_2 (\alpha \Delta_2 + \beta \Delta_1) = 0. \tag{9}$$

To sum up, a solution (Δ_1, Δ_2) exists and satisfies three conditions: (i) $0 < \Delta_2 < \Delta_1 < \psi(e) + \Delta_2$, (ii) equation (9) and (iii) equation (10), which represents the first order condition:

$$-\beta(\Delta_2)^2 + \alpha(\Delta_1)^2 - \frac{\beta - \alpha}{3\psi(e)} \left[-\frac{1}{4} (\psi(e) + \Delta_1 + \Delta_2)^3 + (\Delta_1)^3 + (\Delta_2)^3 \right] = c'(e)\psi(e) \tag{10}$$

tion. This singleton lies to the right of the support under the assumption $\partial U_A(\{\alpha e, 1 + \beta e\}, e)/\partial e < 0$.

We do not know whether there is a unique pair (Δ_1, Δ_2) satisfying these three conditions. However, as we show below, pairs satisfying these three conditions are ranked: if both (Δ_1, Δ_2) and (Δ_1', Δ_2') satisfy them, then either $(\Delta_1, \Delta_2) \gg (\Delta_1', \Delta_2')$ or $(\Delta_1, \Delta_2) \ll (\Delta_1', \Delta_2')$. Thus, the solution to the relaxed problem is unique and corresponds to the smallest pair (Δ_1, Δ_2) satisfying these three conditions.

We show that the pairs satisfying conditions (i), (ii) and (iii) are ranked by proving that indeed, pairs satisfying conditions (i) and (ii) are ranked. To do this, we define the following function for an arbitrary (Δ_1, Δ_2):

$$g(\Delta_1, \Delta_2) \equiv (\psi(e) + \Delta_1 + \Delta_2)^2 (\alpha\Delta_1 + \beta\Delta_2) - 4\Delta_1\Delta_2 (\alpha\Delta_2 + \beta\Delta_1)$$

Thus, (Δ_1, Δ_2) satisfies equation (9) if and only if $g(\Delta_1, \Delta_2) = 0$. For a fixed Δ_2, $g(\cdot, \Delta_2)$ is concave in Δ_1. Moreover, $g(\Delta_2, \Delta_2) > 0$ and $g(\Delta_2 + \psi(e), \Delta_2) < 0$. Thus, there exists a unique $\Delta_1 \in (\Delta_2, \Delta_2 + \psi(e))$ such that $g(\Delta_1, \Delta_2) = 0$. Similarly, for a fixed Δ_1, $g(\Delta_1, \cdot)$ is convex in Δ_2. Moreover, $g(\Delta_1, \Delta_1) > 0$ and $g(\Delta_1, \max\{\Delta_1 - \psi(e), 0\}) < 0$. Thus, there exists a unique $\Delta_2 \in (\max\{\Delta_1 - \psi(e), 0\}, \Delta_1)$ such that $g(\Delta_1, \Delta_2) = 0$.

The argument above shows that there exists a strictly monotonic function $\Delta_1(\Delta_2)$ such that $g(\Delta_1(\Delta_2), \Delta_2) = 0$. Moreover, $g(0,0) = 0$. Thus, $\Delta_1(\Delta_2)$ is strictly increasing, which implies that pairs that solve conditions (i), (ii) and (iii) must be ranked.

Property 1 below summarizes the characterization of the unique solution to the relaxed problem. We say (Δ_1, Δ_2) is the minimal pair in a subset $R \subseteq \mathbb{R}^2$ whenever (i) $(\Delta_1, \Delta_2) \in R$ and (ii) $(\Delta_1, \Delta_2) \ll (\Delta_1', \Delta_2')$ for all $(\Delta_1', \Delta_2') \in R$ with $(\Delta_1', \Delta_2') \neq (\Delta_1, \Delta_2)$.

PROPERTY 1. CASE WITH $\alpha + \beta > 0$. (Δ_1, Δ_2) *is the minimal pair in the following set:*

$$\left\{ \begin{array}{l} (\Delta_1', \Delta_2') : 0 < \Delta_2' < \Delta_1' < \Delta_2' + \psi(e), \\ -\beta(\Delta_2')^2 + \alpha(\Delta_1')^2 - \frac{\beta-\alpha}{3\psi(e)} \left[-\frac{1}{4} (\psi(e) + \Delta_1' + \Delta_2')^3 + (\Delta_1')^3 + (\Delta_2')^3 \right] = c'(e)\psi(e) \\ \textit{and} \ \ (\psi(e) + \Delta_1' + \Delta_2')^2 (\alpha\Delta_1' + \beta\Delta_2') - 4\Delta_1'\Delta_2' (\alpha\Delta_2' + \beta\Delta_1') = 0 \end{array} \right\}$$

We state above Property 1 for the case with $\alpha + \beta > 0$. When instead $\alpha + \beta < 0$, the only difference is that the first condition in Property 1 becomes $0 < \Delta_1' < \Delta_2' < \Delta_1' + \psi(e)$. This follows from Lemma 1.

Step 4. In this final step we show that the solutions from part 1 and 2 also solve the original problem under one of two sufficient conditions. To do this, we let D denote a solution to the relaxed problem from either part 1 or part 2. Thus, either $D = [\alpha e, \underline{a}] \cup [\overline{a}, 1 + \beta e]$ or $D = \{\underline{a}, \overline{a}\}$.

Consider first the case with $0 \leqslant \underline{a} < \overline{a} \leqslant 1$ and build $D' = (-\infty, \underline{a}] \cup [\overline{a}, +\infty)$. For any effort level e', the agent's utility under the delegation set D' is

$$U_A(D', e') = -\frac{1}{1 + (\beta - \alpha)e'} \overbrace{\left[\int_{\underline{a}}^{(\underline{a}+\overline{a})/2} (\omega - \underline{a})^2 \, d\omega + \int_{(\underline{a}+\overline{a})/2}^{\overline{a}} (\omega - \overline{a})^2 \, d\omega \right]}^{\text{constant in } e'} - c(e'),$$

which is strictly concave in e'. Moreover, the original effort level e satisfies the first order condition also under D'. Thus, it is optimal for the agent to choose the effort level e under D'. Finally, note that $U_A(D, e') \leqslant U_A(D', e')$ for all e' and $U_A(D, e') = U_A(D', e)$. Thus, the agent chooses e also under D. We conclude that D solves the principal's original problem.

Consider next the second sufficient condition from Proposition 3. Let $\check{e}$ be the smallest effort level e' for which $(\underline{a} + \overline{a})/2 \in \Gamma(e')$. Note that $\check{e} < e$. As before build $D' = (-\infty, \underline{a}] \cup [\overline{a}, +\infty)$. For all $e' \leqslant e$, $U_A(D', e') = U_A(D, e')$. Thus, in particular, $\partial U_A(D', \check{e})/\partial e' = \partial U_A(D, \check{e})/\partial e' > 0$. Under the delegation set D', for any effort level lower than $\check{e}$, the agent chooses the same unique action (either $\underline{a}$ or $\overline{a}$) for all states. Then, the agent's utility is strictly concave for all effort levels lower than $\check{e}$. These facts together imply that

$$\partial U_A(D', e')/\partial e' > 0 \quad \forall e' \leqslant \check{e}. \tag{11}$$

Next, let $\check{\check{e}}$ denote the lowest effort level e' with $\partial U_A(D', e')/\partial e' = 0$ and $\check{\check{e}} > \check{e}$. Such effort level exists since $\partial U_A(D', e)/\partial e' = 0$. We show next that $\partial U_A(D', e')/\partial e' < 0$ for any $e' > \check{\check{e}}$. This implies that $\check{\check{e}} = e$ and, together with equation (11), that it is optimal for the agent to choose the effort level e under D'. Thus, as before, it is also optimal for the agent to choose the effort level e under the original delegation set D.

We now show that $\partial U_A(D', e')/\partial e' < 0$ for any $e' > \check{\check{e}}$. For any effort level e', let $z(e')$

and $m(e')$ be defined by

$$z(e') = U_A(D', e')\psi(e') = \Phi(D', e') - \psi(e')c(e')$$
$$m(e') = z'(e')\psi(e') - (\beta - \alpha)z(e),$$

and notice that $\partial U_A(D', e')/\partial e' = m(e')/(\psi(e'))^2$. Therefore, it is enough to show that $m(e') < 0$ for any $e' > \check{\check{e}}$. Since $\partial U_A(D', \check{\check{e}})/\partial e' = 0$ and $\partial^2 U_A(D', \check{\check{e}})/\partial (e')^2 \leqslant 0$, then $m(\check{\check{e}}) = 0$ and $m'(\check{\check{e}}) \leqslant 0$. Next, note that $m'(e') = z''(e')\psi(e')$. This, together with $m'(\check{\check{e}}) \leqslant 0$, imply that $z''(\check{\check{e}}) \leqslant 0$. For any $e' \geqslant \check{\check{e}}$,

$$z'''(e') = -2\beta^3 \mathbb{1}_{\{\overline{a}-(1+\beta e')>0\}} + 2\alpha^3 \mathbb{1}_{\{\alpha e' - \underline{a} > 0\}} - 3(\beta - \alpha)c''(e') - \psi(e')c'''(e') < 0.$$

Thus for every $e' > \check{\check{e}}$, we have $z''(e') < 0$, which in turn implies that $m'(e') < 0$. This, together with $m(\check{\check{e}}) = 0$, imply that $m(e') < 0$ for every $e' > \check{\check{e}}$. ■

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