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APPEARANCE OF A NON-SATURATION GAP AND QUANTITATIVE CONVERGENCE FOR THE LANDAU-FERMI-DIRAC EQUATION

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Abstract. This work presents a quantitative analysis of the thermalisation of solutions to the Landau-Fermi-Dirac equation toward the Fermi-Dirac equilibrium. This is accomplished by developing an entropy/entropy production functional inequality and applying a non-saturation gap result satisfied by the solutions to the equation. The appearance in finite time of such gap is proven using a diminishing oscillation argument displayed by diffusive phenomena.

1. Introduction

This document is dedicated to the treatment of the long-time behaviour of solutions to the Landau-Fermi-Dirac (LFD) equation. It establishes the first result exposing a quantitative rate of convergence to equilibrium for the collision dynamic of spatially homogenous Fermi pseudo-particles without any restriction on the size of the Planck constant or the saturation of the initial configuration.

1.1. **Landau-Fermi-Dirac equation.**

Let us consider the dynamics of the velocity distribution $f(t,v) \geqslant 0$, $(t \geqslant 0, v \in \mathbb{R}^3)$, of pseudo-particles experiencing binary collisions in a plasma subjected to the quantum effect of the Pauli's exclusion principle. The evolution of $f(t,v)$ is governed, in the spatially homogeneous setting, by the *Landau-Fermi-Dirac equation* given by

$$\partial_t f(t,v) = \mathscr{C}(f)(t,v), \qquad (t,v) \in (0,\infty) \times \mathbb{R}^3, \qquad f(0) = f_{\text{in}}, \tag{1.1}$$

where the collision operator $\mathscr{C}$ is given by a ε-correction of the classical Landau operator, defined as

$$\mathscr{C}(f)(v) := \nabla_v \cdot \int_{\mathbb{R}^3} \Psi(v-v_*)\,\Pi(v-v_*)\Big\{ f_*(1-\varepsilon f_*)\nabla f - f(1-\varepsilon f)\nabla f_* \Big\}\,\mathrm{d}v_*, \tag{1.2}$$

with the common shorthand $f := f(v)$, $f_* := f(v_*)$, and potential Ψ and projection Π

$$\Psi(z) = |z|^{\gamma+2}, \qquad \Pi(z) = \mathrm{Id} - \frac{z \otimes z}{|z|^2}, \qquad z \in \mathbb{R}^3,\ z \neq 0. \tag{1.3}$$

The Pauli exclusion principle implies that a solution to (1.1) must *a priori* satisfy the natural bound

$$0 \leqslant f(t,v) \leqslant \varepsilon^{-1}, \tag{1.4}$$

where the *quantum parameter*

$$\varepsilon := \frac{(2\pi\hbar)^3}{m^3\beta} > 0$$

depends on the reduced Planck constant $\hbar \approx 1.054 \times 10^{-34}\mathrm{m}^2\mathrm{kg}\,\mathrm{s}^{-1}$, the mass m and the statistical weight β of the particles species, see [16, Chapter 17]. In the case of electrons $\varepsilon \approx$

$1.93 \times 10^{-10} \ll 1$. The parameter ε quantifies the quantum effects of the model. The case $\varepsilon = 0$ corresponds to the classical Landau equation.

In addition to the pointwise bound (1.4), the dynamics enjoys conservation of mass, momentum, and energy. That is

$$\begin{gathered}\int_{\mathbb{R}^3} f(t,v)\mathrm{d}v = \int_{\mathbb{R}^3} f_{\mathrm{in}}(v)\mathrm{d}v = \varrho, \qquad \int_{\mathbb{R}^3} f(t,v)\, v\mathrm{d}v = \int_{\mathbb{R}^3} f_{\mathrm{in}}(v)\, v\mathrm{d}v = \varrho \boldsymbol{u},\\ \int_{\mathbb{R}^3} f(t,v)|v-\boldsymbol{u}|^2\mathrm{d}v = \int_{\mathbb{R}^3} f_{\mathrm{in}}(v)|v-\boldsymbol{u}|^2\mathrm{d}v = 3\varrho\vartheta, \qquad \forall t \geqslant 0.\end{gathered} \tag{1.5}$$

As observed in [26, Proposition 4], for $\varrho > 0$, the constraint $0 \leqslant f_{\mathrm{in}} \leqslant \varepsilon^{-1}$ implies that

$$\vartheta \geqslant \frac{1}{5}\left(\frac{3\varrho\varepsilon}{4\pi}\right)^{\frac{2}{3}}, \tag{1.6}$$

that is, solutions have a positive minimal energy.

The LFD equation can be derived as a limit of the Boltzmann-Fermi-Dirac (BFD) equation in the so-called grazing collision limit. The BFD is itself a ε-correction of the classical Boltzmann equation that takes into account the Pauli's exclusion principle. It reads

$$\partial_t f(t,v) = \mathscr{C}_B(f)(t,v), \qquad (t,v) \in (0,\infty)\times\mathbb{R}^3\,, \qquad f(0) = f_{\mathrm{in}}\,,$$

where the Boltzmann-Fermi-Dirac operator $\mathscr{C}_B$ is given by

$$\mathscr{C}_B(f)(v) = \int_{\mathbb{R}^3\times\mathbb{S}^2}\Big[f'f'_*(1-\varepsilon f)(1-\varepsilon f_*) - ff_*(1-\varepsilon f')(1-\varepsilon f'_*)\Big]B(v,v_*,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_*.$$

We use the common shorthands $f := f(v)$, $f_* := f(v_*)$, $f' := f(v')$, $f'_* := f(v'_*)$ and

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \qquad \sigma \in \mathbb{S}^2.$$

The quantum Boltzmann equation was first derived by Nordheim [29], and Uehling & Uhlenbeck [30] on the basis of heuristic arguments. In contrast with the classical Boltzmann equation valid for a rarefied gas in the so-called Boltzmann-Grad limit, the Fermi-Dirac counterpart should be derived from the evolution of fermions in the so-called weak-coupling limit [8, 9]. We refer also to [18] for a derivation of the spatially homogeneous BFD equation from a stochastic N-particle system which is a modification of the Kac's model that includes the Pauli exclusion principle. A physical explanation of the derivation to the BFD equation is given in [16, Chapter 17], also see [25]. Essentially, the probability of an incoming pseudo-particle to the velocity box $\mathrm{d}v$ is reduced by a factor $1-\varepsilon f(v)$ due to the Pauli's exclusion principle. This explains the presence of such terms in $\mathscr{C}$ and $\mathscr{C}_B$.

The mathematical investigation of the BFD equation was initiated in [21, 26, 27]. The reader can refer to [25, 23] for a review of the recent literature on the field. The study of the LFD equation (1.1) is more recent. We refer the reader to [7] for the derivation of $\mathscr{C}$ from $\mathscr{C}_B$ whereas the Cauchy theory for hard-potentials interactions, corresponding to $\gamma > 0$ in (1.3), has been investigated in [6] – see also [4, 17] for additional results on the hard potential case. A study of (1.1) for soft potentials $-2 \leqslant \gamma < 0$ was initiated in [5]. The present contribution fills the missing details in [5] for the quantitative convergence analysis under generic initial data or, equivalently, unrestricted $\varepsilon > 0$. Finally, we mention that the well-posedness of the LFD equation for the Coulomb case, corresponding to $\gamma = -3$, has been established in [22] and can be extended to the case $-3 < \gamma < 0$ considered here.

1.2. **Fermi-Dirac statistics and saturated (or frozen) states.** An interesting aspect of the LFD equation is that it has two type of possible stationary states which are nonnegative and non trivial distributions $F(v)$ such that

$$\mathscr{C}(F) = 0.$$

In one hand, the saturated or frozen state happens when particles fill up, or saturate, the lowest energy levels possible. In this case the stationary state takes the simple form

$$F_{\varepsilon}(v) = \begin{cases} \varepsilon^{-1} & \text{if} \quad |v - \boldsymbol{u}| \leqslant \left(\dfrac{3\varrho\,\varepsilon}{4\pi}\right)^{\frac{1}{3}}, \\ 0 & \text{if} \quad |v - \boldsymbol{u}| > \left(\dfrac{3\varrho\,\varepsilon}{4\pi}\right)^{\frac{1}{3}}, \end{cases} \tag{1.7}$$

and F_{ε} is a stationary state with mass ϱ and momentum $\boldsymbol{u}$. Notice also that such a stationary state has temperature $E_F = \varrho\,\vartheta_F$ with

$$\vartheta_F := \frac{1}{5}\left(\frac{3\varepsilon\varrho}{4\pi}\right)^{\frac{2}{3}},$$

that is, the inequality (1.6) becomes an equality so that F_{ε} has the minimal admissible temperature. Such a stationary state can only be reached when the gas has been cooled down to such temperature. The temperature E_F is the minimal temperature that a Fermi gas can reach, when the mean distance between neighbouring pseudo-particles is comparable with their quantum wave fields such a state of saturation is formed. In the other hand, if the gas has higher temperature $E > E_F$, particle collisions will occur and induce another type of stationary state known as the Fermi-Dirac statistics

$$\mathcal{M}_{\varepsilon}(v) := \frac{a_{\varepsilon}\exp(-b_{\varepsilon}|v - \boldsymbol{u}|^2)}{1 + \varepsilon\,a_{\varepsilon}\exp(-b_{\varepsilon}|v - \boldsymbol{u}|^2)}, \tag{1.8}$$

where the unique coefficients a_{ε} and b_{ε} are such that $\mathcal{M}_{\varepsilon}(v)$ satisfies (1.5). We refer to [26, Proposition 3] for additional properties of the Fermi-Dirac statistics. An important characterisation of the aforementioned stationary states can be made using the Fermi-Dirac entropy, defined by

$$\mathcal{H}_{\varepsilon}(\phi) := \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\Big(\varepsilon\phi\log(\varepsilon\phi) + (1 - \varepsilon\phi)\log(1 - \varepsilon\phi)\Big)\mathrm{d}v\,, \qquad 0 \leqslant \phi \leqslant \varepsilon^{-1}. \tag{1.9}$$

One can show that the saturated state F_{ε} is characterized as the unique, for given $(\varrho, \boldsymbol{u}, \vartheta)$, state with zero entropy

$$\mathcal{H}_{\varepsilon}(f) = 0 \iff f = F_{\varepsilon}.$$

Furthermore, the Fermi-Dirac entropy plays a crucial role in the dynamics of the LFD equation since for solutions $f(t)$ of (1.1) it follows that $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\varepsilon}(f(t)) \leqslant 0$, that is,

$$\mathcal{H}_{\varepsilon}(f(t)) \leqslant \mathcal{H}_{\varepsilon}(f_{\mathrm{in}}) \qquad \forall t \geqslant 0.$$

Since $\mathcal{H}_\varepsilon(f_{\rm in}) < 0$, unless $f_{\rm in}$ is the saturated state, the Fermi-Dirac statistics $f(t)$ will never evolve into a froze state, rather, it will evolve to a minimiser of the Fermi-Dirac entropy with strictly negative entropy value.

1.3. **Important notations and definitions.** The following is used throughout the paper.

1) *Statistical Moments:* For a measurable $g \geqslant 0$ the η-statistical moment is defined as

$$\boldsymbol{m}_\eta(g) := \int_{\mathbb{R}^3} g(v)\langle v\rangle^\eta \mathrm{d}v\,, \qquad \langle v\rangle := \sqrt{1+|v|^2}\,, \qquad \eta \in \mathbb{R}\,.$$

If $f = f(t)$ is a solution to (1.1) we use the shorthand $\boldsymbol{m}_\eta(t) = \boldsymbol{m}_\eta(f(t))$.

2) *Weighted Lebesgue's spaces:* For $p \geqslant 1$ and $\eta \in \mathbb{R}$ we define the Lebesgue's space $L^p_\eta(\mathbb{R}^3)$ as

$$L^p_\eta(\mathbb{R}^3) := \left\{g\;:\mathbb{R}^3 \to \mathbb{R}\,;\,\|g\|_{L^p_\eta} < \infty\right\}, \qquad \|g\|_{L^p_\eta} := \left(\int_{\mathbb{R}^3} \left|g(v)\right|^p \langle v\rangle^\eta\,\mathrm{d}v\right)^{\frac{1}{p}}\,.$$

For $\eta = 0$ one simply writes $L^p(\mathbb{R}^3) = L^p_0(\mathbb{R}^3)$.

3) H^1*-Sobolev space:* We use the standard definition for H^1, that is, a measurable function $g \in H^1(\mathbb{R}^3)$ if

$$\|g\|^2_{L^2} + \|\nabla g\|^2_{L^2} =: \|g\|^2_{H^1} < \infty\,.$$

4) *Entropy production:* For any $\varepsilon \geqslant 0$ and smooth function $0 < g < \varepsilon^{-1}$ introduce

$$\begin{aligned}\boldsymbol{\Xi}_\varepsilon[g](v,v_*) := gg_*(1-\varepsilon g)(1-\varepsilon g_*)&\\ \times\left|\Pi(v-v_*)\left(\frac{\nabla g}{g(1-\varepsilon g)} - \frac{\nabla g_*}{g_*(1-\varepsilon g_*)}\right)\right|^2 &\geqslant 0\,,\end{aligned} \tag{1.10}$$

and denote by $\mathscr{D}^{[\Psi]}_\varepsilon(g)$ the entropy production associated to the interaction kernel $\Psi(z) \geqslant 0$ defined as

$$\mathscr{D}^{[\Psi]}_\varepsilon(g) := \frac{1}{2}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \Psi(v-v_*)\,\boldsymbol{\Xi}_\varepsilon[g](v,v_*)\mathrm{d}v\mathrm{d}v_*\,. \tag{1.11}$$

Of special interest are the entropy production functionals associated to $\Psi(z) = |z|^{2+\eta}$ with $\eta \in \mathbb{R}$ that will be denoted as $\mathscr{D}^{(\eta)}_\varepsilon(g)$.

5) *Weighted Fisher information and weighted entropy:* The LFD entropy production is related to the so-called Fisher information,

$$\mathscr{F}_\eta[f] := \int_{\mathbb{R}^3} \langle v\rangle^\eta \left|\nabla\sqrt{f(v)}\right|^2 \mathrm{d}v, \qquad \eta \in \mathbb{R}\,. \tag{1.12}$$

In Appendix A.2 is observed that the evolution of $\mathscr{F}_\eta[f(t)]$ is related to the weighted Boltzmann entropy

$$S_\eta[f] := \int_{\mathbb{R}^3} \langle v\rangle^\eta f(v)\log f(v)\mathrm{d}v, \qquad \eta \in \mathbb{R}\,. \tag{1.13}$$

The most general functional framework in which we will consider solutions to (1.1) is given in the following definition.

Definition 1.1. *Given $\varrho > 0$, $\vartheta > 0$, $\boldsymbol{u} \in \mathbb{R}^3$, $\varepsilon > 0$, we define $\mathcal{Y}_\varepsilon[\varrho, \boldsymbol{u}, \vartheta]$ as the class of all $f \in L^1_2(\mathbb{R}^3)$ such that*

$$\int_{\mathbb{R}^3} f(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v = \begin{pmatrix} \varrho \\ \varrho \boldsymbol{u} \\ 3\varrho\vartheta + \varrho|\boldsymbol{u}|^2 \end{pmatrix}, \qquad 0 \leqslant f \leqslant \varepsilon^{-1} \tag{1.14}$$

and $\mathcal{H}_\varepsilon(f) < 0$.

Remark 1.2. *From* [26, Proposition 1.4], *given ϱ , $\vartheta > 0$ the aforementioned conditions are met only when ϑ and ε satisfy the constraint (1.6). In the sequel we assume ϱ, ϑ to be fixed, $\boldsymbol{u} = 0$* [1], *set $\mathcal{Y}_\varepsilon := \mathcal{Y}_\varepsilon[\varrho, 0, \vartheta]$, and consider*

$$0 < \varepsilon < \varepsilon_{\mathrm{sat}}, \qquad \varepsilon_{\mathrm{sat}} := \frac{4\pi(5\vartheta)^{\frac{3}{2}}}{3\varrho}. \tag{1.15}$$

Condition (1.15) is equivalent to $\mathcal{H}_\varepsilon(f_{\mathrm{in}}) < 0$ which is the requirement to have a diffusive process.

Next, adopting the notations in [3], introduce

$$\begin{cases} a(z) &=& (a_{i,j}(z))_{i,j} \quad \text{with} \quad a_{i,j}(z) = |z|^{\gamma+2}\Pi_{i,j}(z) = |z|^{\gamma+2}\left(\delta_{i,j} - \frac{z_i z_j}{|z|^2}\right), \\ b_i(z) &=& \sum_k \partial_k a_{i,k}(z) = -2\, z_i\, |z|^\gamma, \\ c(z) &=& -\nabla \cdot b(z) = -\sum_{k,l} \partial^2_{kl} a_{k,l}(z) = 2\,(\gamma+3)\,|z|^\gamma. \end{cases}$$

For any $f \in L^1_2(\mathbb{R}^3)$, define the vector-valued mapping $\boldsymbol{b}[f]$ and the matrix-valued $\boldsymbol{\Sigma}[f]$ as

$$\boldsymbol{\Sigma}[f] := \big(\boldsymbol{\Sigma}_{ij}[f]\big)_{ij} = \big(a_{ij} * f(1-\varepsilon f)\big)_{ij}, \qquad \boldsymbol{b}_i[f] := \big(b_i * f\big), \qquad i,j = 1,2,3.$$

Also, introduce the scalar mapping $\boldsymbol{c}_\gamma[f] := c * f$. The dependency with respect to the parameter γ in $\boldsymbol{c}_\gamma[f]$ is kept since the same definition with $\gamma + 1$ replacing γ is used.

The LFD equation can be written alternatively under the parabolic-type form

$$\begin{cases} \partial_t f &=& \nabla \cdot \big(\boldsymbol{\Sigma}[f]\,\nabla f - \boldsymbol{b}[f]\,f(1-\varepsilon f)\big), \\ f(t=0) &=& f_{\mathrm{in}}\,. \end{cases} \tag{1.16}$$

We adopt the following notion of solution to (1.1) used in the document.

Definition 1.3. *Given $\varrho > 0$, $\boldsymbol{u} \in \mathbb{R}^3$ and $\vartheta > 0$ and $\varepsilon \in (0, \varepsilon_{\mathrm{sat}})$, consider a non trivial initial datum $f_{\mathrm{in}} \in \mathcal{Y}_\varepsilon[\varrho, \boldsymbol{u}, \vartheta]$. A weak solution to the LFD equation (1.16) is a function $f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^+$ satisfying the following conditions:*

(1) $f \in L^\infty(\mathbb{R}^+; L^1_2(\mathbb{R}^3)) \bigcap \mathscr{C}(\mathbb{R}^+, \mathscr{D}'(\mathbb{R}^3))$,
(2) $f(t) \in \mathcal{Y}_\varepsilon[\varrho, \boldsymbol{u}, \vartheta]$ for any $t \geqslant 0$ and $f(0) = f_{\mathrm{in}}$,
(3) the mapping $t \mapsto \mathcal{H}_\varepsilon(f(t))$ is non-increasing,

[1]This is without loss of generality since if $\boldsymbol{u} \neq 0$ we can replace $f(v)$ with $f(v + \boldsymbol{u})$.

(4) and, for any $\varphi = \varphi(t,v) \in \mathscr{C}^2_c([0,T)\times\mathbb{R}^3)$,

$$-\int_0^T \mathrm{d}t \int_{\mathbb{R}^3} f(t,v)\partial_t\varphi(t,v)\mathrm{d}v - \int_{\mathbb{R}^3} f_{\mathrm{in}}(v)\varphi(0,v)\mathrm{d}v = \int_0^T \mathrm{d}t \int_{\mathbb{R}^3} \boldsymbol{\Sigma}_{i,j}[f(t)]f(t,v)\partial^2_{v_i,v_j}\varphi(t,v)\mathrm{d}v + \int_0^T \mathrm{d}t \int_{\mathbb{R}^6} f(t,v)f(t,w)(1-\varepsilon f(t,w))\boldsymbol{b}_i(v-w)\left[\partial_{v_i}\varphi(t,v) - \partial_{w_i}\varphi(t,w)\right]\mathrm{d}v\mathrm{d}w\,, \tag{1.17}$$

where we used Einstein's convention of summations over repeated indices.

Remark 1.4. *We point out that, by definition, $f(t) \in \mathcal{Y}_\varepsilon[\varrho,\boldsymbol{u},\vartheta]$ and, in particular, $0 \leqslant f(t) \leqslant \varepsilon^{-1}$. Since φ has compact support, all the terms in (1.17) are well defined.*

For $\gamma > 0$ solutions to (1.1) have been constructed in [6] and the method described therein has been extended to cover the case $-2 < \gamma < 0$ in the contribution [5]. In the Coulomb case corresponding to $\gamma = -3$, solutions have been constructed in [22] and the methods used in [5, 22] can be adapted to derive the well-posedness of (1.1) for $\gamma \in (-3,-2]$. We do not elaborate here on this in aspect and rather focus on deriving quantitative properties of solutions to (1.1).

1.4. **Entropy and convergence to Fermi Dirac statistics.** Besides its fundamental use to discriminate steady solutions to (1.1), the Fermi-Dirac entropy plays a fundamental role for the study of the long time behavior of kinetic equation. Quoting our recent contribution [4], entropy acts in this case as a natural *Lyapunov functional* which brings the system towards its equilibrium state (which is a minimizer of the entropy) through some LaSalle's invariance principle. More precisely, introduce the relative entropy $\mathcal{H}_\varepsilon[f\,|\,\mathcal{M}^f_\varepsilon]$ defined as

$$\mathcal{H}_\varepsilon[f|\mathcal{M}^f_\varepsilon] := \mathcal{H}_\varepsilon(f) - \mathcal{H}_\varepsilon(\mathcal{M}^f_\varepsilon)\,,$$

where f satisfies (1.14) and $\mathcal{M}^f_\varepsilon$ is the unique Fermi-Dirac statistics satisfying (1.14). One can check then that the following holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\varepsilon\left[f(t)|\mathcal{M}^{f(t)}_\varepsilon\right] = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\varepsilon(f(t)) = -\mathscr{D}^{(\gamma)}_\varepsilon(f(t)) \leqslant 0\,, \tag{1.18}$$

for any solution $f(t) = f(t,v)$ to (1.1) associated to a collision kernel $\Psi(z) = |z|^{2+\gamma}$. Importantly, $\mathcal{M}^{f(t)}_\varepsilon = \mathcal{M}^{f_{\mathrm{in}}}_\varepsilon$ is independent of $t \geqslant 0$. An entropy/entropy production estimate is, thus, an essential tool to prove convergence towards equilibrium. We prove the following theorem.

Theorem 1.5. *Assume that $\Psi(|z|) \geqslant |z|^2$. Then, for any $\kappa_0 \in (0,1)$, $\varepsilon \in (0,\varepsilon_{\mathrm{sat}})$ and $f \in \mathcal{Y}_\varepsilon$ such that*

$$\inf_{v\in\mathbb{R}^3}(1-\varepsilon f(v)) \geqslant \kappa_0,$$

there exists $C > 0$ depending only on $\|f\|_{L^1_2}$ and $\mathcal{H}_\varepsilon(f)$, not on ε or κ_0, such that

$$\mathscr{D}^{[\Psi]}_\varepsilon(f) \geqslant C\,\kappa_0\,\mathcal{H}_\varepsilon[f\,|\,\mathcal{M}^f_\varepsilon] \geqslant 0\,.$$

Now, in reference [5] the following interpolation between different entropy production functionals was introduced

$$\mathscr{D}_{\varepsilon}^{(0)}(g) \leqslant \left(\mathscr{D}_{\varepsilon}^{(\gamma)}(g)\right)^{\frac{\eta}{\eta-\gamma}} \left(\mathscr{D}_{\varepsilon}^{(\eta)}(g)\right)^{\frac{|\gamma|}{\eta-\gamma}}, \qquad \gamma \leqslant 0 < \eta,$$

or equivalently,

$$\mathscr{D}_{\varepsilon}^{(\gamma)}(g) \geqslant \left(\mathscr{D}_{\varepsilon}^{(0)}(g)\right)^{1-\frac{\gamma}{\eta}} \left(\mathscr{D}_{\varepsilon}^{(\eta)}(g)\right)^{\frac{\gamma}{\eta}}. \tag{1.19}$$

Noticing that $1-\frac{\gamma}{\eta}>0$ we can invoke Theorem 1.5, with $\Psi(|z|)=|z|^2$, to bound from below $\mathscr{D}_{\varepsilon}^{(0)}(g)$ in terms of $\mathcal{H}_{\varepsilon}(g|\mathcal{M}_{\varepsilon})$. Such *lower* bound will be meaningful if we have a good *upper* bound for $\mathscr{D}_{\varepsilon}^{(\eta)}(g)$ since $\frac{\gamma}{\eta}\leqslant 0$. It is in this upper bound estimate where the control of high statistical moments plays a central role. Consequently, Theorem 1.5, equation (1.18), and inequality (1.19) readily imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\varepsilon}\left[f(t)|\mathcal{M}_{\varepsilon}\right] \leqslant -\left(\mathscr{D}_{\varepsilon}^{(\eta)}(f(t))\right)^{\frac{\gamma}{\eta}}\left(C\,\kappa_0\,\mathcal{H}_{\varepsilon}\left[f(t)\middle|\mathcal{M}_{\varepsilon}^{f_{\mathrm{in}}}\right]\right)^{1-\frac{\gamma}{\eta}}, \qquad t\geqslant T.$$

A direct integration of this inequality yields an algebraic rate of convergence towards 0 for $\mathcal{H}_{\varepsilon}\left[f(t)\middle|\mathcal{M}_{\varepsilon}^{f_{\mathrm{in}}}\right]$ which, in turn, yields a quantitative convergence of $f(t)$ towards $\mathcal{M}_{\varepsilon}^{f_{\mathrm{in}}}$ in a L^1-sense due to the Csiszár-Kullback-Pinsker inequality, see [26, 10]. The entropy/entropy production method has been applied with success in classical kinetic theory due to the natural entropic framework of the theory, see for example the pioneering works [33] and subsequently [19, 15]. Theorem 1.5 improves a recent result in [10, Proposition 8] where a similar control have been obtained with a constant proportional to κ_0^4.

A central difficulty in using the entropic method for the LFD equation is showing the pointwise estimate

$$\inf_{v\in\mathbb{R}^3}(1-\varepsilon f(v)) \geqslant \kappa_0 > 0,$$

which has prevented, until now, to give a complete picture of the problem.

1.5. **Main results and method of proof.** For the LFD equation the entropy/entropy production method has been used in contributions [3, 4, 5] and, more recently for the Boltzmann Fermi Dirac equation in [11]. The critical missing piece of the puzzle is related to showing the appearance of a non saturation gap under a general initial configuration. That is, for an explicit time $T_*>0$ is must be the case that for solutions $f(t,v)$ of the LFD equation

$$\inf_{v\in\mathbb{R}^3}(1-\varepsilon f(t,v)) \geqslant \kappa_0, \qquad t\geqslant T_*,$$

for some time uniform non saturation gap κ_0. As long as the initial state is not the frozen state, this property must hold regardless of how saturated the initial configuration is.

In all previous contributions tackling quantitative convergence, see for example [5], the idea was to show L^∞-bounds independent of the Planck's parameter ε. This allows, up to a possible reduction of ε, to prove that

$$f(t,v) \leqslant \sup_{t\geqslant T_*}\|f(t)\|_{L^\infty} < \varepsilon^{-1} \qquad t\geqslant T_*,$$

which shows the existence of a non saturation gap after time t_*. This reduction of ε can be interpreted as to effectively reducing the maximum degree of saturation allowed in the solutions we deal with. Such a strategy yields quantitative convergence to equilibrium for (1.1), however it is not completely satisfactory since only works in a regime of moderately saturated, or relatively warm, solutions.

The main result of the current paper, presenting this missing piece in the study of the thermalisation mechanism for the homogeneous LFD equation is the following theorem.

Theorem 1.6 (***Appearance of a non saturation gap***). *Fix $\gamma \in (-3,0)$, $\varepsilon \in (0,\varepsilon_{\rm sat})$, and $f_{\rm in} \in L^1_{3|\gamma|}(\mathbb{R}^3) \cap \mathcal{Y}_\varepsilon$. Then, there exists an explicit function $\kappa_0 : \mathbb{R}^+ \to \mathbb{R}^+$ and an explicit $T_* > 0$ such that*

$$\inf_{v\in\mathbb{R}^3} \big(1-\varepsilon f(t,v)\big) \geqslant \kappa_0(t) > 0 \qquad \forall t \geqslant T_*$$

holds true for any weak solution $f(t)=f(t,v)$ to (1.1).

Remark 1.7. *In the range $\gamma \in (-\frac{4}{3},0)$ one can choose κ_0 independent of time, that is,*

$$\inf_{t\geqslant T_*} \kappa_0(t) = \underline{\kappa}_0 > 0\,.$$

In the range $\gamma \in \big(-3,-\frac{4}{3}\big]$, although not optimal, it is always possible to use the lower bound

$$\kappa_0(t) \geqslant C\,\boldsymbol{m}_{3|\gamma|}(t)^{-\frac{1}{2}}, \qquad t \geqslant T_*\,,$$

for $C>0$ and $T_*>0$ explicit and depending on $\|f_{\rm in}\|_{L^1_2}$, $\mathcal{H}_\varepsilon(f_{\rm in})$, and $\boldsymbol{m}_{3|\gamma|}(0)$. A complete discussion is given in Section 3.3.

We emphasise that the parameter $\varepsilon \in (0,\varepsilon_{\rm sat})$ is *fixed* and as close to $\varepsilon_{\rm sat}$ as desired hence the initial configurations can be as saturated as desired. Furthermore, the appearance of the saturation gap holds for the whole range of soft potentials, however, for $\gamma \in (-3,-\frac{4}{3}]$ the estimate does not rules out that the gap $\kappa_0(t)$ may be vanishing as time evolves.

For the proof of Theorem 1.6 we use a variant of a De Giorgi's type of approach, refer to [20] for an original implementation in the classical elliptic framework. The method was adapted to the spatially homogeneous Boltzmann equation in [1] and later used for the homogeneous LFD equation in [5]. Here we introduce a novel variant of the method designed to control the oscillation of $f(t,v)$ instead of the absolute size of $f(t,v)$. The diminishing of the solution's oscillation is a well-known property in diffusive phenomena which is quantified in the so-called Oscillation lemma for elliptic and parabolic equations, see for example [12, 32]. This property is used to show Hölder's regularity for solutions of such equations. In our case, as the time passes, the oscillation in a set $[T_*,\infty)\times\mathbb{R}^3$, $T_*>0$, should be quantifiable smaller than the oscillation in the set $[0,\infty)\times\mathbb{R}^3$ which is ε^{-1}, thus, creating a non-saturation gap. We remark that the central point of the argument is that the LFD equation is shown to be uniformly parabolic only with an integral non-saturation condition, that is, only strictly negative entropy $\mathcal{H}_\varepsilon(f_{\rm in})<0$ is needed for diffusiveness. Theorem 1.6 applies to $\gamma \in (-3,0)$ and can, likely, extend to the Coulomb case $\gamma=-3$ which requires a different analysis. Such analysis will be done elsewhere.

The pointwise non-saturation gap appearance combined with slowly increasing bounds on statistical moments has the following fundamental consequence for the long time behaviour of the (1.1) equation.

Theorem 1.8. *Fix $\gamma \in (-2, 0)$, $\varepsilon \in (0, \varepsilon_{\text{sat}})$, and $f_{\text{in}} \in \mathcal{Y}_\varepsilon$. Let $f(t) = f(t, \cdot)$ be a weak-solution to* (1.1)*. Assuming additionally that $f_{\text{in}} \in L^1_\eta(\mathbb{R}^3)$ with $\eta > 2$ sufficiently large, there exists an increasing $\alpha := \alpha(\eta) > 0$ and constant $C > 0$, explicit in terms of f_{in}, such that*

$$\mathcal{H}_\varepsilon(f(t)|\mathcal{M}_\varepsilon) \leqslant C\, t^{-\alpha}, \qquad t \geqslant 1.$$

Remark 1.9. *Furthermore, Theorem 1.8 yields an explicit convergence rate of $f(t)$ to $\mathcal{M}_\varepsilon$ in weighted L^p-topologies $p \in [1, 2]$ thanks to a (Ferrmi-Dirac) version of Csiszár-Kullback-Pinsker inequality, refer to* [10, Proposition 4].

Since Theorem 1.8 applies under general initial data, this result supersedes our previous contribution [5, Theorem 1.7] which dealt only with weakly saturated regimes. Refer to Section 5 for a precise statement in terms of the number of required moments and the expression for $\alpha(\eta)$. We notice that Theorem 1.6 is valid for $\gamma \in (-3, 0)$, yet the thermalization result is restricted to moderately soft potentials $\gamma \in (-2, 0)$. This is a technical constraint due to our inability to shown moderate time growth for $\boldsymbol{m}_{3|\gamma|}(t)$ in the range $\gamma \in (-3, -2]$.

1.6. **Organization of the paper.** Section 2 collects important properties for solutions to the LFD equation used in the sequel. Section 3 implements the De Giorgi argument to prove Theorem 1.6. Section 4 proves Theorem 1.5 while Section 5 is devoted to the proof of Theorem 1.8. The paper ends with important appendices: Appendix A establishes some results announced in Section 2. More precisely, Appendix A.1 gives the technicalities of convolution inequalities latter to be used in Section 3, Appendix A.2 gives some entropy estimates which will be later used to prove Theorem 1.8 and Appendix A.3 gives the full proof of moments growths for the case $\gamma \in (-3, -2]$, Theorem 2.6 which, it is a new result with its own interest. Finally, Appendix B is devoted to technical estimates used in Section 3.

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2. Preliminary results

2.1. **Some properties of solutions to equation (1.1).** We recall the following result, derived first in [3, Lemma 2.3 & 2.4], which quantifies the fact that non saturated states generate diffusion due to particle interactions.

Lemma 2.1 (***Quantitative diffusion lemma***)**.** *Let $\varepsilon \in (0, \varepsilon_{\text{sat}})$ and $f \in \mathcal{Y}_\varepsilon$ be fixed. Then,*

(1) there exists $R_\star > 0$ and $\eta_* > 0$ depending only on $\|f\|_{L^1_2}$ and $\mathcal{H}_\varepsilon(f)$ such that

$$\int_{|v|\leqslant R_\star} f(1 - \varepsilon f)\,\mathrm{d}v \geqslant \eta_\star > 0\,. \tag{2.1}$$

(2) *For any $\delta > 0$ there exists $\eta(\delta) > 0$ depending only on $\|f\|_{L^1_2}$ and $\mathcal{H}_\varepsilon(f)$ such that, for any measurable set $A \subset \mathbb{R}^3$,*

$$|A| \leqslant \eta(\delta) \Longrightarrow \int_A f(1-\varepsilon f)\,\mathrm{d}v \leqslant \delta. \tag{2.2}$$

(3) *There exists a constant $K_0 > 0$ depending only on $\|f\|_{L^1_2}$ and $\mathcal{H}_\varepsilon(f)$ such that*

$$\mathbf{\Sigma}_{i,j}[f](v)\,\xi_i\,\xi_j \geqslant K_0 \langle v\rangle^\gamma\,|\xi|^2 \tag{2.3}$$

holds for any $v, \xi \in \mathbb{R}^3$.

Remark 2.2. *In [5] the aforementioned result has been stated for constants depending on the classical entropy $\int_{\mathbb{R}^3} f\log f \mathrm{d}v$ instead of the LFD entropy $\mathcal{H}_\varepsilon(f)$. Is such a case the result remains valid for $\varepsilon = 0$. In this document $\varepsilon \in (0, \varepsilon_{\rm sat})$ is fixed, thus, it is natural to work directly with $\mathcal{H}_\varepsilon(f)$.*

Recall the following derived in [5, Prop. 2.5] controlling the so-called Fisher information of solutions using the entropy production. It is important in the $L^1 \cap L^2$ theory for the case of moderately soft potentials.

Proposition 2.3. *Fix $\gamma \in (-2,0)$, $\varepsilon \in (0,\varepsilon_{\rm sat})$ and $f \in \mathcal{Y}_\varepsilon$. Then, there exists a positive constant $C_0(\gamma)$ depending only on $\|f\|_{L^1_2}$ and $\mathcal{H}_\varepsilon(f)$ such that*

$$\mathscr{F}_\gamma[f] = \int_{\mathbb{R}^3} \langle v\rangle^\gamma \left|\nabla\sqrt{f(v)}\right|^2 \mathrm{d}v \leqslant C_0(\gamma)\big(1 + \mathscr{D}_\varepsilon(f)\big)\,.$$

The aforementioned results, stated for generic functions in the class $\mathcal{Y}_\varepsilon$, will be applied to solutions $f(t) = f(t,v)$ to (1.1) since, recalling Remark 1.4, we consider solutions $f(t) \in \mathcal{Y}_\varepsilon$.

2.2. **Statistical moments and entropy production estimates for $\gamma \in (-2,0)$.** We provide now an important estimate controlling the statistical moments of solutions to (1.1) for moderately soft potentials, roughly stated as

$$\boldsymbol{m}_\eta(0) < \infty \implies \boldsymbol{m}_\eta(t) \lesssim (1+t) \qquad \forall \eta > 2.$$

This estimate has been derived in [5, Theorem 1.9]. Here we bring some new elements that will sharpen the dependence with respect to the initial datum and the time algebraic growth. The universal algebraic growth is an essential feature of the entropy/entropy production analysis since convergence relies on interpolation with possibly large order moments. This part of the theory is still lacking for $\gamma \in (-3,-2]$ which prevents the consideration of such range.

Theorem 2.4. *Fix $\gamma \in (-2,0)$, $\varepsilon \in (0,\varepsilon_{\rm sat})$, and initial datum $f_{\rm in} \in \mathcal{Y}_\varepsilon$. Let $f(t,\cdot)$ be a weak-solution to (1.1) and assume that*

$$\boldsymbol{m}_\eta(0) < \infty, \qquad \eta > 4 + |\gamma|.$$

Then, there exists a constant $\boldsymbol{C}_\eta > 0$ depending on η and $\boldsymbol{m}_\eta(0)$, $\|f_{\rm in}\|_{L^1_2}$, $\mathcal{H}_\varepsilon(f_{\rm in})$, such that

$$\boldsymbol{m}_\eta(t) \leqslant \boldsymbol{C}_\eta\big(1+t\big) \qquad \forall t \geqslant 0\,. \tag{2.4}$$

Moreover, for any $t_0 > 0$ there exists $C_\eta(t_0) > 0$ such that

$$\int_{t_0}^t \boldsymbol{m}_\eta(\tau)\mathrm{d}\tau \leqslant C_\eta(t_0)\big(1+t\big). \tag{2.5}$$

Proof. Estimate (2.4) has been established in [5, Theorem 1.9] using an argument that considers both L^1 and L^2 moment theory. In fact, it was shown in [5, Theorem 1.9] using a regularisation argument that

$$\boldsymbol{E}_\eta(t) := \boldsymbol{m}_\eta(t) + \frac{1}{2}\int_{\mathbb{R}^3} f(t,v)^2 \langle v\rangle^\eta \mathrm{d}v \leqslant C_\eta \left(t^{-\frac{3}{2}} + t\right) \qquad \forall t > 0\,. \tag{2.6}$$

Recalling [5, Eq. (3.36)] there exist positive constants $c_0(\eta)$, $c_1(\eta)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}_\eta(t) + c_0(\eta)\,\boldsymbol{m}_{\eta+\gamma}(t) \leqslant c_1(\eta) \qquad \eta > 4 + |\gamma|\,, \tag{2.7}$$

for any $t > 0$. Integrating (2.7) over (t_0, t) and using (2.6) and (2.4) yields

$$\int_{t_0}^t \boldsymbol{m}_{\eta+\gamma}(\tau)\mathrm{d}\tau \leqslant \frac{1}{c_0(\eta)}\big(c_1(\eta)\,t + \boldsymbol{E}_\eta(t_0)\big) \leqslant \frac{1}{c_0(\eta)}\Big(c_1(\eta)\,t + C_\eta t_0^{-\frac{3}{2}} + C_\eta t_0\Big),$$

which leads to (2.5). □

We can deduce from this result and suitable estimates for weighted Fisher information the following result about the entropy production for solutions to (1.1) in the range of moderately soft potentials $\gamma \in (-2, 0)$. Such a result improves the corresponding estimates derived in our previous contribution [5, Proposition 5.5] based on the study of the weighted Fisher information (A.13). Refer to Appendix A.2 for a proof.

Proposition 2.5. *Fix $\gamma \in (-2, 0)$, $\varepsilon \in (0, \varepsilon_{\mathrm{sat}})$ and $f_{\mathrm{in}} \in \mathcal{Y}_\varepsilon$. Let $f(t,\cdot)$ be a weak-solution to* (1.1)*. Additionally, assume that*

$$f_{\mathrm{in}} \in L^1_{\tilde\eta}(\mathbb{R}^3) \qquad \textit{for some } \ \tilde\eta > \eta + 2 + |\gamma|, \quad \eta \geqslant 0,$$

then, there exists $C_{\tilde\eta}(f_{\mathrm{in}})$ depending only on $\|f_{\mathrm{in}}\|_{L^1_{\tilde\eta}}$, $\mathcal{H}_\varepsilon(f_{\mathrm{in}})$, and η, such that

$$\int_{t_0}^t \mathscr{D}_\varepsilon^{(\eta)}(f(\tau))\mathrm{d}\tau \leqslant C_{\tilde\eta}(f_{\mathrm{in}})\frac{1+t}{\kappa_0(t)}\,, \qquad 0 \leqslant t_0 < t\,,$$

where $\kappa_0(t) = \inf\{\,1 - \varepsilon\,\|f(\tau)\|_{L^\infty}\ ;\ \tau \in (0,t)\}$.

2.3. **Statistical moments estimates for very soft potentials.** For the range of very soft potentials $\gamma \in (-3, -2]$ it is possible to establish the following statistical moment growth rate. Although insufficient for a long time convergence analysis, it is used for the implementation of the De Giorgi method in such a range in Section 3. Refer to Appendix A.3 for a proof.

Theorem 2.6. *Fix $\gamma \in (-3, -2]$, $\varepsilon \in (0, \varepsilon_{\mathrm{sat}})$, and initial datum $f_{\mathrm{in}} \in \mathcal{Y}_\varepsilon$. Let $f(t) = f(t,\cdot)$ be a weak-solution to* (1.1) *and assume that $\boldsymbol{m}_{\max\{5,\eta\}}(0) < \infty$ for $\eta > 2$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}_\eta(f(t)) + c_{0,\eta}\,\boldsymbol{m}_{\eta+\gamma}\big(f(t)(1-\varepsilon f(t))\big) \leqslant c_{1,\eta}\,\boldsymbol{m}_{\eta+\gamma-1}(f(t)) + c_{2,\eta}, \tag{2.8}$$

for constants $c_{j,\eta} > 0$ depending only on $\|f_{\mathrm{in}}\|_{L^1_2}$, $\mathcal{H}_\varepsilon(f_{\mathrm{in}})$, and η. As a consequence,

$$\boldsymbol{m}_\eta(f(t)) \leqslant \boldsymbol{C}_\eta\,(1+t)^{\frac{\eta-2}{1+|\gamma|}} \qquad \forall t > 0\,, \tag{2.9}$$

for a positive constant $\boldsymbol{C}_\eta$ depending only on $\boldsymbol{m}_{\max\{5,\eta\}}(0)$, γ, η and ε.

Remark 2.7. *The term $\boldsymbol{m}_{\eta+\gamma}(f(1-\varepsilon f))$ in (2.8) is the central difference relative to the classical Landau equation where $\varepsilon=0$. In classical Landau such term can be used readily as an absorption term leading to a linear time growth for any statistical moment if assumed initially finite.*

3. De Giorgi's approach for the appearance of a saturation gap $\kappa_0>0$

The scope of this section is to prove Theorem 1.6 and show the appearance of a saturation gap

$$\kappa_0(t)=\inf_{v\in\mathbb{R}^3}(1-\varepsilon f(t,v))>0, \qquad \forall t\geqslant T_*$$

after an *explicit* time $T_*>0$ depending only on the initial data. To do so, we use a diminishing oscillation argument for solutions of the LFD equation based on the analysis of the solution's level sets near the saturation level ε^{-1}. The argument is valid in the whole range of soft potentials $\gamma\in(-3,0)$, yet, for the thermalisation analysis, Theorem 1.6 will be used for the case of moderately soft potentials.

3.1. **Level-set energy method.** For a solution $f=f(t,v)$ to the LFD equation (1.1), its levels are given by

$$f_\ell^+(t,v):=f_\ell(t,v)\,\mathbf{1}_{\{f\geqslant \ell\}}, \qquad \ell\geqslant 0\,,$$

where $f_\ell(t,v):=f(t,v)-\ell$. Furthermore, for any times $0\leqslant T_1\leqslant T_2$ we introduce the energy functional

$$\mathscr{E}_\ell(T_1,T_2):=2\sup_{t\in[T_1,T_2)}\left[\frac{1}{2}\left\|f_\ell^+(t)\right\|_{L^2}^2+\alpha_0\int_{T_1}^t\left\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_\ell^+(\tau)\right\|_{H^1}^2\mathrm{d}\tau\right], \tag{3.1}$$

where $\alpha_0>0$ is a suitable explicit constant depending only on $\|f_{\rm in}\|_{L^1_2}$ and $\mathcal{H}_\varepsilon(f_{\rm in})$, see Lemma B.1. Since the analysis is done near the saturation level, we assume $\ell\in\left[\frac{3}{4\varepsilon},\frac{1}{\varepsilon}\right]$. The following holds, referring its proof to Appendix B.

Lemma 3.1. *Fix $\gamma\in(-3,0)$, $\varepsilon\in(0,\varepsilon_{\rm sat})$, and $f_{\rm in}\in\mathcal{Y}_\varepsilon$. Let $f(t,\cdot)$ be a weak-solution to (1.1). Then, for any $0\leqslant T_1<T_2\leqslant T_3$*

$$\mathscr{E}_\ell(T_2,T_3)\leqslant\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\|f_\ell^+(t)\|_{L^2}^2\mathrm{d}t \\ +2\ell(1-\varepsilon\ell)\int_{T_1}^{T_3}\mathrm{d}\tau\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+(\tau)](v)\,f_\ell^+(\tau,v)\mathrm{d}v\,, \quad \forall\ell\in\left[\frac{3}{4\varepsilon},\frac{1}{\varepsilon}\right]. \tag{3.2}$$

As a consequence, there exists a constant $C>0$ such that, for all $0\leqslant T_2<T_3$,

$$\mathscr{E}_\ell(T_2,T_3)\leqslant C\,\frac{(1-\varepsilon\ell)^2}{\varepsilon}\,\|f\|_{L^1}\left(1+(T_3-T_2)\frac{1-\varepsilon\ell}{\varepsilon}\,(1+\varepsilon\|f\|_{L^1})\right). \tag{3.3}$$

Moreover, for any $1\leqslant\eta<\min\left(\frac{3}{|\gamma|},\frac{3}{2}\right)$, there exists $C>0$ depending only on $\varepsilon,\eta,\|f_{\rm in}\|_{L^1_2}$ and $\mathcal{H}_\varepsilon(f_{\rm in})$ such that

$$\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+(t)]\,f_\ell^+(t,v)\mathrm{d}v \\ \leqslant\frac{C}{\ell-k}\boldsymbol{m}_s(t)^\beta\left(1+\ell^{-\beta}(1-\varepsilon\ell)^{(2\beta-1)^+}\right)\|f_k^+(t)\|_{L^2}^{1-\beta}\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+(t)\|_{H^1}^2\,, \quad t\geqslant 0, \tag{3.4}$$

holds for any $0 < k < \ell \in \left[\frac{3}{4\varepsilon}, \frac{1}{\varepsilon}\right]$*, and with* $s = \frac{3|\gamma|\eta}{7\eta-6}$, $\beta = \frac{|\gamma|}{s} = \frac{7\eta-6}{3\eta}$.

The fundamental result for the implementation of the level set analysis is given by the following proposition.

Proposition 3.2. *Fix* $\gamma \in (-3,0)$, $\varepsilon \in (0, \varepsilon_{\rm sat})$*, and* $f_{\rm in} \in \mathcal{Y}_\varepsilon$*. Let* $f(t,\cdot)$ *be a weak-solution to* (1.1)*. Then, for any*

$$0 < \theta_1 < 2, \quad \text{and} \quad \left(\frac{2|\gamma|-4}{7-2|\gamma|}\right)^+ < \theta_2 \leqslant 2,$$

there exists constant $C > 0$ *depending only on* θ_1, θ_2, γ, $\|f_{\rm in}\|_{L^1_2}$ *and* $\mathcal{H}_\varepsilon(f_{\rm in})$*, but not on* ε*, such that*

$$\mathscr{E}_\ell(T_2,T_3) \leqslant \frac{C}{T_2-T_1}(\ell-k)^{-\frac{2+\theta_1}{3}}\left[\sup_{\tau\in(T_1,T_2)} \boldsymbol{m}_{\frac{3|\gamma|}{2-\theta_1}}(\tau)\right]^{\frac{2-\theta_1}{3}} \mathscr{E}_k(T_1,T_2)^{1+\frac{\theta_1}{3}}$$
$$+ C\left[\sup_{\tau\in[T_1,T_3]} \boldsymbol{m}_{|\gamma|(1+\theta_2)}(\tau)\right]^{\frac{1}{1+\theta_2}} \frac{\ell(1-\varepsilon\ell)}{\ell-k}\left(1+\ell^{-\frac{1}{1+\theta_2}}\varepsilon^{\frac{(1-\theta_2)^+}{1+\theta_2}}\right)\mathscr{E}_k(T_1,T_3)^{1+\frac{\theta_2}{2(1+\theta_2)}} \tag{3.5}$$

for any times $0 \leqslant T_1 < T_2 \leqslant T_3$ *and levels* $0 < k \leqslant \ell \in \left[\frac{3}{4\varepsilon}, \frac{1}{\varepsilon}\right]$.

Proof. Fix $0 \leqslant T_1 < T_2 \leqslant T_3$ and estimate the terms in (3.2). For the first term in the right side use the estimate derived in [5, Lemma 4.2, Eq. (4.5)] [2], namely, for any $q \in \left(\frac{8}{3}, \frac{10}{3}\right)$, there exists $c_q > 0$ such that

$$\|f_\ell^+\|_{L^2}^2 \leqslant \frac{c_q}{(\ell-k)^{q-2}}\left\|\langle\cdot\rangle^{\frac{3|\gamma|}{10-3q}} f_k^+\right\|_{L^1}^{\frac{10}{3}-q} \|f_k^+\|_{L^2}^{2(q-\frac{8}{3})}\left\|\nabla\left(\langle\cdot\rangle^{\frac{\gamma}{2}} f_k^+\right)\right\|_{L^2}^2, \quad 0 \leqslant k < \ell. \tag{3.6}$$

Writing $q = \frac{8+\theta_1}{3}$ and $\theta_1 \in (0,2)$ readily yields to

$$\frac{1}{T_2-T_1}\int_{T_1}^{T_2} \|f_\ell^+(\tau)\|_{L^2}^2\,\mathrm{d}\tau$$
$$\leqslant \frac{C_1}{(T_2-T_1)(\ell-k)^{\frac{\theta_1+2}{3}}}\left[\sup_{\tau\in(T_1,T_2)} \boldsymbol{m}_\nu(\tau)\right]^{2-\theta_1} \mathscr{E}_k(T_1,T_2)^{1+\frac{\theta_1}{3}}, \qquad \nu = \frac{3|\gamma|}{2-\theta_1}, \tag{3.7}$$

with $C_1 > 0$ depending only on θ, γ, and α_0. In regard of the second term in (3.2), we resort to estimate (3.4) to deduce that

$$2\ell(1-\varepsilon\ell)\int_{T_1}^{T_3}\mathrm{d}\tau\int_{\mathbb{R}^3} \boldsymbol{c}_\gamma[f_\ell^+(\tau)](v)\, f_\ell^+(\tau,v)\mathrm{d}v$$
$$\leqslant C_2\left[\sup_{\tau\in[T_1,T_3]} \boldsymbol{m}_s(\tau)\right]^\beta \frac{\ell(1-\varepsilon\ell)}{\ell-k}\left(1+\ell^{-\beta}(1-\varepsilon\ell)^{(2\beta-1)^+}\right)\mathscr{E}_k(T_1,T_3)^{1+\frac{1-\beta}{2}} \tag{3.8}$$

[2]The work [5] deals with the case $\gamma \in (-2,0)$, yet, estimate (4.5) is valid for any $\gamma \in (-3,0)$.

with β, s defined in (B.7). As a consequence, there exists a constant $C>0$ depending on η, γ, θ_1, and $f_{\rm in}$, through $\|f_{\rm in}\|_{L^1_2}$ and $\mathcal{H}_\varepsilon(f_{\rm in})$ such that

$$\mathscr{E}_\ell(T_2,T_3)\leqslant \frac{C_1(\ell-k)^{-\frac{\theta_1+2}{3}}}{(T_2-T_1)}\left[\sup_{\tau\in(T_1,T_2)}\boldsymbol{m}_\nu(\tau)\right]^{2-\theta_1}\mathscr{E}_k(T_1,T_2)^{1+\frac{\theta_1}{3}}$$
$$+C_2\left[\sup_{\tau\in[T_1,T_3]}\boldsymbol{m}_s(\tau)\right]^{\beta}\frac{\ell(1-\varepsilon\ell)}{\ell-k}\left(1+\ell^{-\beta}\left(1-\varepsilon\ell\right)^{(2\beta-1)^+}\right)\mathscr{E}_k(T_1,T_3)^{1+\frac{1-\beta}{2}}$$

for any $0<k\leqslant\ell\in\left[\frac{3}{4\varepsilon},\frac{1}{\varepsilon}\right]$. Recall that s, β are given by (B.7) and $\nu=\frac{3|\gamma|}{\theta_1+2}$. Writing $s=(1+\theta_2)|\gamma|$, that is, $\theta_2=\frac{6-4\eta}{7\eta-6}$, one deduces the result. Notice also that the condition $1\leqslant\eta<\min\left(\frac{3}{|\gamma|},\frac{3}{2}\right)$ translates into $\left(\frac{2|\gamma|-4}{7-2|\gamma|}\right)^+<\theta_2\leqslant 3$ while $\beta=\frac{|\gamma|}{s}=\frac{1}{1+\theta_2}$. □

3.2. **Proof of Theorem 1.6: Appearance of a non saturation gap.** The proof follow three main steps.

First step: Construction of a solution's sequence $(\mathscr{E}_n)_n$ ***of energies.*** Fix $0<\delta\leqslant\frac{1}{4}$, $0<\Delta\leqslant T$, and define the following sequences of gaps, levels, and times

$$\delta_n:=\frac{\delta}{2}\left(1+\frac{1}{2^n}\right),\qquad k_n:=\frac{1-\delta_n}{\varepsilon},\qquad\text{and}\qquad t_n:=T-\frac{\Delta}{2}\left(1+\frac{1}{2^n}\right),\qquad n\in\mathbb{N}.$$

Define the sequence of energies

$$\mathscr{E}_n:=\mathscr{E}_{k_n}(t_n,T)\,,\qquad n\in\mathbb{N}\,. \tag{3.9}$$

Apply Proposition 3.2 with the choices

$$k=k_n,\qquad \ell=k_{n+1},\qquad T_1=t_n,\qquad T_2=t_{n+1},\qquad T_3=T$$

so that

$$\ell-k=\frac{\delta}{\varepsilon\,2^{n+2}},\qquad T_2-T_1=\frac{\Delta}{2^{n+2}},$$

while

$$\frac{3}{4\varepsilon}\leqslant\ell\leqslant\frac{1}{\varepsilon},\qquad 1-\varepsilon\ell=1-\varepsilon k_{n+1}=\delta_{n+1}\leqslant\delta,\qquad \ell(1-\varepsilon\ell)\leqslant\frac{\delta}{\varepsilon}.$$

With the notations of Proposition 3.2, introduce the *non decreasing* functions $\boldsymbol{\phi}_i\;:\;[0,T]\to\mathbb{R}^+$, with $i=1,2$, given by

$$\boldsymbol{\phi}_1(t):=\sup_{\tau\in[0,t]}\left(\boldsymbol{m}_{\frac{3|\gamma|}{2-\theta_1}}(\tau)+1\right),\qquad \boldsymbol{\phi}_2(t):=\sup_{\tau\in[0,t]}\left(\boldsymbol{m}_{|\gamma|(1+\theta_2)}(\tau)+1\right),\qquad t\in[0,T], \tag{3.10}$$

where $0<\theta_1<2$, $\left(\frac{2|\gamma|-4}{7-2|\gamma|}\right)^+<\theta_2\leqslant 2$. Therefore, it readily follows that

$$\mathscr{E}_{n+1}\leqslant\frac{C\boldsymbol{\phi}_1(T)^{\frac{2-\theta_1}{3}}}{\Delta}2^{\frac{5+\theta_1}{3}(n+2)}\left(\frac{\varepsilon}{\delta}\right)^{\frac{2+\theta_1}{3}}\mathscr{E}_n^{1+\frac{\theta}{3}}$$
$$+C\boldsymbol{\phi}_2(T)^{\frac{1}{1+\theta_2}}2^{n+2}\left(1+\left(\frac{4}{3}\right)^{\frac{1}{1+\theta_2}}\varepsilon^{\frac{1+(1-\theta_2)^+}{1+\theta_2}}\right)\mathscr{E}_n^{1+\frac{\theta_2}{2(1+\theta_2)}}.$$

Since $1 \leqslant \frac{5+\theta_1}{3} \leqslant 3$, $1 \leqslant \frac{1}{1+\theta_2} \leqslant \frac{1}{3}$, and $\varepsilon \in (0, \varepsilon_{\rm sat})$, there exists $C_\gamma > 0$ depending on γ, $\varepsilon_{\rm sat}$, $\|f_{\rm in}\|_{L^1_2}$, and $\mathcal{H}_\varepsilon(f_{\rm in})$, such that

$$\mathscr{E}_{n+1} \leqslant C_\gamma 2^{3n} \left(\Delta^{-1} \delta^{-\frac{2+\theta_1}{3}} \mathscr{E}_n^{1+\frac{\theta_1}{3}} \boldsymbol{\phi}_1(T)^{\frac{2-\theta_1}{3}} + \mathscr{E}_n^{1+\frac{\theta_2}{2(1+\theta_2)}} \boldsymbol{\phi}_2(T)^{\frac{1}{1+\theta_2}} \right), \quad n \in \mathbb{N}. \tag{3.11}$$

Second step: Construction of an upper barrier sequence. Introduce the sequence

$$\mathscr{U}_n := \mathscr{E}_0\, Q^{-n}, \qquad n \in \mathbb{N},$$

with $Q > 1$ to be determined in such a way that, for sufficiently small $\delta > 0$ and sufficiently large $\Delta > 0$, the sequence $(\mathscr{U}_n)_n$ satisfies (3.11) with the reverse inequality, that is,

$$\mathscr{U}_{n+1} \geqslant C_\gamma 2^{3n} \left(\Delta^{-1} \delta^{-\frac{2+\theta_1}{3}} \mathscr{U}_n^{1+\frac{\theta_1}{3}} \boldsymbol{\phi}_1(T)^{\frac{2-\theta_1}{3}} + \mathscr{U}_n^{1+\frac{\theta_2}{2(1+\theta_2)}} \boldsymbol{\phi}_2(T)^{\frac{1}{1+\theta_2}} \right), \quad n \in \mathbb{N}.$$

Such an inequality is equivalent to

$$1 \geqslant C_\gamma 2^{3n} \left(\Delta^{-1} \delta^{-\frac{2+\theta_1}{3}} \mathscr{E}_0^{\frac{\theta_1}{3}} Q^{1-n\frac{\theta_1}{3}} \boldsymbol{\phi}_1(T)^{\frac{2-\theta_1}{3}} + Q^{1-\frac{n\theta_2}{2(1+\theta_2)}} \mathscr{E}_0^{\frac{\theta_2}{2(1+\theta_2)}} \boldsymbol{\phi}_2(T)^{\frac{1}{1+\theta_2}} \right). \tag{3.12}$$

Choosing $Q > \max\left(8^{\frac{3}{\theta_1}}, 8^{2+\frac{2}{\theta_2}}\right)$, one sees that (3.12) will be true provided that

$$1 \geqslant C_\gamma Q \left(\Delta^{-1} \delta^{-\frac{2+\theta_1}{3}} \mathscr{E}_0^{\frac{\theta_1}{3}} \boldsymbol{\phi}_1(T)^{\frac{2-\theta_1}{3}} + \mathscr{E}_0^{\frac{\theta_2}{2(1+\theta_2)}} \boldsymbol{\phi}_2(T)^{\frac{1}{1+\theta_2}} \right), \tag{3.13}$$

which is seen as a condition on δ, Δ, and $\mathscr{E}_0 := \mathscr{E}_{k_0}(t_0, T)$. One deduces from (3.3) with $\ell = k_0$, $T_2 = t_0$ and $T_3 = T$ that

$$\mathscr{E}_{k_0}(t_0, T) \leqslant C \frac{\delta^2}{\varepsilon} \|f\|_{L^1} \left(1 + (T - t_0) \frac{\delta}{\varepsilon} \left(1 + \varepsilon \|f\|_{L^1}\right) \right),$$

where we used that $1 - \varepsilon\, k_0 = \delta$. Moreover, since $\Delta = T - t_0$ it follows that

$$\mathscr{E}_0 \leqslant C \left(\frac{\delta}{\varepsilon}\right)^2 \|f\|_{L^1} \left(\varepsilon + \Delta\, \delta\, (1 + \varepsilon \|f\|_{L^1})\right) \leqslant \boldsymbol{C}[f_{\rm in}, \varepsilon]\, \max(1, \delta\, \Delta)\, \delta^2$$

with

$$\boldsymbol{C}[f_{\rm in}, \varepsilon] := \max\left(1, C\varepsilon^{-2}\left(\varepsilon + 1 + \varepsilon \|f_{\rm in}\|_{L^1}\right) \|f_{\rm in}\|_{L^1}\right).$$

Therefore, condition (3.13) will be met if

$$1 \geqslant C_\gamma Q \Bigg(\left(\boldsymbol{C}[f_{\rm in}, \varepsilon] \max(1, \delta\, \Delta)\right)^{\frac{\theta_1}{3}} \Delta^{-1} \delta^{\frac{\theta_1 - 2}{3}} \boldsymbol{\phi}_1(T)^{\frac{2-\theta_1}{3}} + \left(\boldsymbol{C}[f_{\rm in}, \varepsilon] \max(1, \delta\, \Delta)\right)^{\frac{\theta_2}{2(1+\theta_2)}} \delta^{\frac{\theta_2}{1+\theta_2}} \boldsymbol{\phi}_2(T)^{\frac{1}{1+\theta_2}} \Bigg).$$

At this point, choose

$$\delta = \frac{1}{\Delta},$$

and leave $T \geqslant \Delta > 0$ as free parameter to be suitably chosen, together with θ_1 and θ_2, so that $0 \leqslant \delta \leqslant \frac{1}{4}$. Then, setting [3]

$$K := K[f_{\rm in}, \boldsymbol{\varepsilon}, Q] = C_\gamma\, Q\, \boldsymbol{C}[f_{\rm in}, \boldsymbol{\varepsilon}]^{\frac{2}{3}},$$

which is independent of Δ and T, we observe that (3.13) will be met if

$$1 \geqslant K\left(\Delta^{-\frac{\theta_1+1}{3}}\,\boldsymbol{\phi}_1(T)^{\frac{2-\theta_1}{3}} + \Delta^{-\frac{\theta_2}{1+\theta_2}}\,\boldsymbol{\phi}_2(T)^{\frac{1}{1+\theta_2}}\right),$$

which is satisfied, for instance, whenever

$$\Delta \geqslant \max\left((2K)^{\frac{3}{\theta_1+1}}\,\boldsymbol{\phi}_1(T)^{\frac{2-\theta_1}{1+\theta_1}}\,,\,(2K)^{\frac{\theta_2+1}{\theta_2}}\,\boldsymbol{\phi}_2(T)^{\frac{1}{\theta_2}}\right) > 0.$$

Notice that choosing $Q > C_\gamma^{-1}$ one has that $K > 1$ and the aforementioned inequality holds if

$$\Delta \geqslant \Delta(T) := (2K)^{\max(1+\frac{1}{\theta_2},3)}\max\left(\boldsymbol{\phi}_1(T)^{\frac{2-\theta_1}{1+\theta_1}}, \boldsymbol{\phi}_2(T)^{\frac{1}{\theta_2}}\right) > 0. \tag{3.14}$$

Since $K > 1$ and $\boldsymbol{\phi}_i(T) \geqslant 1$, it holds that $\Delta(T) \geqslant 4$ which ensures that $0 \leqslant \delta = \Delta^{-1} \leqslant \frac{1}{4}$.

Now, recall that we have the constraint $T \geqslant \Delta > 0$. Thus, define

$$T_* := \inf\left\{T > 0 \;:\; \Delta(T) \leqslant T\right\}. \tag{3.15}$$

The functions $\boldsymbol{\phi}_i(\cdot)$ are nondecreasing with a sub-linear time growth that can be estimated thanks to Theorems 2.4 and 2.6 ensuring that $T_* < \infty$. In Section 3.3 we present explicit estimates of these parameters.

In this way, for any time $T > T_*$ and setting $\delta = \delta(T) = \Delta^{-1}(T) = \Delta^{-1}$, the sequence $(\mathscr{U}_n)_n$ given by

$$\mathscr{U}_n = \mathscr{E}_0\, Q^{-n}, \qquad \mathscr{E}_0 = \mathscr{E}_{\frac{1-\delta(T)}{\boldsymbol{\varepsilon}}}\left(T - \Delta(T), T\right),$$

with $Q > 1$ sufficiently large and explicit dependent of $f_{\rm in}$, satisfies (3.11) with reverse inequality.

Final step: A comparison argument. The sequences $(\mathscr{E}_n)_n$ and $(\mathscr{U}_n)_n$ satisfy (3.11) with a reverse inequality while coinciding for $n = 0$, therefore,

$$0 \leqslant \mathscr{E}_n \leqslant \mathscr{U}_n \qquad \forall n \in \mathbb{N}.$$

Since $Q > 1$, it holds that $0 \leqslant \lim_n \mathscr{E}_n \leqslant \lim_n \mathscr{U}_n = 0$. Furthermore, since the sequence of levels $(k_n)_n$ and the sequence of times $(t_n)_n$ are such that

$$\lim_n k_n = \boldsymbol{\ell}(T) = \frac{1}{\boldsymbol{\varepsilon}}\left(1 - \frac{\delta(T)}{2}\right), \qquad \lim_n t_n = T - \frac{\Delta(T)}{2}, \qquad \delta(T) = \Delta^{-1}(T),$$

this implies that

$$\sup_{\tau\in[T-\frac{\Delta(T)}{2},T]} \|f^+_{\boldsymbol{\ell}(T)}(\tau)\|^2_{L^2} = 0\,, \qquad T \geqslant T_* > 0,$$

[3] Recall that $\boldsymbol{C}[f_{\rm in}, \boldsymbol{\varepsilon}] \geqslant 1$ so that $\boldsymbol{C}[f_{\rm in}, \boldsymbol{\varepsilon}]^{\frac{\theta_1}{3}} \leqslant \boldsymbol{C}[f_{\rm in}, \boldsymbol{\varepsilon}]^{\frac{2}{3}}$. Similarly, $\boldsymbol{C}[f_{\rm in}, \boldsymbol{\varepsilon}]^{\frac{\theta_2}{2(1+\theta_2)}} \leqslant \boldsymbol{C}[f_{\rm in}, \boldsymbol{\varepsilon}]^{\frac{1}{3}} \leqslant \boldsymbol{C}[f_{\rm in}, \boldsymbol{\varepsilon}]^{\frac{2}{3}}$ since $\theta_1, \theta_2 \in (0, 2)$.

that is

$$\|f(\tau)\|_{L^\infty} \leqslant \frac{1}{\varepsilon}\Big(1 - \frac{\delta(T)}{2}\Big) \quad \text{for any} \quad \tau \in \left[T - \frac{\Delta(T)}{2}, T\right], \quad T \geqslant T_* > 0,$$

which proves Theorem 1.6 by simply choosing $t = T$ and setting $\kappa_0(t) = \frac{\delta(t)}{2}$.

3.3. **Explicit estimates.** Let us explore more into explicit estimates for the aforementioned $\Delta(T)$ and T_* by recalling the definitions of ϕ_1, ϕ_2 in (3.10). We distinguish several cases.

• *Mildly soft potentials* $\gamma \in (-\frac{4}{3}, 0)$*:* Choose, for instance, $\theta_1 = 2 - \frac{3|\gamma|}{2}$ and $\theta_2 = \frac{1}{2}$ so that $|\gamma|(1+\theta_2) \leqslant 2$ and $\frac{3|\gamma|}{2-\theta_1} = 2$. One deduces from (3.10) that

$$\phi_2(T) \leqslant \phi_1(T) = \boldsymbol{m}_2(T) + 1 = \boldsymbol{m}_2(f_{\rm in}) + 1 \qquad \forall T > 0.$$

Therefore, $\Delta(T)$ in (3.14) can be defined as

$$\Delta(T) = \Delta_* = (2K)^3\big(\boldsymbol{m}_2 f_{\rm in}) + 1\big)^2 \tag{3.16}$$

which is independent of T. Using (3.15) it follows that $T_* = \Delta_*$ and, consequently,

$$\kappa_0(t) = \underline{\kappa_0} = \frac{1}{2}\Delta_*^{-1}, \qquad t > T_*. \tag{3.17}$$

• *Intermediate soft potentials* $\gamma \in \big(-\frac{5}{2}, -\frac{4}{3}\big]$*:* Note that $\left(\frac{2|\gamma|-4}{7-2|\gamma|}\right)^+ \leqslant \frac{1}{2}$, therefore, choosing $\theta_1 \in (0,1]$ and $\left(\frac{2|\gamma|-4}{7-2|\gamma|}\right)^+ < \theta_2 = \frac{1+\theta_1}{2-\theta_1} \leqslant 2$ in (3.10), it follows that

$$\phi_2(T) = \phi_1(T) = \sup_{\tau\in[0,T]}\left(\boldsymbol{m}_{\frac{3|\gamma|}{2-\theta_1}}(\tau) + 1\right) \qquad \forall T > 0.$$

With this, $\Delta(T)$ as defined in (3.14) reads

$$\Delta(T) := (2K)^3 \sup_{t\in[0,T]}\left(\boldsymbol{m}_{\frac{3|\gamma|}{2-\theta_1}}(t) + 1\right)^{\frac{2-\theta_1}{1+\theta_1}}. \tag{3.18}$$

Considering the definition (3.15), to ensure $T_* < \infty$ one needs to find $\theta_1 \in (0,1]$ such that

$$\lim_{T\to\infty} T^{-\frac{1+\theta_1}{2-\theta_1}}\boldsymbol{m}_{\frac{3|\gamma|}{2-\theta_1}}(T) = 0. \tag{3.19}$$

When $\gamma \in (-2, -\frac{4}{3}]$ Theorem 2.4 shows that (3.19) holds if $\frac{1+\theta_1}{2-\theta_1} > 1$, that is, for any $\theta_1 \in \big(\frac{1}{2}, 1\big]$. When $\gamma \in (-\frac{5}{2}, -2]$ Theorem 2.6 shows that (3.19) holds if $\left(\frac{\frac{3|\gamma|}{2-\theta_1} - 2}{1+|\gamma|}\right) > \frac{1+\theta_1}{2-\theta_1}$, that is, for any $\theta_1 \in (0,1]$. Consequently,

$$\kappa_0(t) = \frac{1}{2^4K^3}\left(\sup_{\tau\in[0,t]} \boldsymbol{m}_{\frac{3|\gamma|}{2-\theta_1}}(\tau) + 1\right)^{-\frac{2-\theta_1}{1+\theta_1}}, \tag{3.20}$$

where $\theta_1 \in \big[\frac{1}{2}, 1\big]$ if $\gamma \in \big(-2, -\frac{4}{3}\big]$ and $\theta_1 \in (0,1]$ for $\gamma \in (-\frac{5}{2}, -2]$. From here, we can use Theorem 2.4 and 2.6 to estimate $\boldsymbol{m}_{\frac{3|\gamma|}{2-\theta_1}}(t)$ to explore $\kappa_0(t)$ as $t \to \infty$. Indeed, choosing $\theta_1 = 1$

we obtain the best rates given the moments available. It follows that

$$\gamma \in (-2,-\tfrac{4}{3}] : \; \kappa_0(t) \geqslant C_0\, t^{-\frac{1}{2}} \quad \text{and} \quad \gamma \in (-\tfrac{5}{2}, 2] : \; \kappa_0(t) \geqslant C_1\, t^{-\frac{3|\gamma|-2}{2(1+|\gamma|)}} \quad t \geqslant T_* \tag{3.21}$$

with $C_0 > 0$, $C_1 > 0$, and $T_* > 0$ explicitly depending on K and $\boldsymbol{m}_{3|\gamma|}(0)$.

• *Very soft potential* $\gamma \in (-3, -\frac{5}{2}]$: In this range $\frac{2|\gamma|-4}{7-2|\gamma|} \geqslant \frac{1}{2}$, thus, choosing $\theta_1 > 0$ sufficiency small it holds that $\phi_2(T) \geqslant \phi_1(T)$. Then (3.14) yields

$$\Delta(T) = (2K)^3 \phi_2(T)^{\frac{1}{\theta_2}} = (2K)^3 \sup_{t\in[0,T]} \left(\boldsymbol{m}_{|\gamma|(1+\theta_2)}(t) + 1\right)^{\frac{1}{\theta_2}}. \tag{3.22}$$

Furthermore, the condition $T_* < \infty$ in the definition (3.15) reads

$$\lim_{T\to\infty} T^{-\theta_2} \boldsymbol{m}_{|\gamma|(1+\theta_2)}(T) = 0 \tag{3.23}$$

which holds for any $\theta_2 > |\gamma| - 2$ thanks to Theorem 2.6. Since $\frac{2|\gamma|-4}{7-2|\gamma|} \geqslant |\gamma| - 2$, (3.23) holds for any $\theta_2 \in \left(\frac{2|\gamma|-4}{7-2|\gamma|}\, , 2\right]$ and the following non saturation gap estimate holds

$$\kappa_0(t) = \frac{1}{2^4 K^3} \left(\sup_{\tau\in[0,t]} \boldsymbol{m}_{|\gamma|(1+\theta_2)}(\tau) + 1 \right)^{-\frac{1}{\theta_2}}, \qquad \theta_2 \in \left(\frac{2|\gamma|-4}{7-2|\gamma|}\, , 2\right]. \tag{3.24}$$

Using Theorem 2.6 with the choice $\theta_2 = 2$ one obtains

$$\kappa_0(t) \geqslant C_2\, t^{-\frac{3|\gamma|-2}{2(1+|\gamma|)}}, \qquad t \geqslant T_*$$

with C_2 and T_* explicitly depending on $\boldsymbol{m}_{3|\gamma|}(0)$ and K.

Remark 3.3. *Obseve that in the whole range of soft potentials*

$$\kappa_0(t) \geqslant C_1 \boldsymbol{m}_{3|\gamma|}(t)^{-\frac{1}{2}}, \qquad t > T_*$$

with constant $C_1 > 0$ and $T_* > 0$ depending only on $\|f_{\rm in}\|_{L^1_2}$, $\mathcal{H}_\varepsilon(f_{\rm in})$ and $\boldsymbol{m}_{3|\gamma|}(0)$.

4. Entropy production estimate: Proof of Theorem 1.5

The scope of this section is to prove the entropy production estimate in Theorem 1.5. We consider the class of functions $f = f(v)$ belonging to $\mathcal{Y}_\varepsilon$ where we recall $\mathcal{Y}_\varepsilon = \mathcal{Y}_\varepsilon[\varrho, 0, \vartheta]$ with fixed $\varrho > 0$, $\vartheta > 0$. Assume that

$$m := \int_{\mathbb{R}^3} f(v)(1-\varepsilon f(v))\mathrm{d}v > 0, \qquad \xi := \frac{1}{m}\int_{\mathbb{R}^3} f(v)(1-\varepsilon f(v)) v \mathrm{d}v,$$
$$\text{and} \qquad \boldsymbol{e} := \int_{\mathbb{R}^3} f(v)(1-\varepsilon f(v))\, |v-\xi|^2 \mathrm{d}v > 0. \tag{4.1}$$

Recall that, for collision interaction associated to Ψ, we introduced the entropy production $\mathscr{D}_\varepsilon^{[\Psi]}(\cdot)$ in (5). Since Theorem 1.5 assumes $\Psi(|z|) \geqslant |z|^2$, it is enough to bound from below the entropy production $\mathscr{D}_{\varepsilon}^{(0)}(\cdot)$ associated to the interaction $\Psi(z)=|z|^2$, that is,

$$\mathscr{D}_{\varepsilon}^{(0)}(f)=\frac{1}{2}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}|v-v_*|^2\, f\, f_*(1-\varepsilon f)(1-\varepsilon f_*)$$
$$\times\left|\Pi(v-v_*)\left(\frac{\nabla f}{f(1-\varepsilon f)}-\frac{\nabla f_*}{f_*(1-\varepsilon f_*)}\right)\right|^2\mathrm{d}v\mathrm{d}v_*\,,$$

where $\Pi(z)=\mathrm{Id}-\widehat{z}\otimes\widehat{z}$, $z\in\mathbb{R}^3\setminus\{0\}$, $\widehat{z}=\frac{z}{|z|}$. We have the following observation.

Lemma 4.1. *Assume $f\in L^1_2(\mathbb{R}^3)$ satisfies (1.14). Then*

$$\mathscr{D}_{\varepsilon}^{(0)}(f)=\mathcal{R}_{\varepsilon}(f)-6\varrho^2$$

with

$$\mathcal{R}_{\varepsilon}(f)=\int_{\mathbb{R}^3\times\mathbb{R}^3}f_*(1-\varepsilon f_*)|v-v_*|^2\frac{\Pi(v-v_*)\nabla f}{f(1-\varepsilon f)}\cdot\nabla f\mathrm{d}v\mathrm{d}v_*.$$

Furthermore,

$$\mathcal{R}_{\varepsilon}(f)\geqslant\mathcal{R}^1_{\varepsilon}(f):=\int_{\mathbb{R}^3}\mathcal{A}_f\,\frac{\nabla f(v)}{F(v)}\cdot\nabla f(v)\mathrm{d}v$$

with

$$F=f(1-\varepsilon f),\qquad \mathcal{A}_f=\int_{\mathbb{R}^3}F_*|v_*-\xi|^2\,\Pi(v_*-\xi)\mathrm{d}v_*.$$

Proof. Let $f\in L^1_2(\mathbb{R}^3)$ be fixed. Setting $F=f(1-\varepsilon f)$, one has that

$$\mathscr{D}_{\varepsilon}^{(0)}(f)=\frac{1}{2}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}|v-v_*|^2\,F\,F_*\left|\Pi(v-v_*)\left(\frac{\nabla f}{F}-\frac{\nabla f_*}{F_*}\right)\right|^2\mathrm{d}v\mathrm{d}v_*\,. \tag{4.2}$$

Since Π is an orthogonal projection, one checks after expanding the square in (4.2) and using a symmetry argument that

$$\begin{aligned}\mathscr{D}_{\varepsilon}^{(0)}(f)&=\int_{\mathbb{R}^3\times\mathbb{R}^3}|v-v_*|^2\,F\,F_*\left|\Pi(v-v_*)\frac{\nabla f}{F}\right|^2\mathrm{d}v\mathrm{d}v_*\\&\qquad-\int_{\mathbb{R}^3\times\mathbb{R}^3}|v-v_*|^2\Pi(v-v_*)\nabla f\cdot\nabla f_*\mathrm{d}v\mathrm{d}v_*\\&=\mathcal{R}_{\varepsilon}(f)-\int_{\mathbb{R}^3\times\mathbb{R}^3}|v-v_*|^2\Pi(v-v_*)\nabla f\cdot\nabla f_*\mathrm{d}v\mathrm{d}v_*\,.\end{aligned}$$

The latter integral is easily computed using the definition of $\Pi(v-v_*)$

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3}|v-v_*|^2\Pi(v-v_*)\nabla f\cdot\nabla f_*\mathrm{d}v\mathrm{d}v_*&=\sum_{i=1}^3\int_{\mathbb{R}^3\times\mathbb{R}^3}\left(|v-v_*|^2-(v-v_*)_i^2\right)\partial_i f\partial_{i*}f_*\mathrm{d}v\mathrm{d}v_*\\&\quad-\sum_{i\neq j}\int_{\mathbb{R}^3\times\mathbb{R}^3}(v-v_*)_i(v-v_*)_j\partial_i f\partial_{j*}f_*\mathrm{d}v\mathrm{d}v_*\end{aligned}$$

where, for $z\in\mathbb{R}^3$, z_i denotes the i-th coordinate of z whereas $\partial_i=\frac{\partial}{\partial v_i}$, $\partial_{i*}=\frac{\partial}{\partial v_{*i}}$. Using integration by parts, the first integral vanishes while the second is given by

$$\sum_{i\neq j}\int_{\mathbb{R}^3\times\mathbb{R}^3}(v-v_*)_i(v-v_*)_j\partial_i f\partial_{j*}f_*\mathrm{d}v\mathrm{d}v_*=-\sum_{i\neq j}\int_{\mathbb{R}^3\times\mathbb{R}^3}f f_*\mathrm{d}v\mathrm{d}v_*=-6\varrho^2$$

where we used (1.14). This proves the first part of the result. The rest of our analysis will be to split then $\mathscr{D}_\varepsilon^{(0)}(f)$. Observe that, for the "centered" velocities $\bar{v} := v - \xi$ and $\bar{v}_* = v_* - \xi$, one has

$$|v-v_*|^2\,\Pi(v-v_*) = |\bar{v}-\bar{v}_*|^2\,\Pi(\bar{v}-\bar{v}_*) = \Big(|\bar{v}|^2\,\mathrm{Id} - \bar{v}\otimes\bar{v}\Big) + \Big(|\bar{v}_*|^2\,\mathrm{Id} - \bar{v}_*\otimes\bar{v}_*\Big)$$
$$- \Big(2\bar{v}\cdot\bar{v}_*\,\mathrm{Id} - \bar{v}\otimes\bar{v}_* - \bar{v}_*\otimes\bar{v}\Big).$$

Since $\displaystyle\int_{\mathbb{R}^3}\bar{v}_*F(v_*)\mathrm{d}v_* = 0$, it holds

$$\int_{\mathbb{R}^3}F(v_*)|v-v_*|^2\,\Pi(v-v_*)\mathrm{d}v_* = m\,\Big(|\bar{v}|^2\,\mathrm{Id} - \bar{v}\otimes\bar{v}\Big) + \int_{\mathbb{R}^3}F(v_*)\Big(|\bar{v}_*|^2\,\mathrm{Id} - \bar{v}_*\otimes\bar{v}_*\Big)\mathrm{d}v_*$$
$$= m\,|\bar{v}|^2\,\Pi(\bar{v}) + \int_{\mathbb{R}^3}F(v_*)\,|\bar{v}_*|^2\,\Pi(\bar{v}_*)\mathrm{d}v_*\,.$$

Notice that last integral is independent of v and equal to $\mathcal{A}_f$. We deduce that

$$\mathcal{R}_\varepsilon(f) = m\int_{\mathbb{R}^3}|\bar{v}|^2\,\Pi(\bar{v})\frac{\nabla f}{F}\cdot\nabla f\mathrm{d}v + \int_{\mathbb{R}^3}\mathcal{A}_f\frac{\nabla f}{F}\cdot\nabla f\mathrm{d}v = \mathcal{R}^2_\varepsilon(f) + \mathcal{R}^1_\varepsilon(f)\,.$$

Since $\Pi(\bar{v})\frac{\nabla f}{F}\cdot\nabla f = F\left|\Pi(\bar{v})\frac{\nabla f}{F}\right|^2 \geqslant 0$, we deduce that $\mathcal{R}^2_\varepsilon(f)\geqslant 0$ and get the result. □

Notice that both $\mathcal{R}^2_\varepsilon(f)$ and $\mathcal{R}^1_\varepsilon(f)$ are non-negative. In particular, $\mathcal{R}^2_\varepsilon(f)$ vanishes whenever ∇f is parallel to $\bar{v}$, that is, $\nabla f(v) = \phi(v)\bar{v}$, with $\phi\in\mathbb{R}$. Let us check next that $\mathcal{R}^1_\varepsilon(f)$ is in fact positive definite.

4.1. **Diagonalization and lower bounds.** To investigate the properties of $\mathcal{R}^1_\varepsilon$ it is convenient to introduce a suitable diagonalization of $\mathcal{A}_f$. It is clear that, for a given f, the matrix

$$\mathcal{A}_f = \int_{\mathbb{R}^3}f_*(1-\varepsilon f_*)|v_*-\xi|^2\Pi(v_*-\xi)\mathrm{d}v_* = \int_{\mathbb{R}^3}F_*|v_*-\xi|^2\,\Pi(v_*-\xi)\mathrm{d}v_*$$

is symmetric and therefore there is a orthogonal change of variables Θ_f such that

$$\mathcal{A}_f = \Theta_f\,\mathbb{D}_f\Theta_f^{-1}$$

where $\mathbb{D}_f$ is a diagonal matrix. Notice all matrices $\Theta_f,\mathbb{D}_f$ are independent of v. With such a splitting

$$\mathcal{R}^1_\varepsilon(f) = \int_{\mathbb{R}^3}\mathbb{D}_f\,\frac{\Theta_f^{-1}\nabla f(v)}{F(v)}\cdot\Theta_f^{-1}\nabla f(v)\mathrm{d}v.$$

Writing now

$$\widetilde{f}(v) = f\left(\Theta_f v\right), \qquad \nabla\widetilde{f}(v) = \Theta_f^{-1}\left(\nabla f\right)\left(\Theta_f v\right)$$

and recalling that $\det(\Theta_f) = 1$, the change of variable $v = \Theta_f^{-1}w$ in the above expression of $\mathcal{R}^1_\varepsilon(f)$ yields

$$\mathcal{R}^1_\varepsilon(f) = \int_{\mathbb{R}^3}\mathbb{D}_f\frac{\nabla\widetilde{f}(w)}{\widetilde{F}(w)}\cdot\nabla\widetilde{f}(w)\mathrm{d}w\,. \tag{4.3}$$

We compute first the diagonal matrix $\mathbb{D}_f$.

Lemma 4.2. *With the above notations,*

$$\mathbb{D}_f = \mathrm{diag}(\boldsymbol{e} - \boldsymbol{e}_i)_{i=1,2,3}$$

with

$$\boldsymbol{e}_i := \int_{\mathbb{R}^3} \widetilde{F}(w)\,(w_i - U_i)^2\,\mathrm{d}w = \int_{\mathbb{R}^3} F(v)\,(v_i - \xi_i)^2\,\mathrm{d}v, \qquad U := \Theta_f^{-1}\xi \tag{4.4}$$

and $\sum_{i=1}^3 \boldsymbol{e}_i = \boldsymbol{e}$.

Proof. Observe that with the change of variable $v = \Theta_f w$

$$\sum_{i=1}^3 \boldsymbol{e}_i = \int_{\mathbb{R}^3} \widetilde{F}(w)|w - U|^2\mathrm{d}w = \int_{\mathbb{R}^3} F(v)|\Theta_f^{-1}v - \Theta_f^{-1}\xi|^2\mathrm{d}v = \int_{\mathbb{R}^3} F(v)|v-\xi|^2\mathrm{d}v = \boldsymbol{e}.$$

The rest of the proof is by direct inspection based on the definition of $\mathcal{A}_f$. Namely, noticing that $\Theta_f^{-1}\Pi(z)\Theta_f = \Pi\left(\Theta_f^{-1}z\right)$, one deduces that

$$\begin{aligned}\mathbb{D}_f = \Theta_f^{-1}\mathcal{A}_f\Theta_f &= \int_{\mathbb{R}^3} F_*\left[|\Theta_f^{-1}\bar{v}_*|^2\,\mathrm{Id} - \left(\Theta_f^{-1}\bar{v}_*\right)\otimes\left(\Theta_f^{-1}\bar{v}_*\right)\right]\mathrm{d}v_*\\ &= \int_{\mathbb{R}^3} F_*\left[|\Theta_f^{-1}v_* - U|^2\,\mathrm{Id} - \left(\Theta_f^{-1}v_* - U\right)\otimes\left(\Theta_f^{-1}v_* - U\right)\right]\mathrm{d}v_*\,,\end{aligned}$$

recalling that $\bar{v}_* = v_* - \xi$ and $\Theta_f^{-1}\xi = U$. Performing again the change of variable $w = \Theta_f^{-1}v_*$ and with the previous notations

$$\mathbb{D}_f = \int_{\mathbb{R}^3} \widetilde{F}(w)\left[|w-U|^2\,\mathrm{Id} - (w-U)\otimes(w-U)\right]\mathrm{d}w = \int_{\mathbb{R}^3}\widetilde{F}(w)|w-U|^2\Pi(w-U)\mathrm{d}w.$$

Additionally, notice that

$$U = \Theta_f^{-1}\xi = \frac{1}{m}\int_{\mathbb{R}^3} F(v)\Theta_f^{-1}v\mathrm{d}v = \frac{1}{m}\int_{\mathbb{R}^3}\widetilde{F}(w)\,w\mathrm{d}w$$

which ensures that $\int_{\mathbb{R}^3}\widetilde{F}(w)(w-U)_i(w-U)_j\mathrm{d}w = 0$ if $i \neq j$ showing $\mathbb{D}_f$ to be diagonal. Moreover,

$$(\mathbb{D}_f)_{ii} = \int_{\mathbb{R}^3}\widetilde{F}(w)\left(|w-U|^2 - (w-U)_i^2\right)\mathrm{d}w = \boldsymbol{e} - \boldsymbol{e}_i$$

which gives the result. □

Proposition 4.3. *For $f \in \mathcal{Y}_\varepsilon$ one has that*

$$\mathscr{D}_\varepsilon^{(0)}(f) \geqslant \min_{i=1,2,3}(\boldsymbol{e} - \boldsymbol{e}_i)\int_{\mathbb{R}^3} F(v)\left|\frac{\nabla f(v)}{F(v)} + \frac{3\varrho}{T}(v-\xi)\right|^2\mathrm{d}v \tag{4.5}$$

where $F = f(1-\varepsilon f)$ and $\boldsymbol{e}_i$ is defined in (4.4).

Proof. According to (4.3) and the previous Lemma one has that

$$\mathcal{R}_\varepsilon^1(f) = \int_{\mathbb{R}^3}\mathbb{D}_f\frac{\nabla\widetilde{f}(w)}{\widetilde{F}(w)}\cdot\nabla\widetilde{f}(w)\mathrm{d}w = \sum_{i=1}^3(\boldsymbol{e} - \boldsymbol{e}_i)\int_{\mathbb{R}^3}\widetilde{F}(w)\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}\right)^2\mathrm{d}w. \tag{4.6}$$

For $R>0$ to be chosen, we expand for all $i=1,2,3$

$$\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}\right)^2=\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}+\frac{w_i-U_i}{R}-\frac{w_i-U_i}{R}\right)^2$$
$$=\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}+\frac{w_i-U_i}{R}\right)^2-\frac{2\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}\frac{w_i-U_i}{R}-\left(\frac{w_i-U_i}{R}\right)^2.$$

Multiplying this identity with $\widetilde{F}(w)$ and integrating over $\mathbb{R}^3$ one sees, after integration by parts for the second term, that

$$\int_{\mathbb{R}^3}\widetilde{F}(w)\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}\right)^2\mathrm{d}w=\int_{\mathbb{R}^3}\widetilde{F}(w)\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}+\frac{w_i-U_i}{R}\right)^2\mathrm{d}w$$
$$+\frac{2}{R}\int_{\mathbb{R}^3}\widetilde{f}(w)\mathrm{d}w-\int_{\mathbb{R}^3}\widetilde{F}(w)\left(\frac{w_i-U_i}{R}\right)^2\mathrm{d}w$$
$$=\int_{\mathbb{R}^3}\widetilde{F}(w)\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}+\frac{w_i-U_i}{R}\right)^2\mathrm{d}w+\frac{2\varrho}{R}-\frac{\boldsymbol{e}_i}{R^2},$$

where we used that $\int_{\mathbb{R}^3}\widetilde{f}(w)\mathrm{d}w=\int_{\mathbb{R}^3}f(v)\mathrm{d}v=\varrho$. Consequently,

$$\mathcal{R}^1_{\boldsymbol{\varepsilon}}(f)=\sum_{i=1}^3(\boldsymbol{e}-\boldsymbol{e}_i)\int_{\mathbb{R}^3}\widetilde{F}(w)\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}+\frac{w_i-U_i}{R}\right)^2\mathrm{d}w+\sum_{i=1}^3(\boldsymbol{e}-\boldsymbol{e}_i)\left(\frac{2\varrho}{R}-\frac{\boldsymbol{e}_i}{R^2}\right).\tag{4.7}$$

Recalling that $\boldsymbol{e}=\sum_{i=1}^3\boldsymbol{e}_i$, it follows that

$$\sum_{i=1}^3(\boldsymbol{e}-\boldsymbol{e}_i)\left(\frac{2\varrho}{R}-\frac{\boldsymbol{e}_i}{R^2}\right)=\frac{4\varrho\boldsymbol{e}}{R}-\frac{\boldsymbol{e}^2}{R^2}+\sum_{i=1}^3\frac{\boldsymbol{e}_i^2}{R^2}.$$

Since $\sum_{i=1}^3\boldsymbol{e}_i^2\geqslant\frac{\boldsymbol{e}^2}{3}$ according to Cauchy-Schwarz inequality, we deduce that

$$\sum_{i=1}^3(\boldsymbol{e}-\boldsymbol{e}_i)\left(\frac{2}{R}-\frac{\boldsymbol{e}_i}{R^2}\right)\geqslant\frac{2\boldsymbol{e}}{R}\left(2\varrho-\frac{\boldsymbol{e}}{3R}\right).$$

Choose $R=\frac{\boldsymbol{e}}{3}\varrho$ so that $\sum_{i=1}^3(\boldsymbol{e}-\boldsymbol{e}_i)\left(\frac{2\varrho}{R}-\frac{\boldsymbol{e}_i}{R^2}\right)\geqslant6\varrho^2$ and deduce from (4.7) that

$$\mathcal{R}^1_{\boldsymbol{\varepsilon}}(f)\geqslant\sum_{i=1}^3(\boldsymbol{e}-\boldsymbol{e}_i)\int_{\mathbb{R}^3}\widetilde{F}(w)\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}+\frac{w_i-U_i}{R}\right)^2\mathrm{d}w+6\varrho^2.$$

Recalling from Lemma 4.1 that $\mathscr{D}_{\varepsilon}^{(0)}(f) = \mathcal{R}_{\varepsilon}(f) - 6\varrho^2 \geqslant \mathcal{R}^1_{\varepsilon}(f) - 6\varrho^2$, we deduce that

$$\begin{aligned}\mathscr{D}_{\varepsilon}^{(0)}(f) &\geqslant \sum_{i=1}^{3}(\boldsymbol{e}-\boldsymbol{e}_i)\int_{\mathbb{R}^3}\widetilde{F}(w)\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}+\frac{w_i-U_i}{R}\right)^2\mathrm{d}w\\ &\geqslant \min_i(\boldsymbol{e}-\boldsymbol{e}_i)\sum_{i=1}^{3}\int_{\mathbb{R}^3}\widetilde{F}(w)\left(\frac{\partial_i\widetilde{f}(w)}{\widetilde{F}(w)}+\frac{w_i-U_i}{R}\right)^2\mathrm{d}w\\ &= \min_i(\boldsymbol{e}-\boldsymbol{e}_i)\int_{\mathbb{R}^3}\widetilde{F}(w)\left|\frac{\nabla\widetilde{f}(w)}{\widetilde{F}(w)}+\frac{w-U}{R}\right|^2\mathrm{d}w\,.\end{aligned}$$

Changing back to the variable $v = \Theta_f w$, we get the desired result recalling that $R = \frac{\boldsymbol{e}}{3\varrho}$ and $\xi = \Theta_f U$. □

4.2. **Modified Logarithmic Sobolev inequality.**

In the classical case of the Landau equation for which $\varepsilon = 0$, one has $\xi = U = 0$, $m = \varrho$, $\boldsymbol{e} = 3\varrho\vartheta$ and $R = \vartheta$. Thus, the aforementioned inequality reads

$$\begin{aligned}\mathscr{D}_{0}^{(0)}(f) &\geqslant \min_i(3-\boldsymbol{e}_i)\int_{\mathbb{R}^3}f(v)\left|\frac{\nabla f(v)}{f(v)}+\frac{v}{\theta}\right|^2\mathrm{d}v\\ &= \min_i(3-\boldsymbol{e}_i)\int_{\mathbb{R}^3}f(v)\left|\frac{\nabla f(v)}{f(v)}-\frac{\nabla\mathcal{M}(v)}{\mathcal{M}(v)}\right|^2\mathrm{d}v,\end{aligned}$$

where

$$\mathcal{M}(v) = \varrho(2\pi)^{-\frac{3}{2}}\exp\left(-\frac{|v|^2}{2\vartheta}\right)$$

is the standard Maxwellian distribution satisfying (1.14) with $\boldsymbol{u} = 0$. We recognise in the latter integral a relative Fisher information so that, according to the standard logarithmic Sobolev inequality, it follows that

$$\mathscr{D}_{0}^{(0)}(f) \geqslant \min_i(3-\boldsymbol{e}_i)\mathcal{H}_0(f|\mathcal{M})\,.$$

We shall adopt a similar approach in the LFD case and resort to a quantum version of the logarithmic Sobolev inequality using the following result intuited from [13, 14].

Proposition 4.4 (***Modified logarithmic Sobolev inequality***)**.** *For $f \in \mathcal{Y}_{\varepsilon}$ satisfying*

$$\inf_{v\in\mathbb{R}^3}(1-\varepsilon f(v)) \geqslant \kappa_0 > 0\,, \tag{4.8}$$

it holds that

$$\int_{\mathbb{R}^3}f(v)(1-\varepsilon f(v))\left|\frac{\nabla f(v)}{f(v)(1-\varepsilon f(v))}+\frac{3\varrho}{\boldsymbol{e}}(v-\xi)\right|^2\mathrm{d}v \geqslant \frac{6\kappa_0\,\varrho}{\boldsymbol{e}}\mathcal{H}_{\varepsilon}(f|\mathcal{M}_{\varepsilon})$$

where $\mathcal{M}_{\varepsilon}$ is the unique Fermi-Dirac statistics satisfying (1.14).

Remark 4.5. *Notice that, under (4.8) and (1.14), the parameters $m, \boldsymbol{e}$ in (4.1) are indeed positive.*

Proof. For any $\mu \in \mathbb{R}^3$ and $R > 0$ introduce the generalized Fisher information

$$\mathscr{I}_\varepsilon(f) := \int_{\mathbb{R}^3} f(1-\varepsilon f)\left|\frac{\nabla f}{f(1-\varepsilon f)} + \frac{v-\mu}{R}\right|^2 \mathrm{d}v = \int_{\mathbb{R}^3} f(1-\varepsilon f)\left|\nabla\left(s_\varepsilon'(f) + \frac{|v-\mu|^2}{2R}\right)\right|^2 \mathrm{d}v$$

where

$$s_\varepsilon(r) := \frac{1}{\varepsilon}\left[\varepsilon r\log(\varepsilon r) + (1-\varepsilon r)\log(1-\varepsilon r)\right], \qquad r \in (0,\varepsilon),$$

and $s_\varepsilon'(\cdot)$ denotes its derivative. As noticed in [14, 13], $\mathscr{I}_\varepsilon$ is the entropy production of the Fermi-Fokker-Planck equation

$$\begin{aligned}\partial_t\phi(t,v) &= \nabla\cdot\left[\phi(t,v)\,(1-\varepsilon\phi(t,v))\nabla\left(s_\varepsilon'(\phi(t,v)) + \frac{|v-\mu|^2}{2R}\right)\right]\\ &= \Delta\phi(t,v) + \nabla\cdot\left(\frac{v-\mu}{R}\,\phi(t,v)\,(1-\varepsilon\phi(t,v))\right).\end{aligned} \tag{4.9}$$

Indeed, multiplying (4.9) with $s_\varepsilon'(\phi(t,v)) + \frac{|v-\mu|^2}{2R}$ and integrating over $\mathbb{R}^3$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left[s_\varepsilon(\phi(t,v)) + \frac{|v-\mu|^2}{2R}\phi(t,v)\right]\mathrm{d}v = -\mathscr{I}_\varepsilon(\phi(t)) \qquad \forall t \geqslant 0.$$

Introduce for any R, μ, ε, the generalized entropy functional

$$\begin{aligned}\mathcal{S}_\varepsilon(f) &:= \int_{\mathbb{R}^3} s_\varepsilon(f(v)) + \frac{|v-\mu|^2}{2R}f(v)\mathrm{d}v\\ &= \frac{1}{2R}\int_{\mathbb{R}^3}|v-\mu|^2 f(v)\mathrm{d}v + \mathcal{H}_\varepsilon(f) =: E_{R,\mu}(f) + \mathcal{H}_\varepsilon(f),\end{aligned} \tag{4.10}$$

where $\mathcal{H}_\varepsilon$ is defined in (1.9) and $E_{R,\mu}(f)$ the variance of f. It is also deduced from [14, Lemma 3.2] that the Fermi distribution

$$G_{R,\mu}(v) = \frac{1}{1+\beta\exp\left(\frac{|v-\mu|^2}{2R}\right)},$$

is the unique stationary state to the Fermi-Fokker-Planck equation (4.9) where $\beta = \beta(R)$ is a positive normalising constant ensuring $G_{R,\mu}$ to have $\int_{\mathbb{R}^3} G_{R,\mu}(v)\mathrm{d}v = \varrho$. Moreover, if f is a given distribution with mass ϱ one has that

$$\mathcal{S}_\varepsilon(G_{R,\mu}) \leqslant \mathcal{S}_\varepsilon(f). \tag{4.11}$$

Next, as explained in [14, Theorem 3.5], it follows from a generalized logarithmic Sobolev inequality established in [13, Theorem 17] studying the parabolic problem

$$\partial_t u(t,x) = \nabla\cdot\left[u(t,x)\nabla\big(\mathcal{V}(x) + h(u(t,x))\big)\right], \qquad \mathcal{V}(x) = \frac{|x-\mu|^2}{2R},$$

that

$$\frac{1}{2\alpha_1}\mathcal{J}_h(u) \geqslant \mathcal{I}_h(u) - \mathcal{I}_h(G_{R,\mu}), \quad \text{for functions} \quad \int_{\mathbb{R}^3} u\,\mathrm{d}x = \varrho,$$

where

$$\mathcal{J}_h(u) = \int_{\mathbb{R}^3} u\big|\nabla\mathcal{V} + \nabla h\big|^2\mathrm{d}x, \qquad \mathcal{I}_h(u) = \int_{\mathbb{R}^3}\big(\mathcal{V}u + \Phi(u)\big)\mathrm{d}x, \qquad \Phi'(r) = h(r),$$

and where $\mathrm{Hess}(\mathcal{V}) \geqslant \alpha_1 \mathrm{Id}$. This translates, when $h(r) = s'_\varepsilon(r)$, into the following functional inequality valid for any $g \in L^1_2(\mathbb{R}^3)$ with mass ϱ

$$\int_{\mathbb{R}^3} g(v) \left| \frac{\nabla g(v)}{g(v)(1-\varepsilon g(v))} + \frac{v-\mu}{R} \right|^2 \mathrm{d}v \geqslant \frac{2}{R} \left[\mathcal{S}_\varepsilon(g) - \mathcal{S}_\varepsilon(G_{R,\mu}) \right].$$

Consequently, under the assumption

$$\inf_v \left(1 - \varepsilon g(v)\right) \geqslant \kappa_0 > 0$$

we deduce that

$$\mathscr{I}_\varepsilon(g) \geqslant \frac{2}{R} \kappa_0 \left[\mathcal{S}_\varepsilon(g) - \mathcal{S}_\varepsilon(G_{R,\mu}) \right].$$

Considering the Fermi-Dirac statistics $\mathcal{M}_\varepsilon \in \mathcal{Y}_\varepsilon$, as observed already in (4.11),

$$-\mathcal{S}_\varepsilon(G_{R,\mu}) \geqslant -\mathcal{S}_\varepsilon(\mathcal{M}_\varepsilon)$$

and, therefore, for any $g \in L^1_2(\mathbb{R}^3)$ with mass ϱ

$$\mathscr{I}_\varepsilon(g) \geqslant \frac{2}{R} \kappa_0 \left[\mathcal{S}_\varepsilon(g) - \mathcal{S}_\varepsilon(\mathcal{M}_\varepsilon) \right].$$

For $f \in \mathcal{Y}_\varepsilon$, observe in particular that

$$\mathcal{S}_\varepsilon(f) - \mathcal{S}_\varepsilon(\mathcal{M}_\varepsilon) = \mathcal{H}_\varepsilon(f) - \mathcal{H}_\varepsilon(\mathcal{M}_\varepsilon) = \mathcal{H}_\varepsilon(f|\mathcal{M}_\varepsilon)$$

since $f, \mathcal{M}_\varepsilon$ share mass, momentum, and energy. The choices $g = f$ and $R = \frac{\boldsymbol{e}}{3\varrho}$ give the desired inequality. □

The proof of Theorem 1.5 is now ready.

Proof of Theorem 1.5. Propositions 4.3 and 4.4 readily give that

$$\mathscr{D}^{(0)}_\varepsilon(f) \geqslant \frac{6\kappa_0\, \varrho}{\boldsymbol{e}} \min_{i=1,2,3} \left(\boldsymbol{e} - \boldsymbol{e}_i\right) \mathcal{H}_\varepsilon(f|\mathcal{M}_\varepsilon).$$

Furthermore, the quantities $\boldsymbol{e}$ and $\min_i(\boldsymbol{e} - \boldsymbol{e}_i)$ can be bounded away from zero by a constant independent of κ_0. To do so, use Lemma 2.1 and recall the argument of [19, Proposition 2]. Indeed, with the notation of Lemma 2.1 and with $F = f(1-\varepsilon f)$, set $B_\star = \{v \in \mathbb{R}^3\ ;\ |v| \leqslant R_\star\}$ and

$$A_i := \{v \in B_\star\ ;\ |v_i - \xi_i| < \alpha\} \qquad i = 1,2,3,$$

where $\alpha > 0$ is such that $|A_i|$ is sufficiently small to ensure from (2.2) that $\displaystyle\int_{A_i} F(v)\mathrm{d}v \leqslant \frac{\eta_\star}{2}$.
Then,

$$\begin{aligned}\boldsymbol{e}_i := \int_{\mathbb{R}^3} F(v)(v_i - \xi_i)^2 \mathrm{d}v &\geqslant \int_{v \in B_\star\,;\,|v_i - \xi| \geqslant \alpha} F(v)(v_i-\xi_i)^2 \mathrm{d}v \\ &\geqslant \alpha^2 \int_{v \in B_\star\,;\,|v_i-\xi|\geqslant\alpha} F(v)\mathrm{d}v = \alpha^2 \left(\int_{B_\star} F(v)\mathrm{d}v - \int_{A_i} F(v)\mathrm{d}v \right) \geqslant \alpha^2 \left(\eta_\star - \frac{\eta_\star}{2}\right)\end{aligned}$$

thanks to (2.1) and (2.2). Since $\alpha_\star$ depends only on $R_\star$, this proves that there exists $C_0 > 0$ depending only on $\|f\|_{L^1_2}$ and $\mathcal{H}_\varepsilon(f)$ such that $\boldsymbol{e}_i \geqslant C_0$. In particular, $\boldsymbol{e} \geqslant 3C_0$ and $\min_i(\boldsymbol{e} - \boldsymbol{e}_i) = \sum_{j \neq i} \boldsymbol{e}_j \geqslant 2C_0$. This gives the result where we also observe that $\boldsymbol{e} \leqslant 3\varrho\vartheta$. □

5. Explicit rate of convergence to equilibrium

We deduce from the previous analysis an explicit rate of thermalisation for solutions to the LFD equation (1.1). As expressed in the introduction, we focus on the case of moderately soft potentials $\gamma \in (-2,0)$ noticing that the same method applies to the hard potential case $\gamma \in [0,1)$. The case $\gamma \in (-3,-2)$ is currently an open topic.

Recall the space $\mathcal{Y}_\varepsilon = \mathcal{Y}_\varepsilon[\varrho, 0, \vartheta]$ for fixed $\varrho > 0$, $\vartheta > 0$. Consider an initial datum $f_{\rm in} \in \mathcal{Y}_\varepsilon$, solutions $f(t) = f(t,v) \in \mathcal{Y}_\varepsilon$ to (1.1), and denote by $\mathcal{M}_\varepsilon$ the Fermi-Dirac statistics with same first moments of $f_{\rm in}$, that is, $\mathcal{M}_\varepsilon$ satisfies (1.14). In addition, recall the strategy to prove the converge towards $\mathcal{M}_\varepsilon$ described in the introduction based on an entropy/entropy production estimate. We denoted by $\mathscr{D}_\varepsilon^{(\eta)}(g)$ the entropy production associated to the interaction kernel $\Psi(z) = |z|^{\eta+2}$

$$\mathscr{D}_\varepsilon^{(\eta)}(g) := \frac{1}{2}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} |v-v_*|^{\eta+2}\boldsymbol{\Xi}_\varepsilon[g](v,v_*)\mathrm{d}v\mathrm{d}v_*\,,$$

where $\boldsymbol{\Xi}_\varepsilon[g](v,v_*)$ is defined by (1.10). For solutions to (1.1) associated to $\gamma \in (-2,0)$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\varepsilon(f(t)|\mathcal{M}_\varepsilon) = -\mathscr{D}_\varepsilon^{(\gamma)}(f(t)).$$

And, using the approach introduced in [5] based on the interpolation inequality (1.19), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\varepsilon(f(t)|\mathcal{M}_\varepsilon) \leqslant -\left(\mathscr{D}_\varepsilon^{(0)}(f(t))\right)^{1-\frac{\gamma}{\eta}}\left(\mathscr{D}_\varepsilon^{(\eta)}(f(t))\right)^{\frac{\gamma}{\eta}} \tag{5.1}$$

where a lower bound for $\mathscr{D}_\varepsilon^{(0)}(f(t))$ can be deduced from Theorems 1.6 and 1.5 and an upper bound for $\mathscr{D}_\varepsilon^{(\eta)}(f(t))$ is, then, required. This is the role of the following precise statement of Theorem 1.8.

Theorem 5.1. *Fix $\gamma \in (-2,0)$, $\varepsilon \in (0,\varepsilon_{\rm sat})$, and an initial datum $f_{\rm in} \in \mathcal{Y}_\varepsilon$. Let $f(t,\cdot)$ be a weak-solution to (1.1). For $\eta \geqslant 0$, assume that*

$$f_{\rm in} \in L^1_k(\mathbb{R}^3) \quad \textit{for some} \quad k > \eta + 2 + |\gamma|.$$

Then,

$$\mathcal{H}_\varepsilon(f(t)|\mathcal{M}_\varepsilon) \leqslant \boldsymbol{C}(f_{\rm in})\big(\kappa_0(t)\big)^{\frac{\eta}{\gamma}-2}\,t^{-\frac{\eta}{|\gamma|}}, \qquad \forall t \geqslant \max(2T_*, 1) \tag{5.2}$$

where $T_* > 0$, $\kappa_0 : \mathbb{R}^+ \to \mathbb{R}^+$ are defined in Theorem 1.6 and $\boldsymbol{C}(f_{\rm in})$ is depending on $\|f_{\rm in}\|_{L^1_k}$, $\mathcal{H}_\varepsilon(f_{\rm in})$. In particular,

(1) if $\gamma \in (-\frac{4}{3}, 0)$, then

$$\mathcal{H}_\varepsilon(f(t)|\mathcal{M}_\varepsilon) \leqslant \boldsymbol{C}_0\,t^{-\frac{\eta}{|\gamma|}}, \qquad \forall t \geqslant \max(2T_*, 1)$$

for some $\boldsymbol{C}_0 > 0$ depending on $\|f_{\rm in}\|_{L^1_k}$ and $\mathcal{H}_\varepsilon(f_{\rm in})$.

(2) If $\gamma \in (-2, -\frac{4}{3}]$ and $\eta > 2|\gamma|$, then

$$\mathcal{H}_\varepsilon(f(t)|\mathcal{M}_\varepsilon) \leqslant \boldsymbol{C}_1\,t^{1-\frac{\eta}{2|\gamma|}}, \qquad \forall t \geqslant \max(2T_*, 1)$$

for some $\boldsymbol{C}_1 > 0$ depending on $\|f_{\rm in}\|_{L^1_k}$ and $\mathcal{H}_\varepsilon(f_{\rm in})$.

Proof. The proof starts from (5.1) where $\gamma \in (-2,0)$ is fixed and $\eta > 0$. According to Theorem 1.6 and 1.5, it holds

$$\mathscr{D}_{\boldsymbol{\varepsilon}}^{(0)}(f(t)) \geqslant C\,\kappa_0(t)\,\mathcal{H}_{\boldsymbol{\varepsilon}}(f(t)|\mathcal{M}_{\boldsymbol{\varepsilon}}) \qquad \forall t \geqslant T_*$$

where lower bounds on $\kappa_0 \,:\, \mathbb{R}^+ \to \mathbb{R}^+$ are given in Section 3.3. Setting,

$$A_\eta(t) := C^{1+\frac{|\gamma|}{\eta}}\,\kappa_0(t)^{1+\frac{|\gamma|}{\eta}}\left(\mathscr{D}_{\boldsymbol{\varepsilon}}^{(\eta)}(f(t))\right)^{\frac{\gamma}{\eta}} \quad \text{and} \quad \boldsymbol{y}(t) := \mathcal{H}_{\boldsymbol{\varepsilon}}(f(t)|\mathcal{M}_{\boldsymbol{\varepsilon}})\,,$$

we deduce from (5.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y}(t) + A_\eta(t)\boldsymbol{y}(t)^{1-\frac{\gamma}{\eta}} \leqslant 0, \qquad t \geqslant T_*.$$

Integrating this inequality, we deduce that

$$\boldsymbol{y}(t)^{\frac{\gamma}{\eta}} \geqslant \boldsymbol{y}(T_*)^{\frac{\gamma}{\eta}} - \frac{\gamma}{\eta}\int_{T_*}^t A_\eta(\tau)\mathrm{d}\tau \geqslant \boldsymbol{y}(0)^{\frac{\gamma}{\eta}} - \frac{\gamma}{\eta}\int_{T_*}^t A_\eta(\tau)\mathrm{d}\tau,$$

that is,

$$\mathcal{H}_{\boldsymbol{\varepsilon}}(f(t)|\mathcal{M}_{\boldsymbol{\varepsilon}}) \leqslant \left(\mathcal{H}_{\boldsymbol{\varepsilon}}(f_{\mathrm{in}}|\mathcal{M}_{\boldsymbol{\varepsilon}})^{\frac{\gamma}{\eta}} + \frac{|\gamma|}{\eta}\int_{T_*}^t A_\eta(\tau)\mathrm{d}\tau\right)^{\frac{\eta}{\gamma}}, \qquad t \geqslant T_*. \tag{5.3}$$

Estimate the integral of $A_\eta(\tau)$ from below recalling that $\kappa_0(\cdot)$ is non increasing. It follows that

$$\begin{aligned}\int_{T_*}^t A_\eta(\tau)\,\mathrm{d}\tau &= C^{1+\frac{|\gamma|}{\eta}}(t-T_*)\kappa_0(t)^{1+\frac{|\gamma|}{\eta}}\int_{T_*}^t \mathscr{D}_{\boldsymbol{\varepsilon}}^{(\eta)}(f(\tau))^{\frac{\gamma}{\eta}}\frac{\mathrm{d}\tau}{t-T_*}\\ &\geqslant C^{1+\frac{|\gamma|}{\eta}}(t-T_*)\kappa_0(t)^{1+\frac{|\gamma|}{\eta}}\left(\int_{T_*}^t \mathscr{D}_{\boldsymbol{\varepsilon}}^{(\eta)}(f(\tau))\frac{\mathrm{d}\tau}{t-T_*}\right)^{\frac{\gamma}{\eta}},\end{aligned}$$

where Jensen's inequality was used together with the convexity of the mapping $x > 0 \mapsto x^{\frac{\gamma}{\eta}}$. Therefore,

$$\int_{T_*}^t A_\eta(\tau)\mathrm{d}\tau \geqslant (C(t-T_*)\kappa_0(t))^{1+\frac{|\gamma|}{\eta}}\left(\int_{T_*}^t \mathscr{D}_{\boldsymbol{\varepsilon}}^{(\eta)}(f(\tau))\mathrm{d}\tau\right)^{\frac{\gamma}{\eta}}.$$

Deduce from Proposition 2.5 that there exists $C_\eta > 0$ depending on η, $\|f_{\mathrm{in}}\|_{L^1_k}$, and $\mathcal{H}_{\boldsymbol{\varepsilon}}(f_{\mathrm{in}})$ such that

$$\int_{T_*}^t A_\eta(\tau)\,\mathrm{d}\tau \geqslant C_\eta\big(t-T_*\big)^{1+\frac{|\gamma|}{\eta}}\big(\kappa_0(t)\big)^{1+\frac{2|\gamma|}{\eta}}(1+t)^{\frac{\gamma}{\eta}} \qquad t \geqslant T_*\,.$$

Choosing $t \geqslant \max(2T_*, 1)$, so that $t - T_* \geqslant \frac{t}{2}$, we deduce (5.2) from (5.3). Regarding the final part of the theorem use the estimates for κ_0 derived in Section 3.3. If $\gamma \in (-\frac{4}{3}, 0)$, as observed (3.17), κ_0 is independent of time and the conclusion follows from (5.2). If $\in (-2, -\frac{4}{3}]$ use (3.21) to deduce the result. □

Appendix A. Technical results from Section

A.1. Convolutions inequalities involving $\mathbf{c}_\gamma[\cdot]$.

Lemma A.1. *Fix $\gamma \in (-3,0)$ and $0 \leqslant f \in L^1_2(\mathbb{R}^3)$ satisfying (1.14). Then, there exists $c_0 > 0$ depending only on $\|f\|_{L^1_2}$ and γ such that*

$$\boldsymbol{c}_\gamma[f](v) \geqslant c_0 \langle v\rangle^\gamma \qquad \forall v \in \mathbb{R}^3. \tag{A.1}$$

Moreover, for $f \in L^1_{|\gamma|}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $1 < p < \frac{3}{|\gamma|}$, there exists $C_{p,\gamma}$ depending on γ and p such that

$$\boldsymbol{c}_\gamma[f](v) \leqslant C_{p,\gamma}\left(\boldsymbol{m}_{|\gamma|}(f) + \|f\|_{L^\infty}^{\frac{1}{p}} \boldsymbol{m}_{|\gamma|}(f)^{\frac{p-1}{p}}\right) \langle v\rangle^{\gamma + \frac{|\gamma|}{p}}, \qquad v \in \mathbb{R}^3. \tag{A.2}$$

Proof. Similar to [5, Lemma 2.2]. Observe that for $\gamma > -3$

$$\boldsymbol{c}_\gamma[f](v) = 2(\gamma+3)\int_{\mathbb{R}^3} f(v_*)|v-v_*|^\gamma \mathrm{d}v_* \geqslant 2\int_{\mathbb{R}^3}\left(1+|v-v_*|^2\right)^{\frac{\gamma}{2}} f(v_*)\mathrm{d}v_*.$$

Let $v \in \mathbb{R}^3$ be fixed and define the probability measure $\mathrm{d}\mu$ over $\mathbb{R}^3$ by

$$\mu(\mathrm{d}v_*) = f(v-v_*)\frac{\mathrm{d}v_*}{\varrho}.$$

Thanks to Jensen's inequality, it follows for the convex function $\Phi(r) = (1+r)^{\frac{\gamma}{2}}$, $r > 0$, that

$$\int_{\mathbb{R}^3}\left(1+|v-v_*|^2\right)^{\frac{\gamma}{2}} f(v_*)\mathrm{d}v_* = \varrho\int_{\mathbb{R}^3}\Phi(|v_*|^2)\mu(\mathrm{d}v_*) \geqslant \varrho\,\Phi\left(\int_{\mathbb{R}^3}|v_*|^2\mu(\mathrm{d}v_*)\right).$$

In addition,

$$\int_{\mathbb{R}^3}|v_*|^2\mu(\mathrm{d}v_*) = \frac{1}{\varrho}\int_{\mathbb{R}^3}|v-v_*|^2 f(v_*)\mathrm{d}v_* \leqslant 2|v|^2 + 6\theta + 2|\boldsymbol{u}|^2,$$

thanks to (1.14). And, since Φ is non increasing,

$$\varrho\int_{\mathbb{R}^3}\Phi(|v_*|^2)\mu(\mathrm{d}v_*) \geqslant \varrho\Phi\left(2|v|^2 + 6\theta + 2|\boldsymbol{u}|^2\right) \geqslant \frac{c_0}{2}\langle v\rangle^\gamma,$$

with $c_0 = 2\varrho\left(2+6\theta+2|\boldsymbol{u}|^2\right)^{\frac{\gamma}{2}} = 2^{1+\frac{\gamma}{2}}\|f\|_{L^1}^{1-\frac{\gamma}{2}}\|f\|_{L^1_2}^{\frac{\gamma}{2}}$ which proves (A.1).

For the proof of (A.2), we resort to Peetre's inequality $\langle v_*\rangle^\gamma \leqslant 2^{\frac{|\gamma|}{2}}\langle v-v_*\rangle^{|\gamma|}\langle v\rangle^\gamma$, valid for any $v, v_* \in \mathbb{R}^3$, to deduce that

$$I := 2^{\frac{|\gamma|+2}{2}}(\gamma+3)\int_{\mathbb{R}^3}|v-v_*|^\gamma\langle v-v_*\rangle^{|\gamma|}\langle v_*\rangle^{|\gamma|} f(v_*)\mathrm{d}v_* \geqslant \langle v\rangle^{|\gamma|}\boldsymbol{c}_\gamma[f].$$

Split $I = I_1 + I_2$ according to the regions $|v-v_*| \geqslant 1$ and $|v-v_*| \leqslant 1$. On the one hand, for $|u| > 1$, $\langle u\rangle^{|\gamma|} = (1+|u|^2)^{\frac{|\gamma|}{2}} \leqslant (2|u|^2)^{\frac{|\gamma|}{2}} = 2^{\frac{|\gamma|}{2}}|u|^{|\gamma|}$. Then, it follows that

$$I_1 := \int_{|v-v_*|>1}|v-v_*|^\gamma\langle v-v_*\rangle^{|\gamma|}\langle v_*\rangle^{|\gamma|} f(v_*)\mathrm{d}v_* \leqslant 2^{\frac{|\gamma|}{2}}\boldsymbol{m}_{|\gamma|}(f).$$

On the other hand, for $|u| \leqslant 1$, one has that $\langle u\rangle^{|\gamma|} \leqslant 2^{\frac{|\gamma|}{2}}$. Consequently,

$$I_2 := \int_{|v-v_*|\leqslant 1}|v-v_*|^\gamma\langle v-v_*\rangle^{|\gamma|}\langle v_*\rangle^{|\gamma|} f(v_*)\mathrm{d}v_* \leqslant 2^{\frac{|\gamma|}{2}}\int_{|v-v_*|\leqslant 1}|v-v_*|^\gamma\langle v_*\rangle^{|\gamma|} f(v_*)\mathrm{d}v_*.$$

Furthermore, for $p > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$, one can write $\langle v_*\rangle^{|\gamma|} = \langle v_*\rangle^{\frac{|\gamma|}{p}} \langle v_*\rangle^{\frac{|\gamma|}{q}}$ and, in the region $|v - v_*| \leqslant 1$, it holds that $\langle v_*\rangle \leqslant \langle v\rangle$. Therefore,

$$I_2 \leqslant 2^{\frac{|\gamma|}{2}} \langle v\rangle^{\frac{|\gamma|}{p}} \int_{|v-v_*|\leqslant 1} |v - v_*|^\gamma \langle v_*\rangle^{\frac{|\gamma|}{q}} f(v_*) \mathrm{d}v_*.$$

Using Hölder inequality gives that

$$I_2 \leqslant 2^{\frac{|\gamma|}{2}} \langle v\rangle^{\frac{|\gamma|}{p}} \left(\int_{|v-v_*|\leqslant 1} |v - v_*|^{\gamma\, p} \mathrm{d}v_*\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^3} \langle v_*\rangle^{|\gamma|} f^q(v_*) \mathrm{d}v_*\right)^{\frac{1}{q}}.$$

The first integral is finite for $1 \leqslant p < \frac{3}{|\gamma|}$ and the second is bounded by $\|f\|_{L^\infty}^{q-1} \boldsymbol{m}_{|\gamma|}(f)$. It yields that

$$I_2 \leqslant C(p,\gamma)\langle v\rangle^{\frac{|\gamma|}{p}} \|f\|_{L^\infty}^{\frac{q-1}{q}} \boldsymbol{m}_{|\gamma|}(f)^{\frac{1}{q}}, \qquad C(p,\gamma) = 2^{\frac{|\gamma|}{2}} \left(\frac{|\mathbb{S}^2|}{3+\gamma\, p}\right)^{\frac{1}{p}}.$$

Gathering the estimates show the desired estimate. □

Proposition A.2. *Assume $\gamma \in (-3, 0)$ and let f, g be sufficiently regular. For any $1 \leqslant \eta < \min\left(\frac{3}{|\gamma|}, \frac{3}{2}\right)$ there exists $C_\gamma(\eta) > 0$ depending only on γ and η such that*

$$\int_{\mathbb{R}^3} f^2 \boldsymbol{c}_\gamma[g] \mathrm{d}v \leqslant C_\gamma(\eta) \left(\boldsymbol{m}_s(g)^\beta \|g\|_{L^2}^{1-\beta} \|\langle\cdot\rangle^{\frac{\gamma}{2}} f\|_{H^1}^2 + \boldsymbol{m}_{|\gamma|}(g) \|\langle\cdot\rangle^{\frac{\gamma}{2}} f\|_{L^2}^2\right), \tag{A.3}$$

with $s = \dfrac{3|\gamma|\eta}{7\eta - 6}$ *and* $\beta = \dfrac{|\gamma|}{s} = \dfrac{7\eta - 6}{3\eta}$.

Proof. We recall that

$$\int_{\mathbb{R}^3} f^2 \boldsymbol{c}_\gamma[g] \mathrm{d}v = 2\,(\gamma + 3) \int_{\mathbb{R}^6} |v - v_*|^\gamma\, g(v_*) f^2(v) \mathrm{d}v \mathrm{d}v_*.$$

Using Peetre's inequality we deduce that

$$\int_{\mathbb{R}^3} f^2 \boldsymbol{c}_\gamma[g] \mathrm{d}v \leqslant C(\gamma) \int_{\mathbb{R}^6} |v - v_*|^\gamma \langle v - v_*\rangle^{|\gamma|} \langle v_*\rangle^{|\gamma|} g(v_*) \langle v\rangle^\gamma f^2(v) \mathrm{d}v \mathrm{d}v_*$$

with $C(\gamma) = 2(\gamma + 3)2^{\frac{|\gamma|}{2}}$. Setting $F(v) = \langle v\rangle^{\frac{\gamma}{2}} f(v)$ and $G(v_*) = \langle v_*\rangle^{|\gamma|} g(v_*)$, we rewrite the aforementioned inequality as

$$\int_{\mathbb{R}^3} f^2 \boldsymbol{c}_\gamma[g] \mathrm{d}v \leqslant \mathcal{A}_{\gamma,1}[F,G] + \mathcal{A}_{\gamma,2}[F,G],$$

where

$$\mathcal{A}_{\gamma,i}[F,G] := C(\gamma) \int_{\mathbb{R}^3} \boldsymbol{\kappa}_i(v - v_*) G(v_*) F^2(v) \mathrm{d}v \mathrm{d}v_*, \qquad i = 1, 2,$$

with functions

$$\boldsymbol{\kappa}_1(u) = |u|^\gamma \langle u\rangle^\gamma \mathbf{1}_{|u|\leqslant 1}, \qquad \boldsymbol{\kappa}_2(u) = |u|^\gamma \langle u\rangle^\gamma \mathbf{1}_{|u|>1}, \qquad u \in \mathbb{R}^3.$$

Both, $\mathcal{A}_{\gamma,1}[F,G]$ and $\mathcal{A}_{\gamma,2}[F,G]$, can be estimated using Young's convolution inequality, see for example [24, Theorem 4.2],

$$\mathcal{A}_{\gamma,i}[F,G] \leqslant C(p,r)\|\boldsymbol{\kappa}_i\|_{L^q} \|G\|_{L^p} \|F^2\|_r, \qquad \frac{1}{p} + \frac{1}{r} + \frac{1}{q} = 2, \qquad 1 \leqslant p,q,r \leqslant \infty.$$

With regard of $\mathcal{A}_{\gamma,2}[F,G]$ choose $r=1$, $q=\infty$, $p=1$ to obtain

$$\mathcal{A}_{\gamma,2}[F,G] \leqslant C(\gamma)\|\boldsymbol{\kappa}_2\|_{L^\infty}\|G\|_{L^1}\|F\|_{L^2}^2,$$

observing that $\|\boldsymbol{\kappa}_2\|_{L^\infty} \leqslant 2^{\frac{|\gamma|}{2}}$. Thus, with $C_\gamma = 2^{\frac{|\gamma|}{2}}\, C(\gamma)$, we deduce that

$$\mathcal{A}_{\gamma,2}[F,G] \leqslant C_\gamma \|\langle\cdot\rangle^{|\gamma|} g\|_{L^1}\|F\|_{L^2}^2\,. \tag{A.4}$$

With regard of $\mathcal{A}_{\gamma,1}[F,G]$, observe that

$$\boldsymbol{\kappa}_1 \in L^q(\mathbb{R}^3) \quad \text{for any} \quad 1 \leqslant q < \frac{3}{|\gamma|}\,.$$

Choosing $r=3$ in Young's convolution inequality and using Sobolev inequality in dimension 3, it follows that there exists $C_0(p)>0$ such that

$$\mathcal{A}_{\gamma,1}[F,G] \leqslant C_0(p)\|\boldsymbol{\kappa}_1\|_{L^q}\|G\|_{L^p}\|F\|_{H^1}^2, \qquad \frac{1}{p}+\frac{1}{q}=\frac{5}{3}. \tag{A.5}$$

The constraint implies that $p,q \in \left[1,\frac{3}{2}\right]$ and recall that $1 \leqslant q < \frac{3}{|\gamma|}$ as well. Choose, then, $1 \leqslant q < \min\left(\frac{3}{|\gamma|},\frac{3}{2}\right)$ with associated $p=\frac{3q}{5q-3} \in \left(1,\frac{3}{2}\right]$. Interpolation shows that

$$\|G\|_{L^p} = \|\langle\cdot\rangle^{|\gamma|} g\|_{L^p} \leqslant \|\langle\cdot\rangle^{s} g\|_{L^1}^{\frac{|\gamma|}{s}}\, \|g\|_{L^2}^{\frac{s-|\gamma|}{s}}, \qquad s=\frac{|\gamma|p}{2-p}\,.$$

Therefore, according to (A.5), for any $1 \leqslant q < \max\left(\frac{3}{|\gamma|},\frac{3}{2}\right)$ there exists $C_\gamma(q)>0$ such that

$$\mathcal{A}_{\gamma,1}[F,G] \leqslant C_\gamma(q)\boldsymbol{m}_s(g)^{\frac{7q-6}{3q}}\, \|g\|_{L^2}^{\frac{4q-6}{3q}}\, \|\langle\cdot\rangle^{\frac{\gamma}{2}} f\|_{H^1}^2, \qquad s=\frac{3|\gamma|q}{7q-6}\,. \tag{A.6}$$

Combining (A.6) with (A.4) yields the result after renaming $\eta=q$. ☐

Lemma A.3. *Fix $\gamma \in (-2,0)$, $\boldsymbol{\varepsilon} \in (0,\boldsymbol{\varepsilon}_{\text{sat}})$, and $f_{\text{in}} \in \mathcal{Y}_{\boldsymbol{\varepsilon}}$. Let $f(t,\cdot)$ be a weak-solution to (1.1). Then, for any $\eta \geqslant 0$ and $\beta<0$ there exists $\boldsymbol{C}_{\eta,\beta}(\boldsymbol{\varepsilon})>0$ depending only on $\eta,\beta,\boldsymbol{\varepsilon}$ and $\|f_{\text{in}}\|_{L^1}$ such that*

$$\int_{\mathbb{R}^3}\langle v\rangle^{\eta+\beta} f(t,v)\big(1+|\log f(t,v)|\,\big)\mathrm{d}v \leqslant \boldsymbol{C}_{\eta,\beta}(\boldsymbol{\varepsilon})\,\boldsymbol{m}_\eta(t) \qquad \forall t \geqslant 0. \tag{A.7}$$

Consequently, for any $\eta \geqslant 0$ there exists $\boldsymbol{C}'_{\eta,\gamma}(\boldsymbol{\varepsilon})>0$ depending only on η, γ, $\boldsymbol{\varepsilon}$ and $\|f_{\text{in}}\|_{L^1_\eta}$ such that

$$\int_{\mathbb{R}^3}\langle v\rangle^{\eta}\boldsymbol{c}_\gamma[f(t)]\, f(t,v)\big(1+|\log f(t,v)|\big)\mathrm{d}v \leqslant \boldsymbol{C}'_{\eta,\gamma}(\boldsymbol{\varepsilon})\,\boldsymbol{m}_\eta(t) \qquad \forall t \geqslant 0. \tag{A.8}$$

Proof. Improvement of [5, Lemma 5.3]. In regard of the estimate (A.7), split the integral

$$\int_{\mathbb{R}^3}\langle v\rangle^{\eta+\beta} f(v)\,(1+|\log f(v)|)\,\mathrm{d}v$$

according to $A=\{v \in \mathbb{R}^3 \;:\; \langle v\rangle^\eta f(v) \geqslant 1\}$ and its complement A^c. Writing $|\log f(v)| = |\log(\langle v\rangle^\eta f(v)) - \eta\log\langle v\rangle|$ and using that $\log(\langle v\rangle^\eta f(v)) \geqslant 0$ for $v \in A$ and $\langle v\rangle \geqslant 1$, we have that

$$\int_A \langle v\rangle^{\eta+\beta} f(v)\,(1+|\log f(v)|)\,\mathrm{d}v \leqslant \int_A \langle v\rangle^{\eta+\beta} f(v)\,(1+\log(\langle v\rangle^\eta f(v))+\eta\log\langle v\rangle)\,\mathrm{d}v\,.$$

Since $1 \leqslant \langle v\rangle^\eta f(v) \leqslant \varepsilon^{-1}\langle v\rangle^\eta$ for any $v \in A$, we deduce that

$$\int_A \langle v\rangle^{\eta+\beta} f(v)\,(1+|\log f(v)|)\,\mathrm{d}v \leqslant \int_{\mathbb{R}^3} \langle v\rangle^{\eta+\beta} f(v)\left(1+\eta\left(1+\varepsilon^{-1}\right)\log\langle v\rangle\right)\mathrm{d}v.$$

For $\beta<0$ there is an explicit constant $C_\beta>0$ such that $\langle v\rangle^\beta \log\langle v\rangle \leqslant C_\beta$, thus

$$\int_A \langle v\rangle^{\eta+\beta} f(v)\,(1+|\log f(v)|)\,\mathrm{d}v \leqslant \int_{\mathbb{R}^3} \langle v\rangle^{\eta} f(v)\left(1+\eta\, C_\beta\left(1+\varepsilon^{-1}\right)\right)\mathrm{d}v$$

which shows that, for some $C_\eta(\beta,\varepsilon)>0$,

$$\int_A \langle v\rangle^{\eta+\beta} f(v)\,(1+|\log f(v)|)\,\mathrm{d}v \leqslant C_\eta(\beta,\varepsilon)\boldsymbol{m}_\eta(t). \tag{A.9}$$

In the same way,

$$\int_{A^c} \langle v\rangle^{\eta+\beta} f(v)\,(1+|\log f(v)|)\,\mathrm{d}v \leqslant \int_{A^c} \langle v\rangle^{\eta+\beta} f(v)\,(1-\log\left(\langle v\rangle^\eta f(v)\right)+\eta\log\langle v\rangle)\,\mathrm{d}v\,.$$

Using that for all $p>1$ it holds that $x\,(1-\log x) \leqslant C_p\, x^{\frac{1}{p}}$ for any $x\in(0,1)$, we deduce, choosing $x=\langle v\rangle^\eta f(v)$, that

$$\int_{A^c} \langle v\rangle^{\eta+\beta} f(v)\,(1+|\log f(v)|)\,\mathrm{d}v \leqslant C_p\int_{\mathbb{R}^3} \langle v\rangle^{\beta}\left[\langle v\rangle^\eta f(v)\right]^{\frac{1}{p}}\mathrm{d}v + \eta\int_{\mathbb{R}^3}\langle v\rangle^{\eta+\beta} f(v)\log\langle v\rangle \mathrm{d}v\,,$$

where the latter integral can be bounded by $C_\beta \boldsymbol{m}_\eta(f)$. Thus,

$$\int_{A^c} \langle v\rangle^{\eta+\beta} f(v)\,(1+|\log f(v)|)\,\mathrm{d}v \leqslant C_p\int_{\mathbb{R}^3} \langle v\rangle^{\beta}\left[\langle v\rangle^\eta f(v)\right]^{\frac{1}{p}}\mathrm{d}v + \eta\, C_\beta\, \boldsymbol{m}_\eta(t).$$

Choosing $1<p<1-\frac{3}{|\beta|}$ and using Hölder inequality it follows that

$$\int_{\mathbb{R}^3} \langle v\rangle^{\beta}\left[\langle v\rangle^\eta f(v)\right]^{\frac{1}{p}}\mathrm{d}v \leqslant \boldsymbol{m}_\eta(t)^{\frac{1}{p}}\left[\int_{\mathbb{R}^3}\langle v\rangle^{q\beta}\mathrm{d}v\right]^{\frac{1}{q}}.$$

Observe that the conjugate exponent $q=\frac{p}{p-1}<\frac{3}{|\beta|}$, hence the last integral is finite. Thus, there exists $C(p,\beta)>0$ such that

$$\int_{A^c} \langle v\rangle^{\eta+\beta} f(v)\,(1+|\log f(v)|)\,\mathrm{d}v \leqslant C(p,\beta)\left[\boldsymbol{m}_\eta(t)\right]^{\frac{1}{p}} + \eta\, C_\beta \boldsymbol{m}_\eta(t).$$

Combining this last estimate with Young's inequality and (A.9) we deduce (A.7). The estimate for (A.8) follows from (A.7) combined with the decay of $\boldsymbol{c}_\gamma[f(t)]$ provided by (A.2). More explicitly,

$$\int_{\mathbb{R}^3} \langle v\rangle^{\eta} \boldsymbol{c}_\gamma[f(t)]\, f(t,v)\,(1+|\log f(t,v)|)\,\mathrm{d}v$$
$$\leqslant C_{p,\gamma}(\varepsilon, f_{\rm in})\int_{\mathbb{R}^3} \langle v\rangle^{\eta+\gamma+\frac{|\gamma|}{p}} f(t,v)\,(1+|\log f(t,v)|)\,\mathrm{d}v\,,\qquad p\in\left(1,\tfrac{3}{|\gamma|}\right).$$

Here $C_{p,\gamma}(\varepsilon, f_{\rm in})>0$ depends only on p, γ, ε, and $\|f_{\rm in}\|_{L^1_2}$ since $\boldsymbol{m}_{|\gamma|}(f(t)) \leqslant \boldsymbol{m}_2(f(t)) = \boldsymbol{m}_2(f_{\rm in})$. Finally, using (A.7) with $\beta=\gamma+\frac{|\gamma|}{p}<0$ we deduce (A.8). □

A.2. **Entropy and Fisher information estimates.** The scope of this Appendix is to prove Proposition 2.5 which is an improvement of [5, Lemma 5.3].

Lemma A.4. *For any* $0 \leqslant g \leqslant \varepsilon^{-1}$ *and* $\eta \geqslant -2$ *it holds that*

$$\mathscr{D}_{\varepsilon}^{(\eta)}(g) \leqslant \frac{3^{\eta+4}}{\kappa_0(g)} \left(\boldsymbol{m}_0(g)\mathscr{F}_{\eta+2}[g] + \boldsymbol{m}_{\eta+2+|\gamma|}(g)\mathscr{F}_{\gamma}[g]\right) . \tag{A.10}$$

Recall that $\kappa_0(g) = \inf_{v\in\mathbb{R}^3}(1-\varepsilon g(v)) = 1-\varepsilon\|g\|_{L^\infty}$ *and the weighted Fisher information* $\mathscr{F}_\eta[\cdot]$ *are defined in* (1.12)*.*

Proof. We follow [4, 5]. Recall (1.11)

$$\mathscr{D}_{\varepsilon}^{(\eta)}(g) = \frac{1}{2}\int_{\mathbb{R}^6}|v-v_*|^{\eta+2}\, g\, g_*(1-\varepsilon g)\,(1-\varepsilon g_*)\,|\Pi(v-v_*)\,[\nabla h - \nabla h_*]|^2\,\mathrm{d}v\mathrm{d}v_*,$$

where $h(v) = \log(g(v)) - \log(1-\varepsilon g(v))$. Using the estimate

$$\left|\Pi(v-v_*)\,[\nabla h - \nabla h_*]\,\right|^2 \leqslant 2|\nabla h|^2 + 2|\nabla h_*|^2\,,$$

one has that

$$\begin{aligned}\mathscr{D}_{\varepsilon}^{(\eta)}(g) &\leqslant 2\int_{\mathbb{R}^6}|v-v_*|^{\eta+2}\, g\, g_*(1-\varepsilon g)\,(1-\varepsilon g_*)\left|\frac{\nabla g(v)}{g(v)(1-\varepsilon g(v))}\right|^2\mathrm{d}v\mathrm{d}v_*\\ &\leqslant 2\int_{\mathbb{R}^3}\frac{|\nabla g(v)|^2}{g(1-\varepsilon g)}\mathrm{d}v\int_{\mathbb{R}^3}|v-v_*|^{\eta+2}\, g_*\mathrm{d}v_*.\end{aligned}$$

Rewrite the inner integral as

$$\int_{\mathbb{R}^3}|v-v_*|^{\eta+2}\, g_*\mathrm{d}v_* = \langle v\rangle^{\gamma}\int_{\mathbb{R}^3}|v-v_*|^{\eta+2}\, g_*\langle v\rangle^{|\gamma|}\mathrm{d}v_*$$

and split it according to the regions $\{|v-v_*| \geqslant 2|v_*|\}$ and its complement. In the first region, $|v| \geqslant |v_*|$ and $|u| \leqslant 2|v_*| \leqslant 2\langle v\rangle$, thus

$$\int_{|v-v_*|\geqslant 2|v_*|}|v-v_*|^{\eta+2}g_*\mathrm{d}v_* \leqslant 2^{\eta+2}\langle v\rangle^{\eta+2}\boldsymbol{m}_0(g).$$

In the second region, $|v| \leqslant 3|v_*|$ and $|v-v_*|^{\eta+2} \leqslant 2^{\eta+2}\langle v_*\rangle^{\eta+2}$, therefore

$$\int_{|v-v_*|\leqslant 2|v_*|}|v-v_*|^{\gamma+2}g_*\langle v\rangle^{|\gamma|}\mathrm{d}v \leqslant 3^{|\gamma|}2^{\eta+2}\int_{\mathbb{R}^3}\langle v_*\rangle^{\eta+2+|\gamma|}g_*\mathrm{d}v_*.$$

Consequently,

$$\int_{\mathbb{R}^3}|v-v_*|^{\eta+2}\, g_*\mathrm{d}v_* \leqslant 2^{\eta+2}\left(\langle v\rangle^{\eta+2}\boldsymbol{m}_0(g) + 3^{|\gamma|}\langle v\rangle^{\gamma}\boldsymbol{m}_{\eta+2+|\gamma|}(g)\right),$$

leading to

$$\mathscr{D}_{\varepsilon}^{(\eta)}(g) \leqslant 2^{\eta+3}\left(\boldsymbol{m}_0(g)\int_{\mathbb{R}^3}\langle v\rangle^{\eta+2}\frac{|\nabla g(v)|^2}{g(1-\varepsilon g)}\mathrm{d}v + 3^{|\gamma|}\boldsymbol{m}_{\eta+2+|\gamma|}(g)\int_{\mathbb{R}^3}\frac{|\nabla g(v)|^2}{g(1-\varepsilon g)}\mathrm{d}v\right).$$

The result follows since $1-\varepsilon g(v) \geqslant \kappa_0(g)$. □

We have the following improvement of [5, Proposition 5.5].

Lemma A.5. *Fix $\gamma \in (-2,0)$, $\varepsilon \in (0,\varepsilon_{\rm sat})$, and initial datum $f_{\rm in} \in \mathcal{Y}_\varepsilon$. Let $f(t,\cdot)$ be a weak-solution to equation (1.1). In addition assume that*

$$f_{\rm in} \in L^1_{\eta+\beta}(\mathbb{R}^3) \qquad \eta \geqslant 0, \quad \beta > 0\,.$$

Then, for any $t_0 > 0$ there exists $C_\eta(t_0) > 0$ depending also on γ, $\|f_{\rm in}\|_{L^1_2}$, $\boldsymbol{m}_{\eta+\beta}(0)$ such that

$$c_0 \int_{t_0}^t \mathscr{F}_{\eta+\gamma}[f(\tau)]\mathrm{d}\tau \leqslant C_\eta(t_0)\,(1+t)\,, \qquad t \geqslant t_0 > 0\,. \tag{A.11}$$

Proof. Recall the weighted Boltzmann entropy defined in (1.13), written here simply as $S_\eta(t) = S_\eta[f(t)]$, for a solution $f = f(t,v)$ to (1.1). It was noted in [5, Eq. (5.18)] that there exist $c_0 > 0$ depending only on $\|f_{\rm in}\|_{L^1_2}$ and $S_0(f_{\rm in})$, and $C_\eta(f_{\rm in}) > 0$ depending on η and $\|f_{\rm in}\|_{L^1_2}$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S_\eta(t) + c_0\,\mathscr{F}_{\eta+\gamma}[f(t)] \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}_\eta(t) + C_\eta(f_{\rm in})\int_{\mathbb{R}^3}\langle v\rangle^{\eta+\gamma} f(t,v)\,(1+|\log f(t,v)|)\,\mathrm{d}v\\
+ C_\eta(f_{\rm in})\int_{\mathbb{R}^3}\langle v\rangle^\eta \boldsymbol{c}_\gamma[f(t)](v) f(t,v)\,(1+|\log f(t,v)|)\,\mathrm{d}v\,.\end{aligned} \tag{A.12}$$

Using (A.8) and (A.7) it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}S_\eta(t) + c_0\,\mathscr{F}_{\eta+\gamma}[f(t)] \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}_\eta(t) + C_\eta(f_{\rm in},\gamma,\varepsilon)\boldsymbol{m}_\eta(t), \qquad \forall t \geqslant 0.$$

Integrating this inequality over (t_0,t) yields

$$c_0\int_{t_0}^t \mathscr{F}_{\eta+\gamma}[f(\tau)]\mathrm{d}\tau \leqslant \boldsymbol{m}_\eta(t) + S_\eta(t_0) - S_\eta(t) + C_\eta(f_{\rm in},\varepsilon)\int_{t_0}^t \boldsymbol{m}_\eta(\tau)\mathrm{d}\tau\,, \quad \forall t > t_0 > 0.$$

Furthermore, from (A.7) it follows that for any $\beta > 0$

$$-S_\eta(t) \leqslant \int_{\mathbb{R}^3}\langle v\rangle^\eta f(t,v)\,|\log f(t,v)|\,\mathrm{d}v \leqslant C_{\eta,\beta}(\varepsilon)\boldsymbol{m}_{\eta+\beta}(t)\,,$$

and, therefore,

$$c_0\int_{t_0}^t \mathscr{F}_{\eta+\gamma}[f(\tau)]\mathrm{d}\tau \leqslant C(\eta,\beta)\left(\boldsymbol{m}_{\eta+\beta}(t) + 1 + \int_{t_0}^t \boldsymbol{m}_\eta(\tau)\mathrm{d}\tau\right). \tag{A.13}$$

Consequently, estimate (A.11) follows from Theorem 2.4. □

Proof of Proposition 2.5. According to Lemma A.4, recalling that $\kappa_0(\cdot)$ is nondecreasing, it holds that

$$\int_{t_0}^t \mathscr{D}_\varepsilon^{(\eta)}(f(\tau))\mathrm{d}\tau \leqslant \frac{3^{|\gamma|+\eta+2}}{\kappa_0(t)}\left(\boldsymbol{m}_0(f_{\rm in})\int_{t_0}^t \mathscr{F}_{\eta+2}[f(\tau)]\mathrm{d}\tau + \int_{t_0}^t \boldsymbol{m}_{\eta+2+|\gamma|}(\tau)\mathscr{F}_\gamma[f(\tau)]\mathrm{d}\tau\right),$$

holds for any $t \geqslant t_0 > 0$. From Lemma A.5, we estimate the first integral as

$$\int_{t_0}^t \mathscr{F}_{\eta+2}[f(\tau)]\mathrm{d}\tau \leqslant C_0(\tilde{\eta})\,(1+t)$$

for a constant $C_0(\tilde{\eta})>0$ depending on $\|f_{\rm in}\|_{L^1_{\tilde{\eta}}}$. Regarding the second integral use Proposition 2.3 to deduce that

$$\begin{aligned}\int_{t_0}^t &\boldsymbol{m}_{\eta+2+|\gamma|}(\tau)\mathscr{F}_\gamma[f(\tau)]\mathrm{d}\tau\\ &\leqslant C_0(\gamma)\left(\int_{t_0}^t \boldsymbol{m}_{\eta+2+|\gamma|}(\tau)\mathrm{d}\tau+\left[\sup_{\tau\in(t_0,t]}\boldsymbol{m}_{\eta+2+|\gamma|}(\tau)\right]\int_{t_0}^t\mathscr{D}^{(\eta)}_{\boldsymbol{\varepsilon}}(f(\tau))\mathrm{d}\tau\right)\\ &\qquad\leqslant C_0(\gamma)\int_{t_0}^t \boldsymbol{m}_{\eta+2+|\gamma|}(\tau)\mathrm{d}\tau+C_0(\gamma)\left[\sup_{\tau\in(t_0,t]}\boldsymbol{m}_{\eta+2+|\gamma|}(\tau)\right]\mathcal{H}_{\boldsymbol{\varepsilon}}(f_{\rm in}|\mathcal{M}_{\boldsymbol{\varepsilon}})\,,\end{aligned}$$

where we used tha $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\boldsymbol{\varepsilon}}(f(t)|\mathcal{M}_{\boldsymbol{\varepsilon}})=-\mathscr{D}^{(\eta)}_{\boldsymbol{\varepsilon}}(f(t))$. Thus, by virtue of Theorem 2.4, there exists $C(t_0)>0$ depending additionally on $\mathcal{H}_{\boldsymbol{\varepsilon}}(f_{\rm in})$ and $\|f_{\rm in}\|_{L^1_{\eta+2+|\gamma|}}$ such that

$$\int_{t_0}^t \boldsymbol{m}_{\eta+2+|\gamma|}(\tau)\mathscr{F}_\gamma[f(\tau)]\mathrm{d}\tau\leqslant C(t_0)\,(1+t).$$

This proves the result. □

A.3. **Proof of Theorem 2.6.** The evolution of moments was established in [5, Lemma 3.1] valid for the whole range $\gamma\in(-3,0)$. We use the shorthand $F(t,v):=f(t,v)(1-\boldsymbol{\varepsilon}f(t,v))$ along the section. For simplicity of notations, we will always omit the time dependence and write simply $f=f(t,v)$, $F=F(t,v)=f(1-\boldsymbol{\varepsilon}f)$.

Lemma A.6. *Fix $\gamma\in(-3,0)$, $\boldsymbol{\varepsilon}\in(0,\boldsymbol{\varepsilon}_{\rm sat})$, and $f_{\rm in}\in\mathcal{Y}_{\boldsymbol{\varepsilon}}$. Let $f(t,\cdot)$ be a weak solution to (1.1). For any $s>2$ one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}f(t,v)\langle v\rangle^s\mathrm{d}v=\mathscr{J}_s(f(t),F(t))=\mathscr{J}_{s,1}(f(t),F(t))+\mathscr{J}_{s,2}(f(t),F(t)),\tag{A.14}$$

where, for any nonnegative measurable mappings $h,g\geqslant 0$ and $s>2$, we use the notations

$$\mathscr{J}_{s,1}(h,g)=2s\int\int_{\mathbb{R}^3\times\mathbb{R}^3}h(v)g(v_*)\,|v-v_*|^\gamma\left(\langle v\rangle^{s-2}-\langle v_*\rangle^{s-2}\right)\left(|v_*|^2-(v\cdot v_*)\right)\mathrm{d}v\mathrm{d}v_*,$$
$$\mathscr{J}_{s,2}(h,g)=s(s-2)\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\langle v\rangle^{s-4}h(v)g(v_*)|v-v_*|^\gamma\left(|v|^2\,|v_*|^2-(v\cdot v_*)^2\right)\mathrm{d}v\mathrm{d}v_*.$$

Lemma A.7. *Fix $\gamma\in(-3,-2]$. Under the assumptions of Theorem 2.6, for any $s\geqslant 5$ there exists $C_s>0$ depending on $s,\boldsymbol{\varepsilon},\gamma,\|f_{\rm in}\|_{L^1_2}$ such that*

$$\mathscr{J}_{s,2}(f,F)\leqslant C_{\boldsymbol{\varepsilon}}(\alpha)\boldsymbol{m}_{s+\gamma-\alpha}(F)+C_s\boldsymbol{m}_{s-4}(f)\qquad\forall\alpha\in(0,\tfrac{16+5\gamma}{3})\tag{A.15}$$

for some constant $C_{\boldsymbol{\varepsilon}}(\alpha)>0$ depending on $s,\boldsymbol{\varepsilon},\gamma,\|f_{\rm in}\|_{L^1_2}$ and α.

Proof. We observe that

$$\mathscr{J}_{s,2}(f,F)=\mathscr{J}_{s,2,1}(f,F)+\mathscr{J}_{s,2,2}(f,F)$$

with

$$\mathscr{J}_{s,2,1}(f,F) = s(s-2)\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\Big(\langle v\rangle^{s-4} - \langle v_*\rangle^{s-4}\Big)F(v_*)|v-v_*|^\gamma\big(|v|^2|v_*|^2 - (v\cdot v_*)^2\big)$$

$$\mathscr{J}_{s,2,2}(f,F) = s(s-2)\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\langle v_*\rangle^{s-4}F(v_*)|v-v_*|^\gamma\big(|v|^2|v_*|^2 - (v\cdot v_*)^2\big)\mathrm{d}v\mathrm{d}v_*\,.$$

To estimate the first integral observe that for $s>5$,

$$\left|\langle v\rangle^{s-4} - \langle v_*\rangle^{s-4}\right| \leqslant (s-4)\max\left(\langle v\rangle^{s-5},\langle v_*\rangle^{s-5}\right)|v-v_*|$$

together with

$$|v|^2|v_*|^2 - (v\cdot v_*)^2 \leqslant \min(|v|^2,|v_*|^2)|v-v_*|^2 \leqslant |v||v_*|\,|v-v_*|^2. \tag{A.16}$$

Thus,

$$|\mathscr{J}_{s,2,1}(f,F)| \leqslant s(s-2)(s-4)\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)F(v_*)\max\left(\langle v\rangle^{s-5},\langle v_*\rangle^{s-5}\right)|v||v_*||v-v_*|^{\gamma+3}\mathrm{d}v\mathrm{d}v_*\,.$$

Using that

$$|v-v_*|^{\gamma+3} \leqslant 2^{\gamma+3}\max\left(\langle v\rangle^{\gamma+3},\langle v_*\rangle^{\gamma+3}\right)$$

and estimating the integral in the regions $|v|\leqslant|v_*|$ or $|v|\geqslant|v_*|$, one deduces that

$$|\mathscr{J}_{s,2,1}(f,F)| \leqslant 2^{\gamma+3}s(s-2)(s-4)\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)F(v_*)\left(\langle v_*\rangle^{s+\gamma+2}\langle v\rangle^2 + \langle v\rangle^{s+\gamma-2}\langle v_*\rangle^2\right)\mathrm{d}v\mathrm{d}v_*.$$

Thus, there exists a constant $C_\gamma(s)>0$ such that

$$|\mathscr{J}_{s,2,1}(f,F)| \leqslant C_\gamma(s)\boldsymbol{m}_2(f)\boldsymbol{m}_{s+\gamma-2}(f)$$

where we used $F\leqslant f$. For the second integral introduce

$$\mathcal{R} = \left\{(v,v_*)\in\mathbb{R}^3\times\mathbb{R}^3\,;\;|v-v_*|\leqslant 1\right\}$$

denoting by $\mathcal{R}^c$ its complement. Splitting the term $\mathscr{J}_{s,2,2}(f,F)$ in $\mathcal{R}$ and its complement it follows that

$$\mathscr{J}_{s,2,2}(f,F) = \mathscr{J}_{2,\mathcal{R}}(f,F) + \mathscr{J}_{2,\mathcal{R}^c}(f,F)\,.$$

One deduces from (A.16) that

$$|\mathscr{J}_{2,\mathcal{R}}(f,F)| \leqslant s(s-2)\int_{\mathcal{R}} f(v)F(v_*)\langle v\rangle\langle v_*\rangle^{s-3}|v-v_*|^{\gamma+2}\mathrm{d}v\mathrm{d}v_*.$$

On $\mathcal{R}$, one has $\langle v\rangle\simeq\langle v_*\rangle$ which means that moments can be distributed on $f(v)$ or $F(v_*)$ as desired. Therefore, for any $0<\alpha<4+\gamma$, there exists $C_\alpha>0$ such that

$$|\mathscr{J}_{2,\mathcal{R}}(f,F)| \leqslant C\int_{\mathbb{R}^3} F(v_*)\langle v_*\rangle^{s+\gamma-\alpha}\mathrm{d}v_*\int_{|v-v_*|\leqslant 1} f(v)\langle v\rangle^{|\gamma|+\alpha-2}|v-v_*|^{\gamma+2}\mathrm{d}v\,. \tag{A.17}$$

Estimate the integral over v using Young's convolution inequality. For $\frac{1}{p}+\frac{1}{q}=1$, it holds that

$$\sup_{v_*}\int_{|v-v_*|\leqslant 1} f(v)\langle v\rangle^{|\gamma|+\alpha-2}|v-v_*|^{\gamma+2}\mathrm{d}v \leqslant C_0\left\|f\langle\cdot\rangle^{|\gamma|+\alpha-2}\right\|_{L^p}\left\||\cdot|^{\gamma+2}\mathbf{1}_{\{|\cdot|\leqslant 1\}}\right\|_{L^q},$$

where we observe that $|\cdot|^{\gamma+2}\mathbf{1}_{\{|\cdot|\leqslant 1\}} \in L^q(\mathbb{R}^3)$ for any $q < \frac{3}{|\gamma+2|}$. One chooses $p = \frac{2}{|\gamma|+\alpha-2} > 1$ (recall $\alpha < 4+\gamma$) so that $q = \frac{2}{4+\gamma-\alpha}$. Also, choose $0 < \alpha < \frac{16+5\gamma}{3}$ so that $q < \frac{2}{|\gamma+2|}$ and deduce that there exists $C_{\alpha,\gamma} > 0$ such that

$$\sup_{v_*}\int_{|v-v_*|\leqslant 1} f(v)\langle v\rangle^{\alpha-2}|v-v_*|^{\gamma+2}\mathrm{d}v \leqslant C_{\alpha,\gamma}\|f\langle\cdot\rangle^{|\gamma|+\alpha-2}\|_{L^p}, \qquad p = \frac{2}{|\gamma|+\alpha-2}.$$

Now, using that $\|f\|_\infty \leqslant \boldsymbol{\varepsilon}^{-1}$, simple interpolation shows that

$$\left\|f\langle\cdot\rangle^{|\gamma|+\alpha-2}\right\|_{L^p} \leqslant \boldsymbol{\varepsilon}^{-\frac{p-1}{p}}\|f\langle\cdot\rangle^2\|_{L^1}^{\frac{1}{p}} = \boldsymbol{\varepsilon}^{-\frac{4+\gamma-\alpha}{2}}\boldsymbol{m}_2(f_{\rm in})^{\frac{|\gamma|+\alpha-2}{2}}.$$

Using (A.17), we deduce the existence of $C_{\boldsymbol{\varepsilon}} > 0$ depending on $\boldsymbol{\varepsilon}, \boldsymbol{m}_2(f_{\rm in}), s, \gamma$ and α such that

$$|\mathscr{I}_{2,\mathcal{R}}(f,F)| \leqslant C_{\boldsymbol{\varepsilon}}\boldsymbol{m}_{s+\gamma-\alpha}(F), \qquad \forall \alpha \in \left(0, \tfrac{16+5\gamma}{3}\right). \tag{A.18}$$

Regarding the integral $\mathscr{I}_{\mathcal{R}^c}(f,F)$, use again (A.16) to deduce

$$|\mathscr{I}_{2,\mathcal{R}^c}(f,F)| \leqslant s(s-2)\int_{\mathcal{R}^c} f(v)\langle v\rangle^2 F(v_*)\langle v_*\rangle^{s-4}|v-v_*|^{\gamma+2}\mathrm{d}v\mathrm{d}v_*$$

where, for $(v,v_*) \in \mathcal{R}^c$, $|v-v_*|^{\gamma+2} \leqslant 1$ since $\gamma+2 \leqslant 0$. Thus,

$$|\mathscr{I}_{2,\mathcal{R}^c}(f,F)| \leqslant s(s-2)\boldsymbol{m}_2(f)\boldsymbol{m}_{s-4}(F) \leqslant s(s-2)\boldsymbol{m}_2(f)\boldsymbol{m}_{s-4}(f).$$

Gathering these estimates we deduce the result since $\boldsymbol{m}_{s+\gamma-2}(f) \leqslant \boldsymbol{m}_{s-4}(f)$. □

Lemma A.8. *Fix $\gamma \in (-3,-2]$. Under the assumptions of Theorem 2.6, for any $s \geqslant 5$, there exist $C_s > 0$ and $C_{\boldsymbol{\varepsilon}}(\alpha) > 0$ depending on $s, \boldsymbol{\varepsilon}, \gamma, \|f_{\rm in}\|_{L^1_2}$ such that*

$$\mathscr{I}_{s,1}(f,F) \leqslant -C_1\boldsymbol{m}_{s+\gamma}(F) + C_1\boldsymbol{m}_{s+\gamma-1}(f) + C_{\boldsymbol{\varepsilon}}(\alpha)\boldsymbol{m}_{s+\gamma-\alpha}(F) \quad \forall \alpha \in (0, \tfrac{16+5\gamma}{3}), \tag{A.19}$$

where $C_{\boldsymbol{\varepsilon}}(\alpha) > 0$ depends additionally on α.

Proof. Using the regions $\mathcal{A} = \{(v,v_*) \in \mathbb{R}^3 \times \mathbb{R}^3\,;\, \langle v_*\rangle \geqslant 2\langle v\rangle\}$ and its complement $\mathcal{A}^c$ write

$$\mathscr{I}_{s,1}(f,F) = \mathscr{I}_{1,\mathcal{A}}(f,F) + \mathscr{I}_{1,\mathcal{A}^c}(f,F).$$

Recall that

$$\mathscr{I}_{1,\mathcal{A}}(f,F) = 2s\int_{\mathcal{A}} f(v)F(v_*)\,|v-v_*|^\gamma\left(\langle v\rangle^{s-2} - \langle v_*\rangle^{s-2}\right)\left(|v_*|^2 - (v\cdot v_*)\right)\mathrm{d}v\mathrm{d}v_*\,.$$

The proof is based on the basic observation that, for $\langle v\rangle \leqslant \frac{1}{2}\langle v_*\rangle$, it holds $|v| \leqslant \frac{1}{2}|v_*|$ and

$$\langle v_*\rangle^{s-2} - \langle v\rangle^{s-2} \geqslant \frac{1}{2}\langle v_*\rangle^{s-2}, \qquad |v_*|^2 - (v\cdot v_*) \geqslant \frac{1}{2}|v_*|^2$$

and $|v-v_*| \leqslant \frac{3}{2}|v_*| \leqslant \frac{3}{2}\langle v_*\rangle$. In particular, $|v_*|^2 \geqslant \frac{4}{9}|v-v_*|^2$ and

$$\begin{aligned}
-\mathscr{I}_{1,\mathcal{A}}(f,F) &\geqslant \frac{s}{4}\int_{\mathbb{R}^3} F(v_*)\langle v_*\rangle^{s-2}\,|v_*|^2\mathrm{d}v_*\int_{\langle v_*\rangle\geqslant 2\langle v\rangle} f(v)|v-v_*|^\gamma\mathrm{d}v \\
&\geqslant \frac{s}{9}\int_{\mathbb{R}^3} F(v_*)\langle v_*\rangle^{s-2}\mathrm{d}v_*\int_{\langle v_*\rangle\geqslant 2\langle v\rangle} f(v)|v-v_*|^{\gamma+2}\mathrm{d}v \\
&\geqslant \frac{s}{9}\left(\frac{3}{2}\right)^\gamma\int_{\mathbb{R}^3} F(v_*)\langle v_*\rangle^{s+\gamma}\int_{\langle v_*\rangle\geqslant 2\langle v\rangle} f(v)\mathrm{d}v
\end{aligned}$$

where we recall $\gamma+2\leqslant 0$ and $|v-v_*|\leqslant \frac{3}{2}\langle v_*\rangle$ on $\mathcal{A}$. Since for any $v_*\in\mathbb{R}^3$

$$\int_{\langle v_*\rangle\geqslant 2\langle v\rangle} f(v)\mathrm{d}v = \|f\|_{L^1} - \int_{\langle v_*\rangle<2\langle v\rangle} f(v)\frac{\langle v\rangle^2}{\langle v\rangle^2}\mathrm{d}v \geqslant \|f\|_{L^1} - \frac{4}{\langle v_*\rangle^2}\boldsymbol{m}_2(f),$$

one deduces that there exists $C_{s,\gamma}>0$ such that

$$-\mathscr{J}_{1,\mathcal{A}}(f,F)\geqslant C_{s,\gamma}\left(\boldsymbol{m}_{s+\gamma}(F)-\boldsymbol{m}_{s+\gamma-2}(F)\right).$$

We focus now on

$$\mathscr{J}_{1,\mathcal{A}^c}(f,F)=2s\int_{\langle v_*\rangle<2\langle v\rangle} f(v)F(v_*)\,|v-v_*|^\gamma\left(\langle v\rangle^{s-2}-\langle v_*\rangle^{s-2}\right)\left(|v_*|^2-(v\cdot v_*)\right)\mathrm{d}v\mathrm{d}v_*\,.$$

On the set $\mathcal{A}^c$, one has that

$$\left|\langle v\rangle^{s-2}-\langle v_*\rangle^{s-2}\right|\leqslant (s-2)\max\{\langle v_*\rangle^{s-3},\langle v\rangle^{s-3}\}\,|v-v_*|\leqslant 2^{s-3}(s-2)\langle v\rangle^{s-3}\,|v-v_*|$$

and $\big||v_*|^2-(v\cdot v_*)\big|=|(v_*-v)\cdot v_*|\leqslant |v_*|\,|v-v_*|$, so that

$$|\mathscr{J}_{1,\mathcal{A}^c}(f,F)|\leqslant 2^{s-2}s(s-2)\int_{\mathcal{A}^c} f(v)F(v_*)|v_*|\,|v-v_*|^{\gamma+2}\langle v\rangle^{s-3}\mathrm{d}v\mathrm{d}v_*\,.$$

Split the integral in the regions $\mathcal{A}^c\cap\mathcal{R}$ and $\mathcal{A}^c\cap\mathcal{R}^c$. As in the proof of (A.18), there exists $C_{\boldsymbol{\varepsilon}}>0$ depending on $\boldsymbol{\varepsilon},\boldsymbol{m}_2(f_{\mathrm{in}}),s,\gamma$ and α such that

$$\int_{\mathcal{A}^c\cap\mathcal{R}} f(v)F(v_*)|v_*|\,|v-v_*|^{\gamma+2}\langle v\rangle^{s-3}\mathrm{d}v\mathrm{d}v_*\leqslant C_{\boldsymbol{\varepsilon}}\boldsymbol{m}_{s+\gamma-\alpha}(F)\qquad\forall\alpha\in(0,\tfrac{16+5\gamma}{3})$$

since, on $\mathcal{R}$, $\langle v\rangle\simeq\langle v_*\rangle$. Regarding the region $\mathcal{A}^c\cap\mathcal{R}^c$ we split it further as $\mathcal{A}^c=\mathcal{B}_1\cup\mathcal{B}_2$ with

$$\mathcal{B}_1=\left\{\langle v\rangle>2\langle v_*\rangle\right\}\qquad\text{and}\qquad\mathcal{B}_2=\left\{\tfrac{1}{2}\langle v\rangle\leqslant\langle v_*\rangle<2\langle v\rangle\right\}.$$

Since $\langle v\rangle\leqslant 2|v-v_*|$ for any $(v,v_*)\in\mathcal{B}_1$, one deduces that

$$\int_{\mathcal{B}_1\cap\mathcal{R}^c} f(v)F(v_*)|v_*|\,|v-v_*|^{\gamma+2}\langle v\rangle^{s-3}\mathrm{d}v\mathrm{d}v_*$$
$$\leqslant 2^{\gamma+2}\int_{\mathcal{B}_1} f(v)F(v_*)\langle v\rangle^{s+\gamma-1}|v_*|\mathrm{d}v\mathrm{d}v_*\leqslant 2^{\gamma+2}\boldsymbol{m}_{s+\gamma-1}(f)\boldsymbol{m}_1(F)\,.$$

In addition, since $|v-v_*|^{\gamma+2}\leqslant 1$ on $\mathcal{R}^c$ while, on $\mathcal{B}_2$, $\langle v\rangle^{s-3}\langle v_*\rangle\leqslant 2\langle v\rangle^{s-4}\langle v_*\rangle^2$, one deduces that

$$\int_{\mathcal{B}_1\cap\mathcal{R}^c} f(v)F(v_*)|v_*|\,|v-v_*|^{\gamma+2}\langle v\rangle^{s-3}\mathrm{d}v\mathrm{d}v_*\leqslant 2\boldsymbol{m}_{s-4}(f)\boldsymbol{m}_2(F).$$

Using that $F\leqslant f$ so that $\boldsymbol{m}_1(F)\leqslant\frac{1}{2}\boldsymbol{m}_0(f)+\frac{1}{2}\boldsymbol{m}_2(f)$ and $\boldsymbol{m}_2(F)\leqslant\boldsymbol{m}_2(f)$ the result follows since $\boldsymbol{m}_{s-4}(f)\leqslant\boldsymbol{m}_{s-2+\gamma}(f)\leqslant\boldsymbol{m}_{s+\gamma-1}(f)$. □

Proof of Theorem 2.6. We start from (A.14) and estimate $\mathscr{J}_{s,1}(f(t),F(t))$ and $\mathscr{J}_{s,2}(f(t),F(t))$ with Lemmas A.8 and A.7 respectively to deduce that there exist positive $\boldsymbol{C}_{1,s},\boldsymbol{C}_{s,2},\boldsymbol{C}_{\boldsymbol{\varepsilon}}(\alpha)$ depending on $\boldsymbol{m}_0(f_{\mathrm{in}}),\boldsymbol{m}_2(f_{\mathrm{in}})$ and s,γ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}_s(t)+\boldsymbol{C}_{1,s}\boldsymbol{m}_{s+\gamma}(F(t))\leqslant\boldsymbol{C}_{s,2}\boldsymbol{m}_{s+\gamma-1}(t)+\boldsymbol{C}_{\boldsymbol{\varepsilon}}(\alpha)\boldsymbol{m}_{s+\gamma-\alpha}(F(t))\qquad\alpha\in(0,\tfrac{16+5\gamma}{3})$$

where $\boldsymbol{C}_{\boldsymbol{\varepsilon}}(\alpha)$ depends additionally on α. A simple use of Young's inequality allows to compare $\boldsymbol{m}_{s+\gamma-\alpha}(F(t))$ to $\boldsymbol{m}_{s+\gamma}(F(t))$ and deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}_s(t) + \frac{1}{2}\boldsymbol{C}_{1,s}\boldsymbol{m}_{s+\gamma}(F(t)) \leqslant \boldsymbol{C}_{s,2}\boldsymbol{m}_{s+\gamma-1}(t) + \boldsymbol{C}_{s,3} \tag{A.20}$$

for some $\boldsymbol{C}_{s,3}$ depending on $s, \gamma, \|f_{\rm in}\|_{L^1_2}$ and $\boldsymbol{\varepsilon}$. Interpolation gives

$$\boldsymbol{m}_{s+\gamma-1}(t) \leqslant \boldsymbol{m}_2(t)^{\theta}\boldsymbol{m}_s(t)^{1-\theta} \leqslant C(f_{\rm in})\boldsymbol{m}_s(t)^{1-\theta}, \qquad \theta = \frac{1+|\gamma|}{s-2}\,.$$

Therefore, neglecting the term $\boldsymbol{m}_{s+\gamma}(F(t))$, we deduce that there is $\boldsymbol{C}_{s,4} > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}_s(t) \leqslant \boldsymbol{C}_{s,3} + \boldsymbol{C}_{s,4}\boldsymbol{m}_s(t)^{1-\theta}\,, \tag{A.21}$$

and conclude thanks to Gronwall Lemma recalling that $\theta = \frac{1+|\gamma|}{s-2}$. □

Appendix B. Level set estimates

We revisit the evolution of level sets f_ℓ^+ derived in [5] to add the information $f \leqslant \frac{1}{\boldsymbol{\varepsilon}}$ and incorporate the term $1 - \boldsymbol{\varepsilon} f(t,v)$.

Lemma B.1. *Fix $\gamma \in (-3,0)$, $\boldsymbol{\varepsilon} \in (0, \boldsymbol{\varepsilon}_{\rm sat})$, and let an initial datum $f_{\rm in} \in \mathcal{Y}_{\boldsymbol{\varepsilon}}$ be given. Let $f(t,\cdot)$ be a weak-solution to (1.1). There exists $\alpha_0 > 0$ depending only on $\|f_{\rm in}\|_{L^1_2}$ and $\mathcal{H}_{\boldsymbol{\varepsilon}}(f_{\rm in})$ such that*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\ell^+(t)\|_{L^2}^2 + \alpha_0\left\|\langle\cdot\rangle^{\frac{\gamma}{2}} f_\ell^+(t)\right\|_{H^1}^2 \leqslant \ell(1-\boldsymbol{\varepsilon}\ell)\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+(t)](v)\,f_\ell^+(t,v)\mathrm{d}v, \qquad \ell \in \left[\frac{3}{4\boldsymbol{\varepsilon}}, \frac{1}{\boldsymbol{\varepsilon}}\right]. \tag{B.1}$$

Proof. We follow and modify [5]. Fix $\ell \in [\frac{3}{4\boldsymbol{\varepsilon}}, \frac{1}{\boldsymbol{\varepsilon}}]$ and note that

$$\partial_t\left(f_\ell^+\right)^2 = 2f_\ell^+\partial_t f_\ell^+ = 2f_\ell^+\partial_t f \qquad \text{and} \qquad \nabla f_\ell^+ = \mathbf{1}_{\{f\geqslant\ell\}}\nabla f,$$

so that, multiplying (1.1) with f_ℓ^+ and integrating over $\mathbb{R}^3$ we get that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\ell^+(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^3}\boldsymbol{\Sigma}[f]\nabla f\cdot\nabla f_\ell^+\mathrm{d}v + \int_{\mathbb{R}^3}f(1-\boldsymbol{\varepsilon}f)\boldsymbol{b}[f]\cdot\nabla f_\ell^+\mathrm{d}v\\ &= -\int_{\mathbb{R}^3}\boldsymbol{\Sigma}[f]\nabla f_\ell^+\cdot\nabla f_\ell^+\mathrm{d}v + \int_{\mathbb{R}^3}f(1-\boldsymbol{\varepsilon}f)\boldsymbol{b}[f]\cdot\nabla f_\ell^+\mathrm{d}v.\end{aligned}$$

Observing the identity

$$f(1-\boldsymbol{\varepsilon}f)\nabla f_\ell^+ = \left(\tfrac{1}{2}-\boldsymbol{\varepsilon}\ell\right)\nabla\left(f_\ell^+\right)^2 - \frac{\boldsymbol{\varepsilon}}{3}\nabla\left(f_\ell^+\right)^3 + \ell(1-\boldsymbol{\varepsilon}\ell)\nabla f_\ell^+\,,$$

and recalling that $\boldsymbol{c}_\gamma[f] = -\nabla\cdot\boldsymbol{b}[f]$, we deduce after integration by parts that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\ell^+(t)\|_{L^2}^2 + \int_{\mathbb{R}^3}\boldsymbol{\Sigma}[f]\nabla f_\ell^+\cdot\nabla f_\ell^+\mathrm{d}v &= \left(\tfrac{1}{2}-\boldsymbol{\varepsilon}\ell\right)\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f](f_\ell^+)^2\mathrm{d}v\\ &\quad + \ell(1-\boldsymbol{\varepsilon}\ell)\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f]f_\ell^+\mathrm{d}v - \frac{\boldsymbol{\varepsilon}}{3}\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f]\,(f_\ell^+)^3\mathrm{d}v.\end{aligned}$$

Recalling that $\boldsymbol{c}_\gamma[f] \geqslant 0$, we disregard the last term and, using that $\frac{3}{4\varepsilon} \leqslant \ell \leqslant \frac{1}{\varepsilon}$, we deduce that

$$(\tfrac{1}{2} - \varepsilon\ell) \int_{\mathbb{R}^3} \boldsymbol{c}_\gamma[f](f_\ell^+)^2 \mathrm{d}v \leqslant -\frac{1}{4} \int_{\mathbb{R}^3} \boldsymbol{c}_\gamma[f](f_\ell^+)^2 \mathrm{d}v.$$

Using Lemma A.1 it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\ell^+(t)\|_{L^2}^2 + \int_{\mathbb{R}^3} \boldsymbol{\Sigma}[f]\nabla f_\ell^+ \cdot \nabla f_\ell^+ \mathrm{d}v + \frac{c_0}{4}\left\|\langle\cdot\rangle^{\frac{\gamma}{2}} f_\ell^+(t)\right\|_2^2 \leqslant +\ell(1-\varepsilon\ell)\int_{\mathbb{R}^3} \boldsymbol{c}_\gamma[f] f_\ell^+ \mathrm{d}v\,. \tag{B.2}$$

Using then Eq. (2.3) in Lemma 2.1 we have that

$$\boldsymbol{\Sigma}[f]\nabla f_\ell^+ \cdot \nabla f_\ell^+ \;\geqslant\; K_0 \left|\langle v\rangle^{\frac{\gamma}{2}} \nabla f_\ell^+(v)\right|^2 \;=\; K_0 \left|\nabla\left(\langle v\rangle^{\frac{\gamma}{2}} f_\ell^+(v)\right) - \frac{\gamma}{2} v \langle v\rangle^{\frac{\gamma}{2}-2} f_\ell^+(v)\right|^2 .$$

Recall the inequality $(a-b)^2 \geqslant (1-\eta)a^2 + (1-\frac{1}{\eta})b^2$, valid for any a, b, and $\eta \in (0,1)$ so that

$$\boldsymbol{\Sigma}[f]\nabla f_\ell^+ \cdot \nabla f_\ell^+ \geqslant K_0(1-\eta)\left|\nabla\left(\langle v\rangle^{\frac{\gamma}{2}} f_\ell^+(v)\right)\right|^2 + \frac{K_0\gamma^2}{4}\left(1 - \frac{1}{\eta}\right)\left|v\langle v\rangle^{\frac{\gamma}{2}-2} f_\ell^+(v)\right|^2 .$$

Choosing $\eta = \frac{2K_0\gamma^2}{2K_0\gamma^2 + c_0} \in (0,1)$, that is, $\frac{K_0\gamma^2}{4}\left(1-\frac{1}{\eta}\right) = -\frac{c_0}{8}$, we deduce that

$$\int_{\mathbb{R}^3} \boldsymbol{\Sigma}[f]\nabla f_\ell^+ \cdot \nabla f_\ell^+ \mathrm{d}v \geqslant K_0 \int_{\mathbb{R}^3}\left|\nabla\left(\langle v\rangle^{\frac{\gamma}{2}} f_\ell^+(t,v)\right)\right|^2 \mathrm{d}v - \frac{c_0}{8}\left\|\langle\cdot\rangle^{\frac{\gamma}{2}} f_\ell^+\right\|_{L^2}^2 ,$$

where we used that $|v\langle v\rangle^{\frac{\gamma}{2}-2}| \leqslant \langle v\rangle^{\frac{\gamma}{2}}$. Inserting into (B.2) we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\ell^+(t)\|_{L^2}^2 + K_0\int_{\mathbb{R}^3}\left|\nabla\left(\langle v\rangle^{\frac{\gamma}{2}} f_\ell^+(t,v)\right)\right|^2 \mathrm{d}v + \frac{c_0}{8}\left\|\langle\cdot\rangle^{\frac{\gamma}{2}} f_\ell^+(t)\right\|_2^2 \leqslant \ell(1-\varepsilon\ell)\int_{\mathbb{R}^3} \boldsymbol{c}_\gamma[f] f_\ell^+ \mathrm{d}v\,.$$

Setting $\alpha_0 := \min\left(K_0, \frac{c_0}{8}\right)$ it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\ell^+(t)\|_{L^2}^2 + \alpha_0\left\|\langle\cdot\rangle^{\frac{\gamma}{2}} f_\ell^+(t)\right\|_{H^1}^2 \leqslant \ell(1-\varepsilon\ell)\int_{\mathbb{R}^3} \boldsymbol{c}_\gamma[f] f_\ell^+ \mathrm{d}v.$$

Finally, since $\boldsymbol{c}_\gamma[\cdot]$ is a divergence operator, it holds that

$$\begin{aligned}\boldsymbol{c}_\gamma[f] = -\nabla\cdot\boldsymbol{b}[f] = -\nabla\cdot\int_{\mathbb{R}^3} b(v-v_*)f(v_*)\mathrm{d}v_* &= -\int_{\mathbb{R}^3} b(v-v_*)\cdot\nabla f(v_*)\,\mathrm{d}v_* \\ = -\int_{\mathbb{R}^3} b(v-v_*)\cdot\nabla f_\ell(v_*)\,\mathrm{d}v_* &= \boldsymbol{c}_\gamma[f_\ell] \leqslant \boldsymbol{c}_\gamma[f_\ell^+]\,,\end{aligned}$$

where the latter follows since $f_\ell \leqslant f_\ell^+$. This gives the result. □

Proof of Lemma 3.1. Fix $0 \leqslant T_1 < T_2 \leqslant T_3$. Integrating inequality (B.1) over (t_1, t_2) with $T_1 \leqslant t_1 \leqslant T_2 \leqslant t_2 \leqslant T_3$ we deduce that

$$\begin{aligned}\|f_\ell^+(t_2)\|_{L^2}^2 + 2\alpha_0\int_{T_2}^{t_2}\left\|\langle\cdot\rangle^{\frac{\gamma}{2}} f_\ell^+(\tau)\right\|_{H^1}^2 \mathrm{d}\tau \leqslant \left\|f_\ell^+(t_1)\right\|_{L^2}^2 \\ + 2\ell(1-\varepsilon\ell)\int_{T_1}^{t_2}\mathrm{d}\tau\int_{\mathbb{R}^3} \boldsymbol{c}_\gamma[f_\ell^+(\tau)](v)\, f_\ell^+(\tau,v)\mathrm{d}v\,.\end{aligned}$$

Taking the supremum over $t_2 \in [T_2, T_3]$ it follows that

$$\mathscr{E}_\ell(T_2,T_3) \leqslant \left\|f_\ell^+(t_1)\right\|_{L^2}^2 + 2\ell(1-\varepsilon\ell)\int_{T_1}^{T_3}\mathrm{d}\tau\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+(\tau)](v)\, f_\ell^+(\tau,v)\mathrm{d}v \qquad t_1 \in [T_1,T_2].$$

Integrating with respect to $t_1 \in [T_1, T_2]$ we deduce (3.2). Let us prove now (3.3) by observing that, since $f \leqslant \frac{1}{\varepsilon}$,

$$0 \leqslant f_\ell^+ \leqslant \frac{1-\varepsilon\ell}{\varepsilon}\mathbf{1}_{\{f\geqslant\ell\}} \tag{B.3}$$

and therefore

$$\|f_\ell^+\|_{L^2}^2 = \int_{\{f\geqslant\ell\}}(f_\ell^+)^2\mathrm{d}v \leqslant \left(\frac{1-\varepsilon\ell}{\varepsilon}\right)^2\int_{\mathbb{R}^3}\frac{f}{\ell}\mathrm{d}v = \frac{1}{\ell}\left(\frac{1-\varepsilon\ell}{\varepsilon}\right)^2\|f\|_1.$$

Since $\frac{3}{4\varepsilon} \leqslant \ell \leqslant \frac{1}{\varepsilon}$ one deduces that

$$\|f_\ell^+\|_{L^2}^2 \leqslant \frac{4(1-\varepsilon\ell)^2}{3\varepsilon}\|f\|_{L^1}\,.$$

Due to the conservation of mass, for any $0 \leqslant T_1 < T_2$ it holds

$$\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\|f_\ell^+(\tau)\|_{L^2}^2\mathrm{d}\tau \leqslant \frac{4(1-\varepsilon\ell)^2}{3\varepsilon}\|f\|_{L^1}. \tag{B.4}$$

Moreover, for all $\tau \geqslant 0$, considering the regions $|v-v_*| \geqslant 1$ and $|v-v_*| < 1$, we have that

$$\begin{aligned}\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+(\tau)]f_\ell^+\mathrm{d}v &\leqslant 2(\gamma+3)\|f_\ell^+(\tau)\|_{L^1}^2 + 2(\gamma+3)\|f_\ell^+(\tau)\|_{L^\infty}\sup_v\int_{|v-v_*|\leqslant 1}|v-v_*|^\gamma\mathrm{d}v_*\\ &\leqslant 2(\gamma+3)\|f_\ell^+(\tau)\|_{L^1}\left(\|f_\ell^+(\tau)\|_{L^1} + \frac{|\mathbb{S}^2|}{3+\gamma}\|f_\ell^+(\tau)\|_{L^\infty}\right).\end{aligned}$$

And using (B.3)

$$\|f_\ell^+\|_{L^1} = \int_{\{f\geqslant\ell\}}f_\ell^+\mathrm{d}v \leqslant \int_{\{f\geqslant\ell\}}f_\ell^+\frac{f}{\ell}\mathrm{d}v \leqslant \frac{1-\varepsilon\ell}{\varepsilon\ell}\|f\|_{L^1}, \tag{B.5}$$

and we deduce that there exists $C_0 > 0$ such that

$$2(\gamma+3)\left(\|f_\ell^+(\tau)\|_{L^1} + \frac{|\mathbb{S}^2|}{3+\gamma}\|f_\ell^+(\tau)\|_{L^\infty}\right) \leqslant C_0\left(\frac{1-\varepsilon\ell}{\varepsilon\ell}\|f\|_{L^1} + \frac{1-\varepsilon\ell}{\varepsilon}\right),$$

so that

$$\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+(\tau)](v)f_\ell^+(\tau,v)\mathrm{d}v \leqslant C_0\frac{1-\varepsilon\ell}{\varepsilon}\left(1+\frac{1}{\ell}\|f\|_{L^1}\right)\|f_\ell^+(\tau)\|_{L^1} \qquad \forall \tau \geqslant 0.$$

Using (B.5), since $\frac{1}{\ell} \leqslant \frac{4}{3}\varepsilon$, there exists a universal constant $\tilde{C}_0 > 0$ such that

$$\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+(\tau)](v)f_\ell^+(\tau,v)\mathrm{d}v \leqslant \tilde{C}_0\frac{(1-\varepsilon\ell)^2}{\varepsilon^2\ell}\left(1+\varepsilon\|f\|_{L^1}\right)\|f\|_{L^1} \qquad \forall \tau \geqslant 0.$$

Consequently,

$$2\ell(1-\varepsilon\ell)\int_{T_1}^{T_3}\mathrm{d}\tau\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+(\tau)](v)f_\ell^+(\tau,v)\mathrm{d}v$$
$$\leqslant 2\tilde{C}_0(T_3-T_1)\frac{(1-\varepsilon\ell)^3}{\varepsilon^2}\big(1+\varepsilon\|f\|_{L^1}\big)\|f\|_{L^1}\,. \tag{B.6}$$

Combining (B.4)–(B.6) to (3.2) gives for all $0\leqslant T_1<T_2\leqslant T_3$

$$\mathscr{E}_\ell(T_2,T_3)\leqslant\frac{4(1-\varepsilon\ell)^2}{3\varepsilon}\|f\|_{L^1}+2\tilde{C}_0(T_3-T_1)\frac{(1-\varepsilon\ell)^3}{\varepsilon^2}\big(1+\varepsilon\|f\|_{L^1}\big)\|f\|_{L^1}$$

which, letting T_1 converge to T_2 gives (3.3). Regarding estimate (3.4), as in Proposition A.2, let

$$1\leqslant\eta<\min\left(\frac{3}{|\gamma|},\frac{3}{2}\right),\qquad s=\frac{3|\gamma|\eta}{7\eta-6},\qquad \beta=\frac{|\gamma|}{s}=\frac{7\eta-6}{3\eta}\,. \tag{B.7}$$

Observation that if $0\leqslant k<\ell$ then $0\leqslant f_\ell^+\leqslant f_k^+$ and $\mathbf{1}_{\{f_\ell\geqslant 0\}}\leqslant\frac{f_k^+}{\ell-k}$. Thus,

$$f_\ell^+\leqslant(\ell-k)^{-\alpha}\left(f_k^+\right)^{1+\alpha}\qquad\forall\,\alpha\geqslant 0,\qquad 0\leqslant k<\ell. \tag{B.8}$$

Therefore,

$$\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+]\,f_\ell^+\mathrm{d}v\leqslant\frac{1}{\ell-k}\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+]\,\left(f_k^+\right)^2\mathrm{d}v\,.$$

Use Proposition A.2 to estimate the right side to obtain that for any $1\leqslant\eta<\min\left(\frac{3}{|\gamma|},\frac{3}{2}\right)$ there exists $C_\gamma(\eta)>0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+]\,f_\ell^+\mathrm{d}v&\leqslant\frac{C_\gamma(\eta)}{\ell-k}\left(\boldsymbol{m}_s(f_\ell^+)^\beta\|f_\ell^+\|_{L^2}^{1-\beta}\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{H^1}^2+\boldsymbol{m}_{|\gamma|}(f_\ell^+)\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{L^2}^2\right)\\&\leqslant\frac{C_\gamma(\eta)}{\ell-k}\left(\boldsymbol{m}_s(f)^\beta\|f_k^+\|_{L^2}^{1-\beta}\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{H^1}^2+\boldsymbol{m}_{|\gamma|}(f_\ell^+)\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{L^2}^2\right)\end{aligned} \tag{B.9}$$

with s,β given by (B.7). We used that for any $0\leqslant k<\ell$, it holds $f_\ell^+\leqslant f_k^+\leqslant f$.

Let us focus in the term $\boldsymbol{m}_{|\gamma|}(f_\ell^+)\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{L^2}^2$. On the set $\{f\geqslant\ell\}$ it follows that for all $a>0$, $f^a\ell^{-a}\geqslant 1$, thus

$$\boldsymbol{m}_{|\gamma|}(f_\ell^+)=\|\langle\cdot\rangle^{|\gamma|}f_\ell^+\|_{L^1}\leqslant\ell^{-a}\|\langle\cdot\rangle^{|\gamma|}f_\ell^+f^a\|_{L^1}.$$

For any $p>1,a>0$, we deduce from Hölder inequality that

$$\|\langle\cdot\rangle^{|\gamma|}f_\ell^+f^a\|_{L^1}\leqslant\|\langle\cdot\rangle^{|\gamma|}f^a\|_{L^r}\,\|f_\ell^+\|_{L^p},\qquad\frac{1}{p}+\frac{1}{r}=1.$$

Recalling (B.7) we choose $r=\frac{s}{|\gamma|}=\frac{1}{\beta}$, $a=\frac{1}{r}=\beta$ to deduce that

$$\boldsymbol{m}_{|\gamma|}(f_\ell^+)\leqslant\ell^{-\beta}\,\boldsymbol{m}_s(f)^\beta\,\|f_\ell^+\|_{L^p}\qquad p=\frac{1}{1-\beta}=\frac{3\eta}{6-4\eta}.$$

We distinguish here two cases according to $1<p<2$ or $p\geqslant 2$ or equivalently $\beta<\frac{1}{2}$ or $\beta\geqslant\frac{1}{2}$.

• *Case* $\gamma \in \left(-3,-\frac{11}{4}\right]$: Since $1 \leqslant \eta < \min\left(\frac{3}{|\gamma|},\frac{3}{2}\right) \leqslant \frac{12}{11}$, $\beta < \frac{1}{2}$, and $1 < p = \frac{1}{1-\beta} < 2$, using interpolation gives

$$\|f_\ell^+\|_{L^p} \leqslant \|f_\ell^+\|_{L^1}^{1-2\beta}\,\|f_\ell^+\|_{L^2}^{2\beta} \leqslant \|f\|_{L^1}^{1-2\beta}\,\|f_k^+\|_{L^2}^{2\beta}.$$

Therefore,

$$\boldsymbol{m}_{|\gamma|}(f_\ell^+)\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{L^2}^2 \leqslant \ell^{-\beta}\,\boldsymbol{m}_s(f)^\beta\|f\|_{L^1}^{1-2\beta}\,\|f_k^+\|_{L^2}^{2\beta}\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{H^1}^2.$$

We deduce from (B.9) that

$$\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+]\,f_\ell^+\mathrm{d}v \leqslant \frac{C_1}{\ell-k}\boldsymbol{m}_s(f)^\beta\left(1+\ell^{-\beta}\right)\left(\|f_k^+\|_{L^2}^{1-\beta}+\|f_k^+\|_{L^2}^{2\beta}\right)\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{H^1}^2.$$

Since $\|f_k^+\|_{L^2} \leqslant \frac{\|f\|_{L^1}}{\varepsilon}$, there exists $C_2 > 0$ depending only on $\varepsilon, \|f\|_{L^1}$ such that

$$\|f_k^+\|_{L^2}^{2\beta}+\|f_k^+\|_{L^2}^{1-\beta} \leqslant C_2\|f_k^+\|_{L^2}^{\min(1-\beta,2\beta)} = C_2\|f_k^+\|_{L^2}^{1-\beta}$$

where we notice that $\eta \geqslant 1$ implies $2\beta \geqslant 1-\beta$. Therefore, there exists $C = C(\gamma,\eta,\varepsilon,f_{\rm in})$ depending only on γ,η,ε and $\|f_{\rm in}\|_{L^1_2}$ such that

$$\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+]\,f_\ell^+\mathrm{d}v \leqslant \frac{C_2}{\ell-k}\boldsymbol{m}_s(f)^\beta\left(1+\ell^{-\beta}\right)\|f_k^+\|_{L^2}^{1-\beta}\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{H^1}^2. \tag{B.10}$$

• *Case* $\gamma \in \left(-\frac{11}{4},0\right)$: This case corresponds to $\min\left(\frac{3}{|\gamma|},\frac{3}{2}\right) > \frac{12}{11}$, thus, choose $\eta \geqslant \frac{12}{11}$ so that $\beta \geqslant \frac{1}{2}$ and $p = \frac{1}{1-\beta} \geqslant 2$. Use the interpolation

$$\|f_\ell^+\|_{L^p} \leqslant \|f_\ell^+\|_{L^2}^{\frac{2}{p}}\|f_\ell^+\|_{L^\infty}^{\frac{p-2}{p}}.$$

Since $f_\ell^+ \leqslant f-\ell \leqslant \frac{1-\varepsilon\ell}{\varepsilon}$ and $\frac{2}{p} = 2(1-\beta)$ one deduces that

$$\boldsymbol{m}_{|\gamma|}(f_\ell^+)\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{L^2}^2 \leqslant \ell^{-\beta}\left(\frac{1-\varepsilon\ell}{\varepsilon}\right)^{2\beta-1}\|f_k^+\|_{L^2}^{2(1-\beta)}\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{H^1}^2$$

since $\|\cdot\|_{L^2} \leqslant \|\cdot\|_{H^1}$. Moreover, $s \geqslant 2$ so that $\boldsymbol{m}_s(f) \geqslant \boldsymbol{m}_2(f) = \boldsymbol{m}_2(f_{\rm in})$, thus, it follows from (B.9) that there exists $C := C(f_{\rm in}) > 0$ such that

$$\int_{\mathbb{R}^3}\boldsymbol{c}_\gamma[f_\ell^+]\,f_\ell^+\mathrm{d}v \leqslant \frac{C}{\ell-k}\boldsymbol{m}_s(f)^\beta\left(1+\ell^{-\beta}\left(1-\varepsilon\ell\right)^{2\beta-1}\right)\|f_k^+\|_{L^2}^{1-\beta}\|\langle\cdot\rangle^{\frac{\gamma}{2}}f_k^+\|_{H^1}^2. \tag{B.11}$$

We used that $\|f_k^+\|_{L^2}^{1-\beta}+\|f_k^+\|_{L^2}^{2(1-\beta)} \leqslant (1+(\varepsilon^{-1}\|f\|_{L^1})^{1-\beta})\|f_k^+\|_{L^2}^{1-\beta}$.

Combining (B.10) and (B.11) yields the result observing that, since (B.10) holds for $\beta < \frac{1}{2}$, one has that $\left(1+\ell^{-\beta}\right) = \left(1+\ell^{-\beta}\left(1-\varepsilon\ell\right)^{(2\beta-1)^+}\right)$ where $x^+ = \max(0,x)$. □

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