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A redistributive GSA scheme to cope with socio-economic mortality differentials

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Abstract

Longevity risk is threatening the sustainability of traditional pension systems. To deal with this issue, decumulation strategies alternative to annuities have been proposed in the literature. However, heterogeneity in mortality experiences in the pool of policyholders due to socio-economic classes generates inequity, because of implicit wealth transfers from the more disadvantaged to the wealthier classes. We address this issue in a GSA (Group Self-Annuitization) scheme in the presence of stochastic mortality by proposing a redistributive GSA scheme where benefits are optimally shared across classes. The expected present values of the benefits in a standard GSA scheme show relevant gaps across socio-economic groups, which are reduced in the redistributive GSA Scheme. We explore sensitivity to pool size and interest rates.

Keywords: group self-annuitization, mortality differentials, socio-economic classes, stochastic mortality, redistribution.

JEL: C63, G22, G23, G52.

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1 Introduction

In light of the increasing pressure imposed on traditional pension systems by longevity risk, both the actuarial academic literature and practice have started exploring alternative schemes, especially in the decumulation phase. Tontines and Group Self Annuitization (GSA) schemes (Winter and Planchet [2022]) are currently the most debated ones, because they can provide feasible and improved risk sharing. In these schemes, which are referred to as self-insurance schemes, a group of policyholders agrees to collect funds in a unique pool jointly managed. Investment strategies, withdrawal rules, sharing mechanisms are common to all policyholders and pre-agreed among them and, at their death, the benefits of deceased members are shared among the survivors. In tontines and GSA schemes there is a different sharing mechanism. In the context of tontines there is the maximization of an expected utility while in the context of GSA schemes there is an annuitization of the current fund which is performed yearly based on the number of survivors and on their age.

Self-insurance schemes have several advantages, which have been identified and studied in the literature. Their main one vis-à-vis traditional pension schemes lies in their ability to pool and mitigate longevity risk, as pointed out by a vast literature, since the seminal contribution by Piggott et al. [2005]. While standard retirement products expose the issuer to potential losses and solvency concerns because of longevity risk, i.e. unexpected changes from the expected mortality of the policyholders, in self-insurance schemes longevity risk is shared among participants. In some cases an intermediary may intervene, but its cost in terms of capital requirements is lower than with traditional pension solutions ([Chen et al., 2019]). Another advantage is that it is easy to cope with moral hazard issues within the pool, by adopting a pre-determined withdrawal structure [Fullmer and Sabin, 2018]. Finally, self-insurance schemes can indeed constitute a valuable alternative for policyholders (see Denuit et al. [2022]). It has been shown that in several instances the tontines may be the preferred option for a policyholder (see for instance Chen et al. [2021]), especially if optimally designed in combination with other products, such as annuities (see Chen et al. [2019], Chen et al. [2020]), or Long-Term Care benefits (Hieber and Lucas [2022]).

However, there are some aspects of self-insurance schemes which need further scrutiny. In particular, it has been pointed out that the longevity risk sharing mechanism may lead to unfairness if the policyholders are heterogeneous in their survival probabilities. Indeed, while previous works typically assume homogeneity in the mortality dynamics of policyholders, some authors (see, for instance Donnelly [2015]) have raised the issue that pooling together individuals with different characteristics (age, contributions to the fund, health status) cannot be performed in a standard GSA scheme without discriminating at least some individuals.

Inequality among policyholders arising from heterogeneous pooling has always been a common problem in insurance. Self-insurance products are clearly not exempt from this issue. This problem has even become more urgent recently since the promotion of UN's 2030 Agenda for sustainable development goals. Among these, goal 10.4 stresses the importance of implementing policies aimed at reducing inequalities.

Heterogeneity in self-insurance schemes has been treated in a few previous works. When mortality is deterministic, a GSA-type scheme can be designed to be actuarially fair (Donnelly et al. [2014]), even when its members are heterogeneous in terms of wealth and a-priori mortality rates. However, this is in general not true when mortality is stochastic, unless one allows for different contributions or benefit calculation rules for the different groups that are pooled together. In the context of tontines Milevsky and Salisbury [2016] introduced the notion of equitability to capture the idea that a self-annuitizing pool of heterogeneous cohorts should be designed so that they would all be equally (un)-happy. Building on this, Chen and Rach [2023] find that, by allowing the participation rate (i.e. the price for an individual to participate in the scheme) to differ among cohorts, the scheme can be both individually (to each policyholder) and collectively (in aggregate, to the whole scheme) fair.

Inspired by the importance of goal 10.4 of UN's 2030 Agenda, we contribute to this stream of literature as well, addressing the issue of reducing inequalities among different socio-economic classes in a GSA scheme with stochastic mortality. We propose a GSA scheme with heterogeneous policyholders whose mortalities are described by different stochastic processes in which, while contributions are equal across individuals, benefits differ, as if they were set by a planner (as in Dhaene and Milevsky [2024]). We call this scheme a "Redistributive GSA scheme". We set each group's benefit re-scaling optimally the baseline benefit of the reference group, with the objective of minimizing the squared distance between the actuarially fair expected value of the benefits for the reference population and that of each policyholder in the group (similar to Bernard et al. [2024]). The main contribution of this paper consists in implementing a redistributive mechanism in a GSA scheme in the presence of stochastic mortalities that are heterogeneous due to the members' socio-economic status. Our work is close to Qiao and Sherris, 2013, who studied a GSA scheme with stochastic mortality. While they consider an open fund where new cohorts enter, we consider a closed fund and tackle the issue of redistribution among different socio-economic classes.

In detail, our analysis unfolds as follows. We start by considering a traditional GSA scheme where (baseline) benefits are set equal among policyholders, and are actuarially fair for a reference group. We consider that the scheme pools together groups of individuals with (stochastic) mortalities different than that of the reference group, and that each group belongs to a different socio-economic class. Our analysis is motivated by the presence of relevant mortality differentials across socio-economic classes. These differentials are beyond the cohort-based ones considered by Milevsky and Salisbury [2016] and Chen and Rach [2023] and may prevent the viability of self-insurance schemes. Since Antonovsky, 1967, many papers (Cairns et al., 2022, Wen et al., 2021, Chetty et al. [2016] among others), stress how the socio-economic class heavily affects the mortality experience. Accordingly, there will be expected gains/losses for different groups who enroll in the same scheme. We use a British dataset on historical mortality by socio-economic classes previously used in the actuarial literature (see Cairns et al., 2022, Wen et al., 2021, Millossovich et al., 2014) and calibrate a stochastic mortality model for three different socio-economic classes. Then, we adopt a simulative approach and analyze the distribution of the Expected Present Value (EPV) of the benefits obtained from the GSA scheme for the different groups of individuals.

Our results show that large differences appear across policyholders belonging to different socio-economic groups when they are pooled together in the same traditional GSA scheme. Indeed, as Donnelly [2015] points out, unfairness arises since those who on average die earlier (the poorest) subsidize those who on average die later (the wealthiest). This is a socially undesirable outcome as pointed out also by UN's 2030 Agenda. We quantify such transfer to be in the order of 30%. This happens because the actuarial fairness principle, which would imply differentials in pricing the GSA product to different sub-groups, is violated.

We then evaluate to what extent our proposed Redistributive GSA scheme is able to restore fairness, comparing the distribution of the EPV of the different sub-groups when the benefits are optimally set. The scheme achieves the objective of improving equity across sub-groups, because the simulated distributions of their EPVs become more similar.

Finally, we study how our results depend on two key factors: the size of the pool, which matters because idiosyncratic mortality can be perfectly diversified only in large samples, and the level of interest rates. While the former affects the dispersion of EPVs within each group, the latter is inversely linked to inequity. Indeed, when interest rates are lower, longevity differences matter more and thus

inequity is more pronounced.
The paper is organized as follows: Section 2 describes the decumulation products and Section 2.2 introduces our proposed redistributive GSA Scheme, Section 3 describes the mortality modelling approach, Section 4 provides the empirical application, Section 5 discusses sensitivity of the results. Finally, Section 6 concludes.

2 The decumulation products

2.1 GSA scheme

We assume that a community of l_x workers retire at time $t = 0$ aged x and are free to decide the decumulation strategy to follow. They evaluate the competing strategies according to their expected present values at time $t = 0$. As a benchmark decumulation strategy, we consider an immediate annuity product with annual benefit b_A paid at the beginning of each year. In the following and in the remaining of the paper, we will be assuming that there is a *reference population* whose survival probabilities are used by the insurance company to price the lifetime annuity. Assuming that

$$\{{}_tp^r_x\}_{t=0,\dots,\ \omega-x-1} \tag{1}$$

is the vector that collects all the survival probabilities for a head aged x over t years for the reference population, and ignoring commission expenses and safety loadings, the single premium of the unitary immediate lifetime annuity paid in advance sold to a policyholder aged x is

$$\ddot{a}_x = \sum_{t=0}^{\omega-x-1} {}_tp^r_x v^t$$

where $v^t = (1+i)^{-t}$ is the $t-$years financial discount factor and ω denotes the limiting age (i.e. $p_\omega = 0$). Then, the expected present value at time $t = 0$ of the benefits paid by the annuity for a policyholder whose survival probabilities are $\{{}_tp_x\}_{t=0,\dots,\ \omega-x-1}$ is

$$EPV_A(0) = \sum_{t=0}^{\omega-x-1} {}_tp_x v^t b_A$$

The alternative we focus on in this paper is a collective self-insurance scheme, namely the group self-annuitization scheme proposed by Piggott et al., 2005. The scheme works as follows. When it is setup, the scheme pools together into a fund the resources collected from the policyholders. To make fair the comparison with the annuity, we assume that each individual contributes $b_A\ddot{a}_x$ to the fund. Hence, the total fund at time $t = 0$ is

$$F(0) = l_x b_A \ddot{a}_x.$$

At time 0, each individual receives a benefit b_A, equal to the annuity benefit, therefore

$$b_{GSA}(0) = b_A. \tag{2}$$

Assuming investment at the risk-free interest rate level i, at time $t = 1$ the fund is valued

$$F(1) = (F(0) - l_x b_A)(1+i) = l_x b_A(\ddot{a}_x - 1)(1+i),$$

where the annuity price $\ddot{a}_x$ (as well as all the other annuity prices $\ddot{a}_{x+t}$ for all $t \geq 1$) is computed using the survival probabilities (1) of the reference population. Given the actual number of survivors at age $x+1$, l^*_{x+1}, the benefit at time $t = 1$ received by each survivor in the scheme, $b_{GSA}(1)$, obtained sharing among the survivors the annuitized value of the fund at time 1, is

$$b_{GSA}(1) = \frac{1}{l^*_{x+1}}\left(\frac{F(1)}{\ddot{a}_{x+1}}\right) = \frac{1}{l^*_{x+1}}\left(\frac{l_x b_A(\ddot{a}_x - 1)(1+i)}{\ddot{a}_{x+1}}\right).$$

Considering that (i) the recursive relationship for the annuity prices is

$$\ddot{a}_{x+1} = (\ddot{a}_x - 1)(1+i)/\, p^r_x$$

and that (ii) the number of survivors at time 1, l^*_{x+1}, is given by $l_x p^*_x, p^*_x$ being the realized 1-year survival probability at age x, $b_{GSA}(1)$ can be rewritten as

$$b_{GSA}(1) = \frac{1}{l^*_{x+1}}\left(\frac{l_x b_A(\ddot{a}_x - 1)(1+i)}{(\ddot{a}_x - 1)(1+i)/\, p^r_x}\right) = b_A\left(\frac{p^r_x}{p^*_x}\right).$$

Observing that at a generic time $t \geq 1$

$$l^*_{x+t} = l^*_{x+t-1}p^*_{x+t-1}, \tag{3}$$

the benefit of the GSA scheme at time t is

$$\begin{aligned} b_{GSA}(t) &= \frac{F(t)}{l^*_{x+t}\ddot{a}_{x+t}} = \frac{l^*_{x+t-1}b_{GSA}(t-1)\,(\ddot{a}_{x+t-1} - 1)\,(1+i)}{l^*_{x+t}\ddot{a}_{x+t}} = \\ &= b_{GSA}(t-1)\frac{l^*_{x+t-1}}{l^*_{x+t}}\frac{(\ddot{a}_{x+t-1} - 1)\,(1+i)}{(\ddot{a}_{x+t-1} - 1)\,(1+i)\,/\,p^r_{x+t-1}} = \end{aligned}$$

$$= b_{GSA}(t-1)\left(\frac{p^r_{x+t-1}}{p^*_{x+t-1}}\right).$$

Hence:

$$b_{GSA}(t) = b_{GSA}(t-1) * MEA_t, \tag{4}$$

where

$$MEA_t = \frac{p^r_{x+t-1}}{p^*_{x+t-1}}. \tag{5}$$

MEA_t, $t = 1,2,...,\omega - x - 1$ is the Mortality Experience Adjustment, i.e. the ratio of the expected and the actual survival rates in year $[t-1,t]$. The realized survival probabilities $p^*_{x+t}, t = 0,1,...$ are found via simulation of the number of deaths in the pool, and therefore simulation of the realized number of survivors l^*_{x+t}. Piggott et al., 2005 assume the randomness in mortality to increase linearly with age. Differently from Piggott et al., 2005, we model the number of survivors in each time period by simulating the evolution of the stochastic mortality intensity process λ and the time of death of each individual in the pool as the first jump time of a *doubly stochastic process* with intensity λ (see Section 3).

The time-0 expected present value for an individual enrolled in the GSA scheme aged x and whose survival probabilities are $\{_t p_x\}_{t=0,...,\ \omega-x-1}$ is

$$EPV_{GSA}(0) = \sum_{t=0}^{\omega-x-1} v^t\, {}_t p_x b_{GSA}(t)$$

where the GSA benefits $b_{GSA}(t)$ are as in (4).

2.2 The Redistributive GSA Scheme

In the description of the GSA scheme, we have neglected the fact that the scheme can pool together individuals belonging to different sub-populations, which can display different mortality patterns. We consider three categories: high risk (HR) individuals, medium risk (MR) individuals, low risk (LR) individuals, with survival probabilities $\{_t p^{HR}_x\}_{t=0,...,\ \omega-x-1}$, $\{_t p^{MR}_x\}_{t=0,...,\ \omega-x-1}$ and $\{_t p^{LR}_x\}_{t=0,...,\ \omega-x-1}$, respectively. In this context, by "high (low) risk individual" we refer to an individual with lower (higher) survival probabilities with respect to the medium ones at every time horizon:

$${}_t p^{HR}_x < {}_t p^{MR}_x < {}_t p^{LR}_x \qquad \text{for all } t = 0,\dots,\omega - x - 1.$$

If we calculate the expected present values at $t = 0$ of the GSA benefits for the three categories, $EPV_{GSA}^{j}(0)$, $j = HR, MR, LR$, using their own respective survival probabilities, we obtain that

$$\sum_{t=0}^{\omega-x-1} {}_tp_x^{HR} v^t b_{GSA}(t) < \sum_{t=0}^{\omega-x-1} {}_tp_x^{MR} v^t b_{GSA}(t) < \sum_{t=0}^{\omega-x-1} {}_tp_x^{LR} v^t b_{GSA}(t)$$

that is

$$EPV_{GSA}^{HR}(0) < EPV_{GSA}^{MR}(0) < EPV_{GSA}^{LR}(0). \tag{6}$$

It is clear that, due to the different survival probabilities of each sub-population, distributing the same benefit to each sub-group leads to a solidarity transfer across individuals, in particular from high risk to low risk individuals. If, as usual, high risk individuals belong to lower socio-economic classes, such transfer exhacerbates inequality. It is thus interesting to study redistribution mechanisms within the scheme to reduce those inequalities. We consider a redistribution mechanism obtained with a simulative approach. In particular, we simulate 10 000 scenarios of mortality patterns for each sub-population and implement an optimal redistribution policy by minimizing for each simulation k ($k = 1, \ldots, 10000$) the squared distance between a re-scaling of $EPV_{GSA}^{j,k}(0)$ ($j = HR, MR, LR$) through the use of redistributive shares α^j and the baseline $EPV_{GSA}(0)$ level of the medium policyholder in the absence of other sub-populations in the scheme. The redistributive scheme works as follows.

We assume that each retiree pays $b_A \ddot{a}_x = 1000$ in the pooled fund at time $t = 0$. The unique pool has value $N_0 b_A \ddot{a}_x = 1000 N_0$, where N_0 is the total number of retirees aged x given by the sum of the number N_0^j of retirees in each sub-population $j \in \{HR, MR, LR\}$:

$$N_0 = N_0^{HR} + N_0^{MR} + N_0^{LR}.$$

The optimal *redistributive shares*, α_*^j, are found via a simulative approach by solving the following problem:

$$\min_{\{\alpha^j\}_{j \in \{LR, MR, HR\}}} \left\{ \frac{1}{10000} \sum_{k=1}^{10000} \frac{1}{3} \sum_{j \in \{LR, MR, HR\}} \left(1000 - \alpha^j EPV_{GSA}^{j,k}(0)\right)^2 \right\} \tag{7}$$

$$s.t. \sum_{j=1}^{3} \alpha^j = 3. \tag{8}$$

where $EPV_{GSA}^{j,k}(0)$ is the expected present value of GSA benefits for sub-population j in simulation k. At time $t = 0$, the *optimal redistributive share* α_*^j defines the benefits paid to sub-group j in the redistributive scheme:

$$b_{RE}^j(0) = \alpha_*^j b_{GSA}^j(0) = \alpha_*^j b_A. \tag{9}$$

Due to the recursive mechanism (4), for each $t > 0$, it holds

$$b_{RE}^j(t) = b_{RE}^j(t-1) * MEA_t \tag{10}$$

for MEA_t defined as in eq. (5).

The approach results to be financially sustainable since the constraint in (8) ensures that the funds awarded to each individual are a fraction of the total funds available at each time t.

The effect of the redistribution can be measured evaluating the EPV of the benefits obtained by the groups after the redistribution:

$$EPV_{RE}^j(0) = \sum_{t=0}^{\omega-x-1} {}_t p_x^j v^t b_{RE}^j(t) = \alpha_*^j EPV_{GSA}^j(0). \tag{11}$$

The goal after the redistribution is to obtain a reduced gap among the $EPV_{RE}(0)$ of the different sub-populations.

3 Mortality modelling

3.1 The theoretical framework

We consider a complete filtered probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a filtration $\{\mathscr{G}_t : t \geq 0\}$ of sub-σ-algebras of $\mathscr{F}$. We introduce a nonexplosive counting process N with intensity $\lambda(t)$ and another filtration $\{\mathscr{F}_t : t \geq 0\}$ such that $\mathscr{F}_t \subset \mathscr{G}_t$.

The process N is said to be *doubly stochastic* driven by $\{\mathscr{F}_t : t \geq 0\}$, if $\lambda(t)$ is $(\mathscr{F}_t)$-predictable and for all t, s with $t < s$, conditional on the σ-algebra $\mathscr{G}_t \vee \mathscr{F}_s$ generated by $\mathscr{G}_t \cup \mathscr{F}_s$, the process $N_s - N_t$ is Poisson distributed with parameter

$$\int_t^s \lambda(u) du.$$

Conditional on the knowledge of a particular trajectory $t \to \lambda(t, \widetilde{\omega}) = \widetilde{\lambda(t)}$ for fixed $\widetilde{\omega} \in \Omega$, the counting process N becomes Poisson with (conditionally deterministic) time varying intensity $\widetilde{\lambda(t)}$.

A stopping time τ is said to be *doubly stochastic with intensity* λ if the underlying counting process whose first jump time is τ is doubly stochastic with intensity λ. The specification of the sub-filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ is meant to indicate that the first jump time of N is a stopping time with respect to $(\mathscr{G}_t)_{t\geq 0}$, but outside the span of $(\mathscr{F}_t)_{t\geq 0}$, which carries sufficient information to reveal the intensity $\widetilde{\lambda(t)}$ (i.e. the likelihood that the jump will happen) but not enough to predict the occurrence of the jump. Doubly stochastic processes with a stopping time are typically exploited to model the survival process of individuals. In particular, the time of death is typically modelled as a doubly stochastic stopping time with intensity given by the mortality intensity λ that is a stochastic force of mortality. The mortality intensity is typically modelled as an affine process in order to exploit well-known analytical results for the computation of the survival function (see Duffie et al., 2000). For the specification of the affine stochastic mortality intensity we consider non-mean reverting processes among those introduced by Luciano and Vigna, 2008. In particular, we model the mortality intensity $\lambda_x(t)$ of an individual aged $x+t$ of a specific cohort with initial age x with a Feller process with dynamics given by:

$$d\lambda_x(t) = a\lambda_x(t)dt + \sigma\sqrt{\lambda_x(t)}dW_x(t) \tag{12}$$

where $a > 0$ and $\sigma \geq 0$ represent the drift and diffusions parameters associated to the processes, $W(t)$ being a Brownian motion. Following Duffie et al., 2000, given an affine mortality intensity $\lambda_x(t)$, the survival probability $S_x(t)$ describing the probability of an individual aged x to survive t years is:

$$S_x(t) = {}_tp_x = \mathbb{E}[e^{-\int_0^t \lambda_x(u)du}|\mathscr{G}_0] = e^{\alpha(t)+\beta(t)\lambda_x(0)}, \tag{13}$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ solve appropriate ODEs and the initial observed intensity $\lambda_x(0)$ is taken to be equal to $-ln(\hat{p}_x)$, where $\hat{p}_x$ is the observed survival rate at age x.
In the case of the *Feller process* we have

$$\begin{cases} \alpha(t) = 0 \\ \beta(t) = \frac{1-e^{bt}}{c+de^{bt}} \end{cases} \tag{14}$$

with $b = -\sqrt{a^2+2\sigma^2}$, $c = \frac{b+a}{2}$, $d = \frac{b-a}{2}$.
The inequality constraint

$$e^{bt}\left(\sigma^2 + 2d^2\right) > \sigma^2 - 2dc \tag{15}$$

must hold in order for the survival process $S_x(t)$ to be a decreasing function of t.

3.2 Methodology for the simulation of the number of deaths in one simulated scenario

Let τ be the time of death of the individual modelled as a doubly stochastic stopping time with intensity λ. For the death time of each individual in the group self-annuitization pool it holds

$$\mathbb{P}(\tau > t) = \mathbb{P}(N_t = 0) = \mathbb{E}[e^{-\int_0^t \widetilde{\lambda(u)}du} | \mathscr{F}_0] \tag{16}$$

where the knowledge of the parameter $\int_0^t \widetilde{\lambda(u)}du$ is conditional to the filtration $\mathscr{F}_t$. It can also be proven that, conditional on the particular trajectory of the intensity process $\widetilde{\lambda(t)}$, τ satisfies

$$\tau = \inf_{t \geq 0} \left\{ t : E_1 \leq \int_0^t \widetilde{\lambda(u)}du \right\} \tag{17}$$

where $E_1 \sim Exp(1)$.
In order to obtain the number of deaths in the GSA pool and get to the p^*_{x+t-1} probabilities in one simulated scenario, we simulate, for $j \in \{HR,\ MR,\ LR\}$, $N_0^j = 1000$ realizations from an $Exp(1)$ random variable and thanks to (17) every single extraction leads to the death time of a different individual out of the $N_0^j = 1000$ individuals. Thus, for a given j, the exponential random vector

$$\{E_n^j\} = \{E_1^j, E_2^j, ..., , E_{1000}^j\}$$

for $n = 1, ..., N_0^j = 1000$, leads through eq. (17) to the death time random vector for the simulated scenario

$$\{\tau_n^j\} = \{\tau_1^j, \tau_2^j, ..., , \tau_{1000}^j\}.$$

It follows that the number of individuals of sub-population j dying between ages $x+t-1$ and $x+t$ in the simulated scenario is given by

$$d^j_{x+t-1} = \mathbb{E}\left[\sum_{n=1}^{N_0^j} \mathbb{1}_{\left\{t-1<\tau_n^j \leq t\right\}} \right]. \tag{18}$$

This leads to the computation of the survivors l^{*j}_{x+t} at age $x+t$ belonging to sub-population j in the GSA pool as

$$l^{*j}_{x+t} = l^{*j}_{x+t-1} - d^j_{x+t-1} \tag{19}$$

and for each t the total number of survivors in the GSA pool is given by

$$l^*_{x+t} = \sum_{j\in\{HR,MR,LR\}} l^{*j}_{x+t}. \qquad (20)$$

From the sequence $\{l^*_{x+t}\}$ provided by eq. (20) one can compute the realized survival probabilities $\{_tp^*_x\}$ as in (3) in the simulated scenario. Finally, using (4) it is possible to compute in the simulated scenario the benefits $\{b(t)_{GSA}\}_{t=0,1,\ldots,\omega-x-1}$ for all survivors in the GSA scheme, and using (9) and (10) the benefits $\{b^j_{RE}(t)\}_{t=0,1,\ldots,\omega-x-1}$ of the redistributive scheme for each sub-population j.

4 Empirical application

We intend to apply the *Redistributive GSA Scheme* using the methodology illustrated in Sections 2-3 to a specific dataset of different socio-economic classes. Section 4.1 illustrates the dataset.

4.1 The dataset

Individuals belonging to distant socio-economic classes bear a different longevity risk. More disadvantaged social classes are more exposed to riskier environments, riskier habits and to access barriers to healthcare. According to [Ardito et al., 2019], the longevity gap is an increasing function of the degree of deprivation and it is more pronounced for male individuals than for female ones and, in addition, the gap shrinks with the policyholder's age.

To model the mortality differences in socio-economic classes, we use the dataset which groups the English population in ten deciles, according to the English Indices of Deprivation (version of 2015)[a]. The deciles are the result of the subdivision of the 32,844 Lower-Layer Super Output Areas in ten equal-sized groups such that the first decile corresponds to the most deprived areas and the tenth decile corresponds to the least deprived ones. The ranking is based on income deprivation, employment deprivation, education, skills and training deprivation, health deprivation and disability, crime, barriers to housing and services, living environment deprivation.

[a]The dataset is available on the Office for National Statistics website: https://www.ons.gov.uk/peoplepopulationandcommunity/birthsdeathsandmarriages/deaths/adhocs/009299numberofdeathsandpopulationsindeprivationdecileareasbysexandsingleyearofageengland andwalesregisteredyears2001to2017.

The dataset has been used in other papers on mortality differentials by socio-economic classes such as Cairns et al., 2022, Wen et al., 2021, Millossovich et al., 2014. As stressed by Wen et al., 2021, "the health deprivation and disability" factor does not perfectly suit the purposes of a research analysis based on socio-economic mortality differentials. It concerns the quality of the health status rather than a mere economic wealth based discriminant. Still, considering the correlation among the health status and all the remaining factors, the English Indices of Deprivation dataset is a useful and proficient tool to spot differences in mortality experience according to the wealth level.
The dataset provides figures for the number of deaths and exposed to risk (mid-year estimates) in each population decile area by gender and single year of age for calendar years 2001-2017.
We select three sub-populations out of the ten deciles of the dataset: we consider as low socio-economic status representatives the first decile sub-population individuals, as middle class representatives the fifth decile sub-population individuals and as high socio-economic status representative the tenth decile sub-population individuals. Therefore, given the differences in the survival probabilities level, we consider three categories of policyholders by assuming first decile retirees as high risk individuals ($j = HR$), fifth decile retirees as medium risk individuals ($j = MR$) and tenth decile retirees as low risk individuals ($j = LR$).
Although policyholders contribute identically to the scheme funding, high socio-economic status individuals (i.e. LR individuals) will receive on average benefits for a longer period with respect to lower socio-economic groups retirees (i.e. HR individuals). Thus, lower socio-economic classes will bear the longevity risk associated to the richest.

4.2 Calibration

For the calibration procedure we focus our attention on the cohort of male individuals born in 1936, entering the scheme in 2001 at the age of $x = 65$[b] and, as mentioned in Section 4.1, we consider the three sub-populations of HR, MR and LR individuals as the males born in 1936 belonging to the first, fifth and tenth percentile, respectively. The mortality intensity of each sub-population will be given

[b] In the remainder of the paper, we will sometimes use x, sometimes use 65 as the initial age.

by the Feller process as in (12):

$$d\lambda_{65}^{j}(t) = a^{j}\lambda_{65}^{j}(t)dt + \sigma^{j}\sqrt{\lambda_{65}^{j}(t)}dW_{65}^{j}(t) \qquad j = HR, MR, LR \tag{21}$$

We calibrate the Feller processes (21) on seventeen observations, from 2001 to 2017, over the age span 65 to 81 and we obtain closed-form expressions for the remaining survival probabilities up to the limiting age $\omega = 110$.
In detail, we first derive from the dataset described in Section 4.1 three vectors of observed empirical survival probabilities $\{{}_{t}p_{65,emp}^{j}\}$ for $j \in \{HR, MR, LR\}$ relative to males born in 1936 and observed from 2001 to 2017. Then, we calibrate the parameters of the processes $\{\lambda_{65}^{HR}, \lambda_{65}^{MR}, \lambda_{65}^{LR}\}$ by minimizing the Mean Squared Error (MSE), i.e. the average squared differences between the model implied and the observed survival probabilities:

$$\min_{a^{j},\sigma^{j}} \sum_{t=1}^{17} \frac{1}{17}\left({}_{t}p_{65,emp}^{j} - S_{65}^{j}(t)\right)^{2} \tag{22}$$

for $j \in \{HR,\ MR,\ LR\}$, where $S_{65}^{j}(t)$ is defined as in (13)-(14) and applied to the $j-th$ sub-population. From the calibration procedure we get the minimizing parameters a^{j}, σ^{j} that are reported in Table 1. The value of the long term mean parameter a^{j} ranges between 0.073 and 0.075 while the volatility σ^{j} ranges from 0.00025 to 0.00048. The low volatility levels are associated to a low calibration error ranging from 0.004 to 0.0049. The observed mortality intensity at age 65, $\lambda_{65}^{j}(0)$, ranges from 0.011 to 0.025.
Figure 1 reports, for each $j \in \{HR,\ MR,\ LR\}$ the calibrated model implied survival function $S_{65}^{j}(t)$ together with the observed survival curve. Figure 13 in the Appendix reports the difference $\left(S_{65}^{j}(t) - {}_{t}p_{65,\ emp}^{j}\right)$ between the fitted and the observed survival probabilities for every $t = 1,\ \ldots,\ 17$ and every $j \in \{HR,\ MR,\ LR\}$. We can see that the model implied survival curve overlaps quite well with the empirical survival process for all the risk classes and the differences between the model implied and the empirical survival probabilities are below 1% in absolute value.

Sub-population j	a^j	σ^j	$\lambda_{65}^j(0)$	Error
HR	0.074664	0.000254	0.024836	0.004898
MR	0.073965	0.00025	0.015865	0.004491
LR	0.073313	0.000478	0.011364	0.004024

Table 1: Feller model's calibration.

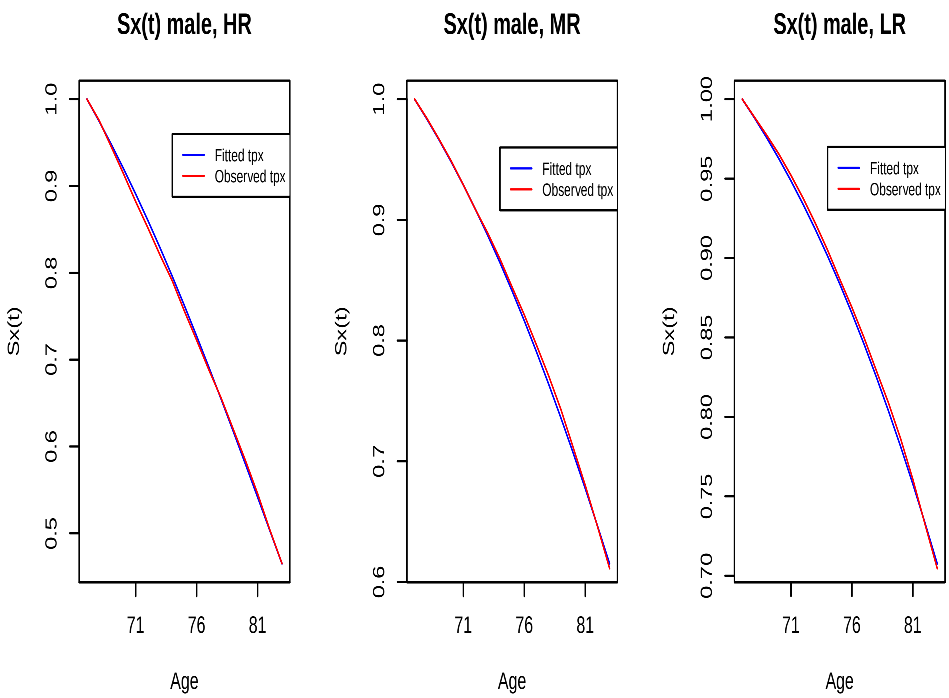

Figure 1: Observed versus fitted $S_{65}^j(t)$, $j = HR, MR, LR$.

4.3 Simulation of the benefits under the GSA scheme

We have run 10 000 Monte Carlo simulations of the processes (21). In order to compute the GSA and the annuity benefits, we need a reference population (see Section 2). We, therefore, set the reference population r to be the fifth decile male policyholders, i.e. $r = MR$, that implies

$$\{{}_tp_x^r\}_{t=0,\dots,\omega-x-1} = \{{}_tp_x^{MR}\}_{t=0,\dots,\omega-x-1} \tag{23}$$

Accordingly, given that the premium paid by the policyholders is 1000, the annuity level benefit is

$$b_A = \frac{1000}{\ddot{a}_x^r} = \frac{1000}{\ddot{a}_x^{MR}}. \tag{24}$$

In the calculation of the annuity we set the interest rate $i = 2\%$. In the case of the GSA scheme, the starting point for the GSA benefits as in eq.(2) is given by

$$b_{GSA}(0) = \frac{1000}{\ddot{a}_x^{MR}} = b_A.$$

Considering a cohort of three equally-sized sub-populations of $N_0^j = 1\,000$ males, $j \in \{LR,\ MR,\ HR\}$ (so that $N_0 = 3\,000$) coming from the first, the fifth and the tenth deciles of the English population, the simulation process is based on the following algorithm:

- For each simulated scenario we simulate the trajectory of $\lambda_{65}^j(t)$ for $t = 1,\dots,45$ for each sub-population j. We run 10000 simulated scenarios.
- For each simulated scenario we use the procedure illustrated in Section 3.2 to simulate the number of deaths and obtain the realized survivors $\{l_{x+t}^*\}$ and the realized survival probabilities ${}_tp_x^*$ for that scenario.
- For each of the 10000 scenarios we compute the benefits for the GSA scheme $b_{GSA}(t)\ \forall t = 1,\dots,45$ using the $\{{}_tp_x^*\}_{t=1,\dots,45}$ as illustrated in Section 2.
- Therefore, we get the distribution of the 10000 paths for the GSA benefits $b_{GSA}(t)\ \forall t = 1,\dots,45$.

Fig. 2 shows over ages 65-110 the level annuity benefit b_A and the 10%, 30%, 50%, 70% and 90% percentiles of the distribution of benefits $b_{GSA}(t)$ out of the 10000 simulations. GSA and annuity benefits are unique across individuals, therefore Figure 2 holds indiscriminately for each sub-population $j = \{HR,\ MR,\ LR\}$. Fig. 3 reports the scheme benefits over ages 65-85, while Fig. 4 reports the expectation and the standard deviation of the GSA benefits for all ages 65-110.

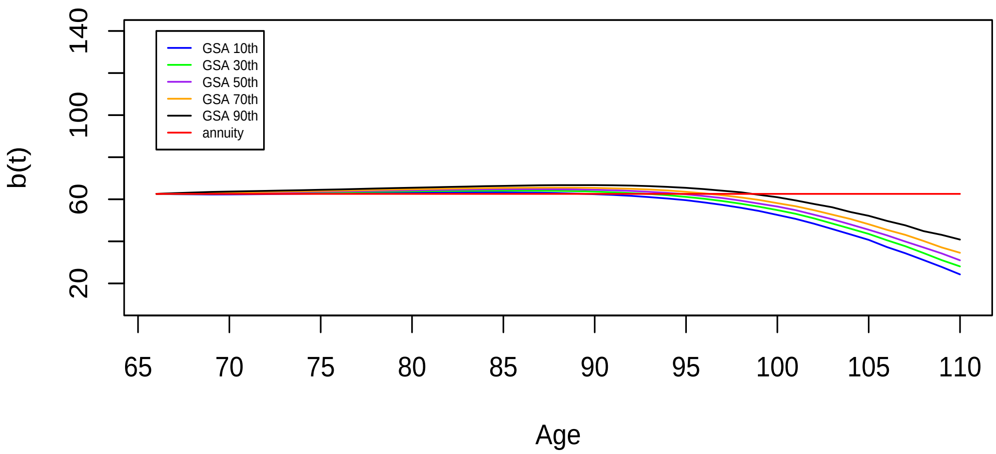

Figure 2: Cash flow streams: GSA versus annuity.

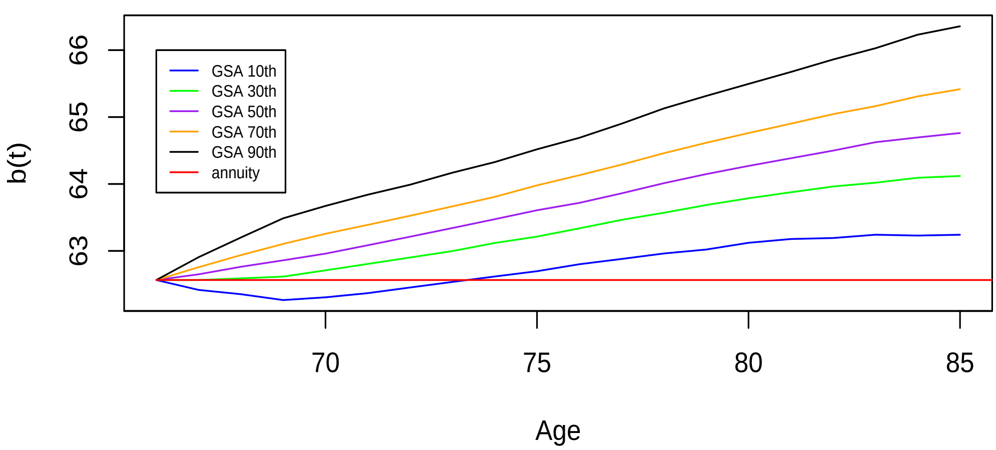

Figure 3: Cash flow streams: GSA versus annuity. Detail, age 65-85.

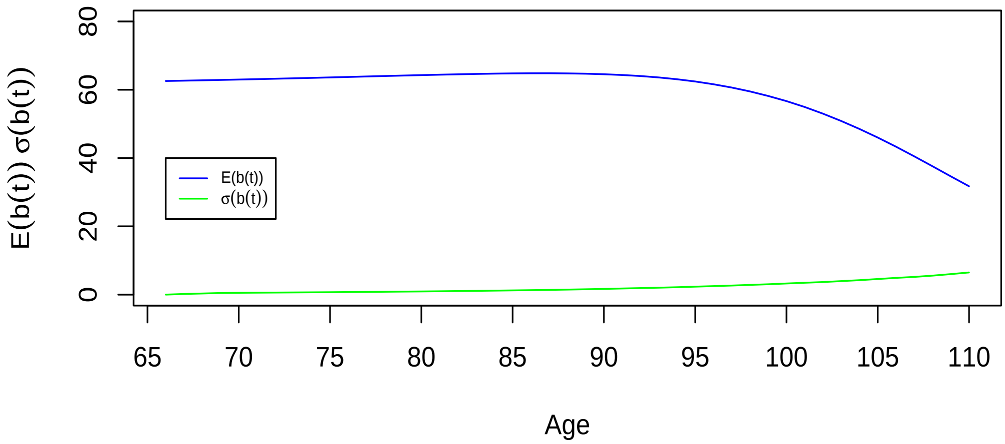

Figure 4: $\mathbb{E}(b(t)_{GSA})$, $\sigma(b(t)_{GSA})$.

At time $t=0$ we clearly have $b_{GSA}(0) = b_A$. In most cases, the $b_{GSA}(t)$ distribution outperforms the level annuity benefit until about the age $x = 90$, when it bends towards a minimum of approximately 20 (see Fig. 2).
From Fig. 4 we see that the expectation of the GSA benefit $\mathbb{E}(b_{GSA}(t))$ is a decreasing function of the policyholder age while the standard deviation of the GSA benefit $\sigma(b_{GSA}(t))$ is an increasing function of the policyholder age.

4.4 Calculation of EPV(0) for each socio-economic class

The benefit from the GSA scheme is the same for all individuals. However, different socio-economic classes experience different lifetime durations. Therefore, the value of the benefit's stream will be different. A natural way to measure such value is the expected present value at time $t = 0$ of the benefits using the class related survival function.
Therefore, for each of the 10000 simulated scenarios of $b_{GSA}(t)$ we calculate $EPV^j_{GSA}(0)$, where

$$EPV^j_{GSA}(0) = \sum_{t=0}^{\omega-x-1} v^t \, {}_tp^j_x b_{GSA}(t),$$

where ${}_t p_x^j = S_x^j(t)$ where in $S_x^j(t)$ we plug the calibrated parameters a^j, σ^j ($j \in \{HR, MR, LR\}$) of Tab. 1. Thus, for each socio-economic class $j \in \{HR, MR, LR\}$, we obtain the distribution of the 10 000 $EPV_{GSA}^j(0)$.
Similarly, we calculate the expected present value of the benefits for the annuity, $EPV_A^j(0)$, where

$$EPV_A^j(0) = \sum_{t=0}^{\omega-x-1} v^t\, {}_t p_x^j b_A.$$

Because of (23) and (24) when $j = MR$ we have $EPV_A^{MR}(0) = 1000$.
Table 2 collects $EPV_A^j(0)$ and some percentiles of the distribution of $EPV_{GSA}^j(0)$.

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
Annuity	831.1444	1000	1130.149
GSA 90^{th}	852.5442	1026.2595	1157.8222
GSA 70^{th}	846.7466	1018.3843	1147.8504
GSA 50^{th}	842.7923	1012.941	1141.095
GSA 30^{th}	838.8706	1007.7905	1134.8674
GSA 10^{th}	833.5499	1000.481	1125.7446

Table 2: $EPV_A^j(0)$ and some percentiles of the distribution of $EPV_{GSA}^j(0)$, $j \in \{HR, MR, LR\}$.

Table 2 shows the effect of ignoring mortality differentials across socio-economic classes: if the same benefit is awarded to all individuals regardless of their heterogeneous contribution to the overall risk, the expected present value of cash flow streams at time $t = 0$ is considerably different across sub-groups when the actualization process is weighted by the specific mortality experience of each sub-population individually. Without redistribution, the more deprived socio-economic individuals (HR) enjoy a value about 15-17% lower than the reference individuals (MR) who, in turn, enjoy a value about 12-13% lower than the one of the least deprived individuals (LR). More importantly, on average, the transfer from HR to LR individuals is about 30% of the benefit enjoyed by the reference individuals. This strong solidarity from the high risk retiree to the low risk retiree can be dealt with by means of the redistributive scheme illustrated in Section 2.2.

4.5 Simulation of redistributive GSA benefits

We implement the redistributive mechanism as described in Section 2.2. In particular, as it is shown in problem (7)-(8), we consider the whole distribution of the $EPV_{GSA}^{j}(0)$ for each sub-population $j \in \{HR,\ MR,\ LR\}$ and, in each simulated scenario $k = 1, \ldots, 10000$, we subtract from the benchmark value $EPV(0) = 1000$ the product between the *redistributive share* α^j ($j \in \{HR,\ MR,\ LR\}$) and the population specific expected present values $\left(EPV_{GSA}^{j,k}(0)\right)$ and we take the square of this difference before averaging over the number of sub-populations and over the number of simulated scenarios. Finally, we find the optimal shares α_*^j that minimize the average of the squared error.
Table 3 reports the optimal redistributive shares α_*^j, $j \in \{HR, MR, LR\}$.

	HR	MR	LR
α_*^j	1.1643	0.9717	0.864
$b_{RE}^j(0)$	72.8451	60.7946	54.0554
b_A	62.565	62.565	62.565

Table 3: α_*^j, $b_{RE}^j(0)$, b_A for $j \in \{HR,\ MR,\ LR\}$.

From Tab. 3 we see that the redistributive mechanism ensures that retirees enjoying a more favourable mortality experience are penalized by a lower initial benefit:

$$b_{RE}^{HR}(0) > b_A > b_{RE}^{MR}(0) > b_{RE}^{LR}(0). \qquad (25)$$

Figures 5, top, medium and bottom panels report the benefits $b_{RE}(t)$ paid by the redistributive scheme to HR, MR and LR members, respectively. Figures 6 and 7 report, respectively, the expectation and the standard deviation of $b_{RE}^j(t)$ for all $j \in \{HR, MR, LR\}$. Table 4 reports the optimal shares and some percentiles of the distribution of $EPV_{RE}^j(0)$.

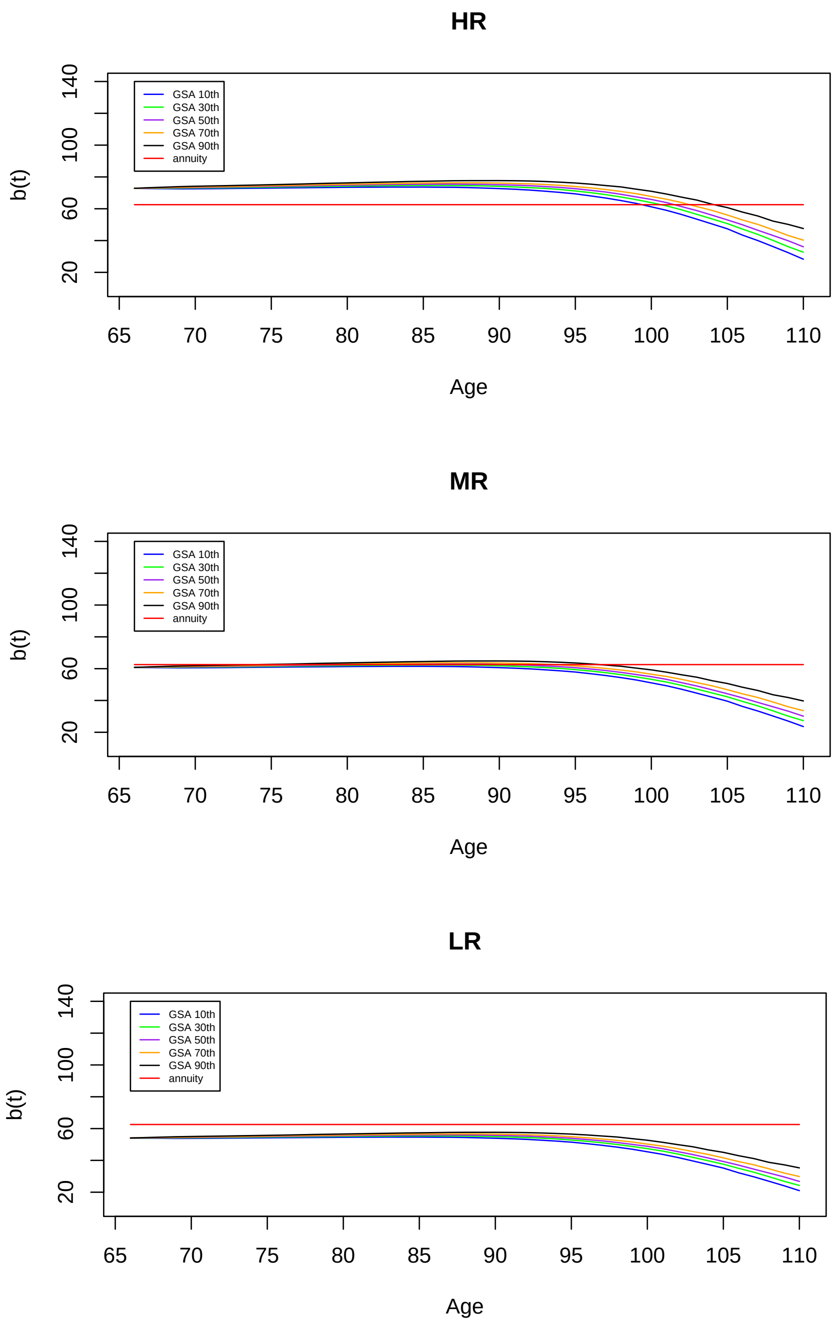

Figure 5: Annuity benefits and relevant percentiles of the GSA scheme benfits distribution. Top panel: HR individuals; Medium panel: MR individuals; Bottom panel: LR individuals.

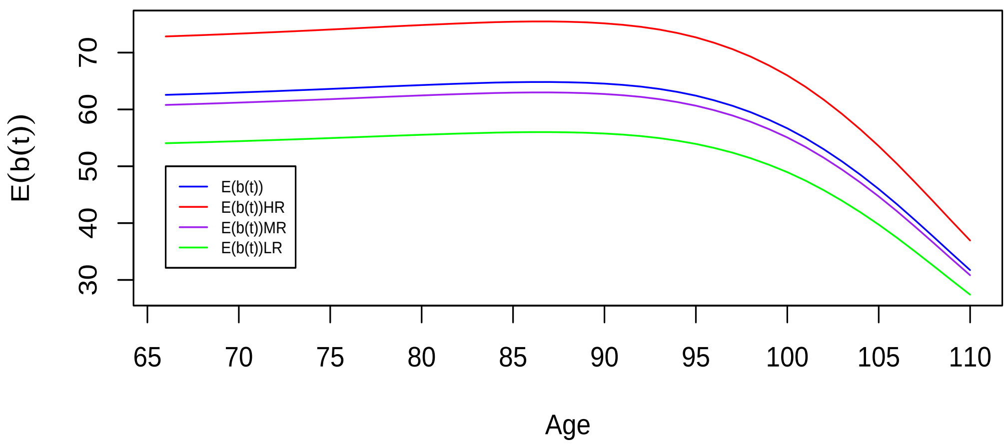

Figure 6: $E(b(t))$.

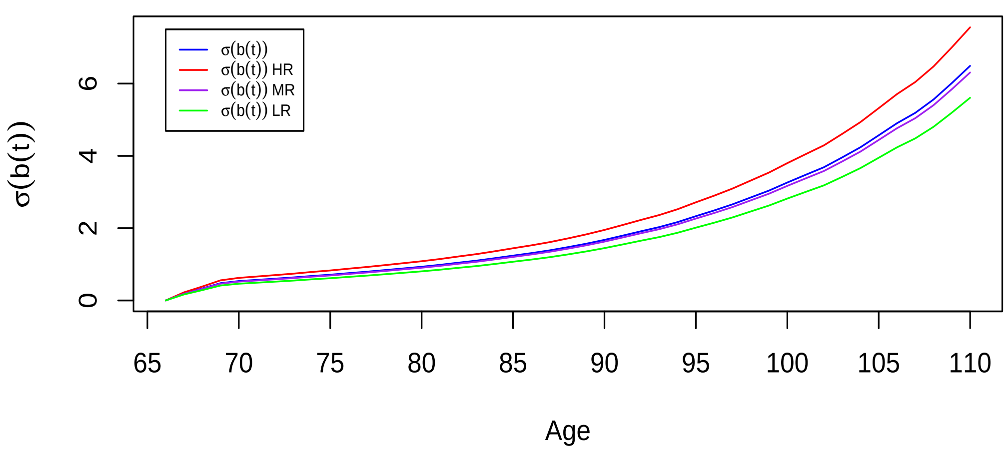

Figure 7: $\sigma(b(t))$.

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
α_*^j	1.1643	0.9717	0.864
Annuity	831.1444	1000	1130.149
GSA 90^{th}	992.6261	997.219	1000.3433
GSA 70^{th}	985.8759	989.5666	991.7278
GSA 50^{th}	981.2719	984.2774	985.8912
GSA 30^{th}	976.7058	979.2726	980.5106
GSA 10^{th}	970.5108	972.1699	972.6287

Table 4: $EPV_A^j(0)$ and some percentiles of the distribution of $EPV_{RE}^j(0)$, $j \in \{HR,\ MR,\ LR\}$ for the Redistributive GSA scheme.

We observe the following:

- Remarkably and as expected, the effect of redistribution becomes evident in Table 4. Thanks to the redistributive mechanism, the EPV(0) gaps across socio-economic groups have been significantly shrunk. The $EPV_{RE}(0)$ of HR individuals is still lower than that of LR individuals, but now the transfer from HR to LR individuals is on average less than $0.5-1\%$.

- The comparison among Figure 2 and Figure 5 in the top, medium and bottom panels shows that the range of $b(t)$ is wider in Figure 5 top panel with respect to Figure 2.

- For high risk policyholders redistribution triggers a wider dispersion of the $b_{RE}(t)$ distribution at each time t with respect to the case with no redistribution, as it emerges by observing Figure 7.

- When analysed over time the dispersion is an increasing function of the policyholder's riskiness. To show this we will consider two types of ranges: first we consider the minimum and the maximum values reached by $b_{GSA}(t)$ over time and over paths, $[\min_t b_{GSA},(t) \max_t b_{GSA}(t)]$, then we consider the minimum value reached by $b_{GSA}(t)$ over time over the 10^{th} percentile and the maximum value reached by $b_{GSA}(t)$ over time over the 90^{th} percentile, $[\min_t b_{GSA}(t)10^{th}, \max_t b_{GSA}(t)90^{th}]$.

1. Before redistribution, $[\min_t b_{GSA}(t), \max_t b_{GSA}(t)]$ ranges in $[16.96, 81.70]$. After redistribution, for high risk individuals $b_{RE}(t)$ ranges in [19.74, 95.12], for medium risk individuals it ranges in [16.48, 79.39], finally for low risk individuals it ranges in [14.65, 70.59].
2. Before redistribution $[\min_t b_{GSA}(t)10^{th}, \max_t b_{GSA}(t)90^{th}]$ ranges in [24.29, 66.73], while after redistribution for high risk individuals $b_{RE}(t)$ ranges in [28.28, 77.70], for medium risk individuals it ranges in [23.60, 64.84], while for low risk individuals it ranges in [20.99, 57.66].

- For all the sub-populations the expectation of the GSA benefit $\mathbb{E}(b_{GSA}(t))$ is a decreasing function of age while the standard deviation of the GSA benefit $\sigma(b_{GSA}(t))$ is an increasing function of the age (Figures 6 and 7).

5 Sensitivity analysis

We perform two different sensitivity analyses, one with respect to the sample size, the other with respect to the risk free interest rate. Recalling that in the base case we had $N_0^j = 1000$ and $i = 2\%$, we will now consider $N_0^j = 500$ and $N_0^j = 2000$, $j \in \{HR, MR, LR\}$, and $i = 1\%$ and $i = 10\%$. In both cases, we show results both for the GSA scheme and for the redistributive scheme. Notice that for the sensitivity analysis with respect to the interest rate we also recompute b_A and $b_{GSA}(0)$ which are dependent on the interest rate level.

5.1 Sensitivity to sample size: GSA scheme

Figures 8 and 9 report the benefit of the GSA scheme for the cases $N_0^j = 500$ and $N_0^j = 2000$, respectively, Figure 10 reports the standard deviation of the benefits over time for the cases $N_0^j = 500$ and $N_0^j = 2000$, while Tables 5 and 6 report the $EPV(0)$ values for the cases $N_0^j = 500$ and $N_0^j = 2000$, respectively.

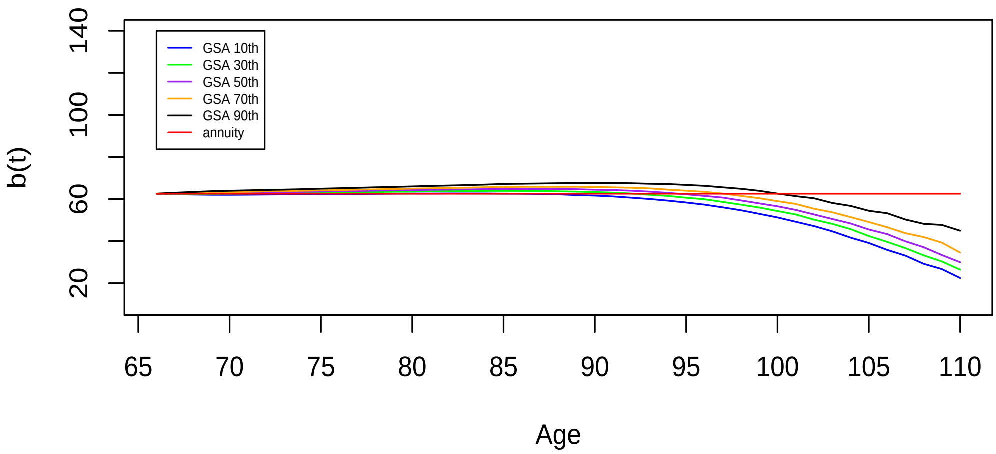

Figure 8: Sensitivity analysis (GSA), $b(t)_{GSA}$ for $N_0^j = 500$, $j \in \{HR,\ MR,\ LR\}$.

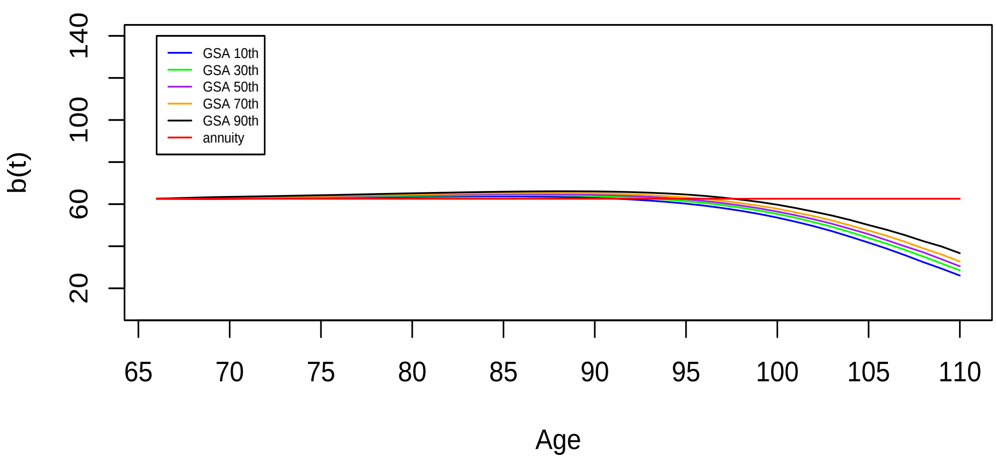

Figure 9: Sensitivity analysis (GSA), $b(t)_{GSA}$ for $N_0^j = 2000$, $j \in \{HR,\ MR,\ LR\}$.

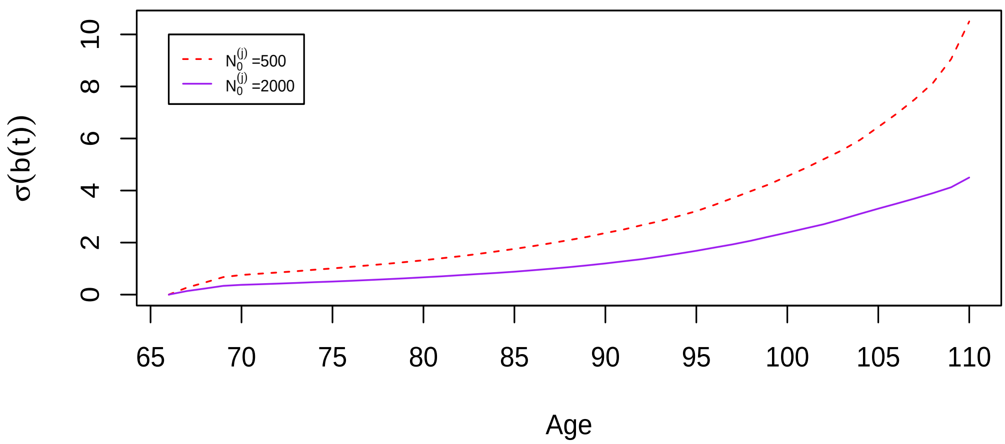

Figure 10: $\sigma(b(t))$ for $N_0^j = 500$ and $N_0^j = 2000$, $j \in \{HR,\ MR,\ LR\}$.

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
Annuity	831.1444	1000	1130.149
GSA 90^{th}	856.8729	1032.0833	1164.8524
GSA 70^{th}	848.4606	1020.7523	1150.7939
GSA 50^{th}	842.7035	1013.073	1141.3652
GSA 30^{th}	837.2744	1005.4926	1132.1717
GSA 10^{th}	829.7536	995.6293	1119.9493

Table 5: Sensitivity analysis (GSA), $EPV(0)$ for $N_0^j = 500$, $j \in \{HR,\ MR,\ LR\}$.

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
Annuity	831.1444	1000	1130.149
GSA 90^{th}	849.7705	1022.4114	1152.8735
GSA 70^{th}	845.5693	1016.7388	1145.8872
GSA 50^{th}	842.7915	1012.9231	1141.1517
GSA 30^{th}	840.0843	1009.2386	1136.5069
GSA 10^{th}	836.188	1004.0471	1130.025

Table 6: Sensitivity analysis (GSA), $EPV(0)$ for $N_0^j = 2000$, $j \in \{HR,\ MR,\ LR\}$.

When the sensitivity analysis is driven by the sample size N_0^j we see that

- for $N_0^j = 500$, $j \in \{HR,\ MR,\ LR\}$ the $EPV(0)$ distribution is expanded for all the sub-populations as it emerges by comparing Table 5 with Table 2;
- for $N_0^j = 2000$, $j \in \{HR,\ MR,\ LR\}$ the $EPV(0)$ distribution is compressed for all the sub-populations as it emerges in the comparison between Table 6 and Table 2;
- the initial benefits b_A and $b(0)_{GSA}$ are unchanged with respect to the base case $N_0^j = 1000$, but the variability of the benefits is larger with a smaller sample size $\left(N_0^j = 500\right)$ than with a larger one $\left(N_0^j = 2000\right)$ as highlighted by Figure 10.

5.2 Sensitivity to sample size: Redistributive GSA

Tables 7 and 8 report for the redistributive GSA scheme the values of $EPV_A(0)$ and the percentiles of $EPV_{RE}^j(0)$ $(j \in \{HR,\ MR,\ LR\})$ for the cases $N_0^j = 500$ and $N_0^j = 2000$, respectively, together with the optimal redistributive shares α_*^j $(j = HR,\ MR,\ LR)$. The three panels of Figures 14 and 15 in Appendix report the benefits $b_{RE}^j(t)$ over time for each sub-population for $N_0^j = 500$ and $N_0^j = 2000$ respectively.

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
α_*^j	1.1645	0.9717	0.8638
Annuity	831.1444	1000	1130.149
GSA 90^{th}	997.8352	1002.8578	1006.2101
GSA 70^{th}	988.0391	991.8476	994.0662
GSA 50^{th}	981.3349	984.3858	985.9216
GSA 30^{th}	975.0127	977.0201	977.9802
GSA 10^{th}	966.2547	967.4361	967.4224

Table 7: Sensitivity analysis (Redistributive GSA), $EPV(0)_{RE}$ for $N_0^j = 500$, $j \in \{HR, MR, LR\}$.

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
α_*^j	1.1642	0.9717	0.8641
Annuity	831.1444	1000	1130.149
GSA 90^{th}	989.32	993.4902	996.1601
GSA 70^{th}	984.4288	987.978	990.1234
GSA 50^{th}	981.1949	984.2703	986.0316
GSA 30^{th}	978.0431	980.69	982.0183
GSA 10^{th}	973.5069	975.6453	976.4174

Table 8: Sensitivity analysis (Redistributive GSA), $EPV(0)_{RE}$ for $N_0^j = 2000$, $j \in \{HR, MR, LR\}$.

The sensitivity analysis in the redistributive scheme driven by the sample size N_0^j shows that

- for $N_0^j = 500$, $j \in \{HR,\ MR,\ LR\}$ leads to an expanded $EPV_{RE}^j(0)$ distribution for all the sub-populations (Table 7 versus Table 4). The benefit level is unchanged (Figures 14 in the Appendix versus Fig. 5 top, medium and bottom panels);

- opposite, for $N_0^j = 2\,000$, $j \in \{HR,\ MR,\ LR\}$ leads to a compressed $EPV_{RE}^j(0)$ distribution for all the sub-populations (Table 8 versus Table 4). The benefit level is unchanged (Figures 15 in the Appendix versus Fig. 5 top, medium and bottom panels).

5.3 Sensitivity to interest rate: GSA scheme

Figures 11 and 12 report the benefit of the GSA scheme for the cases $i = 1\%$ and $i = 10\%$, respectively, while Tables 9 and 10 report the $EPV(0)$ for the cases $i = 1\%$ and $i = 10\%$, respectively.

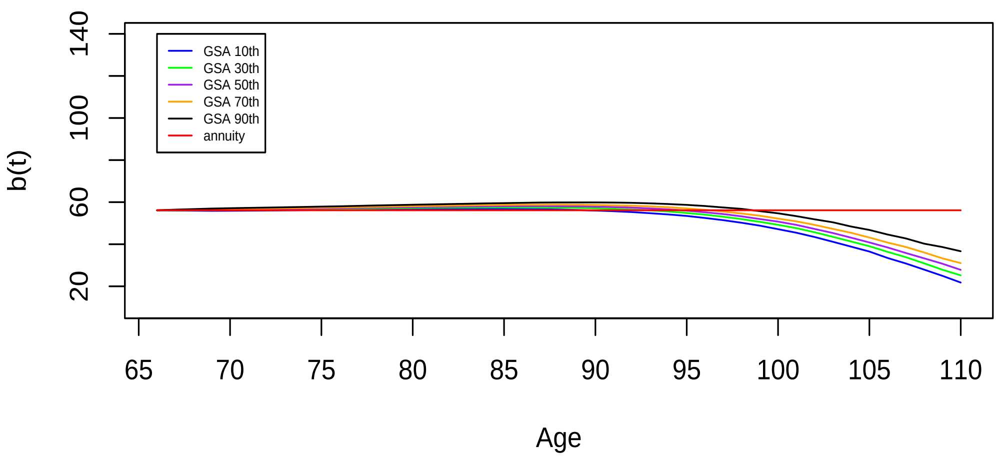

Figure 11: Sensitivity analysis (GSA), $b(t)$ for $i = 1\%$.

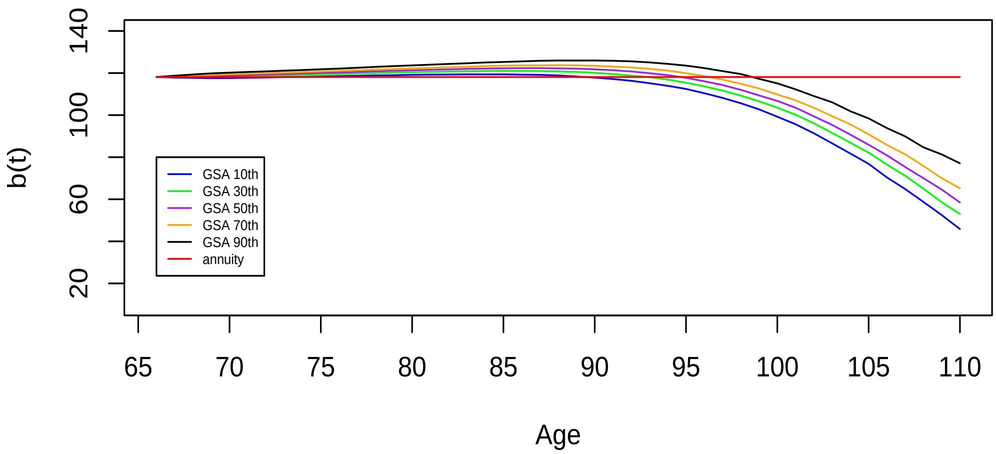

Figure 12: Sensitivity analysis (GSA), $b(t)$ for $i = 10\%$.

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
Annuity	816.1316	1000	1146.1674
GSA 90^{th}	837.9185	1026.7849	1173.9226
GSA 70^{th}	832.0108	1018.4334	1163.0566
GSA 50^{th}	827.9534	1012.7284	1155.7601
GSA 30^{th}	823.948	1007.3549	1149.0237
GSA 10^{th}	818.5084	999.6807	1139.276

Table 9: Sensitivity analysis (GSA), $EPV(0)$ for $i = 1\%$.

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
Annuity	909.112	1000	1058.1765
GSA 90^{th}	926.5199	1020.4112	1080.2816
GSA 70^{th}	921.6044	1014.6675	1073.9317
GSA 50^{th}	918.2522	1010.7838	1069.7018
GSA 30^{th}	914.9722	1006.957	1065.4873
GSA 10^{th}	910.4039	1001.6128	1059.5865

Table 10: Sensitivity analysis (GSA), $EPV(0)$ for $i = 10\%$.

When the sensitivity analysis is driven by changes in the interest rate i we see that

- for $i = 1\%$ the $EPV(0)$ distribution is shifted to the left for the high risk sub-population while it is expanded for the medium risk sub-population and shifted to the right for the low risk one (Table 9 versus Table 2). The intuition is that with a low interest rate the future cash flows become more valuable and this benefits more those who live longer such as the low risk sub-populations and vice versa. The benefit level is reduced (Figure 11 versus Figure 2);

- opposite, for $i = 10\%$ the $EPV(0)$ distribution is shifted to the right for the high risk sub-population, it is compressed for the medium risk sub-population and it is shifted to the left for the low risk one (Table 10 versus Table 2). The benefit level is increased (Figure 12 versus Figure 2).

5.4 Sensitivity to interest rate: Redistributive GSA

Tables 11 and 12 report for the redistributive GSA scheme values of $EPV_A(0)$ and the percentiles of $EPV^j_{RE}(0)$ for the cases $i = 1\%$ and $i = 10\%$, respectively, together with the optimal redistributive shares α^j_*. The three panels of Fig. 16 and 17 in the Appendix report the benefit $b^j_{RE}(t)$ over time for each sub-population j for $i = 1\%$ and $i = 10\%$, respectively.

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
α_*^j	1.1803	0.9689	0.8508
Annuity	816.1316	1000	1146.1674
GSA 90^{th}	988.974	994.8395	998.8171
GSA 70^{th}	982.0014	986.7478	989.572
GSA 50^{th}	977.2124	981.2203	983.3638
GSA 30^{th}	972.485	976.014	977.6323
GSA 10^{th}	966.0647	968.5786	969.3385

Table 11: Sensitivity analysis (Redistributive GSA), $EPV_{RE}(0)$ for $i = 1\%$.

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
α_*^j	1.0839	0.9851	0.931
Annuity	909.112	1000	1058.1765
GSA 90^{th}	1004.2499	1005.1806	1005.7759
GSA 70^{th}	998.9221	999.5227	999.864
GSA 50^{th}	995.2887	995.6969	995.9259
GSA 30^{th}	991.7334	991.9273	992.002
GSA 10^{th}	986.7819	986.6628	986.5081

Table 12: Sensitivity analysis (Redistributive GSA), $EPV_{RE}(0)$ for $i = 10\%$.

The sensitivity analysis driven by changes in the interest rate level in the redistributive GSA scheme shows that

- when $i = 1\%$ there is a shift to the left of the $EPV(0)$ distribution for all the sub-populations (Table 11 versus Table 4). The benefit level is shrunk (Figures 16 versus Fig. 5 top, medium and bottom panels). When the interest rate is reduced α_*^{HR}, the weight given to the high risk population, is higher to compensate the fact that a lower interest rate favours the sub-populations which live longer. Consequently, with a lower interest rate we have more distant values of α_*^j, $j \in \{HR, MR, LR\}$;

- when $i = 10\%$ there is a shift to the right of the $EPV(0)$ distribution for all the sub-populations (Table 12 versus Table 4). The benefit level is increased (Figures 17 versus Fig. 5 top, medium and bottom panels). With a higher interest rate α_*^{HR} is lower because future cash flows are less relevant. Therefore, with a higher interest rate the weights α_*^j, $j \in \{HR, MR, LR\}$, are closer. As an extreme situation, when $i \to \infty$ it does not matter who lives longer because future benefits will count nothing and we will expect $\alpha_*^j = 1$ for every j.

5.5 Sensitivity analysis: discussion

In this section, first, we analyzed sensitivity to the pool size. When the sample size is small, idiosyncratic mortality risk is not well diversified and, fixing any time horizon, the volatility in the distribution of benefits is larger. This happens in both the GSA and the redistributive GSA scheme, because the mortality credits are in both cases shared among a small number of survivors. Consequently, the expected present value of the benefits is more dispersed. Vice versa, a larger sample size leads to a less volatile benefit distribution and to more certainty for the policyholders. As a consequence, the distribution of the expected present values is more compressed. We highlight that the pool size seems to have a small effect on the redistributive share, with a slightly larger redistribution when the sample size is small. We also remark that these results are obtained under the assumption of an equal composition of the pool in terms of its sub-groups.

Second, we analyzed sensitivity to interest rates. Interestingly, changing the interest rate produces different effects on the GSA scheme with or without redistribution. In the traditional GSE scheme when the interest rate is lower the distribution of expected present values shifts to the left for the HR members and to the right for the LR ones. In other words, inequity among different socio-economic classes increases with low interest rates. This is in line with intuition: with a low interest rate the future cash flows become more valuable to the benefit of those who live longer (LR individuals) and to the detriment of those who live shorter (HR individuals). When instead the interest rate is higher, the EPV distribution shifts to the right for HR members and to the left for LR ones because the different life expectancy is less important, so the inequity among socio-economic classes decreases. On the other hand, in the redistributive GSA scheme with a lower interest rate both distributions of EPVs of HR and LR individuals shift to the left and the redistributive shares are more distant from one another. Vice versa, when the interest rate is higher both distributions of EPVs are shifted to the right and the optimal redistributive shares are closer. Thanks to the redistributive mechanism, in the presence of lower interest rates the penalization of HR members is spread among all participants to the pool. This seems to suggest that the redistributive mechanism mitigates the inequity among socio-economic classes especially in times of low interest rates.

6 Conclusions

In this paper we propose a simple way of coping with heterogeneity in closed self-insurance pools. Our Redistributive GSA scheme optimally sets different benefits to each sub-group within the pool by minimizing the distance of the expected present values of policyholders relative to a benchmark, which is the expected present value of a policyholder belonging to a reference group. While heterogeneity in self-insurance schemes has been analyzed by previous works, we contribute by taking a simulative approach and measuring inequity by studying the distributions of EPVs across sub-groups. Up to our knowledge, we are the first who address the problem of reducing inequality among socio-economic classes in a GSA scheme in the presence of stochastic mortality. We study sensitivity of our results to different pool sizes and risk-free interest rates, highlighting in particular that a lower risk-free rate worsens inequity.

Further research is envisaged, with the aim of extending our framework to go beyond the assumption of risk-neutrality and introduce in the valuation of the benefits different risk attitudes for different policyholders. Another interesting avenue of future research is the investigation of how the variabilities of the stochastic mortality intensities affect the inequalities among policyholders of different socio-economic classes. Finally, also the study of how the correlation structure among different mortality intensities impacts the extent of inequalities would be worth exploring. Indeed, the doubly stochastic setup for the mortality intensity makes this possible.

Declarations

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Conflict of interest The authors have no relevant financial or non-financial interests to disclose.
Contribution The authors contributed equally to this work.

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A Appendix

Figure 13 reports the differences between the model implied and the observed survival probabilities for the three subpopulations $j \in \{HR, MR, LR\}$.

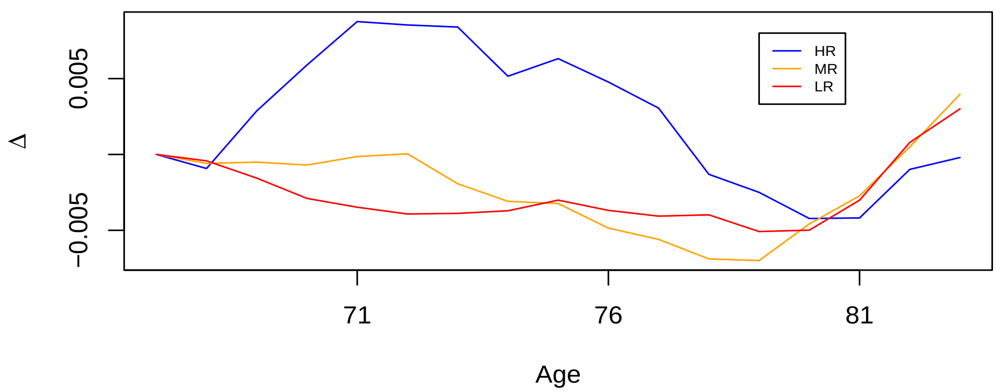

Figure 13: Δ: $S_x^j(t) - {}_t p_{x,\ emp}^j$, for $j \in \{HR,\ MR,\ LR\}$.

Figure 14 reports the benefits of the redistributive scheme with sample size $N_0^j = 500$ for the HR, MR and LR individuals, while Figure 15 reports the benefits of the redistributive scheme with sample size $N_0^j = 2000$ for the HR, MR and LR individuals. Figure 16 reports the benefits of the redistributive scheme with interest rate $i = 1\%$ for the HR, MR and LR individuals, while Figure 17 reports the benefits of the redistributive scheme with interest rate $i = 10\%$ for the HR, MR and LR individuals.

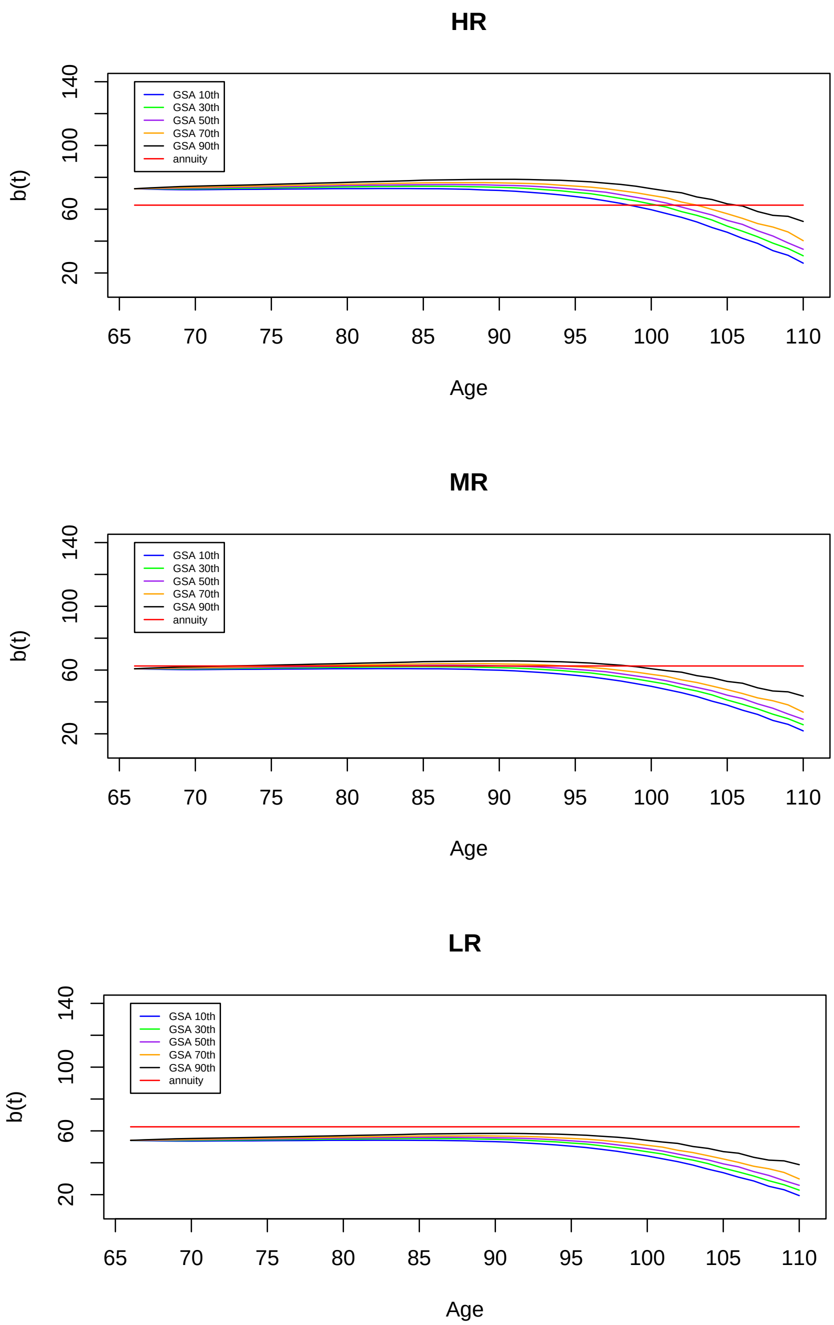

Figure 14: Sensitivity analysis (Redistributive GSA), $b(t)$ for $N_0^j = 500$. Top panel: HR individuals; Medium panel: MR individuals; Bottom panel: LR individuals.

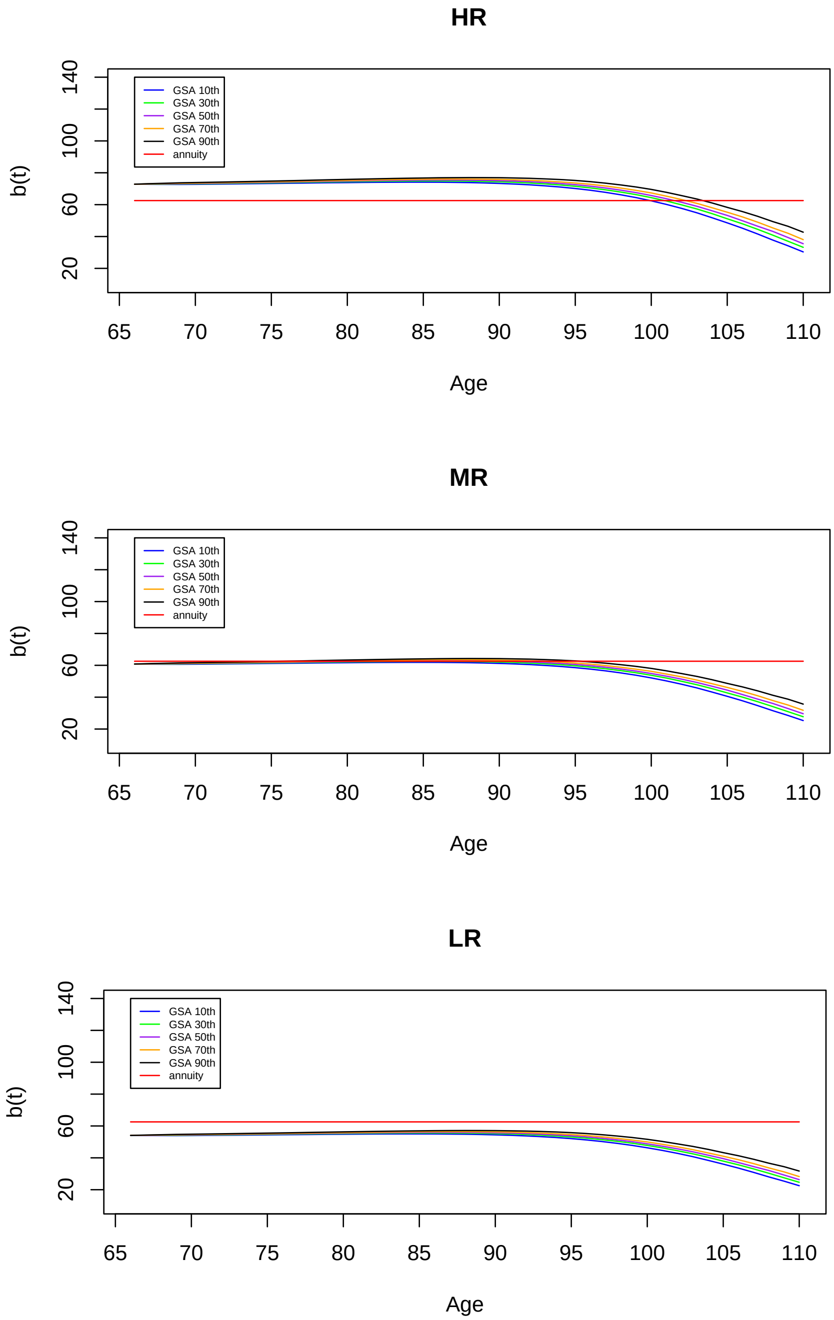

Figure 15: Sensitivity analysis (Redistributive GSA), $b(t)$ for $N_0^j = 2000$. Top panel: HR individuals; Medium panel: MR individuals; Bottom panel: LR individuals.

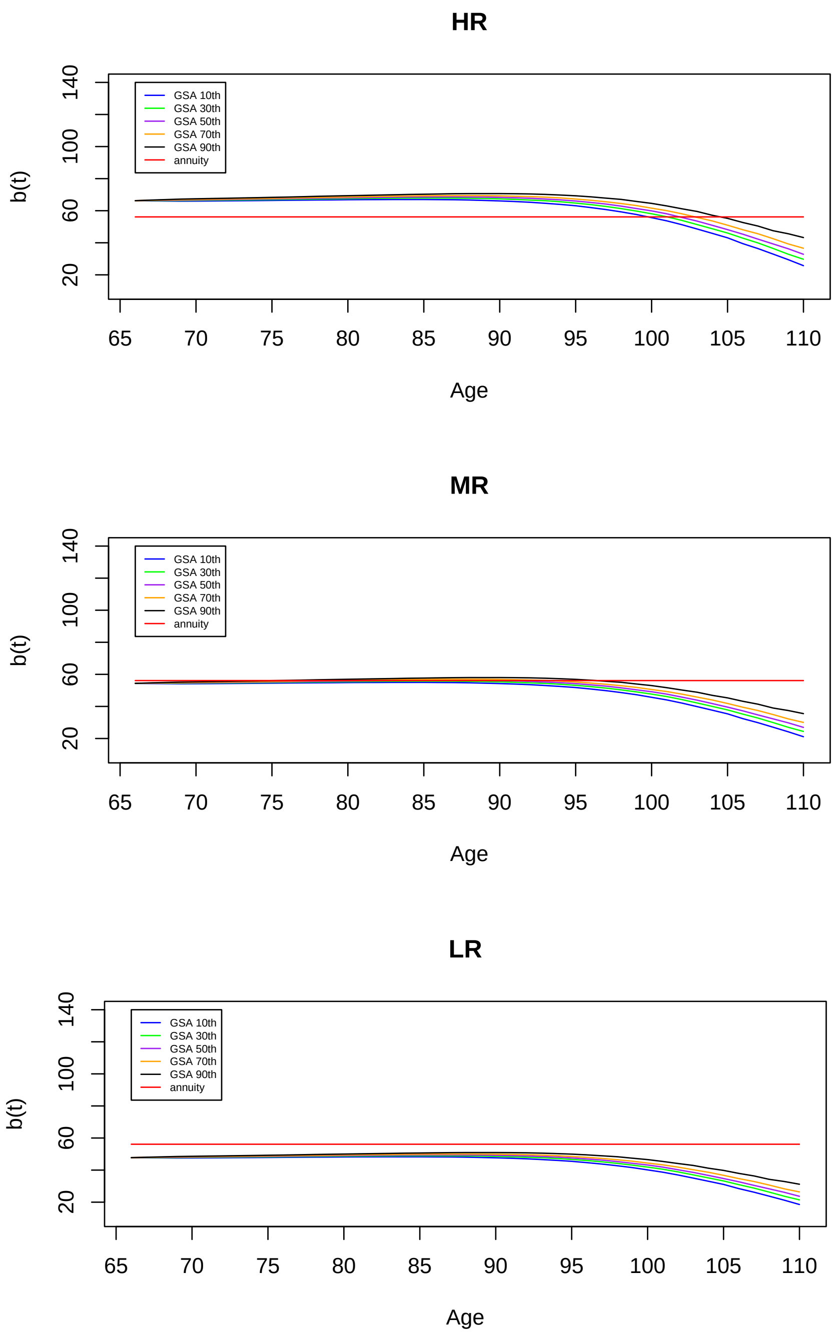

Figure 16: Sensitivity analysis (Redistributive GSA), $b(t)$ for $i = 1\%$. Top panel: HR individuals; Medium panel: MR individuals; Bottom panel: LR individuals.

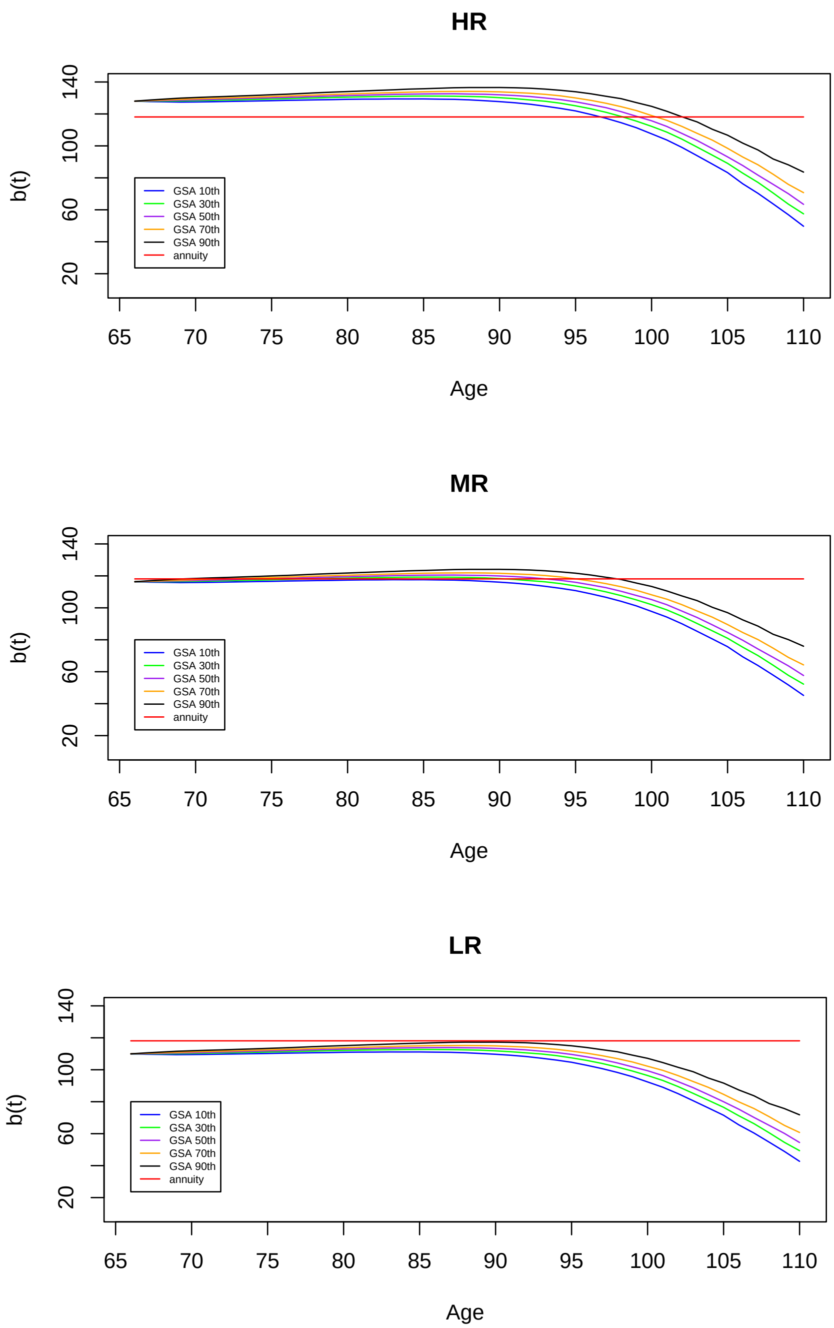

Figure 17: Sensitivity analysis (Redistributive GSA), $b(t)$ for $i = 10\%$. Top panel: HR individuals; Medium panel: MR individuals; Bottom panel: LR individuals.